Math. 513. Homework 1

1. Check whether the following set F forms a field with respect to given operations.

(a) $F = \{a + b\sqrt{2} \in \mathbb{R} : a, b \in \mathbb{Q}\}$ considered as a a subset of $\mathbb{R}$ with respect to the usual operations of addition and multiplication.

(b) $F = \{a + bi \in \mathbb{C} : a, b, c \in \mathbb{Q}\}$ considered as a a subset of $\mathbb{C}$ with respect to the usual operations of addition and multiplication.

(c) $F = \{a+b\sqrt{2}+c\sqrt{3} \in \mathbb{R} : a, b, c \in \mathbb{Q}\}$ considered as a a subset of $\mathbb{R}$ with respect to the usual operations of addition and multiplication.

(d) F is the set of matrices $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$, where $a, b \in \mathbb{R}$ with respect to matrix multiplication and addition.

(e) Let F_1 and F_2 be two fields. Let $F = F_1 \times F_2$ of pairs $(a, b), a \in F_1, b \in F_2$ with operations $(a, b) + (a', b') = (a + a', b + b')$ and $(a, b) \cdot (a', b') = (aa', bb')$.

(f)) $F = \mathbb{R}^2$ is the set of 2-vectors (a, b) with operations $(a, b)+(a', b') = (a+a', b+b')$ and $(a, b)\cdot(a', b') = (aa' - bb', ab' + a'b)$.

2. Let F be a field. Deduce form the axioms that $(-1)a = -a$ and $0 \cdot a = 0$ for any $a \in F$.

3. Let F be a set consisting of 4 elements. Define operations $+$ and $\cdot$ satisfying the axioms of a field.

4. What is the smallest subfield of $\mathbb{R}$ (i.e. a field contained in $\mathbb{R}$ but not containing any other filed)?

5. Solve the following system of linear equations over the given field F.

$$x_1 + x_2 - x_3 = 0$$

$$-x_1 + 4x_2 + x_3 = 0$$

$$x_2 + x_3 = 0.$$

(a) $F = \mathbb{R}$

(b) $F = \mathbb{F}_5$