Math. 512. Plane crystallographic groups

October 13, 2006

Let T be the subgroup of the group of motions $M(2)$ of the plane $\mathcal{P}$ which consists of translations. The subgroup T is a normal subgroup of $M(2)$. For each $g \in M(2)$ and $t_{\mathbf{a}} \in T$ we have

$$g \circ t_{\mathbf{a}} = t_{g(\mathbf{a})} \circ g. \tag{1}$$

Fix a point O on the plane so that any vector is equal to a vector $\mathbf{a} = \vec{OP}$. thus we can identify vectors with their end points P. Then vectors form a vector space of dimension 2. If we fix a rectangular coordinate system, then any vector $\mathbf{a}$ has coordinates (a, b). Let $O(2)$ be the group of motions which fix the point P. Then we have proved in class that any $g \in O(2)$ is a linear transformation and its matrix in the basis $\mathbf{i}, \mathbf{j}$ is an orthogonal matrix A. It is of determinant 1 if g is a rotation, and of determinant -1 if g is reflection with respect to a line ℓ containing O.

Consider a homomorphism

$$\phi : M \to O(2)$$

defined as follows. For any motion g we have

$$t_{-g(O)} \circ g(O) = O.$$

This shows that any element $m \in M(2)$ is of the form $m = t_{\mathbf{a}} \circ g$, where $g \in O(2)$. Since $\mathbf{a} = m(O)$, the end point of the vector $\mathbf{a}$, the vector $\mathbf{a}$ is defined uniquely by g, hence the orthogonal part g is defined uniquely. We set

$$\phi(t_a \circ g) = g.$$

By (1), we have

$$(t_{\mathbf{a}} \circ g) \circ (t_b \circ g') = t_{\mathbf{a}} \circ (g \circ t_b) \circ g' = (t_{\mathbf{a}} \circ t_{g(\mathbf{b})}) \circ (g \circ g') = t_{\mathbf{a}+g(\mathbf{b})} \circ g \circ g'.$$

Thus $\phi((t_{\mathbf{a}} \circ g) \circ (t_b \circ g')) = g \circ g'$. This checks that ϕ is a homomorphism.

By Chasles's Theorem any motion is either rotation $\rho_\theta(c)$ about an angle θ with fixed point c, or a reflection r_ℓ with the mirror line ℓ, or a translation $t_{\mathbf{a}}$, or a glide reflection $t_{\mathbf{a}} \circ r_\ell$. We can write

$$\rho_\theta(c) = t_{\mathbf{c}} \circ \rho_\theta \circ t_{-\mathbf{c}},$$

$$r_\ell = t_{\mathbf{a}} \circ r_{\ell'} \circ t_{-\mathbf{a}},$$

where ℓ' is the line parallell to ℓ and containing O, and $t_{\mathbf{a}}(\ell) = \ell'$. It follows from the definition of ϕ that $\phi(t_{\mathbf{a}}) = 1$, the identity map. Since ϕ is a homomorphism, we get

$$\phi(\rho_\theta(c)) = \rho_\theta, \quad \phi(r_\ell) = r_{\ell'}.$$

It follows from the definition of ϕ that its kernel is T and the image is the whole $O(2)$. Thus

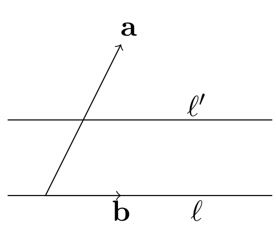
$$M(2)/T \cong O(2).$$

Lemma 1. *Let $g = t_{\mathbf{a}} \circ r_\ell$. Then $g = t_{\mathbf{b}} \circ r_{\ell'}$, where the vector $\mathbf{b}$ is parallel to the line ℓ'. Here $\mathbf{b}$ is equal to the orthogonal projection of $\mathbf{a}$ to ℓ, and ℓ' is the line parallel to ℓ which passes through the end point of the vector $\frac{1}{2}(\mathbf{a} - \mathbf{b})$.*

