

[Preliminary Version]

THE F -RATIONAL SIGNATURE AND DROPS IN THE HILBERT-KUNZ MULTIPLICITY

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ABSTRACT. Let (R, \mathfrak{m}) be a Noetherian local ring of prime characteristic p . We define the F -rational signature of R , denoted by $r(R)$, as the infimum of drops in Hilbert-Kunz multiplicities when one enlarges the ideals generated by systems of parameters. In particular, if R is excellent, then R is F -rational if and only if $r(R) > 0$, the proof of which depends on the following result: Given an \mathfrak{m} -primary ideal I in R , there exists a positive $\delta_I \in \mathbb{R}^+$ such that, for any ideal $J \supseteq I$, $e_{HK}(I, R) - e_{HK}(J, R)$ is either 0 or $\geq \delta_I$. Then we study how F -rational signature behaves under deformations, local (and flat) ring extensions, and localizations.

0. INTRODUCTION

Throughout this paper we assume that (R, \mathfrak{m}, k) is a Noetherian local ring of prime characteristic p , where \mathfrak{m} is the maximal ideal and $k = R/\mathfrak{m}$ is the residue field of R . Then there is the Frobenius homomorphism $F : R \rightarrow R$ defined by $r \mapsto r^p$ for any $r \in R$. Therefore, for any $e \in \mathbb{N}$, we have the iterated Frobenius homomorphism $F^e : R \rightarrow R$ defined by $r \mapsto r^q$ for any $r \in R$, where $q = p^e$. From now on, q will be used to denote the value p^e for various $e \in \mathbb{N}$ in the context. Similarly, we also use $Q = p^E, q_0 = p^{e_0}, q' = p^{e'}, q'' = p^{e''}$, etc. to denote varying powers of p .

Let M be an R -module. Then for any $e \geq 0$, we can derive a left R -module structure on the set M by $r \cdot m := r^{p^e} m$ for any $r \in R$ and $m \in M$. For technical reasons, we keep the original right R -module structure on M by default. We denote the derived R - R -bimodule by ${}^e M$. Thus, in ${}^e M$, we have $r \cdot m = m \cdot r^{p^e}$, which is equal to $r^q m$ in the original M . If R is reduced, then ${}^e R$, as a left R -module, is isomorphic to $R^{1/q}$. We use $\lambda^l(-), \lambda^r(-)$ to denote the left and right lengths of a bimodule. It is easy to see that $\lambda^l({}^e M) = q^{\alpha(R)} \lambda^r({}^e M) = q^{\alpha(R)} \lambda(M)$ for any finite length R -module M , in which $\alpha(R) = \log_p[k : k^p]$.

We say R is F -finite if ${}^1 R$ is a finitely generated left R -module. If this is the case, it is easy to see that ${}^e M$ is a finitely generated left R -module for every $e \in \mathbb{N}$ and for every finitely generated R -modules M .

For an ideal I of R , we denote by $I^{[q]}$ the ideal generated by $\{r^q \mid r \in I\}$. Then $R/I \otimes_R {}^e M \cong {}^e(M/I^{[q]}M) \cong {}^e M \otimes_R R/I^{[q]}$ for every R -module M and every $e \in \mathbb{N}$.

In this paper, we define and study the F -rational signature of a local ring R . Firstly, we would like to introduce the readers to the notion of F -signature:

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The notion of F -signature is first introduced and studied in [HL] by C. Huneke and G. Leuschke for F -finite rings.

Definition 0.1. Let (R, \mathfrak{m}, k) be an F -finite local ring and M a finitely generated R -module. For each $e \in \mathbb{N}$, write ${}^eM \cong R^{a_e} \oplus M_e$ as left R -modules such that M_e has no non-zero free direct summand. Denote $d := \dim R$.

- (1) We may denote a_e by $\#({}^eM, R)$ and $\alpha(R) = \log_p[k : k^p] < \infty$.
- (2) We denote $s^+(M) := \limsup_{e \rightarrow \infty} \frac{\#({}^eM, R)}{q^{\alpha(R)+d}}$, $s^-(M) := \liminf_{e \rightarrow \infty} \frac{\#({}^eM, R)}{q^{\alpha(R)+d}}$ and $s(M) := \lim_{e \rightarrow \infty} \frac{\#({}^eM, R)}{q^{\alpha(R)+d}}$ provided the last limit exists.
- (3) If $M = R$, we call $s(R) = \lim_{e \rightarrow \infty} \frac{\#({}^eR, R)}{q^{\alpha(R)+d}}$ the F -signature of R (see [HL]). In case $s(R)$ does not exist, we may call $s^-(R)$ and $s^+(R)$ the lower and upper F -signature of R respectively.

In [Yao], a definition of the F -signature is given for all Noetherian local rings of prime characteristic p and it is equivalent to Definition 0.1 when R is F -finite.

Definition 0.2. Let (R, \mathfrak{m}, k) be a Noetherian local ring of characteristic p and M a finitely generated R -module. Let $E = E_R(k)$ be the injective hull of k and let $\psi : k \rightarrow E$ and $\phi : E \rightarrow E/k$ be such that $0 \rightarrow k \xrightarrow{\psi} E \xrightarrow{\phi} E/k \rightarrow 0$ is exact.

- (1) Denote $\#({}^eM) := \lambda^r(\ker(\phi \otimes_R 1_{{}^eM})) = \lambda^r(\text{Image}(\psi \otimes_R 1_{{}^eM}))$ for all $e \in \mathbb{N}$.
- (2) We define $s^-(M)$ and $s^+(M)$ to be, respectively, the liminf and limsup of the sequence $\left\{ \frac{\#({}^eM)}{q^{d \dim(R)}} \right\}_{e=0}^{\infty}$ as $e \rightarrow \infty$. If $s^-(M) = s^+(M)$, the limit is denoted by $s(M)$.
- (3) In the case of $M = R$, we call $s^-(R)$, $s^+(R)$ and $s(R)$ the lower F -signature, upper F -signature and F -signature of R respectively.

It is easy to see that $s_R^+(R) > 0$ implies $0_E^* = 0$ and hence the strong R -regularity of R . In fact, it is proved in [AL] that $s^+(R) > 0 \iff R$ is strongly F -regular $\iff s^-(R) > 0$ provided R is excellent (e.g., F -finite).

Given a local ring (R, \mathfrak{m}) of prime characteristic p , recall that R is defined to be F -rational if every ideal generated by a system of parameters of R is tightly closed. Hence it is natural to define the ‘ F -rational signature’ of R as follows:

Definition 0.3. Let (R, \mathfrak{m}, k) be a local ring of prime characteristic p and M a finitely generated R -module. Define (here s.o.p. stands for system of parameters)

$$r_R(M) = \inf \{ e_{HK}(\underline{x}, M) - e_{HK}(J, M) \mid \underline{x} \text{ is a s.o.p. and } (\underline{x}) \subsetneq J \}.$$

In particular, $r_R(R)$ is called the F -rational signature of R .

In the above definition, \underline{x} , a priori, runs over all system of parameters of R . It turns out that any given system of parameters will work.

Theorem (See Theorem 2.5 and Theorem 2.4). *Let (R, \mathfrak{m}, k) be a Noetherian local ring of characteristic p with $\dim(R) = d$ and M a finitely generated R -module. Then*

- (1) *Assume R is an equidimensional homomorphic image of a Cohen-Macaulay ring or \widehat{R} is equidimensional (e.g., R is excellent and equidimensional). Then*

for any given (fixed) system of parameters \underline{x} , we always have

$$r_R(M) = \inf\{e_{HK}((\underline{x}), M) - e_{HK}(J, M) \mid (\underline{x}) \subsetneq J\}.$$

(2) If R and M are both maximal Cohen-Macaulay, then

$$r_R(M) = \inf \left\{ \lim_{e \rightarrow \infty} \frac{\lambda^r(\text{Image}(\psi \otimes_R 1_{eM}))}{q^d} \mid 0 \rightarrow k \xrightarrow{\psi} H_{\mathfrak{m}}^d(R) \text{ is exact} \right\}.$$

The following result describes how Hilbert-Kunz multiplicity $e_{HK}(I, M)$ drops when the ideal I gets enlarged. This result will be used to prove Theorem 4.1.

Theorem (Theorem 3.1). *Let (R, \mathfrak{m}, k) be a Noetherian local ring of characteristic p and let $I \subset R$ be an \mathfrak{m} -primary ideal of R . Then there exists $0 < \delta \in \mathbb{R}$ such that, for any ideal $J \supseteq I$ and for any finitely generated R -module M , $e_{HK}(I, M) - e_{HK}(J, M)$ is either 0 or $\geq \delta$.*

It is easy to see from definition that $r_R(R) > 0$ implies R is F -rational. In fact, more can be said about the F -rational signature. In particular, part (1) of the following result should justify our choice of the term ‘ F -rational signature’ for $r_R(R)$.

Theorem (Theorem 4.1, Lemma 4.2 and Lemma 4.3). *Let (R, \mathfrak{m}, k) be a Noetherian local ring of characteristic p . Then*

- (1) *Suppose R is excellent or there exists a common parameter (weak) test element for R and \widehat{R} . Then R is F -rational if and only if $r_R(R) > 0$;*
- (2) *$s^+(R) \leq r_R(R) \leq e_{HK}(R)$. If $|k| = \infty$ and R is not regular, then $s^+(R) \leq r_R(R) \leq \min\{e_{HK}(R), e(R) - e_{HK}(R)\}$;*
- (3) *If R is Gorenstein, then $r_R(R) = s(R)$. If R is regular, then $r_R(R) = 1$.*

We will also study how the F -rational signature behaves under deformation and local flat extension.

Theorem (Theorem 5.1). *Let (R, \mathfrak{m}, k) be a Noetherian local ring of characteristic p and $\underline{x} = x_1, x_2, \dots, x_h$ an R -regular sequence. Denote $\bar{R} = R/(\underline{x})R$. Then $r_R(R) \geq r_{\bar{R}}(\bar{R})$.*

Theorem (Theorem 5.6 and Theorem 5.7). *Let $(R, \mathfrak{m}, k) \rightarrow (S, \mathfrak{n}, l)$ be a local flat ring homomorphism of Noetherian local rings of prime characteristic p . Denote $\bar{S} := S/\mathfrak{m}S$.*

- (1) *We have $r_S(S) \leq r_R(R) t(\bar{S})$. If \bar{S} is Gorenstein, then $r_S(S) \leq r_R(R)$;*
- (2) *If R is Gorenstein, then $r_S(S) \geq r_R(R) r_{\bar{S}}(\bar{S})$;*
- (3) *If \bar{S} is Gorenstein and the induced map $R/\mathfrak{m} \rightarrow S/\mathfrak{n}$ is an isomorphism, then $r_S(S) \geq r_R(R) r_{\bar{S}}(\bar{S})$;*
- (4) *If \bar{S} is regular and the induced map $R/\mathfrak{m} \rightarrow S/\mathfrak{n}$ is an isomorphism, then $r_S(S) = r_R(R)$.*

Theorem (Theorem 5.9). *Let (R, \mathfrak{m}, k) be a local Noetherian ring of characteristic p , $P, Q \in \text{Spec}(R)$ prime ideals such that $P \subsetneq Q$ and M a finitely generated R -module. Then $r(M_Q) \leq r(M_P) \alpha(P, Q)$, in which $\alpha(P, Q) := \min\{e(\underline{x}, (R/P)_Q)\}$ with \underline{x} running over all systems of parameters of $(R/P)_Q$. If $|k| = \infty$ or $Q \subsetneq \mathfrak{m}$, then $r(M_Q) \leq r(M_P) e((R/P)_Q)$ where $e((R/P)_Q) = e((Q/P)_Q, (R/P)_Q)$ denotes the Hilbert multiplicity of $(R/P)_Q$ considered as a local ring.*

1. REVIEW AND PRELIMINARY RESULTS

This section is allocated for reviewing background and prerequisite results. Most of the displayed results will be quoted in the coming sections.

A very important concept in studying rings of characteristic p is tight closure. Tight closure was first studied and developed by Hochster and Huneke in the 1980's.

Definition 1.1 ([HH1]). Let R be a Noetherian ring of characteristic p and $N \subseteq M$ be R -modules. The tight closure of N in M , denoted by N_M^* , is defined as follow: An element $x \in M$ is said to be in N_M^* if there exists an element $c \in R^\circ$ such that $x \otimes c \in N_M^{[q]} \subseteq M \otimes_R {}^eR$ for all $e \gg 0$, where R° is the complement of the union of all minimal primes of the ring R and $N_M^{[q]}$ denotes the right R -submodule of $M \otimes_R {}^eR$ generated by $\{x \otimes 1 \in M \otimes_R {}^eR \mid x \in N\}$. The element $x \otimes 1 \in M \otimes_R {}^eR$ is denoted by $x_M^{p^e} = x_M^q$ and $M \otimes_R {}^eR$ is denoted by $F^e(M)$.

