Problem 1

(a) (10 pts.) At which points (x_0, y_0) the level curves of the functions $f(x,y) = \frac{1}{2}\ln(1+x^2+y^2)$ and $g(x,y) = 2x^2+3y^2-1$ are tangent to each other?

SOLUTION:
The tangent line to a level set $f(x,y) = c$ at a point (x_0, y_0) is perpendicular to the gradient vector $\nabla(f)(x_0,y_0) = \frac{1}{1+x_0^2+y_0^2}(x_0,y_0)$. The tangent line to a level set $g(x,y) = d$ at a point (x_0, y_0) is perpendicular to the gradient vector $\nabla(g)(x_0,y_0) = 2(2x_0, 3y_0)$. Thus two level sets are tangent at a point (x_0, y_0) if the gradients are parallel. Obvioulsy, $\nabla(f)(x_0,y_0)$ is parallel to (x_0,y_0). We must have $(x_0,y_0)\times(2x_0,3y_0) = 2x_0y_0 - 3x_0y_0 = -x_0y_0 = 0$. Thus either $x_0 = 0$ or $y_0 = 0$. In the first case $f(0,y_0) = \ln|y_0| = c$, $g(0,y_0) = 3y_0^2 = d$. Clearly, y_0 could be aribtrary, we can always choose c and d to satisfy this. Similarly, we see that any point $(x_0, 0)$ could be a point where some level curves of f and g are tangent.

(b)(10 pts) (10 pts) At which points the function $f(x,y)$ has extremum under the constraint defined by $g(x,y) = 0$, where f and g as in (a)?

SOLUTION: The extremum is where $\nabla(f)(x_0,y_0) = \lambda\nabla(x_0,y_0)$ for some λ and $g(x_0,y_0) = 0$. The first condition is exactly the condition that the level sets $f(x,y) = f(x_0,y_0)$ and $g(x_0,y_0) = 0$ are tangent. By the previous problem (x_0,y_0) is either $(x_0, 0)$ or $(0, y_0)$. Plugging in the contraint, we get $x_0 = \pm\sqrt{1/2}, y_0 = \pm\sqrt{1/3}$.

Problem 2 (a) (10 pts) Compute its length of the curve $c(t) = (2t^{3/2}, \cos 2t, \sin 2t), 0 \le t \le 1$.

SOLUTION: $c(t)' = (3t^{1/2}, -\sin 2t, 2\cos 2t), ||c(t)'|| = \sqrt{9t+4}$. Hence $L = \int_0^1 \sqrt{9t+4}dt = \frac{2}{27}(9t+4)^{3/2}|_0^1 = \frac{2}{27}(13^{3/2}-8)$.

Problem 3

(a) (10 pts.) Find the flux of the vector field $\mathbf{F} = (x, y, -z)$ over the surface given parametrically by $\mathbf{r}(u,v) = (u^2-v^2, uv, u^2+v^2), 0 \le u, v \le 1$, oriented inward.

SOLUTION We have

$$\mathbf{r}'_u = (2u, v, 2u), \quad \mathbf{r}'_v = (-2v, u, 2v), \quad \mathbf{r}'_u \times \mathbf{r}'_v = 2(v^2-u^2, -4uv, u^2+v^2).$$

Observe that

$$(u^2-v^2)^2 + 4(uv)^2 = (u^2+v^2)^2.$$

Thus the surface is contained in the cone $x^2+4y^2 = z^2$. The normal vector at the point $\mathbf{r}(1,1) = (0,1,2)$ is equal to $2(0,-4,2)$ and looks inward. Thus

$$Flux = \int_0^1\int_0^1 2(u^2-v^2, uv, -u^2-v^2)\cdot(v^2-u^2, -4uv, u^2+v^2)dudv = -4\int_0^1\int_0^1(u^4+2u^2v^2+v^4)dudv =$$
$$-4\int_0^1(\frac{u^5}{5}+\frac{2u^3}{3}v^2+uv^4|_0^1)dv = -4\int_0^1(\frac{1}{5}+\frac{2}{3}v^2+v^4)dv = -4(\frac{v}{5}+\frac{2}{9}v^3+\frac{v^5}{5})|_0^1dv = -\frac{132}{45}.$$

(b) (10 pts.) Show that the vector field $\mathbf{F} = (x\sin y + 1, \frac{1}{2}x^2\cos y, z)$ is potential and find a potential function.

SOLUTION: One checks immediately that curl$(\mathbf{F}) = 0$. The potential can be found by integrating the first coordinate as as a function in x to get $f(x,y,z) = x^2\sin y/2 + \phi(y,z)$. Then we get $f'_y + \phi(y,z)'_y = x^2\cos y/2$, and hence $\phi'_y = 0$. Finally $\phi'_z = z$ implies that

$$f(x,y,z) = (x^2\sin y + z^2)/2 + c,$$

where c is any constant.