Proof. Choose coordinate system so that ℓ is the x-axis. Let $\mathbf{a} = (a, b)$. Consider the line ℓ' given by the equation $y = b/2$. For any $\mathbf{x} = (x, y)$ we have

$$t_{(a,0)} \circ r_{\ell'}(\mathbf{x}) = (x, y - 2(y - b/2)) + (a, 0) = (x + a, -y + b) = t_{\mathbf{a}} \circ r_\ell(\mathbf{x}).$$

$\square$



A *glide reflection* is the motion g of the form $t_{\mathbf{a}} \circ r_\ell$, where $\mathbf{a}$ is not orthogonal to the line ℓ (equivalently, its orthogonal projection to ℓ is not equal to zero). Note that if $\mathbf{a}$ is orthogonal to ℓ then g is the reflection with respect to a line parallel to ℓ. The main difference between a glide reflection and a reflection is that the former does not have fixed points, and the latter has a whole line of fixed points.

We denote a glide reflection by $s_{\ell,\mathbf{c}}$, where $s_\ell^{\mathbf{c}} = t_{\mathbf{c}} \circ r_\ell$, and $\mathbf{c}$ is parallel to ℓ.

Note that

$$s_{\ell,\mathbf{c}}^2 = (t_{\mathbf{c}} \circ r_\ell) \circ (t_{\mathbf{c}} \circ r_\ell) = t_{\mathbf{c}} \circ t_{r_\ell(\mathbf{c})} \circ r_\ell^2 = t_{2\mathbf{c}}. \tag{2}$$

Let Γ be a discrete subgroup of the group $M(2)$ of motions of the plane. Let L_Γ denote its translation part, i.e. the inetrsection $\Gamma \cap T$, where T is the subgroup of translations of M. Restricting the homomorphism ϕ to Γ we obtain a homomorphism $\phi' : \Gamma \to O(2)$. Its image is called the *point group* of Γ. We denote it by $\bar{\Gamma}$. We have

$$\Gamma / L_\Gamma \cong \bar{\Gamma}.$$

To describe Γ we have to first describe $L_\Gamma, \bar{\Gamma}$.

We have proved the following theorem in class.

Theorem 1. *The group L_Γ is of two kinds:*

(i) There exists a vector $\mathbf{a}$ such that $L_\Gamma = \{t_{m\mathbf{a}}, m \in \mathbb{Z}\}$.

(ii) There exist two non-proportional vectors $\mathbf{a}, \mathbf{b}$ such that

$$L_\Gamma = \{t_{m\mathbf{a}+n\mathbf{b}}, m, n \in \mathbb{Z}\}.$$

A discrete group is called *crystallographic* if its translation part is of the second type.

The set

$$\Omega_\Gamma = \{x\mathbf{a} + y\mathbf{b}, 0 \le x, y \le 1\}$$

is called the *fundamental parallelogram*. It has the following property.

For any point P there exists $g \in L_\Gamma$ such that $g(P) \in \Omega_\Gamma$. In fact we take $g = t_{-m\mathbf{a}-n\mathbf{b}}$, where $OP = x\mathbf{a} + y\mathbf{b}, m = [a], n = [b]$ (the integer parts of a, b.)

Let L be the set of vectors $\mathbf{a}$ such that $t_{\mathbf{a}} \in L_\Gamma$. Identifying vectors with their end-points we consider L as a subset of the plane $\mathcal{P}$ (a *lattice*).

The following lemma is Artin's, Prop. (4.6).

Lemma 2. *Let $g \in \bar{\Gamma}$. Then $g(L) \subset L$.*

Proof. If $g \in \bar{\Gamma}$, then $h = t_{\mathbf{a}} \circ g \in \Gamma$ for some translation $t_{\mathbf{a}} \in T$. For any $\mathbf{x} \in L$ we get

$$t_{g(\mathbf{x})} = g \circ t_{\mathbf{x}} \circ g^{-1} = (t_{-\mathbf{a}} \circ h) \circ t_{\mathbf{x}} \circ (h^{-1} \circ t_{\mathbf{a}})$$

$$= t_{-\mathbf{a}} \circ (h \circ t_{\mathbf{x}} \circ h^{-1}) \circ t_{\mathbf{a}}) = t_{-\mathbf{a}} \circ t_{h(\mathbf{x})} \circ t_{\mathbf{a}} = t_{h(\mathbf{x}} \in \Gamma.$$

This shows that $g(\mathbf{x}) \in L$. □

Note that elements g in the point group $\bar{\Gamma}$ are not necessary in Γ. However, $t_{\mathbf{a}} \circ g \in \Gamma$ for some translation $t_{\mathbf{a}}$ (not necessary in L_Γ).