Given R -modules $N \subseteq M$, N_M^*/N , as an R -submodule of M/N , is exactly $0_{M/N}^*$ and we say N is tightly closed if $N_M^* = N$.

If every ideal of R generated by parameters (i.e., every ideal I such that $\text{height}(I)$ equals to its minimal number of generators) is tightly closed, we say R is F -rational. (We treat 0 as an ideal generated by parameters by convention, which makes sure that an F -rational ring is reduced.) An F -rational ring is always normal. In case (R, \mathfrak{m}) is an equidimensional homomorphic image of a Cohen-Macaulay local ring, it turns out that R is F -rational as long as one ideal generated by a system of parameters is tightly closed. Finally, if (R, \mathfrak{m}) is an F -rational homomorphic image of a Cohen-Macaulay local ring, then (R, \mathfrak{m}) itself is a Cohen-Macaulay normal domain.

The notion of *test element* is very important in studying tight closure.

Definition 1.2 ([HH1], [HH2]). Let R be a Noetherian ring of characteristic p . We say an element $c \in R^\circ$ is a test element if, for every finitely generated R -module M and any $x \in 0_M^*$, we have $0 = x \otimes c \in M \otimes_R {}^eR$ for all $e \geq 0$. We say that a test element $c \in R^\circ$ is a locally (completely) stable if c is a test element for (the completion of) every local ring of R . Also, we say $c \in R^\circ$ is a parameter test element if $cx^q \in I^{[q]}$ for any ideal generated by a system of parameters, all $x \in I^*$, and all $q = p^e$.

Theorem 1.3 (Existence of test elements, [HH2]). *Let R be a reduced algebra of finite type over an excellent local ring (B, \mathfrak{m}) of characteristic p . Let $c \in R^\circ$ be such that $R[c^{-1}]$ is F -regular and Gorenstein (e.g., $R[c^{-1}]$ is regular). Then c has a power which is a completely stable test element for R .*

In particular, if (R, \mathfrak{m}) is a reduced excellent local ring of characteristic p , then every element in the defining ideal of the singular locus of R has a power which is a completely stable test element for R . By [Ku], an F -finite ring is excellent. Therefore completely stable test elements always exist for reduced F -finite local rings of characteristic p .

One of the major open questions in tight closure theory is whether tight closure commutes with localization, i.e., whether $(N_M^*)_P = (N_P^*)_{M_P}$ for every $P \in \text{Spec}(R)$ and every finitely generated R -module $N \subseteq M$. One reason why the question is hard is that the definition of tight closure involves infinitely many equations (i.e., you need

$0 = x \otimes c \in M \otimes_R {}^eR$ for all $q \gg 0$ to make sure that an element $x \in M$ is in 0_M^*). There is a notion of *test exponent*, which has been introduced in [HH3] to avoid the infinitely many equations in the definition of tight closure.

Definition 1.4 ([HH3]). Let R be a (reduced) Noetherian ring of prime characteristic p , $c \in R^\circ$ (a test element), and $N \subseteq M$ R -modules. We say that $q_0 = p^{e_0}$ is a test exponent for c and $N \subseteq M$ if, for any $x \in M$, we have $x \in N_M^*$ whenever $x \otimes c \in N_M^{[q]} \subseteq F^e(M)$ for one single $q \geq q_0$.

If there exists a test exponent for a locally stable test element $c \in R^\circ$ and finitely generated R -modules $N \subseteq M$, then the tight closure of N in M commutes with localization. This result is implicit in [McD] and is explicitly stated in [HH3, Proposition 2.3]. Moreover, Hochster and Huneke showed in [HH3] that the converse is true.

Theorem 1.5 ([HH3]). *Let R be a (reduced) Noetherian ring of prime characteristic p with a given locally stable test element c , and $N \subseteq M$ finitely generated R -modules. Assume that the tight closure of N in M commutes with localization. Then there exists a test exponent for c and $N \subseteq M$.*

In particular, if $\lambda(M/N) < \infty$, then there exists a test exponent for c and $N \subseteq M$ since tight closure commutes with localizations in this case (cf. [HH1, Proposition 8.9]).

Next we review the Hilbert-Kunz multiplicity.

Theorem 1.6. *Let (R, \mathfrak{m}, k) be a Noetherian local ring of prime characteristic p with $\dim(R) = d$ and M a finitely generated R module. Then (with $q = p^e$)*

- (1) *For any R -module L with $\lambda_R(L) < \infty$, the limit*

$$\lim_{e \rightarrow \infty} \frac{\lambda^r(L \otimes_R {}^eM)}{q^d}$$

exists by [Se]. (The statement in [Se, Page 278, Theorem] is more general whose proof requires F -finiteness. However, the result quoted here does not need F -finiteness as it always reduces to the F -finite case.)

- (2) *In particular, if $L = R/I$ with I being any \mathfrak{m} -primary ideal, the limit*

$$\lim_{e \rightarrow \infty} \frac{\lambda_R^r(R/I \otimes_R {}^eM)}{q^d} = \lim_{e \rightarrow \infty} \frac{\lambda_R(M/I^{[q]}M)}{q^d}$$

exists. This particular case was first proved in [Mo].

Notation 1.7. Let (R, \mathfrak{m}) be a Noetherian local ring of prime characteristic p with $\dim(R) = d$, L and M finitely generated R -modules with $\lambda_R(L) < \infty$.

- (1) We denote $e_{HK}(L, M) := \lim_{e \rightarrow \infty} \frac{\lambda_R^r(L \otimes_R {}^eM)}{q^d}$, which is positive if and only if $\dim(M) = d$.
- (2) In case $L = R/I$ with I an \mathfrak{m} -primary ideal, we usually write $e_{HK}(L, M)$ as $e_{HK}(I, M)$, which is called the Hilbert-Kunz multiplicity of M with respect to I . In particular, $e_{HK}(I, M) > 0$ if and only if $\dim(M) = d$.

The following result is referred to as ‘length criterion for tight closure’ in [HH1]. Actually in [HH1, Theorem 8.17], more general results are proved.

Theorem 1.8 ([HH1, Theorem 8.17]). *Let (R, \mathfrak{m}) be a local Noetherian ring, M and $K \subseteq L$ R -modules such that $\dim(M) = \dim(R)$ and $\lambda(L) < \infty$, and $I \subseteq J$ \mathfrak{m} -primary ideals of R .*

- (1) *If $K \subseteq 0_L^*$, then $e_{HK}(L, M) = e_{HK}(L/K, M)$. In particular, if $J \subseteq I^*$, then $e_{HK}(I, M) = e_{HK}(J, M)$.*
- (2) *Conversely, if R is an analytically unramified, quasi-unmixed ring with a completely stable test element (e.g., (R, \mathfrak{m}) is a complete domain) and $e_{HK}(L, R) = e_{HK}(L/K, R)$, then $K \subseteq 0_L^*$. In particular, $e_{HK}(I, R) = e_{HK}(J, R)$ implies $J \subseteq I^*$.*

Next, we quote some results about F -signature.

Theorem 1.9 ([HL], [AL]). *Let (R, \mathfrak{m}, k) be a Noetherian local ring of prime characteristic p . Then the following are true:*

- (1) *If $s^+(R) > 0$, then R is an F -regular, Cohen-Macaulay domain. See [HL].*
- (2) *Actually, if R is excellent (e.g., F -finite), it is proved that $s^+(R) > 0 \iff R$ is strongly F -regular $\iff s^-(R) > 0$ in [AL]. Here we mention that, although the argument in [AL] addresses the F -finite cases only, it can be generalized to all cases where (R, \mathfrak{m}) is excellent. Moreover, we are going to sketch an alternate proof of this claim in Remark 3.6 assuming (R, \mathfrak{m}) is excellent only.*
- (3) *$e_{HK}(I, R) - e_{HK}(J, R) \geq \lambda_R(J/I)s^+(R)$ for any two \mathfrak{m} -primary ideals $I \subseteq J$ of R (see [HL]). Therefore*

$$s^+(R) \leq \inf\{e_{HK}(I_1) - e_{HK}(I_2) \mid I_1 \subset I_2, \sqrt{I_1} = \mathfrak{m}, I_2/I_1 \cong k\}.$$

- (4) *Also, the inequality $(e(R) - 1)(1 - s^+(R)) \geq e_{HK}(R) - 1$ is proved in [HL]. Hence $s^+(R) \geq 1 \implies R$ is regular $\implies s(R) = 1$.*
- (5) *Also in [HL], it is shown that if R is Gorenstein then $s(R) = e_{HK}((\underline{x}), R) - e_{HK}((\underline{x}, u), R)$ for any system of parameters \underline{x} and $u \in ((\underline{x}) :_R \mathfrak{m}) \setminus (\underline{x})$.*

Remark 1.10. The value $\inf\{e_{HK}(I_1) - e_{HK}(I_2) \mid I_1 \subset I_2, \sqrt{I_1} = \mathfrak{m}, I_2/I_1 \cong k\}$ is closely related to the *minimal relative Hilbert-Kunz multiplicity for cyclic modules* of R that is defined in [WY] by K.-i. Watanabe and K. Yoshida.

The next result is used in Section 5. The exact statement of the following theorem can be found in [HH2, Theorem 7.10], which refers the readers to a more general result in [Mat1, 20.F].

Theorem 1.11. *Let $(R, \mathfrak{m}, k) \rightarrow (S, \mathfrak{n}, l)$ be a local flat ring homomorphism. If x_1, x_2, \dots, x_t form a regular sequence on $S/\mathfrak{m}S$, then they form a regular sequence on S and $R \rightarrow S/(x_1, x_2, \dots, x_t)S$ is again a (faithfully) flat local homomorphism.*

2. DEFINING THE F -RATIONAL SIGNATURE

Let us recall the definition (as well as some equivalent definitions) of F -rational signature.

Definition 2.1. Let (R, \mathfrak{m}, k) be a local ring of prime characteristic p and M a finitely generated R -module. Define (here s.o.p. stands for system of parameters)

$$\begin{aligned} r_R(M) &= \inf\{e_{HK}(\underline{x}, M) - e_{HK}(\underline{x}, u), M \mid \underline{x} \text{ is a s.o.p. and } ((\underline{x}) :_R u) = \mathfrak{m}\} \\ &= \inf\{e_{HK}(\underline{x}, M) - e_{HK}(\underline{x}, v), M \mid \underline{x} \text{ is a s.o.p. and } v \notin (\underline{x})\} \\ &= \inf\{e_{HK}(\underline{x}, M) - e_{HK}(J, M) \mid \underline{x} \text{ is a s.o.p. and } (\underline{x}) \subsetneq J\}, \end{aligned}$$

in which \underline{x} , a priori, runs over all (full) systems of parameters of R . If no confusion arises, we may simply write $r_R(M)$ as $r(M)$. In particular, $r(R)$ is called the F -rational signature of R .

Observation 2.2. Let (R, \mathfrak{m}, k) and M be as in Definition 2.1. Denote the \mathfrak{m} -adic completion of R and M by \widehat{R} and \widehat{M} respectively. Then we observe:

- (1) As the \mathfrak{m} -primary ideals (generated by systems of parameters) of R naturally correspond to the $\mathfrak{m}\widehat{R}$ -primary ideals (generated by systems of parameters) of \widehat{R} while Hilbert-Kunz multiplicities are unchanged when passing from R to \widehat{R} , we have $r_R(M) = r_{\widehat{R}}(\widehat{M})$. (See Theorem 5.6(4) for a more general statement.)
- (2) If $\dim(M) < \dim(R)$, then $r(M) = 0$ because $e_{HK}(I, M) = 0$ for any \mathfrak{m} -primary ideal I .
- (3) Suppose R is a domain (for example, R is F -rational) and say the torsion-free rank of M is n (i.e., $M_P \cong R_P^n$ when $P = 0 \in \text{Spec}(R)$). Then, by the additivity of Hilbert-Kunz multiplicity with respect to short exact sequences, $e_{HK}(J, M) = ne_{HK}(J, R)$ for any \mathfrak{m} -primary ideal J of R . This implies $r(M) = nr(R)$ from the definitions of $r(M)$ and $r(R)$. Obviously, $n > 0$ if and only if $\dim(M) = \dim(R)$.
- (4) If \widehat{R} is not F -rational (in particular, if R is not F -rational), then there exists a system of parameters \underline{x} of R (hence of \widehat{R}) and $u \in (((\underline{x})\widehat{R})^* \cap R) \setminus (\underline{x})R$, which gives $e_{HK}(\underline{x}, M) - e_{HK}(\underline{x}, u), M = 0$ implying $r(M) = 0$ for any finitely generated R -module M .
- (5) If R is not a normal Cohen-Macaulay domain, then $r(M) = 0$ for any M . Indeed, if R is not a normal domain, then R can not be F -rational. If R is not Cohen-Macaulay, then \widehat{R} is not Cohen-Macaulay, hence can not be F -rational. So $r(M) = 0$ for any finitely generated R -module M .

