Math 285. Winter 2004/05. Differential forms

A *differential 0-form* on $\mathbb{R}^3$ is a scalar function $\omega : D \to \mathbb{R}^3$ of class C^1 (i.e. all partial derivatives are continuous functions) on a domain D in $\mathbb{R}^3$.

Let S be a set of points $p_1, \ldots, p_k$ in $\mathbb{R}^3$. We assign to each point a sign $+$ or $-$ and call this assignment an *orientation* of the set S.

The *integral* of a differential 0-form ω over an oriented set S of points is the sum

$$\int_S \omega = \epsilon_1 \omega(p_1) + \ldots + \epsilon_k \omega(p_k),$$

where $\epsilon_i = 1$ if the sign attached to p_i is equal to $+$ and $\epsilon_i = -1$ otherwise.

A *differential 1-form* on $\mathbb{R}^3$ is an expression of the form

$$\omega = Adx + Bdy + Cdz,$$

where $\mathbf{F} = (A(x,y,z), B(x,y,z), C(x,y,z)) : D \to \mathbb{R}^3$ is a vector field on a domain D in $\mathbb{R}^3$ for which the functions A, B, C belong to class C^1.

For each point $P \in D$ and a vector $\mathbf{t} = \overline{PQ}$ originated at P the differential 1-form assigns the number

$$\omega(\mathbf{t}; P) = \mathbf{F}(P) \cdot \overline{PQ}.$$

Note we put P in the notation for $\omega(\mathbf{t}; P)$ because the value depends not only on the coordinates of the vector $\mathbf{t}$ but also on the choice of the point P. In particular, the differential 1-form dx just reads off the first coordinate of $\overline{PQ}$, the form dy reads the second coordinate, and dz reads the third coordinate.

Now let C be a oriented simple curve in $\mathbb{R}^3$. We define the *integral* of a differential 1-form ω along C by

$$\int_C \omega = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt = \int_a^b \omega(c'(t); \mathbf{c}(t))dt,$$

where $\mathbf{c} : [a,b] \to C$ is a parametrization which agrees with the orientation of C.

A *differential 2-form* on $\mathbb{R}^3$ is an expression of the form

$$\omega = Ady \wedge dz + Bdz \wedge dx + Cdx \wedge dy,$$

where $\mathbf{F} = (A, B, C) : D \to \mathbb{R}^3$ is a vector field of class C^1 on a domain D in $\mathbb{R}^3$.

Let $P \in \mathbb{R}^3$ and $\mathbf{t}_1 = \overline{PQ}, \mathbf{t}_2 = \overline{PR}$ be two vectors originating at a point $P \in D$. We define

$$\omega(\mathbf{t}_1, \mathbf{t}_2; P) = \mathbf{F}(P) \cdot (\mathbf{t}_1 \times \mathbf{t}_2).$$

If $P = (a,b,c)$, $Q = (x,y,z)$, $R = (x', y', z')$, then $\mathbf{t}_1 = (x-a, y-b, z-c)$, $\mathbf{t}_2 = (x'-a, y'-b, z'-c)$ and

$$\mathbf{t}_1 \times \mathbf{t}_2 = ((y-b)(z'-c)-(y'-b)(z-c), (z-c)(x'-a)-(z'-c)(x-a), (x-a)(y'-b)-(x'-a)(y-b)).$$

In particular, the differential 2-form $dy \wedge dz$ reads off the first coordinate of the cross-product, the form $dz \wedge dx$ reads off the second coordinate, and $dx \wedge dy$ reads off the third coordinate of the cross-product. In other words, if we consider dx, dy, dz as differential 1-forms, we get

$$dy \wedge dz(\mathbf{t}_1, \mathbf{t}_2; P) = \det \begin{pmatrix} dy(\mathbf{t}_1; P) & dz(\mathbf{t}_1; P) \\ dy(\mathbf{t}_2; P) & dz(\mathbf{t}_2; P) \end{pmatrix};$$

$$dz\wedge dx(\mathbf{t}_1,\mathbf{t}_2;P)=\det\begin{pmatrix} dz(\mathbf{t}_1;P) & dx(\mathbf{t}_1;P)\\ dz(\mathbf{t}_2;P) & dx(\mathbf{t}_2;P)\end{pmatrix};$$

$$dx\wedge dy(\mathbf{t}_1,\mathbf{t}_2;P)=\det\begin{pmatrix} dx(\mathbf{t}_1;P) & dy(\mathbf{t}_1;P)\\ dx(\mathbf{t}_2;P) & dy(\mathbf{t}_2;P)\end{pmatrix}.$$