Lemma 3. *The group $\bar{\Gamma}$ is a finite group. Its intersection with $O(2)^+$ (the subgroup of rotations) is a cyclic group C_n of order $n = 1, 2, 3, 4$ or 6. If $\bar{\Gamma}$ contains a reflection, then it is a group D_2, D_3, D_4, or D_6.*

Proof. One proves that $\bar{\Gamma}$ is finite because Γ is discrete (we omit the proof). Thus any $g \in \bar{\Gamma} \cap O(2)^+$ is represented by an orthogonal matrix of finite order. It looks as

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta = 2\pi/n.$$

Let $L = \mathbb{Z}\mathbf{a} + \mathbb{Z}\mathbf{a}$. Since $g(L) \subset L$, we get

$$g(\mathbf{a}) = m_1\mathbf{a} + n_1\mathbf{b}, \quad g(\mathbf{b}) = m_2\mathbf{a} + n_2\mathbf{b},$$

for some integers m, n. The matrix

$$B = \begin{pmatrix} m_1 & n_1 \\ m_2 & n_2 \end{pmatrix}$$

represents g in the basis $(\mathbf{a}, \mathbf{b})$. Hence $B = CAC^{-1}$ for some matrix C (the basis change matrix). Also, this implies that the traces of A and B coincide (it is the sum of the eigenvalues of g). This gives

$$Tr(A) = 2\cos 2\pi i/n = Tr(B) = m_1 + n_2 \in \mathbb{Z}.$$

Thus $2\cos 2\pi i/n = 0, \pm 1, \pm 2$. This gives $2\pi/n = \pi/2, 3\pi/2, \pi/3, 2\pi/3, 0, \pi$. This proves the first assertion. If Γ contains a reflection, then it is a dihedral group with the cyclic subgroup of index 2 of one of C_n from the assertion. □

Definition 1. *A lattice L is called* rectangular *(resp.* hexagonal*) if one can choose a rectangular fundamental parallelogram (resp. parallelogram with two equal side and the angle $\pi/3$ between them). A lattice is called* half-rectangular *if it is spanned by a side of a rectangular and its middle point.*

In the following, if L is rectangular (resp. hexagonal) we will choose a basis $(\mathbf{a}, \mathbf{b})$ of a lattice L which defines a fundamental parallelogram which is a rectangle (resp. has equal length sides with the angle $\pi/3$ between the sides). However, note that there are bases of the same lattice of different shape.

Assume that a reflection $r_\ell \in \bar{\Gamma}$. Then, for any $\mathbf{x} \in L$, $\mathbf{x} + r_\ell(\mathbf{x}) \in L$. But it is easy to see that this vector is equal to $2pr_\ell(\mathbf{x})$, the twice of the orthogonal projection of $\mathbf{x}$ to ℓ.

Thus ℓ is spanned by a vector from the lattice L. By taking a lattice vector on ℓ of smallest length, we may assume that ℓ is spanned by $\mathbf{a}$. Now

$$2pr_\ell(\mathbf{b}) = \mathbf{b} + r_\ell(\mathbf{b}) \in L,$$

hence, after replacing $\mathbf{b}$ with $\mathbf{b} + m\mathbf{a}$ for some $m \in \mathbb{Z}$, we may assume that $pr_\ell(\mathbf{b}) = 0$, or $\frac{1}{2}\mathbf{a}$. In the first case we get a rectangular lattice, in the second case we get a half-rectangular lattice.

Now we are ready to classify all crystallographic groups.

- Type I: No element of Γ fixes a point.