The following result relates $e_{HK}(I, M) - e_{HK}(J, M)$, in which $I \subseteq J$ and I is generated by a system of parameters of a Cohen-Macaulay local ring R and M is a maximal Cohen-Macaulay R -module, to the limit of a sequence determined intrinsically by the image of the natural and injective map $J/I \rightarrow R/I \rightarrow \varinjlim_{\underline{x} \text{ s.o.p.}} R/(\underline{x}) \cong H_{\mathfrak{m}}^{\dim(R)}(R)$.

Proposition 2.3. *Let (R, \mathfrak{m}, k) be a Noetherian local Cohen-Macaulay ring of prime characteristic p with $\dim R = d$ and M be a finitely generated maximal Cohen-Macaulay module. Denote $H = H_{\mathfrak{m}}^d(R)$. Given an R -module L and an R -linear map $\psi : L \rightarrow H$ such that $\lambda_R(\psi(L)) < \infty$. There is an induced bimodule homomorphism $\psi \otimes_R 1_{eM} : L \otimes_R {}^eM \rightarrow H \otimes_R {}^eM$ for every $e \in \mathbb{N}$. Then*

$$\lim_{e \rightarrow \infty} \frac{\lambda^r(\text{Image}(\psi \otimes_R 1_{eM}))}{q^d} = e_{HK}(I, M) - e_{HK}(J, M)$$

for any ideals $I \subseteq J$ such that $I = (\underline{x})$ is generated by a system of parameters and $J/I \cong \psi(L)$ and, moreover, such ideals $I \subseteq J$ always exist.

Conversely, for any \mathfrak{m} -primary ideals $I \subseteq J$ such that I is generated by a system of parameters, there exists an (inclusion) map $\psi : L \rightarrow H$ such that $\psi(L) \cong J/I$ and such that $e_{HK}(I, M) - e_{HK}(J, M) = \lim_{e \rightarrow \infty} \frac{\lambda^r(\text{Image}(\psi \otimes_R 1_{eM}))}{q^d}$.

Proof. To prove the claim in the first paragraph, we may simply replace L by $\psi(L)$ and hence assume $L \subseteq H$, $\lambda_R(L) < \infty$, and ψ is the inclusion map. There exists a system of parameters $\underline{x} := x_1, x_2, \dots, x_d$ of R such that $R/(\underline{x}) \cong K \subseteq H$ and $L \subseteq K$. Say L corresponds to $(\underline{x}, \underline{u})/(\underline{x}) \subseteq R/(\underline{x})$ under the isomorphism $R/(\underline{x}) \cong K$, in which $\underline{u} = u_1, u_2, \dots, u_r$.

Let $\phi : K \rightarrow H$ be the inclusion map. Then there is an induced bimodule homomorphisms $\phi \otimes_R 1_{eM} : K \otimes_R {}^eM \rightarrow H \otimes_R {}^eM$ for every $e \in \mathbb{N}$. As M is Cohen-Macaulay, we see that, for every $e \in \mathbb{N}$, eM is a big (i.e., not necessarily finitely generated) Cohen-Macaulay left R -module. From this it is straightforward to see that the maps $\phi \otimes_R 1_{eM} : K \otimes_R {}^eM \rightarrow H \otimes_R {}^eM$ are injective for all $e \in \mathbb{N}$.

Hence, by our set-up, (and for all $e \in \mathbb{N}$)

$$\begin{aligned} \lambda^r(\text{Image}(\psi \otimes_R 1_{eM})) &= \lambda^r(\text{Image}(L \otimes_R 1_{eM} \rightarrow K \otimes_R 1_{eM})) \\ &= \lambda^r \left(\text{Image} \left(\frac{(\underline{x}, \underline{u})}{(\underline{x})} \otimes 1_{eM} \rightarrow \frac{R}{(\underline{x})} \otimes 1_{eM} \right) \right) \\ &= \lambda^r \left(\frac{R}{(\underline{x})} \otimes_R {}^eM \right) - \lambda^r \left(\frac{R}{(\underline{x}, \underline{u})} \otimes_R {}^eM \right) \\ &= \lambda \left(\frac{M}{(\underline{x})^{[q]}M} \right) - \lambda \left(\frac{M}{(\underline{x}, \underline{u})^{[q]}M} \right), \end{aligned}$$

which gives the existence of

$$\lim_{e \rightarrow \infty} \frac{\lambda^r(\text{Image}(\psi \otimes_R 1_{eM}))}{q^d} = \lim_{e \rightarrow \infty} \frac{\lambda \left(\frac{M}{(\underline{x})^{[q]}M} \right) - \lambda \left(\frac{M}{(\underline{x}, \underline{u})^{[q]}M} \right)}{q^d},$$

which is $e_{HK}((\underline{x}), M) - e_{HK}((\underline{x}, \underline{u}), M)$.

The other direction follows immediately by embedding $J/I \subseteq R/I$ to $H_{\mathfrak{m}}^d(R)$ just as in the above proof. \square

Concerning F -rational signature, Proposition 2.3 shows the following:

Theorem 2.4. *Let (R, \mathfrak{m}, k) be a Noetherian Cohen-Macaulay local ring of characteristic p with $\dim(R) = d$ and M a finitely generated maximal Cohen-Macaulay R -module. Then*

- (1) $r_R(M) = \inf \left\{ \lim_{e \rightarrow \infty} \frac{\lambda^r(\text{Image}(\psi \otimes_R 1_{eM}))}{q^d} \mid 0 \rightarrow k \xrightarrow{\psi} H_{\mathfrak{m}}^d(R) \text{ is exact} \right\}$;
- (2) For any given (fixed) system of parameters \underline{x} , we always have

$$r_R(M) = \inf \{ e_{HK}((\underline{x}), M) - e_{HK}((\underline{x}, \underline{u}), M) \mid ((\underline{x}) :_R \underline{u}) = \mathfrak{m} \}.$$

Proof. (1). Apply Proposition 2.3 to all injective maps $\psi : k \rightarrow H_{\mathfrak{m}}^d(R)$.

(2). For every exact sequence $0 \rightarrow k \xrightarrow{\psi} H_{\mathfrak{m}}^d(R)$, $L = \text{Image}(\psi)$ is contained in $(0 :_{H_{\mathfrak{m}}^d(R)} \mathfrak{m})$, the socle of $H_{\mathfrak{m}}^d(R)$. Say $R/(\underline{x}) \cong K \subseteq H_{\mathfrak{m}}^d(R)$. Then, since R is Cohen-Macaulay, $L \subseteq (0 :_{H_{\mathfrak{m}}^d(R)} \mathfrak{m}) = (0 :_K \mathfrak{m})$, i.e., ψ factors through K . Hence $\lim_{e \rightarrow \infty} \frac{\lambda^r(\text{Image}(\psi \otimes_R 1_{eM}))}{q^d} = e_{HK}((\underline{x}), M) - e_{HK}((\underline{x}, u), M)$ for some $u \in ((\underline{x}) :_R \mathfrak{m}) \setminus (\underline{x})$ by Proposition 2.3. Now (2) follows from (1). \square

It turns out that Theorem 2.4(2) remains true without the Cohen-Macaulay assumption on R or M .

Theorem 2.5. *Suppose (R, \mathfrak{m}, k) is a Noetherian local ring of characteristic p and M is a finitely generated R -module. Say $\dim(R) = d$. Assume R is an equidimensional homomorphic image of a Cohen-Macaulay ring or \widehat{R} is equidimensional (e.g., R is excellent and equidimensional). Then for any given (fixed) system of parameters $\underline{x} = x_1, x_2, \dots, x_d$, we always have*

$$\begin{aligned} r_R(M) &= \inf\{e_{HK}((\underline{x}), M) - e_{HK}((\underline{x}, u), M) \mid ((\underline{x}) :_R u) = \mathfrak{m}\} \\ &= \inf\{e_{HK}((\underline{x}), M) - e_{HK}((\underline{x}, v), M) \mid v \notin (\underline{x})\} \\ &= \inf\{e_{HK}((\underline{x}), M) - e_{HK}(J, M) \mid (\underline{x}) \subsetneq J\}, \end{aligned}$$

Proof. It suffices to show that, for any system of parameters $\underline{y} = y_1, y_2, \dots, y_d$, we always have

$$\begin{aligned} &\inf\{e_{HK}((\underline{y}), M) - e_{HK}((\underline{y}, v), M) \mid ((\underline{y}) :_R v) = \mathfrak{m}\} \\ &= \inf\{e_{HK}((\underline{x}), M) - e_{HK}((\underline{x}, u), M) \mid ((\underline{x}) :_R u) = \mathfrak{m}\}, \end{aligned}$$

which is unaffected if we pass to \widehat{R} and \widehat{M} , the \mathfrak{m} -adic completion of R and M . Therefore, we may assume R is an equidimensional homomorphic image of a Cohen-Macaulay ring without loss of generality.

If $(\underline{x}) \subsetneq (\underline{x})^*$, then there exists $u \in (\underline{x})^* \cap ((\underline{x}) :_R \mathfrak{m}) \setminus (\underline{x})$ giving $e_{HK}((\underline{x}), M) - e_{HK}((\underline{x}, u), M) = 0$. Thus $r(M) = 0$.

On the other hand, if $(\underline{x}) = (\underline{x})^*$, then R is F -rational and hence a Cohen-Macaulay normal domain. Say M has torsion-free rank n over R . Then $r(M) = nr(R)$ (cf. Observation 2.2(3)). Now apply Theorem 2.4(2) and we get $r(R) = \inf\{e_{HK}((\underline{x}), R) - e_{HK}((\underline{x}, u), R) \mid u \in ((\underline{x}) :_R \mathfrak{m}) \setminus (\underline{x})\}$. Consequently,

$$\begin{aligned} r(M) &= nr(R) = \inf\{ne_{HK}((\underline{x}), R) - ne_{HK}((\underline{x}, u), R) \mid u \in ((\underline{x}) :_R \mathfrak{m}) \setminus (\underline{x})\} \\ &= \inf\{e_{HK}((\underline{x}), M) - e_{HK}((\underline{x}, u), M) \mid ((\underline{x}) :_R u) = \mathfrak{m}\} \end{aligned}$$

and the proof is complete. \square

3. DROPS IN THE HILBERT-KUNZ MULTIPLICITY

From the definition of the Hilbert-Kunz multiplicity, one observes that $e_{HK}(I, M) \geq e_{HK}(J, M)$ for any \mathfrak{m} -primary ideals $I \subseteq J$. The following result shows how the Hilbert-Kunz multiplicity decreases when the ideal increases from I to J .

Theorem 3.1. *Let (R, \mathfrak{m}, k) be a Noetherian local ring of characteristic p and let $N \subseteq L$ be R -modules such that $\lambda(L/N) < \infty$. Then*

- (1) *There exists $0 < \delta \in \mathbb{R}$ such that, for any R -submodule K with $N \subseteq K \subseteq L$ and for any finitely generated R -module M , $e_{HK}(L/N, M) - e_{HK}(L/K, M)$ is either 0 or $\geq \delta$.*
- (2) *In case $L = R$ and $N = I$ is an \mathfrak{m} -primary ideal of R , there exists $0 < \delta \in \mathbb{R}$ such that, for any ideal $J \supseteq I$ and for any finitely generated R -module M , $e_{HK}(I, M) - e_{HK}(J, M)$ is either 0 or $\geq \delta$.*

Proof. Clearly, it is enough to prove part (1). First, we may assume R is complete without loss of generality. Say $\min(R) = \{P_1, P_2, \dots, P_n\}$ and say $\dim(R/P_i) = \dim(R)$ exactly when $1 \leq i \leq s$ for some $s \leq n$. Since Hilbert-Kunz multiplicity is additive with respect to short exact sequences, we have, for any R -submodule $K \subseteq L$ and for any finitely generated R -module M ,

$$\begin{aligned} e_{HK}(L/N, M) - e_{HK}(L/K, M) &= \sum_{i=1}^s \lambda_{R_{P_i}}(M_{P_i}) (e_{HK}(L/N, R/P_i) - e_{HK}(L/K, R/P_i)) \\ &= \sum_{i=1}^s \lambda_{R_{P_i}}(M_{P_i}) (e_{HK}(L/(N + P_i L), R/P_i) - e_{HK}(L/(K + P_i L), R/P_i)). \end{aligned}$$

Therefore it suffices to prove the desired result under the assumption that R is a complete local domain and $M = R$.