Problem 4

(a) (5 pts.) Without computation show that the line integral of the vector field $\mathbf{F} = (x^3, y^3, -z^3)$ along the parametric curve $\mathbf{r}(t) = (\sqrt{\cos t}, \sqrt{\sin t}, 1), \frac{\pi}{6} \le t \le \frac{\pi}{3}$, is equal to zero.

SOLUTION: We observe that $\mathbf{F} = \frac{1}{4}\nabla(x^4+y^4-z^4)$. Since $(\sqrt{\cos t})^4 + (\sqrt{\sin t})^4 - 1^4 = 0$, the curve is contained in a level set of the potential function f. Thus the integral must be equal to zero (since f is constant on its level set).
Note that you cannot apply Stokes' theorem because the curve is not a boundary of any surface! It is not a closed curve.

(b) (10 pts.) Let B be a solid with boundary $S = \partial B$, and let (x_0, y_0, z_0) be a fixed point in space. Show that the volume of B is equal to one third of the flux integral of the vector field $\mathbf{F} = (x - x_0, y - y_0, z - z_0)$ over the surface S. Using this show that the volume of a pyramid is equal to one third of its height times the area of the base.

SOLUTION: Applying the divergence theorem, we obtain that

$$\iint_S = \iiint_B \operatorname{div}(\mathbf{F}) dV = 3 \iint_B dV = 3\operatorname{vol}(B).$$

By placing (x_0, y_0, z_0) at the vertex of the pyramid, we find that the restriction of $\mathbf{F}$ to the faces is perpendicular to the normal vector, hence the integral over the faces is equal to zero. The projection of the restriction of $\mathbf{F}$ to the normal vector to the base is equal to the height of the pyramid. Thus the integral over the base is equal to the one third of the height and the area of the base. The total integral over the surface is equal to the integral over the base. It remains to apply the previous formula.

(c) (5 pts.) Prove the integration-by-parts formula:

$$\iint_S f \nabla g d\mathbf{S} = \iiint_V \nabla f \cdot \nabla g dV + \iiint_V f \nabla^2 g dV,$$

where S is the boundary of a solid region V, and $\nabla^2 = \operatorname{div} \circ \nabla$ is the Laplacian operator. Suppose $g \equiv 0$ in S and $\nabla^2(g) \equiv 0$ on V. Show that $g \equiv c$ for some constant c in V.

SOLUTION: Applying the divergence theorem we obtain that the left-hand-side is equal to $\int_V \operatorname{div}(f \nabla g) dV$. We have

$$\operatorname{div}(f\nabla g) = (f g_x)'_x + (f g_y)'_y + (f g_z)'_z = (f_x g_x + f'_y g'_y + f'_z g'_z) + f(g''_{xx} + g''_{yy} + g''_{zz}) = \nabla f \cdot \nabla g + f \nabla^2 g.$$

This proves the formula. For the second part, applying the formula to the case when $f = g$ satisfies the assumptions, we get

$$\int_V \nabla g \cdot \nabla g dV = \int_V |\nabla g|^2 dV = 0.$$

Since $|\nabla g| \geq 0$, this implies that $|\nabla g| \equiv 0$. Thus $g'_x = g'_y = g'_z = 0$ everywhere in V. This implies that g is constant in V.

Problem 5(20 pts.)

(a) (10 pts.) (10 pts.) Compute the mass of the surface which is the part of the plane $2x+y+z = 3$ that lies above the region in xy-plane given by the inequalities $0 \leq x \leq \frac{1}{3}, 0 \leq y \leq 1 - 3x$, if the density function is equal to $\rho(x, y, z) = y + z$equalities $0 \leq x \leq \frac{1}{3}, 0 \leq y \leq 1 - 3x$, if the density function is equal to $\rho(x, y, z) = y + z$.

SOLUTION: The surface can be parametrized by $x = u, y = v, z = 3 - 2u - v$, where $0 \leq u \leq 1/3, 0 \leq v \leq 1 - 3u$. We have $||T_u \times T_v|| = \sqrt{6}$. Thus

$$M = \int_0^1 \int_0^{1-3u} (3 - 2u)\sqrt{6} dv du = \sqrt{6} \int_0^1 (1 - 3u)(3 - 2u) du = \sqrt{6} \int_0^1 (1 - 11u + 6u^2) du = 5\sqrt{6}/2.$$

(b) (10 pts.) Using a change of variables compute the double integral $\int\int_D \sqrt{36 - 9x^2 - 4y^2} dx dy$, where D is the area inside the ellips $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

SOLUTION: Use $x = 2r\cos\theta, y = 3r\sin\theta, 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi$. We have $|\frac{\partial(x, y)}{\partial(r, \theta)}| = 6r$. We get

$$\int\int_D \sqrt{36 - 9x^2 - 4y^2} dx dy = \int_0^{2\pi} \int_0^1 \sqrt{36 - 36r^2} 6r dr d\theta = 72\pi \int_0^1 (1 - r^2)^{1/2} r dr = -24\pi(1 - r^2)|_0^1 = 24\pi.$$