In this case Γ contains only translations and glide reflections. Also the composition of two glide reflections whose glide vectors are not parallel has a fixed poins since

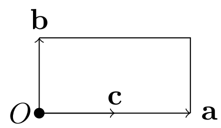
$$(t_{\mathbf{c}} \circ r_\ell) \circ (t_{\mathbf{c}'} \circ r_{\ell'}) = t_{\mathbf{c}} \circ t_{r_\ell(\mathbf{c}')} \circ r_{\ell'} \circ r_{\ell'} = t_{\mathbf{c}+r_\ell(\mathbf{c}')} \circ \rho_{2\theta},$$

where θ is the angle between ℓ and ℓ'. Thus all glide reflections $s_{\ell,\mathbf{c}}$ have lines ℓ parallel to the same vector.

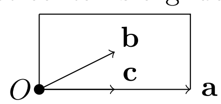
Let $s = s_{\ell,\mathbf{c}} \in \Gamma$. By changing O we may assume that $O \in \ell$. Applying (2) we see that $2\mathbf{c} \in L$. Let $\mathbf{a}'$ be the smallest length vector in L which lies on ℓ. By composing $s = t_{\mathbf{c}} \circ r_\ell$ with $t_{m\mathbf{a}}$ with $m \in \mathbb{Z}$ we may replace $\mathbf{c}$ with a vector equal to $\mathbf{a}'/2$. We know that $r_\ell(L) \subset L$. As we observed before this implies that L is a a rectangular lattice or a half-rectangular lattice. Let Ω_Γ be spanned by $\mathbf{a}'$ and $\mathbf{b}'$.

$I_1 : \bar{\Gamma} = \{1\}, L = \mathbb{Z}\mathbf{a} + \mathbb{Z}\mathbf{b}$ is any lattice.

I_2: $L = \mathbb{Z}\mathbf{a} + \mathbb{Z}\mathbf{b}$ is a rectangular lattice, $\bar{\Gamma} \cong C_2$ and the non-trivial coset contains a glide reflection $s = s_{\ell,\mathbf{a}/2}$.



I_3: $L = \mathbb{Z}\mathbf{a} + \mathbb{Z}\mathbf{b}$ is a *half-rectangular lattice*, $\bar{\Gamma} \cong C_2$ and the non-trivial coset contains a glide reflection $s = s_{\ell,(\mathbf{a}+\mathbf{b})/2}$.



- Type II: $\bar{\Gamma}$ is a cyclic group of rotations.

$II_1 : L$ is any lattice, $\bar{\Gamma} \cong C_2$ is generated by ρ_π.

$II_2 : L$ is square lattice (i.e. Ω_Γ is a square), $\bar{\Gamma} \cong C_4$ is generated by $\rho_{\pi/2}$.

$II_3 : L$ is hexagonal lattice, $\bar{\Gamma} \cong C_3$ is generated by $\rho_{2\pi/3}$.

$II_4 : L$ is hexagonal lattice, $\bar{\Gamma} \cong C_6$ is generated by $\rho_{\pi/3}$.

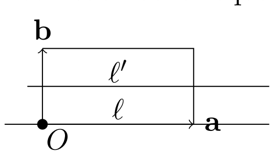
- Type III: Γ contains a reflection r_ℓ. We choose O on ℓ. Since $\mathbf{x}+r_\ell(\mathbf{x}) \in L$ for any $\mathbf{x} \in L$, we may assume that one side, say $\mathbf{a}$ of Ω_Γ spans ℓ.

We will use that the product of two reflections is a rotation about the angle equal twice the angle between the lines. This gives

$$\rho_\theta \circ r_\ell = r_{\ell'}, \tag{3}$$

where the angle between ℓ and ℓ' is equal to $\theta/2$.

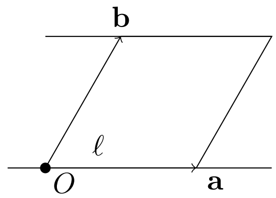
$III_1 : L = \mathbb{Z}\mathbf{a} + \mathbb{Z}\mathbf{b}$ is a rectangular lattice, $\bar{\Gamma} \cong C_2$. It is generated by r_ℓ. The mirror lines of reflections from Γ are lines parallel to ℓ with distance between each other equal to an integer multiple of $||\mathbf{b}||/2$.