Now the proof follows immediately from the following Theorem 3.2. \square

Theorem 3.2. *Let (R, \mathfrak{m}, k) be a Noetherian local ring of characteristic p such that its \mathfrak{m} -adic completion \widehat{R} is a domain. Assume there is a common (weak) test element for R and \widehat{R} (e.g., R is excellent). Let $N \subseteq L$ be R -modules such that $\lambda(L/N) < \infty$. Then there exists $0 < \delta \in \mathbb{R}$ such that, for any R -submodule K with $N \subseteq K \subseteq L$, exactly one of the following holds.*

- (1) *If $K \subseteq N_L^*$, then $e_{HK}(L/N, R) - e_{HK}(L/K, R) = 0$.*
- (2) *If $K \not\subseteq N_L^*$, then $e_{HK}(L/N, R) - e_{HK}(L/K, R) \geq \delta$.*

Proof. (1). This was proved in [HH1, Theorem 8.17] (cf. Theorem 1.8) without any assumption on R at all.

(2) This part is mostly implicit in the proof of [HH1, Theorem 8.17] in light of the existence of a test exponent by [HH3]. Nevertheless, we present a proof here for completeness. As R and \widehat{R} share a common (weak) test element, we see that $K \not\subseteq N_L^*$ over R if and only if $K \otimes \widehat{R} \not\subseteq (N \otimes \widehat{R})_{L \otimes \widehat{R}}^*$ over \widehat{R} . Therefore, we may assume $R = \widehat{R}$ is complete (hence a complete domain) without loss of generality. We may further assume $N = 0$.

Our complete local domain (R, \mathfrak{m}, k) is a module-finite and torsion-free extension over a complete regular local domain A with the same coefficient field. For every q , we can form $R[A^{1/q}] \subseteq R^{1/q}$. Moreover, there exists q' such that $S := R[A^{1/q'}]$ is generically smooth over $A^{1/q'}$ (meaning the fraction field of S is separable over the fraction field of $A^{1/q'}$, given that S is module-finite over $A^{1/q'}$). By [HH1, Section 6], $S[A^{1/qq'}] \cong S \otimes_{A^{1/q'}} A^{1/qq'}$ is flat over $S = R[A^{1/q'}]$ and there exists $c \in A^\circ$ such that

$cS^{1/q} \subseteq S[A^{1/qq'}]$ for all q . Evidently, the fact that $R \subseteq S \subseteq R^{1/q'}$ implies S is integral over A , which implies that $(IS) \cap A \subseteq I_A^F \subseteq I_A^* = I$ for any ideal I of A .

For easy identification, we denote the maximal ideals of complete local domains A and S by \mathfrak{m}_A and \mathfrak{m}_S respectively. Since S is integral over A (and S is local), there exists q'' such that $\mathfrak{m}_S^{[q'']} \subseteq \mathfrak{m}_A S$.

Pick $c' \in A^\circ$ that is a locally stable test element for R . Then $cc' \in A^\circ$ is also a locally stable test element for R . By [HH3] (see Theorem 1.5 and observe the fact that $\lambda(L) < \infty$), there exists a test exponent, say $q''' = p^{e'''}$, for cc' and $N = 0 \subseteq L$. (More generally, by a result in [HY], there exists a test exponent for cc' and $N \subseteq L$ as long as L/N is Artinian over R .)

Let $\delta = (1/q''q''')^d > 0$, in which $d = \dim(R) = \dim(A)$, and let K be an arbitrary R -submodule of L such that $K \not\subseteq 0_L^*$. We need to show $e_{HK}(L, R) - e_{HK}(L/K, R) \geq \delta$.

Choose $x \in K \setminus 0_L^*$. By our choice of test exponent q''' , we have $0 \neq x \otimes c \in L \otimes_R e'''S$. (Indeed, suppose $0 = x \otimes c \in L \otimes_R e'''S$ on the contrary. Then $0 = x \otimes c \in L \otimes_R e'''(R^{1/q'}) \cong L \otimes_R e'''R \otimes_R R^{1/q'}$, which would imply $x \otimes c \in 0_{F_R^{e'''}}^F \subseteq 0_{F_R^{e'''}}^*$. Consequently, $0 = x \otimes cc' \in F_R^{e'''}(L)$, and therefore $x \in 0_L^*$, a contradiction.)

At this point, the desired result that $e_{HK}(L, R) - e_{HK}(L/K, R) \geq \delta$ follows immediately from Theorem 3.3 below with $M = R$; and the proof of Theorem 3.2 will be complete once Theorem 3.3 is proved. \square

Theorem 3.3. *Let (R, \mathfrak{m}, k) be a Noetherian complete local domain of prime characteristic p with $\dim(R) = d$; and let $A, q', S, c \in A^\circ \subseteq R^\circ, c' \in A^\circ \subseteq R^\circ$ and q'' be as in the proof of the above Theorem 3.2 and let M be any finitely generated R -modules with torsion-free rank n . Then, for any R -modules $N \subset K \subseteq L$ with $\lambda(L/N) < \infty$ such that the natural image of $(K/N) \otimes c$ in $(L/N) \otimes_R e'''S$ is not 0 (which is the case if $q''' = p^{e'''}$ is a test exponent for cc' and $N \subseteq L$ while $K \not\subseteq N_L^*$ as seen in the proof of Theorem 3.2), we have $e_{HK}(L/N, M) - e_{HK}(L/K, M) \geq \frac{n}{(q''q''')^d}$.*

Proof. Without loss of generality, we assume $N = 0$ and $M = R$ so that $n = 1$. The assumption on e''' and K says that there is an element $x \in K$ such that $0 \neq x \otimes c \in L \otimes_R e'''S$. Hence $\text{Ann}_S(x \otimes c \in L \otimes_R e'''S) \subseteq \mathfrak{m}_S$.

For any $Q = p^E$ (with $E \in \mathbb{N}$), let $J_Q := \{a \in A \mid 0 = x \otimes a \in L \otimes_R {}^E R\}$, which is an ideal of A . We first note that A/J_Q embeds into $K_L^{[Q]}$ via the A -linear map sending the class of $b \in A$ to $x \otimes b \in L \otimes_R {}^E R$. Thus $\lambda_A(A/J_Q) \leq \lambda_R(K_L^{[Q]})$ for every Q . For any $Q = p^E \geq q''q'''$, and writing $Q = qq''q'''$ with $q = p^e$, we have

$$\begin{aligned}
 a \in J_Q &\implies 0 = x \otimes a \in L \otimes_R {}^E R \\
 &\implies 0 = x \otimes a^{1/qq''} \in L \otimes_R e'''(R^{1/qq''}) \\
 &\implies 0 = x \otimes a^{1/qq''} \in L \otimes_R e'''(S^{1/qq''}) \\
 &\implies 0 = x \otimes a^{1/qq''} c \in L \otimes_R e'''(S^{1/qq''} c) = L \otimes_R e'''(cS^{1/qq''}) \\
 &\implies 0 = x \otimes ca^{1/qq''} \in L \otimes_R e'''(S[A^{1/qq'q''}]) \cong L \otimes_R e'''S \otimes_S S[A^{1/qq'q''}] \\
 &\implies a^{1/qq''} \in \text{Ann}_{S[A^{1/qq'q''}]} \left(x \otimes c \in L \otimes_R e'''(S[A^{1/qq'q''}]) \right)
 \end{aligned}$$

$$\begin{aligned}
&\implies a^{1/qq''} \in \text{Ann}_S(x \otimes c \in L \otimes_R e'''S)S[A^{1/qq'q''}] \quad (\text{by flatness}) \\
&\implies a^{1/qq''} \in \mathfrak{m}_S S[A^{1/qq'q''}] \\
&\implies a \in \mathfrak{m}_S^{[qq'']} S^{qq''} [A^{1/q'}] \subseteq \mathfrak{m}_S^{[qq'']} S \subseteq \mathfrak{m}_A^{[q]} S \\
&\implies a \in (\mathfrak{m}_A^{[q]} S) \cap A \subseteq (\mathfrak{m}_A^{[q]})_A^* = \mathfrak{m}_A^{[q]},
\end{aligned}$$

which implies $J_Q \subseteq \mathfrak{m}_A^{[q]} = \mathfrak{m}_A^{[Q/q''q''']}$. Consequently,

$$\lambda_A(A/J_Q) \geq \lambda_A(A/\mathfrak{m}_A^{[Q/q''q''']}) = (Q/q''q''')^d \quad \text{for all } Q \geq q''q''''.$$

Putting things together, we have (with $Q = p^E$)

$$\begin{aligned}
e_{HK}(L/N, R) - e_{HK}(L/K, R) &= \lim_{Q \rightarrow \infty} \frac{\lambda^r(F^E(L))}{Q^d} - \lim_{Q \rightarrow \infty} \frac{\lambda^r(F^E(L)/K_L^{[Q]})}{Q^d} \\
&= \lim_{Q \rightarrow \infty} \frac{\lambda_R(K_L^{[Q]})}{Q^d} \geq \lim_{Q \rightarrow \infty} \frac{\lambda_A(A/J_Q)}{Q^d} \\
&\geq \lim_{Q \rightarrow \infty} \frac{\left(\frac{Q}{q''q''''}\right)^d}{Q^d} = \left(\frac{1}{q''q''''}\right)^d,
\end{aligned}$$

which finishes the proof. \square

The proofs of Theorem 3.2 and of Theorem 3.3 actually produce the following result, which may be viewed as a refined version of [HH1, Theorem 8.17].

Corollary 3.4. *Let (R, \mathfrak{m}, k) be a Noetherian complete local domain of prime characteristic p with $\dim(R) = d$; and let $A, q', S, c \in A^\circ \subseteq R^\circ$ and q'' be as in the proof of the above Theorem 3.2. Instead, choose $c' \in A^\circ \subseteq R^\circ$ so that c' is a test element for all (not necessarily finitely generated) R -modules (cf. Remark 3.5(1) below). Then, for any R -modules $N \subset K \subseteq L$ with $\lambda(K/N) < \infty$ such that the natural image of $(K/N) \otimes c$ in $(L/N) \otimes_R e'''S$ is not 0 (which is the case if $q''' = p^{e'''}$ is a test exponent for cc' and $N \subseteq L$ while $K \not\subseteq N_L^*$ as seen in the proof of Theorem 3.2), we have $\lambda(K_L^{[Q]}/N_L^{[Q]}) \geq \left(\frac{Q}{q''q''''}\right)^d$ for all $Q = p^E \geq q''q''''$.*

Concerning the assumptions in Corollary 3.4, we remark on the existence of test elements that works for all (not necessarily finitely generated) R -modules and the existence of a test exponent for any given Artinian R -module.

Remark 3.5. Assume (R, \mathfrak{m}, k) be an excellent local ring of prime characteristic p .

- (1) If R is reduced, there exists a completely stable test element that works for all (i.e., not necessarily finitely generated) R -modules. (This was proved in [El] by H. Elitzur under the assumption that R is F -finite. The general case then follows from the F -finite case via a faithfully flat extension $R \rightarrow \widehat{R} \rightarrow \widehat{R}^\Gamma$, where \widehat{R} is the \mathfrak{m} -adic completion of R and \widehat{R}^Γ is a suitable (F -finite) Γ -extension of \widehat{R} . See [HH2] for details about Γ -extensions.)
- (2) For any $d \in R^\circ$ and any R -modules $N \subseteq L$ such that L/N is Artinian, it has been shown recently in [HY] that there exists a test exponent for d and $N \subseteq L$.

Finally, we end this section by pointing out an alternate proof of the claim in Theorem 1.9(2) without the F -finite assumption.

Remark 3.6. Let (R, \mathfrak{m}, k) be an excellent local ring of prime characteristic p and let $E = E_R(k)$ be the injective hull of the residue field $k = R/\mathfrak{m}$. Denote by K the socle of E (so that $E \supseteq K \cong R/\mathfrak{m}$). Recall that a local ring (R, \mathfrak{m}, k) is strongly F -regular if and only if $0 = 0_E^*$, i.e., $K \not\subseteq 0_E^*$ (cf. [Sm, 7.1.2] or [LS, Proposition 2.9]).

