Math.594. Solutions. MIDTERM EXAM. Winter 2004

1a)Let G act 2-transitively on a set X. Show that the stabilizer subgroup G_x of each $x \in X$ is a maximal subgroup of G (i.e. is not contained in any proper subgroup of G).

Solution: Suppose $G_x \subset H$. Let $g \in G \setminus H$ and $h \in H \setminus G_x$. Since G acts 2-transitively, there exists $g' \in G$ such that $g'(gx, x) = (x, hx)$, i.e. $g'gx = x, g'x = hx$. The second equality implies $h^{-1}g' \in G_x \subset H$, i.e. $g' \in H$. The first equality gives $g'g \in G_x$, i.e. $g \in H$, a contradiction.

1b) Show that the alternating group A_n acts $(n-2)$-transitively on a set of n elements. Show also the converse, any proper subgroup of S_n acting $(n-2)$-transitively on the set $\{1, \dots, n\}$ coincides with A_n.

Solution: Let $I = (i_1, \dots, i_{n-2})$ be an ordered subset of $(n-2)$ elements. It is enough to show that there exists $g \in A_n$ such that $g(s) = i_s$, $s = 1, \dots, n-2$. Certainly, there exists $g \in S_n$ satisfying this property. If $g \notin A_n$, we replace g with $g \circ s$, where s is the transposition $(n-1n)$.

Conversely, it is easy to see that A_n is generated by cycles (abc). Indeed, each $g \in A_n$ is a product of even number of transpositions, the product of two transpositions can be always written as the product of cycles (abc). So it suffices to show that $(abc) \in G \subset S_n$. Without loss of generality, we may assume that $(abc) = (123)$. Consider the ordered set of $n-2$-elements $(1234 \dots n-2)$. There exists $g \in G$ such that $g(1) = 3, g(2) = 1, g(3) = 2, g(i) = i, i > 3$. Then $g(n-1) = n-1, g(n) = n$, or $g(n-1) = n, g(n) = n-1$. In any case, $g^2 = (123) \in G$.

2a) Show that the Sylow p-subgroup of the permutation group S_{2p} is abelian.

Solution: $(2p)! = p^2 k$, where $(k, p) = 1$. Thus the p-Sylow subgroup is of order p^2, and we saw that each such group is abelian (its center is of order p, and the quotient is cyclic).

2b) Prove that groups of order 185 and 255 are abelian.

Solution: $185 = 37.5$. Hence the 37-Sylow subgroup is normal and G is a semi-direct product of $\mathbb{Z}/37$ and $\mathbb{Z}/5$. Since $Aut(\mathbb{Z}/37) = \mathbb{Z}/(36)$ does not contain nontrivial elements of order 5. The semi-direct product is the direct product.

$255 = 3.5.17$. The 17-Sylow subgroup H is normal. Since 3 and 5 do not define 16, any element of order 3 or 5 acts on H by the conjugation trivially. Thus H belongs to the center. The quotient G/H is a cyclic group of order 15. Thus G is abelian (by a homework problem).

3a) Let F be a free group with two generators. Does it contain a subgroup of index 2?

Solution: Consider a homomorphism $F(x, y) \to \mathbb{Z}/2$ defined by $x \to 0, y \to 1$. Its kernel is a subgroup of index 2.

3b) Show that the subgroup of a free group F_n with n generators generated by the squares of all elements of F_n has index 2^n.

Solution: The square of each element in the quotient group is equal to 1. Thus the group is 2-elementary abelian, i.e. isomorphic to $(\mathbb{Z}/2)^k$. Obviously, the free generators of F_n are mapped to generators of this group. So, $k = n$.

4a)Prove that the dihedral group D_{2n} of order $2n$ has $(n-1/2$ (n is odd) or $(n-2)/2$ (n is even) isomorphism classes of irreducible representations of dimension >1. Compute their characters. How many non-isomorphic irreducible 2-dimensional representations has the binary dihedral group?

Solution: Let r be the rotation and s be a reflection generating D_n. We have $sr^ks = r^{-k}, rsr^kr^{-1} = sr^{k-2}$. We have $1+[n/2]$ conjugacy classes $C(r^k) = \{r^k, r^{-k}\}, k = 0,\ldots,[n/2]$ of r^k and 2 conjugacy classes $C(s)$ of s and $C(sr)$ of sr.