Since $\bar{\Gamma}$ is of order 2, two reflections in Γ differ by a translation from L_Γ and their reflection lines are parallel to ℓ. If $r_{\ell'} = t_\mathbf{x} \circ r_\ell$, then we apply

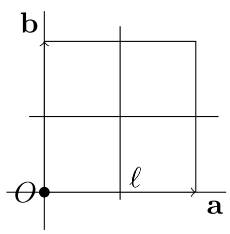
Lemma 2 to write it as $t_{\mathbf{c}} \circ r_{\ell'}$, where ℓ' is parallel to ℓ and the distance between the lines ℓ and ℓ' is equal to the length of the vector $\frac{1}{2}(\mathbf{x} - pr_\ell(\mathbf{x}))$. Write $\mathbf{x} = m\mathbf{a} + n\mathbf{b}$ for some integers m, n. Then $\mathbf{x} - pr_\ell(\mathbf{x}) = n\mathbf{b}$, and hence the length of $\frac{1}{2}(\mathbf{x} - pr_\ell(\mathbf{x}))$ is equal to $n||\mathbf{b}||/2$.

III_2 : $L = \mathbb{Z}\mathbf{a} + \mathbb{Z}\mathbf{b}$ is a hexagonal lattice, $\bar{\Gamma} \cong C_2$ is generated by r_ℓ. The mirror lines of reflections from Γ are lines parallel to ℓ with distance between each other equal to an integer multiple of $\sqrt{3}||\mathbf{b}||/2$.



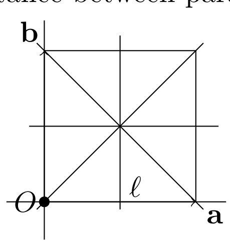
Similar to the previous case. We note that the vector $\mathbf{c} = -\mathbf{a}+\mathbf{b}$ belongs to L. It is perpendicular to $\mathbf{a}$ and its length is $\sqrt{3}||\mathbf{a}|| = \sqrt{3}||\mathbf{b}||$.

III_3 : $L = \mathbb{Z}\mathbf{a} + \mathbb{Z}\mathbf{b}$ is a square lattice, $\bar{\Gamma} \cong D_2$ is generated by ρ_π, r_ℓ. The mirror lines of reflections from Γ are lines parallel to $\mathbf{a}$ or $\mathbf{b}$ with distance between parallel lines equal to an integer multiple of $||\mathbf{a}||/2$.

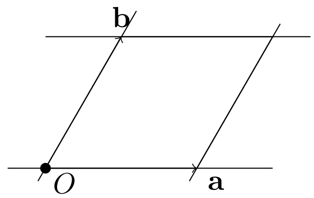


We use (3) to see that lines spanned by $\mathbf{a}$ and $\mathbf{b}$ are mirror lines, and then get more applying the argument from case III_1.

III_4 : $L = \mathbb{Z}\mathbf{a} + \mathbb{Z}\mathbf{b}$ is a square lattice, $\bar{\Gamma} \cong D_4$ is generated by $\rho_{\pi/2}, r_\ell$. The mirror lines of reflections from Γ are lines parallel to $\mathbf{a}, \mathbf{b}, \mathbf{c} = \mathbf{a} + \mathbf{b}$ with distance between parallel lines equal to an integer multiple of $\sqrt{3}||\mathbf{a}||/2$.

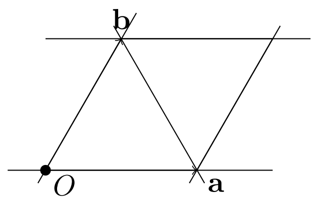


III_5 : $L = \mathbb{Z}\mathbf{a} + \mathbb{Z}\mathbf{b}$ is a hexagonal lattice, $\bar{\Gamma} \cong D_3$ is generated by $\rho_{2\pi/3}, r_\ell$. The mirror lines of reflections from Γ are lines parallel to $\mathbf{a}$ or $\mathbf{b}$with distance between parallel lines equal to an integer multiple of $\sqrt{3}||\mathbf{a}||/2$.



We use (3) to see that lines spanned by $\mathbf{a}$ and $\mathbf{b}$ are mirror lines, and then get more applying the argument from case III_2.