(1). Suppose that R is complete and strongly F -regular (hence a domain). Let $A, S, c \in A^\circ \subseteq R^\circ$, and q'' be as in the proof of the above Theorem 3.3. As R is strongly F -regular, there exists $q''' = p^{e''''}$ such that the natural image of $K \otimes c$ in $L \otimes_R e'''S$ is not 0. (Otherwise, we would have $\text{Image}(K \otimes_R S \rightarrow E \otimes_R S) \subseteq 0_{E \otimes_R S}^*$ over S , which implies $K \subseteq 0_E^*$ since S is integral over R .) Then by Theorem 3.3, we get that $\lambda(K_E^{[q]}) \geq (\frac{q}{q''q'''})^d$ for all $q = p^e$. Also observe that $\lambda(K_E^{[q]}) = \#(eR)$ for any $e \in \mathbb{N}$ (cf. Definition 0.2). Thus

$$\begin{aligned} R \text{ is strongly } F\text{-regular} &\implies s^-(R) > 0 \\ &\implies s^+(R) > 0 \implies R \text{ is strongly } F\text{-regular.} \end{aligned}$$

(2). In general, if R is excellent and strongly F -regular, then it is well-known that \widehat{R} is also strongly F -regular, i.e. $0_E^* = 0$ as modules over \widehat{R} . (For example, this may be seen immediately via the existence of a completely stable test element for $0 \subset E$. See Remark 3.5(1) above.) Also observe that $s^-(R) = s^-(\widehat{R})$ and $s^+(R) = s^+(\widehat{R})$ (see [Yao, Remark 2.3(3)]). This should sketch an alternate proof of the equivalence that

$$R \text{ is strongly } F\text{-regular} \iff s^-(R) > 0 \iff s^+(R) > 0$$

without assuming that R is F -finite. See Theorem 1.9(2).

4. BASIC RESULTS ON F -RATIONAL SIGNATURE

The next theorem explains why $r(R)$ is called the F -rational signature of R . Its proof depends on Theorem 3.1 and Theorem 2.5.

Theorem 4.1. *Let (R, \mathfrak{m}, k) be a Noetherian local ring of characteristic p . And we use M to denote finitely generated R -module(s). Consider*

- (1) $r(R) > 0$;
- (2) $r(M) > 0$ for every M such that $\dim(M) = \dim(R)$;
- (3) $r(M) > 0$ for some M ;
- (4) \widehat{R} is F -rational;
- (5) R is F -rational.

Then (1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Rightarrow (5). If, moreover, R is excellent or there exists a common parameter (weak) test element for R and \widehat{R} , then (1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5).

Proof. The implication (4) \Rightarrow (5) is always true. And (4) \Leftrightarrow (5) holds if there exists a common parameter (weak) test element for R and \widehat{R} , which is the case when R is excellent (cf. Theorem 1.3). It remains to show (1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4). To do this, we assume $R = \widehat{R}$ without loss of generality (cf. Observation 2.2(1)).

If $R = \widehat{R}$ is not F -rational, then there exist a system of parameters \underline{x} and an element $u \in (\underline{x})^* \setminus (\underline{x})$, which will force $r(M) = 0$ for all M . This proves that either one of (1), (2) and (3) will imply (4). (Also see Observation 2.2(4).)

For the remainder of the proof, we assume that (4) holds, i.e., $R = \widehat{R}$ is F -rational and hence a complete domain. Then, for any finitely generated R -module M with $\dim(M) = \dim(R)$, its torsion-free rank, say n , is positive and $r(M) = nr(R)$ as discussed in Observation 2.2(3). Consequently, it is enough to prove (4) \Rightarrow (1) only. Pick a system of parameters, say \underline{x} . By Theorem 2.5, $r_R(R) = \inf\{e_{HK}((\underline{x}), R) - e_{HK}((\underline{x}, v), R) \mid v \notin (\underline{x})\}$. Moreover, by Theorem 3.1, there exists $0 < \delta \in \mathbb{R}$ such that, for any $J \supseteq (\underline{x})$, $e_{HK}((\underline{x}), M) - e_{HK}(J, M)$ is either 0 or $\geq \delta$. By [HH1] (cf. Theorem 1.8), we know $e_{HK}((\underline{x}), R) - e_{HK}((\underline{x}, v), R) > 0$, for any $v \notin (\underline{x}) = (\underline{x})^*$, which forces $e_{HK}((\underline{x}), R) - e_{HK}((\underline{x}, v), R) \geq \delta$. Therefore, $r_R(R) \geq \delta > 0$. \square

We next prove some relations among the invariants $r(M)$, $s^+(M)$, $e(M) = e(\mathfrak{m}, M)$ and $e_{HK}(M) = e_{HK}(\mathfrak{m}, R)$.

Lemma 4.2. *If (R, \mathfrak{m}, k) is a Noetherian local ring (R, \mathfrak{m}, k) of characteristic p and M is a finitely generated R -module, then*

- (1) $s^+(M) \leq r(M) \leq e_{HK}(M)$.
- (2) $s^+(M) \leq r(M) \leq \min\{e_{HK}(M), e(M) - e_{HK}(M)\}$ if $|k| = \infty$ and R is not regular.

Proof. (1). This follows immediately from Definition 2.1 and [Yao, Lemma 2.5(2)].

(2). The assumption $|k| = \infty$ ensures the existence of a system of parameters \underline{x} such that the ideal (\underline{x}) reduces \mathfrak{m} and hence $e_{HK}((\underline{x}), M) = e((\underline{x}), M) = e(M)$. The assumption that R is not regular guarantees $(\underline{x}) \subsetneq \mathfrak{m}$. Thus $e_{HK}((\underline{x}), M) - e_{HK}((\underline{x}, u), M) = e((\underline{x}), M) - e_{HK}((\underline{x}, u), M) \leq e((\underline{x}), M) - e_{HK}(\mathfrak{m}, M)$ for any $u \in ((\underline{x}) :_R \mathfrak{m}) \setminus (\underline{x})$. This implies $r(M) \leq e(M) - e_{HK}(M)$. \square

In case (R, \mathfrak{m}) is a regular local ring, one easily sees that $r(R) = 1$ since $e_{HK}(I, R) = \lambda(R/I)$ for any \mathfrak{m} -primary ideal. Thus, for any R -module M , $r(M)$ equal the torsion-free rank of M . Actually we have a very easy lemma about $r(R)$ when R is Gorenstein.

Lemma 4.3. *If (R, \mathfrak{m}, k) is a Gorenstein Noetherian local ring of characteristic p , then $r(M) = s(M)$ for any finitely generated maximal Cohen-Macaulay R -module M . In particular, $r(R) = s(R)$.*

Proof. It follows from Definition 0.2, Theorem 2.4(1). \square

5. DEFORMATION, FLAT EXTENSION, AND LOCALIZATION

Given a local ring homomorphism $(R, \mathfrak{m}, k) \rightarrow (S, \mathfrak{n}, l)$ of Noetherian local rings of prime characteristic p , a finitely generated module M over R and $P \in \text{Spec}(R)$, we get an S -module $M \otimes_R S$ by scalar extension and an R_P -module M_P by localization. To avoid the cumbersome subscripts, we sometimes simply write $r(M \otimes_R S)$, $r(S/\mathfrak{m}S)$ and $r(M_P)$ etc. instead of $r_S(M \otimes_R S)$, $r_{S/\mathfrak{m}S}(S/\mathfrak{m}S)$ and $r_{S_P}(M_P)$ etc. respectively. As long as no confusion arises, we identify $u \in R$ with its image in S even though the local ring homomorphism $(R, \mathfrak{m}, k) \rightarrow (S, \mathfrak{n}, l)$ is not necessarily injective.

Let us first consider deformation. We know that if, for some non-zero-divisor x , R/xR is F -rational, then so is R . This result can be described in terms of F -rational signature.

Theorem 5.1. *Let (R, \mathfrak{m}, k) be a Noetherian local ring of characteristic p and $\underline{x} = x_1, \dots, x_h$ an R -regular sequence. Denote $\bar{R} = R/(\underline{x})R$. Then*

- (1) $r_R(R) \geq r_{\bar{R}}(\bar{R})$;
- (2) $r_R(M) \geq r_{\bar{R}}(M/(\underline{x})M)$ for any finitely generated maximal Cohen-Macaulay R -module M .

Proof. If $r_{\bar{R}}(\bar{R}) = 0$, then we also have $r_{\bar{R}}(M/(\underline{x})M) = 0$ (cf. Theorem 4.1) and both (1) and (2) are trivial. Therefore, we assume $r_{\bar{R}}(\bar{R}) > 0$, which forces $\bar{R} = R/(\underline{x})R$ to be Cohen-Macaulay. Consequently, we assume R is Cohen-Macaulay and, as a result, it suffices to prove (2).

By induction on h , it suffices to prove $r_R(M) \geq r_{\bar{R}}(M/xM)$ whenever x is a non-zero-divisor on R (and hence on M).

Extend x to a (full) system of parameters $x, y_2, \dots, y_{\dim(R)}$ of R and denote $\underline{y} = y_2, \dots, y_{\dim(R)}$. Also, for any q , denote $\underline{y}^{[q]} = y_2^q, \dots, y_{\dim(R)}^q$.

By the result of Theorem 2.5, it suffices to prove

$$\begin{aligned} \inf\{e_{HK}((x, \underline{y}), M) - e_{HK}((x, \underline{y}, u), M) \mid u \notin (x, \underline{y})\} \\ \geq \inf\{e_{HK}(\underline{y}, M/xM) - e_{HK}(\underline{y}, u), M/xM) \mid u \notin (x, \underline{y})\}. \end{aligned}$$

Hence it is enough to show

$$\lim_{e \rightarrow \infty} \frac{\lambda\left(\frac{(x^q, \underline{y}^{[q]}, u^q)M}{(x^q, \underline{y}^{[q]})M}\right)}{q^{\dim(R)}} \geq \lim_{e \rightarrow \infty} \frac{\lambda\left(\frac{(x, \underline{y}^{[q]}, u^q)M}{(x, \underline{y}^{[q]})M}\right)}{q^{\dim(R)-1}} \quad \text{for any } u \notin (x, \underline{y}).$$

Therefore it suffices to prove, for any given $u \notin (x, \underline{y})$ and q ,

$$\lambda\left(\frac{(x^q, \underline{y}^{[q]}, u^q)M}{(x^q, \underline{y}^{[q]})M}\right) \geq q\lambda\left(\frac{(x, \underline{y}^{[q]}, u^q)M}{(x, \underline{y}^{[q]})M}\right).$$

Thus we may consider both u and q as arbitrarily chosen and then fixed elements. Moreover, to simplify notation, we denote $S = R/(\underline{y}^{[q]})R$ and $N = M/(\underline{y}^{[q]})M$, which are a one-dimensional Cohen-Macaulay local ring and module respectively. The classes of x and u are still denoted by x and u . Denote u^q by v . Now, to finish the proof, it remains only to establish the inequality

$$(*) \quad \lambda_S\left(\frac{(x^q, v)N}{(x^q)N}\right) \geq q\lambda_S\left(\frac{(x, v)N}{(x)N}\right).$$

Since N is Cohen-Macaulay and x is a non-zero-divisor, there is an injective S -linear map $\phi : N/(x)N \rightarrow N/(x^q)N$ sending the class of z to the class of $x^{q-1}z$ for every $z \in N$. It is easy to see that (since ϕ is injective)

$$\lambda\left(\frac{(x, v)N}{(x)N}\right) = \lambda\left(\phi\left(\frac{(x, v)N}{(x)N}\right)\right) = \lambda\left(\frac{(x^q, x^{q-1}v)N}{(x^q)N}\right).$$

The inclusion $(x^q)N \subseteq (x^q, v)N$ may be filtered in the following way:

$$(x^q)N \subseteq (x^q, x^{q-1}v)N \subseteq \cdots \subseteq (x^q, x^i v)N \subseteq (x^q, x^{i-1}v)N \subseteq \cdots \subseteq (x^q, v)N.$$

There is a surjective S -map $f_i : \frac{(x^q, x^{i-1}v)N}{(x^q, x^i v)N} \rightarrow \frac{(x^q, x^{q-1}v)N}{(x^q)N}$ induced by multiplication by x^{q-i} , implying $\lambda\left(\frac{(x^q, x^{i-1}v)N}{(x^q, x^i v)N}\right) \geq \lambda\left(\frac{(x^q, x^{q-1}v)N}{(x^q)N}\right)$ for every $i = 1, 2, \dots, q-1$. As a result, we have

$$\lambda\left(\frac{(x^q, v)N}{(x^q)N}\right) = \sum_{i=1}^q \lambda\left(\frac{(x^q, x^{i-1}v)N}{(x^q, x^i v)N}\right) \geq q\lambda\left(\frac{(x^q, x^{q-1}v)N}{(x^q)N}\right),$$

which is inequality (*), and our proof is complete. \square

We want to study the behavior of the F -rational signature under a local homomorphism $R \rightarrow S$. Recall that an ideal I of R is contracted from S if $I = \{r \in R \mid rS \subseteq IS\}$. We start with a special case of local extension.

