Math. 513. Fall 2003. Final Exam

Part I. True False (30 pts)

Circle the number which represents the true statement (no expanation needs to be given, no partial credit is given). All vector spaces are assumed to be finite-dimensional.

1. A linear operator on a finite-dimensional space over reals is self-adjoint if and only if its matrix with respect to some basis is symmetric.
2. A linear space of positive dimension with an inner product has infinitely many orthonormal bases.
3. A symmetric matrix with entries in a field F has always an eigenvector.
4. The determinant does not change after applying an elementary transformations over its rows.
5. A finite-dimensional vector space always admits an inner product.
6. An unitary linear operator is always invertible.
7. The composition of two normal operators is a normal operator.
8. Any linear operator on a finite-dimensional vector space has always a Jordan form.
9. A linear operator for which any nonzero vector is an eigenvector must be equal to the identity operator multiplied by a constant.
10. An orthogonal projection operator is a normal operator.

Part II. Proofs (25 pts)

1. Suppose a linear operator has two linearly independent eigenvectors with the same eigenvalue. Show that its characteristic polynomial has a multiple root.

2. Let $\{\lambda_1, \dots, \lambda_k\}$ be the set of eigenvalues of a complex matrix A , and $m_1, \dots, m_k$ be the respective multiplicities. Prove that $\lambda_1^{m_1} \dots \lambda_k^{m_k} = \det(A)$.

3. Let A be a complex $n \times n$ matrix such that $A^k = 0$ for some $k > 0$. Show that a Jordan form of A does not contain Jordan blocks of size greater than k.

4. Let A be a skew-symmetric nonzero real matrix (i.e. $A^T = -A$). Show that the complex matrix $\sqrt{-1}A$ is diagonalizable and has real eigenvalues.

5. Prove that the composition of two unitary operators is a unitary operator. Is the same true for self-adjoint operators?

Part III. Computational (45 pts)

1. Find an invertible matrix C and a Jordan matrix J such that

$$\begin{pmatrix} 6 & 6 & -15 \\ 1 & 5 & -5 \\ 1 & 2 & -2 \end{pmatrix} = C^{-1} \cdot J \cdot C.$$

2. Find an orthogonal matrix Q and a diagonal matrix Λ such that

$$\begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix} = Q^T \cdot \Lambda \cdot Q.$$

3. Let $\mathcal{P}_2$ be the linear space of real polynomials of degree ≤ 2. Consider the inner product on $\mathcal{P}_2$ defined by

$$\langle p(x), q(x) \rangle = \int_0^1 p(x)q(x)dx.$$

Let W be the linear subspace spanned by the polynomials 1 and x. Find an orthogonal projection of the polynomial $p(x) = 1 + x^2$ to W. Find a basis in the space $W^{\perp}$.