When $n = 2d$ is even, the group D_n has a normal subgroup H generated by the rotation r^2. The quotient group $G = D_n/H$ is isomorphic to the sum of two cyclic groups of order 2. The 4 one-dimensional representations of G define 4 one-dimensional representations of D_n. We have a natural 2-dimensional representation defined by

$$\rho(r) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \quad \rho(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\zeta = e^{2\pi i/n}$. Its character is defined by $\chi(r^k) = 2\cos 2\pi k/n), \chi(s) = \chi(sr) = 0$. It is easy to verify that $s \mapsto s, r \mapsto r^t$ is a homomorphism $D_n \to D_n$. Composing it with ρ we obtain an irreducible 2-dimensional representation ρ_t. Obviously

$$\rho_t(r) = \begin{pmatrix} \zeta^t & 0 \\ 0 & \zeta^{-t} \end{pmatrix}, \quad \rho_t(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is clear t mod n and $-t$ mod n give isomorphic representations. Thus we have 4 one-dimensional represenatations and $d-1 = (n-2)/2$ 2-dimensional representations. This agrees $2n = 4d = 4 + 4(n-2)/2$. The characters of 2-dimensional representations are

$$\chi_{\rho_t}(r^k) = 2\cos 2\pi tk/n, \chi_{\rho_t}(s) = \chi_{\rho_t}(sr) = 0.$$

When $n = 2d+1$ we have only (r) normal subgroup of index 2. We get 2 one-dimensional representations. The remaining are $d = (n-1)/2$ two-dimensional representations ρ_t. This agrees $2n = 2d+2 = 2+4(n-1)/2$.

The binary dihedral group BD_n is generated by the matrices

$$r = \begin{pmatrix} e^{\pi i/n} & 0 \\ 0 & e^{-\pi i/n} \end{pmatrix}, \quad s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We have always a normal subgroup of index 4 which gives 4 one-dimensional representations. The remaining $n-1$ representations are 2-dimensional. They are defined by the formulas

$$\rho_t(r) = \begin{pmatrix} e^{\pi ti/n} & 0 \\ 0 & e^{-\pi ti/n} \end{pmatrix}, \quad \rho_t(s) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad t = 1,\ldots,n-1.$$

This agrees $4n = 4 + 4(n-1)$.

4b) Let $\rho : G \to GL(V)$ be a linear representation of a finite group G in a finite-dimensional vector space V. Consider the representation of G in $\Lambda^2(V)$ defined by $\Lambda^2(\rho)(g)(x \wedge y) =$

$(\rho(x) \wedge \rho(y))$. Prove the following formula for the character of this representation in terms of the character of the representation ρ: $\chi_{\wedge^2(\rho)}(g) = \frac{1}{2}(\chi_\rho(g)^2 - \chi_\rho(g^2))$.

Solution: Choose a basis $e_1, \ldots, e_n$ in V. Then $e_i \wedge e_j, 1 \leq i < j \leq n$ is a basis in $\Lambda^2(V)$. We have $\rho(g)(e_i) = \sum_k a_{ki}e_k$, $\rho(g)(e_j) = \sum_s a_{sj}e_s$, hence

$$\rho(g)(e_i \wedge e_j) = \sum_{1 \leq k,s \leq n} a_{ki}a_{sj}e_k \wedge e_s = \sum_{1 \leq k < s \leq n} (a_{ki}a_{sj} - a_{si}a_{kj})e_k \wedge e_s.$$

Now

$$Trace(\wedge^2(\rho(g)) = \sum_{1 \leq i < j \leq n} (a_{ii}a_{jj} - a_{ji}a_{ij}) = \frac{1}{2}(\sum_{i=1}^{n} a_{ii})^2 - \frac{1}{2} \sum_{1 \leq i,j \leq n} a_{ji}a_{ij}.$$

Now it is clear that the first summand is equal to $\frac{1}{2}\chi_\rho(g)^2$ and the second one is equal to $\frac{1}{2}\chi_\rho(g^2)$.