Lemma 5.2. *Let $(R, \mathfrak{m}, k) \rightarrow (S, \mathfrak{n}, l)$ be a local ring homomorphism of Noetherian local rings of prime characteristic p . Denote $\bar{S} = S/\mathfrak{m}S$. Assume $\dim(R) + \dim(\bar{S}) = \dim(S)$ and $\underline{y} = y_1, y_2, \dots, y_{\dim(\bar{S})}$ in S such that the images of \underline{y} form a (full) system of parameters of \bar{S} and $(\underline{x})R$ is contracted from $S/(\underline{y})S$ for all systems of parameters $\underline{x} = x_1, \dots, x_{\dim(R)}$ of R . Then $r(M \otimes_R S) \leq r(M)e((\underline{y}), \bar{S})$ for any finitely generated R -module M .*

If R is an equidimensional homomorphic image of a Cohen-Macaulay ring, then the same conclusion still holds under the assumption that $(\underline{x})R$ is contracted from $S/(\underline{y})S$ for one system of parameters $\underline{x} = x_1, \dots, x_{\dim(R)}$ of R .

Proof. The assumption implies that the images of \underline{x} and \underline{y} form a system of parameters of S . Denote $M_S = M \otimes_R S$ (as an S -module). For any $u \in ((\underline{x})R :_R \mathfrak{m}) \setminus (\underline{x})R$, we know $u \notin (\underline{x}, \underline{y})S$ (since $(\underline{x})R$ is contracted from $S/(\underline{y})S$) and, moreover, there is an inequality as follows:

$$\begin{aligned} \lambda_S\left(\frac{(\underline{x}, \underline{y}, u)^{[q]}(M \otimes_R S)}{(\underline{x}, \underline{y})^{[q]}(M \otimes_R S)}\right) &\leq \lambda_S\left(\frac{(\underline{x}, u)^{[q]}M}{(\underline{x})^{[q]}M} \otimes_R \frac{S}{(\underline{y})^{[q]}S}\right) \\ &\leq \lambda_R\left(\frac{(\underline{x}, u)^{[q]}M}{(\underline{x})^{[q]}M}\right) \lambda_{\bar{S}}\left(\frac{\bar{S}}{(\underline{y})^{[q]}\bar{S}}\right). \end{aligned}$$

By definition of Hilbert-Kunz multiplicity, we have

$$\begin{aligned} \lim_{e \rightarrow \infty} \frac{\lambda_S\left(\frac{(\underline{x}, \underline{y}, u)^{[q]}M_S}{(\underline{x}, \underline{y})^{[q]}M_S}\right)}{q^{\dim(S)}} &= e_{HK}((\underline{x}, \underline{y}), M_S) - e_{HK}((\underline{x}, \underline{y}, u), M_S), \\ \lim_{e \rightarrow \infty} \frac{\lambda_S\left(\frac{(\underline{x}, u)^{[q]}M}{(\underline{x})^{[q]}M}\right)}{q^{\dim(R)}} &= e_{HK}((\underline{x}), M) - e_{HK}((\underline{x}, u), M), \\ \lim_{e \rightarrow \infty} \frac{\lambda_{\bar{S}}\left(\frac{\bar{S}}{(\underline{y})^{[q]}\bar{S}}\right)}{q^{\dim(\bar{S})}} &= e_{HK}((\underline{y}), \bar{S}) = e((\underline{y}), \bar{S}). \end{aligned}$$

Thus $r(M \otimes_R S) \leq r(M)e_{HK}(\underline{y}, \bar{S}) = r(M)e(\underline{y}, \bar{S})$, once we run through all system of parameters \underline{x} of R and all $u \in ((\underline{x})R :_R \mathfrak{m}) \setminus (\underline{x})R$.

In case R is an equidimensional homomorphic image of a Cohen-Macaulay ring, the claim follows easily from the argument above and Theorem 2.5. \square

The above lemma applies when $(R, \mathfrak{m}, k) \rightarrow (S, \mathfrak{n}, l)$ is local and flat. In fact, the flatness gives a sharper upper bound for $r(S)$. As a preparatory step, we first study certain special filtrations of an Artinian local ring.

Notation 5.3. Let (S, \mathfrak{n}, l) be a Noetherian local ring and N a finitely generated S -module.

- (1) The type of N , denoted by $t(N)$, is defined as $\text{rank}_l(\text{Ext}_S^h(l, N))$ in which $h = \text{depth}(N)$. It can be shown that $t(H) = \text{rank}_l((0 :_{N/(\underline{y})N} \mathfrak{n}))$ for any maximal N -regular sequence $\underline{y} = y_1, \dots, y_h \in \mathfrak{n}$.
- (2) Suppose S is Artinian. For any ideal I of R , denote $I^\# = (0 :_S I) = \text{Ann}_S(I)$, $I^{\#\#} = (I^\#)^\#$ and $I^{\#\#\#} = (I^\#)^{\#\#}$. It is straightforward to check that $I^{\#\#\#} = I^\#$. Also we denote by $n(S)$ the maximal $n \in \mathbb{N}$ such that there is a filtration

$$0 = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_{i-1} \subsetneq I_i \subsetneq \cdots \subsetneq I_{n-1} \subsetneq I_n = S$$

with $I_i = I_i^{\#\#}$ (or, equivalently, $I_i = J_i^\#$ for some ideal J_i) for all $0 \leq i \leq n$.

For example, if (S, \mathfrak{n}, l) is Artinian, one may choose $0 \neq w_1 \in (0 :_S \mathfrak{n})$ and let $J_1 = w_1 S$ and $I_1 = J_1^{\#\#} = (0 :_S \mathfrak{n})$. As a recursive step, suppose we have chosen I_{i-1} . Then choose any $w_i \in (I_{i-1} :_S \mathfrak{n}) \setminus I_{i-1}$ and let $J_i = I_{i-1} + w_i S$ and $I_i = J_i^{\#\#}$.

Lemma 5.4. *Let (S, \mathfrak{n}, l) be an Artinian local ring which is equicharacteristic and let*

$$0 = I_0 \subsetneq J_1 \subseteq I_1 \subsetneq \cdots \subseteq I_{i-1} \subsetneq J_i \subseteq I_i \subsetneq J_{i+1} \subseteq \cdots \subseteq I_{n-1} \subsetneq J_n \subseteq I_n = S$$

be any filtration (of ideals) of S such that $J_i = I_{i-1} + w_i S$ with $(I_{i-1} :_S w_i) = \mathfrak{n}$ and $I_i = J_i^{\#\#}$ for all $1 \leq i \leq n$. Then $n \geq \lceil \lambda(\mathfrak{n})/t(S) \rceil + 1 \geq \lceil \lambda(S)/t(S) \rceil$. Consequently, $n(S) \geq \lceil \lambda(\mathfrak{n})/t(S) \rceil + 1 \geq \lceil \lambda(S)/t(S) \rceil$. In particular, $n(S) = \lambda(S)$ if S is Gorenstein.

Proof. It suffices to prove $n \geq \lceil \lambda(\mathfrak{n})/t(S) \rceil + 1$. As S is complete, we may just assume that $l \subseteq S$ by the existence of a coefficient field. Therefore, every ideal of S is an l -vector subspace of S . For every $0 \leq i \leq n$, let $V_i = I_i^\#$. Then we have

$$S = V_0 \supsetneq V_1 \supsetneq \cdots \supsetneq V_{i-1} \supsetneq V_i \supsetneq \cdots \supsetneq V_{n-1} \supsetneq V_n = 0.$$

(Indeed, if $V_{i-1} = V_i$ for some $0 \leq i \leq n$, then $I_{i-1} = V_{i-1}^\# = V_i^\# = I_i$, a contradiction.) By construction, $I_1 = (0 :_S \mathfrak{n})$, $I_{n-1} = \mathfrak{n}$ and hence $V_1 = \mathfrak{n}$, $V_{n-1} = (0 :_S \mathfrak{n})$. For every $1 \leq i \leq n$, there exists an l -vector subspace $V_i' \subseteq V_{i-1}$ such that $V_i \oplus V_i' = V_{i-1}$. Now, as $\text{rank}_l(V_1') = 1$, we only need to prove $\text{rank}_l(V_i') \leq t(S)$ for all $2 \leq i \leq n$. For any $i = 1, 2, \dots, n$, we have $w_i V_{i-1} \subseteq (0 :_S \mathfrak{n})$, which gives rise to an l -linear map $h_i : V_i' \rightarrow (0 :_S \mathfrak{n})$ defined by $h_i(x) = w_i x$. For any $i = 1, 2, \dots, n$ and any $x \in V_i'$, if $h_i(x) = w_i x = 0$, then $x \in \text{Ann}_S(J_i) = J_i^\# = J_i^{\#\#\#} = (J_i^{\#\#})^\# = I_i^\# = V_i$, which implies $x \in V_i \cap V_i' = 0$. Therefore, for every $1 \leq i \leq n$, h_i is an injective l -linear map, implying $\text{rank}_l(V_i') \leq \text{rank}_l(0 :_S \mathfrak{n}) = t(S)$. \square

Remark 5.5. Let $S = k[T, X, Y]/(T^n, X^2, XY, Y^2)$, $J = (x)$ and $I = (x, yz^{n-1})$. Then $t(S) = 2$ but $\lambda(J^\# / J) = n$ and $\lambda(I^\# / I) = n - 1$ as $J^\# = I^\# = (x, y)^\# = (x, y)R$.

Theorem 5.6. *Let $(R, \mathfrak{m}, k) \rightarrow (S, \mathfrak{n}, l)$ be a local flat ring homomorphism of Noetherian local rings of prime characteristic p . Denote $\bar{S} := S/\mathfrak{m}S$. Then, for any finitely generated R -module M , we have*

$$(\#) \quad r(M \otimes_R S) \leq r(M) \min \left\{ \frac{e(\underline{y}\bar{S}, \bar{S})}{n(\bar{S}/(\underline{y}\bar{S}))} \mid \underline{y} \text{ is a system of parameters of } \bar{S} \right\}.$$

In particular, we have $r(M \otimes_R S) \leq r(M) t(\bar{S})$. In case \bar{S} is Gorenstein, we have $r(M \otimes_R S) \leq r(M)$.

Proof. If S is not Cohen-Macaulay, then $r(M \otimes_R S) = 0$ (cf. Observation 2.2) and the claims all become trivial. Therefore, we assume S is Cohen-Macaulay without loss of generality and, hence, R and \bar{S} are Cohen-Macaulay.

Clearly, $\dim(S) = \dim(R) + \dim(\bar{S})$. Fix any system of parameters \underline{x} of R . For any elements $\underline{y} = y_1, y_2, \dots, y_{\dim(\bar{S})} \in S$ such that their images form a system of parameters for \bar{S} , we know that $S/(\underline{y})^{[q]}S$ is faithfully flat over R for every $q = p^e$ (cf. Theorem 1.11), which implies that, for any $u \in R \setminus (\underline{x})R$, we have $u \notin (\underline{x}, \underline{y})S$, i.e., $(\underline{x})R$ is contracted from $S/(\underline{y})S$. Also, as \bar{S} is Cohen-Macaulay, we have $e((\underline{y}\bar{S}, \bar{S})/n(\bar{S}/(\underline{y}\bar{S}))) = \lambda(\bar{S}/(\underline{y}\bar{S}))/n(\bar{S}/(\underline{y}\bar{S})) \leq t(\bar{S}/(\underline{y}\bar{S})) = t(\bar{S})$ by Lemma 5.4.

Consequently, we only need to prove $(\#)$. Fix any sequence \underline{y} such that their images form a system of parameters for \bar{S} . Denote $\tilde{S} = S/(\mathfrak{m}, \underline{y})S = \bar{S}/(\underline{y}\bar{S})$ and $\tilde{\mathfrak{n}} = \mathfrak{n}/(\mathfrak{m}, \underline{y})S$. Then $(\tilde{S}, \tilde{\mathfrak{n}})$ is a 0-dimensional (i.e., Artinian) local ring. Say $n(\tilde{S}) = n$. Then there exists a filtration

$$0 = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_{i-1} \subsetneq I_i \subsetneq \cdots \subsetneq I_{n-1} \subsetneq I_n = \tilde{S}$$

such that $I_i = I_i^\#$ for all $0 \leq i \leq n$. For every $i = 1, \dots, n$, choose $w_i \in I_i$ such that $(I_{i-1} :_S w_i) = \tilde{\mathfrak{n}}$ and hence there exists $s_i \in I_{i-1}^\#$ such that $0 \neq s_i w_i = v_i \in (0 : \tilde{\mathfrak{n}})$.