Let S be an oriented surface. We define *integral* of a differential 2-form ω along S by

$$\int_S \omega=\int_{\mathbf{\Phi}}\mathbf{F}(\mathbf{\Phi}(\mathbf{u},\mathbf{v}))\cdot(\mathbf{T_u}\times\mathbf{T_v})\mathbf{dudv}=\int_{\mathbf{\Phi}}\omega(\mathbf{T_u},\mathbf{T_v};\mathbf{\Phi}(\mathbf{u},\mathbf{v}))\mathbf{dudv},$$

where $\mathbf{\Phi}: U\to S$ be an orientation preserving orientation of S.

A *differential 3-form* on $\mathbb{R}^3$ is an expression of the form

$$\omega=f dx\wedge dy\wedge dz,$$

where $f: D\to\mathbb{R}^3$ is a scalar function of class C^1 defined on a domain D in $\mathbb{R}^3$.

Let $P\in\mathbb{R}^3$ and $\mathbf{t}_1=\overline{PQ},\mathbf{t}_2=\overline{PR},\mathbf{t}_3=\overline{PS}$ be three vectors originating at a point $P\in D$. We define

$$\omega(\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3;P)=f(P)\cdot\det\begin{pmatrix} dx(\mathbf{t}_1;P) & dy(\mathbf{t}_1;P) & dz(\mathbf{t}_1;P)\\ dx(\mathbf{t}_2;P) & dy(\mathbf{t}_2;P) & dz(\mathbf{t}_2;P)\\ dx(\mathbf{t}_3;P) & dy(\mathbf{t}_3;P) & dz(\mathbf{t}_3;P)\end{pmatrix}.$$

Note that the determinant is equal to the volume of the parallelopiped formed by the vectors $\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3$ taken with sign $+$ if the triple of vectors $\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3$, taken in this order, is a right triple (i.e. the vector $\mathbf{t}_1$ looks in the same direction as the cross-product of $\mathbf{t}_2\times\mathbf{t}_3$) and with sign $-$ otherwise.

Let D be a domain in $\mathbb{R}^3$. We define the *integral* of a differential 3-form ω along D by

$$\int_D \omega=\int_D f(x,y,z)dxdydz.$$

Now we shall define the *differentials of differential forms.*

The differential of a differentail 0-form $\omega=f(x,y,z)$ is the 1-form is defined by

$$df=Adx+Bdy+Cdz,$$

where $(A,B,C)=\nabla f=(f'_x,f'_y,f'_z)$.

The differential of a 1-form is the 2-form defined by

$$d\omega=dA\wedge dx+dB\wedge dy+dC\wedge dz=$$

$$(A'_x dx+A'_y dy+A'_z dz)\wedge dx+(B'_x dx+B'_y dy+B'_z dz)\wedge dy+(C'_x dx+C'_y dy+C'_z dz)\wedge dz=$$

$$(C'_y-B'_z)dy\wedge dz+(C'_x-A'_z)dz\wedge dx+(B'_x-A'_y)dx\wedge dy.$$

Here we use the "skew-commutative" rules to open the brackets:

$$du\wedge dv=-dv\wedge du,$$

where $u,v=x,y$ or z. In particular

$$dx\wedge dx=dy\wedge dy=dz\wedge dz=0.$$

The differential of a 2-form is a 3-form

$$d\omega = dA \wedge dy \wedge dz + dB \wedge dz \wedge dx + dC \wedge dx \wedge dy =$$

$$(A'_x dx + A'_y dy + A'_z dz) \wedge dy \wedge dz + (B'_x dx + B'_y dy + B'_z dz) \wedge dz \wedge dx + (C'_x dx + C'_y + C'_z) \wedge dx \wedge dy$$

$$= (A'_x + B'_y + C'_z) dx \wedge dy \wedge dz.$$

Here we have used the associativity law to open the brackets following the skew-commutativity property. In particular,

$$du \wedge dv \wedge dw = 0,$$

whenever two letters coincide.

