

Math. 513. Fall 2003. Midterm Exam

Part I. True False (20 pts) Circle the number which represents the true statement (no expansion needs to be given, no partial credit is given).

1. The inverse of a bijective linear map of linear spaces is a linear map.
2. A system of m linear equations with n unknowns is always solvable if $n > m$.
3. The rank of the sum of two matrices of the same size is always larger than the rank of each matrix.
4. The rank of the product of two square matrices is less or equal than the rank of each matrix.
5. Two linear spaces of the same (finite) dimension are isomorphic if and only if there exists a surjective linear map from one space to another.
6. A finite-dimensional linear space has infinitely many different bases.
7. The intersection of two linear subspaces of a vector space is a linear subspace.
8. A linear space of dimension > 1 over the field of real numbers has infinitely many subspaces.
9. For any two square matrices A, B one has $(A + B)^2 = A^2 + 2AB + B^2$.
10. If $T : V \rightarrow V$ is a linear transformation, then $Null(T) \neq R(T)$.

Part II. Proofs (20 pts)

1. Prove that the number of vectors in a linear space of dimension n over a field F consisting of p elements is equal to p^n .
2. Let v_1, \dots, v_n form a basis of a vector space V and w_1, \dots, w_n are arbitrary n vectors in a vector space W . Show that there exists a linear transformation from $T : V \rightarrow W$ such that $T(v_1) = w_1, \dots, T(v_n) = w_n$.
3. Prove that any 2×2 matrix can be reduced to a row-reduced echelon form by using at most 4 elementary operations.
4. Suppose the product AB of two square matrices is an invertible matrix. Prove that each matrix A and B is invertible.

Part III. Computational (30 pts)

1. Let $T : M_2(F) \rightarrow M_2(F)$ be the map from the space of 2×2 matrices to itself which is defined by $T(A) = A^t$ (the transpose of the matrix). Show that T is a linear transformation. Compute the matrix of T with respect to the bases $\mathcal{B} = \mathcal{B}'$ in $M_2(F)$ formed by the matrices E_{ij} (with 1 at the (ij) -spot and 0 elsewhere).

2. (a) Find the inverse of the matrix

$$A = \begin{pmatrix} 3 & -2 & 4 \\ 5 & 3 & 3 \\ 2 & 5 & -2 \end{pmatrix}$$

Find a solution of the equation

$$A \cdot X = B,$$

where

$$B = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & -1 \end{pmatrix},$$

and the unknown X is a 3×3 matrix.

3. Find a basis in the space of solutions of the system of linear equations $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 2 & 1 & -3 \\ -1 & -2 & 1 & -2 \end{pmatrix}$$

Find a solution of the system $Ax = b$, where $b = (1, 1, 1)$.