Now fix a lifting of s_i, w_i, v_i, I_i up to S for every $1 \leq i \leq n$. For convenience, we continue to denote their liftings by s_i, w_i, v_i, I_i , that is, from now on, $s_i, w_i, v_i \in S$ and I_i are ideals of S for $1 \leq i \leq n$. Set $I_0 = 0$.

For any $u \in (\underline{x})R :_R \mathfrak{m} \setminus (\underline{x})R$, we see that (as $S/(\underline{y})S$ is flat over R)

$$\frac{(\underline{x}, \underline{y}, u)S}{(\underline{x}, \underline{y})S} \cong \frac{(\underline{x}, u)R}{(\underline{x})R} \otimes_R \frac{S}{(\underline{y})S} \cong \tilde{S},$$

which implies that $uv_i \notin (\underline{x}, \underline{y})S$ (actually $uv_i \in ((\underline{x}, \underline{y})S :_S \mathfrak{n}) \setminus (\underline{x}, \underline{y})S$) for every $1 \leq i \leq n$.

For any $q = p^e$, we now have (recall $n = n(\tilde{S})$)

$$\begin{aligned} \sum_{i=1}^n \lambda_S \left(\frac{(\underline{x}, \underline{y}, uI_{i-1}, uw_i)^{[q]}(M \otimes_R S)}{(\underline{x}, \underline{y}, uI_{i-1})^{[q]}(M \otimes_R S)} \right) &\leq \sum_{i=1}^n \lambda_S \left(\frac{(\underline{x}, \underline{y}, uI_i)^{[q]}(M \otimes_R S)}{(\underline{x}, \underline{y}, uI_{i-1})^{[q]}(M \otimes_R S)} \right) \\ &= \lambda_S \left(\frac{(\underline{x}, \underline{y}, u)^{[q]}(M \otimes_R S)}{(\underline{x}, \underline{y})^{[q]}(M \otimes_R S)} \right) \end{aligned}$$

Also notice that, for every q and every $1 \leq i \leq n$, there is an S -linear map

$$\phi_{q,i} : \frac{(\underline{x}, \underline{y}, uI_{i-1}, uw_i)^{[q]}(M \otimes_R S)}{(\underline{x}, \underline{y}, uI_{i-1})^{[q]}(M \otimes_R S)} \longrightarrow \frac{(\underline{x}, \underline{y}, uv_i)^{[q]}(M \otimes_R S)}{(\underline{x}, \underline{y})^{[q]}(M \otimes_R S)}$$

defined by sending a class $[z]$ to $[s_i^q z]$, which is well-defined by our choice of s_i, w_i, v_i . In particular, $\phi_{q,i}([(uw_i)^q]) = [(s_i uw_i)^q] = [(uv_i)^q]$, which means that $\phi_{q,i}$ is surjective for every q and every $1 \leq i \leq n = n(\tilde{S}) = n(\bar{S}/(\underline{y})\bar{S})$.

Thus, for every $q = p^e$, there is an inequality

$$\begin{aligned} \sum_{i=1}^{n(\tilde{S})} \lambda_S \left(\frac{(\underline{x}, \underline{y}, uv_i)^{[q]}(M \otimes_R S)}{(\underline{x}, \underline{y})^{[q]}(M \otimes_R S)} \right) &\leq \lambda_S \left(\frac{(\underline{x}, \underline{y}, u)^{[q]}(M \otimes_R S)}{(\underline{x}, \underline{y})^{[q]}(M \otimes_R S)} \right) \\ &= \lambda_S \left(\frac{(\underline{x}, u)^{[q]}M}{(\underline{x})^{[q]}M} \otimes_R \frac{S}{(\underline{y})^{[q]}S} \right) = \lambda_R \left(\frac{(\underline{x}, u)^{[q]}M}{(\underline{x})^{[q]}M} \right) \lambda_{\bar{S}} \left(\frac{\bar{S}}{(\underline{y})^{[q]}\bar{S}} \right), \end{aligned}$$

which, after divided by $q^{\dim(S)} = q^{\dim(R)}q^{\dim(\bar{S})}$ with $q \rightarrow \infty$, implies

$$\begin{aligned} \sum_{i=1}^{n(\tilde{S})} (e_{HK}((\underline{x}, \underline{y}), M \otimes_R S) - e_{HK}((\underline{x}, \underline{y}, uv_i), M \otimes_R S)) \\ \leq (e_{HK}((\underline{x}), M) - e_{HK}((\underline{x}, u), M))e((\underline{y})\bar{S}, \bar{S}) \end{aligned}$$

which gives $n(\tilde{S})r(M \otimes_R S) \leq r(M)e((\underline{y})\bar{S}, \bar{S})$ as u is arbitrary in $((\underline{x})R :_R \mathfrak{m}) \setminus (\underline{x})R$. That is, $r(M \otimes_R S) \leq r(M)e((\underline{y})\bar{S}, \bar{S})/n(\tilde{S}/(\underline{y})\bar{S})$.

The inequality (#) now follows as we run through all \underline{y} such that their images form a system of parameters for \bar{S} . \square

Under the assumptions of Theorem 5.6, if \bar{S} is furthermore Gorenstein and the induced map $R/\mathfrak{m} \rightarrow S/\mathfrak{n}$ is an isomorphism, then we can bound $r(S)$ below.

Theorem 5.7. *Let $(R, \mathfrak{m}, k) \rightarrow (S, \mathfrak{n}, l)$ be a local flat ring homomorphism of Noetherian local rings of prime characteristic p . Denote $\bar{S} := S/\mathfrak{m}S$. Let M be an arbitrary finitely generated R -module M .*

- (1) *If R is Gorenstein, then $r(M \otimes_R S) \geq r(M)r(\bar{S})$. In particular, $r(S) \geq r(R)r(\bar{S})$;*
- (2) *If \bar{S} is Gorenstein and the induced map $R/\mathfrak{m} \rightarrow S/\mathfrak{n}$ is an isomorphism, then $r(M \otimes_R S) \geq r(M)r(\bar{S})$. In particular, $r(S) \geq r(R)r(\bar{S})$;*
- (3) *If \bar{S} is regular and the induced map $R/\mathfrak{m} \rightarrow S/\mathfrak{n}$ is an isomorphism, then $r(M \otimes_R S) = r(M)$. In particular, $r(S) = r(R)$.*

Proof. It suffices to prove (1) and (2) because, then, (2) and Theorem 5.6(1) will imply (3). However, all of (1), (2) and (3) will be proved from scratch in this proof. Notice that we may assume R and \bar{S} are both Cohen-Macaulay (so is S) without loss of generality (otherwise $r(M) = 0$ or $r(\bar{S}) = 0$ and all the claims become trivial).

Choose a system of parameters \underline{x} for R . Also choose $\underline{y} \in S$ such that their images form a system of parameters for \bar{S} . (In case \bar{S} is regular, make sure the images of \underline{y} form a regular system of parameters for \bar{S} .)

Say $((\underline{x})R :_R \mathfrak{m})/(\underline{x})R \cong k^{\oplus r} \cong \left(\frac{R}{\mathfrak{m}}\right)^{\oplus r}$ as k -vector spaces for some $1 \leq r \in \mathbb{N}$. Similarly, say $((\underline{y})\bar{S} :_{\bar{S}} \mathfrak{n})/(\underline{y})\bar{S} \cong l^{\oplus s} \cong \left(\frac{S}{\mathfrak{n}}\right)^{\oplus s}$ for some $1 \leq s \in \mathbb{N}$.

Since $S/(\underline{y})S$ is flat over R (cf. Theorem 1.11) and hence $S/(\underline{x}, \underline{y})S$ is flat over $R/(\underline{x})R$, we have

$$\begin{aligned} \left(0 :_{\frac{S}{(\underline{x}, \underline{y})S}} \mathfrak{n}\right) &= \left(0 :_{\left(0 :_{\frac{S}{(\underline{x}, \underline{y})S}} \mathfrak{m}\right)} \mathfrak{n}\right) = \left(0 :_{\left(0 :_{\frac{R}{(\underline{x})R}} \mathfrak{m}\right) \frac{S}{(\underline{x}, \underline{y})S}} \mathfrak{n}\right) \cong \left(0 :_{\left(k^{\oplus r} \otimes \frac{S}{(\underline{y})S}\right)} \mathfrak{n}\right) \\ &\cong \left(0 :_{\left(\frac{R}{\mathfrak{m}} \otimes \frac{S}{(\underline{y})S}\right)^{\oplus r}} \mathfrak{n}\right) \cong \left(0 :_{\left(\frac{R}{\mathfrak{m}} \otimes \frac{S}{(\underline{y})S}\right)} \mathfrak{n}\right)^{\oplus r} \cong \left(0 :_{\frac{\bar{S}}{(\underline{y})\bar{S}}} \mathfrak{n}\right)^{\oplus r} \cong \left(\frac{S}{\mathfrak{n}}\right)^{\oplus rs}, \end{aligned}$$

which has the following implications:

- (1) Under the assumption (1), we have $r = 1$. Say $\left(0 :_{\frac{R}{(\underline{x})R}} \mathfrak{m}\right) \cong k$ is generated by the image of $u \in R$. Then there is an isomorphism $\left(0 :_{\frac{\bar{S}}{(\underline{y})\bar{S}}} \mathfrak{n}\right) \cong \left(0 :_{\frac{S}{(\underline{x}, \underline{y})S}} \mathfrak{n}\right)$ as l -vector spaces sending any class $[z] \mapsto [uz]$. Consequently, for any $w \in ((\underline{x}, \underline{y}) :_S \mathfrak{n}) \setminus (\underline{x}, \underline{y})$, there exists $v \in S$ whose image is in $((\underline{y})\bar{S} :_{\bar{S}} \mathfrak{n}) \setminus (\underline{y})\bar{S}$ such that $(\underline{x}, \underline{y}, uv)S = (\underline{x}, \underline{y}, w)S$.
- (2, 3) If \bar{S} is Gorenstein (or regular), then $s = 1$. Say $\left(0 :_{\frac{\bar{S}}{(\underline{y})\bar{S}}} \mathfrak{n}\right)$ is generated by the image of $v \in S$. (In case \bar{S} is regular, we choose $v = 1$.) Then, given the assumption that the induced map $R/\mathfrak{m} \rightarrow S/\mathfrak{n}$ is an isomorphism, there is an isomorphism $\left(0 :_{\frac{R}{(\underline{x})R}} \mathfrak{m}\right) \cong \left(0 :_{\frac{S}{(\underline{x}, \underline{y})S}} \mathfrak{n}\right)$ as k -vector spaces sending any class $[z] \mapsto [zv]$. Consequently, for any $w \in ((\underline{x}, \underline{y}) :_S \mathfrak{n}) \setminus (\underline{x}, \underline{y})$, there exists $u \in ((\underline{x})R :_R \mathfrak{m}) \setminus (\underline{x})R$ such that $(\underline{x}, \underline{y}, uv)S = (\underline{x}, \underline{y}, w)S$ (and vice versa, which is needed in proving (3)).