III_6 : $L = \mathbb{Z}\mathbf{a} + \mathbb{Z}\mathbf{b}$ is a hexagonal lattice, $\bar{\Gamma} \cong D_6$ is generated by $\rho_{\pi/3}, r_\ell$. The mirror lines of reflections from Γ are lines parallel to $\mathbf{a}, \mathbf{b}, \mathbf{a} - \mathbf{b}$ with distance between parallel lines equal to an integer multiple of $\sqrt{3}||\mathbf{a}||/2$.



- Type IV: Γ contains glide reflections $s_{\ell,\mathbf{c}}$ with $\mathbf{c} \notin L_\Gamma$ but not of type I.

Since Γ is not of type I it must contain either rotations or reflections. In fact, if it contains a reflection then the composition with a glide reflection will be a rotation. So, we may assume that Γ contains a non-trivial rotation ρ_θ. We choose a fundamental parallelogram such that $\mathbf{c} = \mathbf{a}/2$ and use that

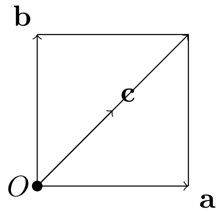
$$s_{\ell,\mathbf{c}} \circ \rho_\theta = (t_\mathbf{c} \circ r_\ell) \circ \rho_\theta = t_\mathbf{c} \circ r_{\rho_{\theta/2}(\ell)}, \tag{4}$$

This shows that the projection of $2\mathbf{c}$ to the line $\rho_{\theta/2}(\ell)$ belongs to the lattice L_Γ.

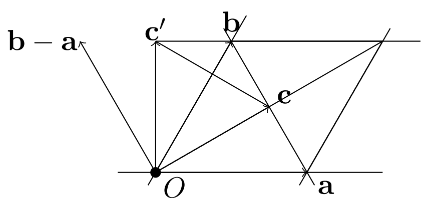
IV_1, IV_2: $L = \mathbb{Z}\mathbf{a} + \mathbb{Z}\mathbf{b}$ is a rectangular lattice, $\bar{\Gamma} \cong D_2$ is generated by ρ_π, r_ℓ. Γ contains glide reflections $s_{\ell,\mathbf{c}}$ parallel to $\mathbf{a}$ reflections with respect to lines parallel to $\mathbf{b}$. The distances between the lines parallel to $\mathbf{a}$ (resp.$\mathbf{b}$) are integer multiple of the half-lengths of $\mathbf{b}$ (resp. $\mathbf{a}$).

In this case $\theta = \pi$, hence in (4) we have $\mathbf{c}$ is perpendicular to $\rho_{\pi/2}(\ell)$. As we had remarked before this means that $s_{\ell,\mathbf{c}} \circ \rho_\theta$ is a reflection with respect to a line parallel to a line perpendicular to ℓ. This implies that L is rectangular (case IV_1) or half-rectangular (case IV_2). The other properties had been explained before.

IV_3: $L = \mathbb{Z}\mathbf{a} + \mathbb{Z}\mathbf{b}$ is a square lattice, $\bar{\Gamma} \cong D_4$ is generated by $\rho_{\pi/2}, r_\ell$. Γ contains glide symmetries $t_{\mathbf{c}} \circ r_\ell$ with ℓ parallel to the sides of the fundamental parallellogram and $\mathbf{c}$ belonging to the obit of $\frac{1}{2}(\mathbf{a} + \mathbf{b})$.



IV_4: $L = \mathbb{Z}\mathbf{a} + \mathbb{Z}\mathbf{b}$ is a hexagonal lattice, $\bar{\Gamma} \cong D_3$ is generated by $\rho_{2\pi/3}, r_\ell$. Γ contains glide reflections $s_{\ell,\mathbf{c}}$, where $\mathbf{c}$ belongs to the L_Γ-orbit of $\frac{1}{2}(\mathbf{a} + \mathbf{b})$ and glide reflections $s_{\ell,\mathbf{c}'}$, where $\mathbf{c}'$ belongs to L_Γ-orbit of $\mathbf{b} - \frac{1}{2}\mathbf{a}$.



So all 17 types of groups are accounted for.

Observe that $\bar{\Gamma} \cong \Gamma / L_\Gamma$ is isomorphic to one of the following 9 groups:

$$\{1\},\ C_2,\ C_3,\ C_4,\ C_6,\ D_2,\ D_3,\ D_4,\ D_6$$

but we got 17 different types of groups.