The differential of a 3-form is defined to be zero.

Observe now that if we write ω_f for the 0-form defined by a function f, $\omega_{\mathbf{F}}$ for 1-form defined by a vector field $\mathbf{F}$ and $\omega^{\mathbf{F}}$ for a 2-form defined by a vector field $\mathbf{F}$, and ω^f for a 3-form defined by a function f, we obtain

$$d\omega_f = \omega_{\nabla f},$$

$$d\omega_{\mathbf{F}} = \omega^{\text{curl } \mathbf{F}},$$

$$d\omega^{\mathbf{F}} = \omega^{\text{div } \mathbf{F}}.$$

Here $\text{div } \mathbf{F}, \text{curl } \mathbf{F}$ are as defined as in the book (section 4.4).

To treat points, curves, surfaces, domains together we define the notion of an oriented *k-manifold.* A *0-manifold* is an oriented set of points. A *1-manifold* is an oriented curve, a *2-manifold* is an oriented surface, a *3-manifold* is a closed domain in $\mathbb{R}^3$.

For any k-manifold M we define its boundary $\partial\Gamma$ as as follows.

When $k = 0$ the boundary is empty.

When $k = 1$, the boundary is the set of two points $\{\mathbf{c}(a), \mathbf{c}(b)\}$ (the endpoints of the path) where the first points is taken with $-$ and the second points is taken with $+$ sign.

When $k = 2$, we additionally assume that the boundary of $\mathcal{D}_0$ is a 1-manifold. We take its one-to-one parametrization $\phi : [a, b] \to \partial\mathcal{D}_0$ and consider the boundary of the surface S as the parametrized curve $\mathbf{c}(\phi(t)) : [a, b] \to S$. We assume that the orientation of the surface and the curve agree in the following sense. When one t changes from a to b the image $\phi(t)$ moves along the boundary of $\mathcal{D}_0$ in counter-clockwise way. We always assume that the coordinates u, v are chosen in $\mathcal{D}_0$ in such a way that the u-axis rotates towards v-axis in counter-clockwise way.

When $k = 3$, we assume that the boundary of $\mathcal{D}$ is a 2-manifold. It can be parametrized by a function $\mathbf{c} : \mathcal{D}_0 \to \mathbb{R}^3$.

Theorem (Stokes). *Let M be a closed $k + 1$-manifold and ∂M be its boundary. Let ω be a differential k-form defined in a domain containing M. Then*

$$\int_{\partial M} \omega = \int_M d\omega.$$

Examples. 1. Take $k = 0, \omega = f$. Then M is a curve C and ∂M are its two boundary points. We have

$$\int_C d\omega = \int_C \nabla f d\mathbf{s} = f(Q) - f(P),$$

where Q is the end-point and P is the origin of the path. So, we obtain the theorem that the integral of the gradient vector field depends only on the endpoints of the path.

2. Take $k = 1$, $\omega = \omega_{\mathbf{F}}$. Then M is a surface S and ∂M is its boundary C. We obtain the Stokes theorem from the book:

$$\int_C \mathbf{F} d\mathbf{s} = \int_S curl(\mathbf{F}) d\mathbf{S}.$$

3. Take $k = 1$, $\omega = \omega_{\mathbf{F}}$. Assume that M is a closed domain $D \subset \mathbb{R}^2$ viewed as a surface in $\mathbb{R}^3$. Then $\mathbf{F} = (A, B, 0)$ and $\omega = Adx + Bdy$. Then $curl\mathbf{F} = (0, 0, B'_x - A'_y)$, and we obtain the Green theorem from the book:

$$\int_C Adx + Bdy = \int_D (B'_x - A'_y) dxdy.$$

4. Take $k = 2$, $\omega = \omega^{\mathbf{F}}$. Then M is a domain D in $\mathbb{R}^3$ and its boundary is a surface S. We obtain the Gauss theorem from the book:

$$\int_S \mathbf{F} d\mathbf{S} = \int_D div(\mathbf{F}) dxdydz.$$

Notice the following corollary of the Stokes Theorem:

Corollary. *Let ω be a differential k-form. Let M be a k-manifold which is a boundary of a closed $k+1$-manifold. Assume that $d\omega = 0$. Then*

$$\int_M \omega = 0.$$

Theorem. *For any differential k-form*

$$d(d\omega) = 0.$$

In particular, if M is a boundary of a closed $(k+2)$-manifold,

$$\int_M d\omega = 0.$$

Finally, let us see how to generalize differential forms to forms of any *order* k in any $\mathbb{R}^n$. Let $x_1, \ldots, x_n$ be the coordinate functions in $\mathbb{R}^n$. For any subset I of k numbers $i_1 < i_2 \ldots < i_k$ from the set $\{1, 2, \ldots, n\}$ (taken with increasing order) we set

$$dx_I = dx_{i_1} \wedge \ldots\ldots \wedge dx_{i_k}.$$

This can be considered as a function on k vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ defined by

$$dx_I(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \det[\mathbf{v}_1 \ldots \mathbf{v}_k]_{i_1, \ldots, i_k},$$

where the subscripts in the $n \times k$ matrix with columns equal to $\mathbf{v}_1, \ldots, \mathbf{v}_k$ means that we have to delete from the matrix the rows with numbers not belonging to the subset I. A differential k-form on $\mathbb{R}^n$ is an expression

$$\omega = \sum_I A_I(x) dx_I,$$

where the sum is taken over the set of all subsets I as above (there will be $\binom{n}{k}$ subsets) and $A_I(x)$ is a function on a domain D in $\mathbb{R}^n$. We understand that ω is a function of k-vectors in $\mathbb{R}^n$ depending on a point $x \in D$:

$$\omega(\mathbf{v}_1, \dots, \mathbf{v}_k; x) = \sum_I A_I(x) dx_I(\mathbf{v}_1, \dots, \mathbf{v}_k).$$

We define the *differential* of a k-form ω by

$$d\omega = \sum_I dA_I(x) \wedge dx_I. \tag{$*$}$$

Here, for any function $f : D \to \mathbb{R}$ its differential df is the 1-form

$$df = \sum \frac{\partial f}{\partial x_i} dx_i$$

and we open the brackets in (*) by following the skew-commutativity rule for the wedge product ($dx_i \wedge dx_j = -dx_j \wedge dx_i$).

Finally one defines the notion of a parametrized closed k-manifold in $\mathbb{R}^n$ as the image M of a one-to-one map of class C^1 from a closed domain $\mathcal{D}_0$ in $\mathbb{R}^k$ to $\mathbb{R}^n$ and proves the Stokes Theorem

$$\int_{\partial M} \omega = \int_M d\omega.$$

Here, we assume that the boundary of $\mathcal{D}_0$ is a $(k-1)$-manifold with a parametrization defining a positive orientation (i.e. the vectors

Exercises 1. Let $\omega = (x^2 + y^2)dx + e^x y^2 dy + xz^2 dz$. Compute
(a) $d\omega$;
(b) the value of $d\omega$ at the vectors $\overline{PQ} = (1, 2, 3), \overline{PR} = (2, 3, 4)$ at the point $P = (-1, 2, 3)$.

2. $\omega = xydy \wedge dz + e^{x+y+z} dz \wedge dx + z^2 dx \wedge dy$. Compute
(a) $d\omega$;
(b) the value of $d\omega$ at the vectors $\overline{PQ} = (1, 2, 3), \overline{PR} = (2, 3, 4), \overline{PS} = (1, 1, 1)$ at the point $P = (-1, 2, 3)$.

3. By direct computation (when $k = 0, 1, 2, 3$ and $n = 3$) prove that

$$dd(\omega) = 0.$$