Therefore, in all the cases (1), (2) or (3) above, we have the following inequality (equality in case \bar{S} is regular). For convenience, we write $M \otimes_R S = M_S$.

$$\begin{aligned} \lambda_S \left(\frac{(\underline{x}, \underline{y}, uv)^{[q]} M_S}{(\underline{x}, \underline{y})^{[q]} M_S} \right) &= \lambda_S \left(\frac{(\underline{x}, \underline{y}, u)^{[q]} M_S}{(\underline{x}, \underline{y})^{[q]} M_S} \right) - \lambda_S \left(\frac{(\underline{x}, \underline{y}, u)^{[q]} M_S}{(\underline{x}, \underline{y}, uv)^{[q]} M_S} \right) \\ &= \lambda_S \left(\frac{(\underline{x}, u)^{[q]} M}{(\underline{x})^{[q]} M} \otimes_R \frac{S}{(\underline{y})^{[q]} S} \right) - \lambda_S \left(\frac{M_S}{((\underline{x}, \underline{y}, uv)^{[q]} M_S :_{M_S} u^q)} \right) \\ &= \lambda_R \left(\frac{(\underline{x}, u)^{[q]} M}{(\underline{x})^{[q]} M} \right) \lambda_{\bar{S}} \left(\frac{\bar{S}}{(\underline{y})^{[q]} \bar{S}} \right) - \lambda_S \left(\frac{M_S}{((\underline{x}, \underline{y})^{[q]} M_S :_{M_S} u^q) + v^q M_S} \right) \\ &= \lambda_R \left(\frac{(\underline{x}, u)^{[q]} M}{(\underline{x})^{[q]} M} \right) \lambda_{\bar{S}} \left(\frac{\bar{S}}{(\underline{y})^{[q]} \bar{S}} \right) - \lambda_S \left(\frac{M_S}{((\underline{x})^{[q]} M_S :_{M_S} u^q) + (\underline{y})^{[q]} M_S + v^q M_S} \right) \\ &= \lambda_R \left(\frac{(\underline{x}, u)^{[q]} M}{(\underline{x})^{[q]} M} \right) \lambda_{\bar{S}} \left(\frac{\bar{S}}{(\underline{y})^{[q]} \bar{S}} \right) - \lambda_S \left(\frac{M_S}{((\underline{x})^{[q]} M_S :_{M_S} u^q)} \otimes_S \frac{S}{(\underline{y}, v)^{[q]} S} \right) \end{aligned}$$

$$\begin{aligned}
 &= \lambda_R \left(\frac{(\underline{x}, u)^{[q]} M}{(\underline{x})^{[q]} M} \right) \lambda_{\bar{S}} \left(\frac{\bar{S}}{(\underline{y})^{[q]} \bar{S}} \right) - \lambda_S \left(\frac{M}{((\underline{x})^{[q]} M :_M u^q)} \otimes_R \frac{S}{(\underline{y}, v)^{[q]} S} \right) \\
 &\geq \lambda_R \left(\frac{(\underline{x}, u)^{[q]} M}{(\underline{x})^{[q]} M} \right) \lambda_{\bar{S}} \left(\frac{\bar{S}}{(\underline{y})^{[q]} \bar{S}} \right) - \lambda_R \left(\frac{M}{((\underline{x})^{[q]} M :_M u^q)} \right) \lambda_{\bar{S}} \left(\frac{\bar{S}}{(\underline{y}, v)^{[q]} \bar{S}} \right) \\
 &= \lambda_R \left(\frac{(\underline{x}, u)^{[q]} M}{(\underline{x})^{[q]} M} \right) \lambda_{\bar{S}} \left(\frac{\bar{S}}{(\underline{y})^{[q]} \bar{S}} \right) - \lambda_R \left(\frac{(\underline{x}, u)^{[q]} M}{(\underline{x})^{[q]} M} \right) \lambda_{\bar{S}} \left(\frac{\bar{S}}{(\underline{y}, v)^{[q]} \bar{S}} \right) \\
 &= \lambda_R \left(\frac{(\underline{x}, u)^{[q]} M}{(\underline{x})^{[q]} M} \right) \lambda_{\bar{S}} \left(\frac{(\underline{y}, v)^{[q]} \bar{S}}{(\underline{y})^{[q]} \bar{S}} \right).
 \end{aligned}$$

(In case \bar{S} is regular, equality holds throughout as $v = 1$.) Dividing both sides by $q^{\dim(S)} = q^{\dim(R)} q^{\dim(\bar{S})}$ and letting $q \rightarrow \infty$, we get

$$\begin{aligned}
 &e_{HK}((\underline{x}, y), M \otimes S) - e_{HK}((\underline{x}, y, w), M \otimes S) \\
 &\geq (e_{HK}((\underline{x}), M) - e_{HK}((\underline{x}, u), M))(e_{HK}((\underline{y})\bar{S}, \bar{S}) - e_{HK}((\underline{y}, v)\bar{S}, \bar{S})).
 \end{aligned}$$

(Equality holds if \bar{S} is regular.) As w exhausts all elements in $((\underline{x}, \underline{y}) :_S \mathfrak{n}) \setminus (\underline{x}, \underline{y})$ we have $r(M \otimes_R S) \geq r(M)r(\bar{S})$. (If \bar{S} is regular, u will also exhaust all elements in $((\underline{x}) :_R \mathfrak{m}) \setminus (\underline{x})$, implying $r(M \otimes_R S) \leq r(M)r(\bar{S}) = r(M)$.) Now (1), (2) and (3) are proved. \square

In particular, under the assumption of Theorem 5.7, if R and \bar{S} are both excellent and F -rational, then S is also F -rational. This is very similar to [En, Proposition 3.1].

Proposition 5.8. *Let (R, \mathfrak{m}, k) be a local Noetherian ring of characteristic p , $P \in \text{Spec}(R)$ a prime ideal such that $\dim(R/P) = 1$ and M a finitely generated R -module. Then $r(M) \leq r(M_P)\alpha(P)$, in which $\alpha(P) := \min \{e(x, R/P) = \lambda_R(\frac{R}{(P, x)R}) \mid x \in \mathfrak{m} \setminus P\}$. If $|k| = \infty$, then $r(M) \leq r(M_P)e(R/P)$ where $e(R/P) = e(\mathfrak{m}, R/P)$ denotes the Hilbert multiplicity of R/P considered as an one-dimensional ring.*

Proof. In case $|k| = \infty$, there exists $y \in \mathfrak{m} \setminus P$ such that $e(R/P) = e(y, R/P)$. Thus it is enough to prove the first claim. If R is not a Cohen-Macaulay domain, then $r(M) = 0$ and the claim is trivially true. Thus we assume R is a Cohen-Macaulay domain. Say $\dim(R) = d$ and therefore $\dim(R_P) = d - 1$. It suffices to prove $r(M) \leq r(M_P)e(x, R/P)$ for any $x \in \mathfrak{m} \setminus P$.

Fix an arbitrary $x \in \mathfrak{m} \setminus P$. Then there exists $\underline{x} = x_1, \dots, x_{d-1} \in P$ such that \underline{x}, x form a full system of parameters of R . Denote $I = (\underline{x})R$. Then IR_P is an ideal generated by a full system of parameters of R_P .

Now, by Theorem 2.5, it is enough to show

$$r(M) \leq (e_{HK}(IR_P, M_P) - e_{HK}((I, u)R_P, M_P))e(x, R/P)$$

for all $u \in R$ such that $(I :_R u)R_P = PR_P$.

Indeed, for any u such that $(I :_R u)R_P = PR_P$, by replacing u with wu for some suitable $w \notin P$, we may assume $(I :_R u) = P$ without loss of generality. Actually, we may further assume $u \notin (\underline{x}, x)R$. (Indeed, if $u \in (\underline{x}, x)R$, then write $u = v + u'x^t$

with $v \in I$, $u' \in R$ such that $t \in \mathbb{N}$ is the largest possible exponent among all such equations. Then $u' \notin (\underline{x}, x)R$, $(I, u)R_P = (I, u')R_P$ and, as x is regular on $R/(\underline{x})R$, we still have $(I :_R u') = P$. Replace u with u' and rename it as u .) Now we have

$$\begin{aligned}
& \lambda_R\left(\frac{R}{(I, x)^{[q]}}\right) - \lambda_R\left(\frac{R}{(I, u, x)^{[q]}}\right) \\
& \leq e\left(x^q, \frac{R}{I^{[q]}}\right) - e\left(x^q, \frac{R}{(I, u)^{[q]}}\right) \\
& = e\left(x^q, \frac{(I, u)^{[q]}}{I^{[q]}}\right) = q \cdot e\left(x, \frac{(I, u)^{[q]}}{I^{[q]}}\right) \\
& = q \cdot \sum_{Q \in \min(R/I)} \lambda_{R_Q}\left(\frac{(I, u)^{[q]}R_Q}{I^{[q]}R_Q}\right) e(x, R/Q) \quad (\text{by associativity formula}) \\
& = q \cdot \lambda_{R_P}\left(\frac{(I, u)^{[q]}R_P}{I^{[q]}R_P}\right) e(x, R/P) \quad (\text{as } (I :_R u) = P).
\end{aligned}$$

Now, dividing the above inequality by q^d and letting $q \rightarrow \infty$, we have

$$e_{HK}((I, x), R) - e_{HK}((I, u, x), R) \leq (e_{HK}(IR_P, R_P) - e_{HK}((I, u)R_P, R_P))e(x, R/P).$$

Finally, let n be the torsion-free rank of M over R . Then by Observation 2.2(3),

$$\begin{aligned}
r(M) & \leq e_{HK}((I, x), M) - e_{HK}((I, u, x), M) \\
& = n \cdot e_{HK}((I, x), R) - e_{HK}((I, u, x), R) \\
& \leq n \cdot (e_{HK}(IR_P, R_P) - e_{HK}((I, u)R_P, R_P))e(x, R/P) \\
& = (e_{HK}(IR_P, M_P) - e_{HK}((I, u)R_P, M_P))e(x, R/P).
\end{aligned}$$

This completes the proof. \square

Theorem 5.9. *Let (R, \mathfrak{m}, k) be a local Noetherian ring of characteristic p , $P, Q \in \text{Spec}(R)$ prime ideals such that $P \subsetneq Q$ and M a finitely generated R -module. Then $r(M_Q) \leq r(M_P)\alpha(P, Q)$, in which $\alpha(P, Q) := \min \{e(\underline{x}, (R/P)_Q)\}$ with \underline{x} running over all systems of parameters of $(R/P)_Q$. If $|k| = \infty$ or $Q \subsetneq \mathfrak{m}$, then $r(M_Q) \leq r(M_P)e((R/P)_Q)$ where $e((R/P)_Q) = e((Q/P)_Q, (R/P)_Q)$ denotes the Hilbert multiplicity of $(R/P)_Q$ considered as a local ring.*

Proof. If $|k| = \infty$ or $Q \subsetneq \mathfrak{m}$, then the residue field of $(R/P)_Q$ is infinite and hence there exists a system of parameters \underline{x} of $(R/P)_Q$ such that $e(\underline{x}, (R/P)_Q) = e((R/P)_Q)$. Thus it suffices to prove the first claim. Without loss of generality, we may assume $Q = \mathfrak{m}$. Then it suffices to prove $r(M) \leq r(M_P)e(\underline{x}, R/P)$ for any system of parameters \underline{x} of R/P , which we proceed by induction on $\dim(R/P)$.

If $\dim(R/P) = 1$, then the claim is proved in Proposition 5.8. Suppose the claim is true for $\dim(R/P) < c$. Now let $\dim(R/P) = c \geq 2$ and write $\underline{x} = x_1, x_2, \dots, x_c$. Also write $\underline{x}' = x_2, \dots, x_c$ and $\Gamma = \{Q \mid Q \in \min(R/(P, \underline{x}')R), \text{height}(Q/P) = c-1\}$. Fix a prime $P' \in \Gamma$. Then x_1 and \underline{x}' are systems of parameters of R/P' and $(R/P)_{P'}$ respectively, which implies $r(M) \leq r(M_{P'})e(x_1, R/P') \leq r(M_P)e(\underline{x}'(R/P)_{P'})e(x_1, R/P')$ by the induction hypothesis. Therefore it is enough to prove $e(x_1, R/P')e(\underline{x}', (R/P)_{P'}) \leq$

$e(x, R/P)$. But, by [Mat2, Exercise 14.6], we have

$$e(x_1, R/P')e(\underline{x}', (R/P)_{P'}) \leq \sum_{Q \in \Gamma} e(x_1, R/Q)e(\underline{x}', (R/P)_Q) = e(x, R/P),$$

which completes the proof. \square

Remark 5.10. Keep the notations as in Theorem 5.9.

- (1) As can be seen from the proof of Theorem 5.9, a result that is possibly sharper than what is stated explicitly in Theorem 5.9 would be that

$$r(M_Q) \leq r(M_P) \prod_{i=1}^c \alpha(P_i, P_{i-1}) = r(M_P) \alpha(Q, P_1) \prod_{i=2}^c e((R/P_i)_{P_{i-1}})$$

for any saturated chain of prime ideals $Q = P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_{c-1} \supsetneq P_c = P$. In case $|k| = \infty$ or $Q \subsetneq \mathfrak{m}$, we have $r(M_Q) \leq r(M_P) \prod_{i=1}^c e((R/P_i)_{P_{i-1}})$.

- (2) Suppose R is a domain (e.g., $r(R) > 0$), $Q = \mathfrak{m}$ and $P = 0$. Then Theorem 5.9 states that $r(R) \leq r(R_P) \alpha(0, \mathfrak{m}) = \alpha(0, \mathfrak{m})$, which is not as sharp as Lemma 4.2 since $\alpha(0, \mathfrak{m}) \geq e_{HK}(R)$. However, the estimate obtained in part (1) above could be sharper than Lemma 4.2.

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