# Groups of order $p^2q$

January 31, 2004

We will use the following lemma, whose proof is a HW problem (stated with $m = 1$).

**Lemma 0.1** *Let $G$ be a group of order $p^nq^m$, where $p, q$ are prime numbers. Assume that $|Syl_p(G)| = 1$. Let $H$ be a Sylow $p$-subgroup and $K$ be any Sylow $q$-subgroup. Then $G \cong H \rtimes K$, where the semi-direct product is defined by*

$$\alpha : K \to Aut(H), \alpha(k)(h) = khk^{-1}.$$

*A semi-direct product $H \rtimes_\beta K$, where $\beta : K \to Aut(H)$, is isomorphic to $G$ if and only if there exists an automorphism $\phi : K \to K$ and an element $\psi \in Aut(H)$ such that, for any $k \in K$,*

$$\beta(k) = \psi \circ \alpha(\phi(k)) \circ \psi^{-1}.$$

Let $G$ be a finite group of order $p^2q$, where $p, q$ are prime numbers.

CASE 1: $p > q$.

By Sylow's Theorem, $|Syl_p(G)|$ is 1 or q and $\equiv 1 \mod p$. Since $q < p$ and not equal to 1, we get $\mid Syl_p(G) \mid = 1$. Thus $G$ contains an invariant p-Sylow subgroup $H$. Let $g$ be an element of order $q$ and $K = (g) \cong \mathbb{Z}/q$. By Lemma, $G \cong H \rtimes K$ with respect to a homomorphism $\alpha : K \to Aut(K)$ defined by $\alpha(g)(h) = ghg^{-1}$.

We have two cases:

a) $H \cong \mathbb{Z}/p^2$, b) $H \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$.

In case a), $Aut(H) \cong (\mathbb{Z}/p^2)^*$ the group of invertible elements of the ring $\mathbb{Z}/p^2$ (each automorphism is uniquely defined by its image of the coset of 1, which must be an invertible element of $\mathbb{Z}/p^2$). The number of such elements is $\phi(p^2) = p(p-1)$ (the set consists of the cosets of numbers $k < p^2$ which are not divisible by $p$). Thus $\alpha$ is defined by sending $1 \mod q$ to an element of

$(\mathbb{Z}/p^2Z)^*$ which is of order $q$. This shows that a non-trivial homomorphism exists only if $q \mid p(p-1)$, thus $q \mid p-1$.

We use the following (we will prove it later, when dealing with cyclotomic field extensions).

**Lemma 0.2** *Assume $p \neq 2$. Then*

$$(\mathbb{Z}/p^n\mathbb{Z})^* \cong \mathbb{Z}/(p^n - p^{n-1}).$$

Notice that $(\mathbb{Z}/2^n)^*$ is not cyclic for $n \geq 3$. For example, $(\mathbb{Z}/8)^* = \{1, 3, 5, 7\}$ and $3^2 = 5^2 = 7^2 = 1$ (modulo 8). Thus the group is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.

So, if $q$ does not divide $p-1$, $G$ is an abelian group isomorphic to $\mathbb{Z}/p^2 \oplus \mathbb{Z}/q$. If $p - 1 = cq$ for some positive integer $c$, then there exists a unique (up to isomorphism) non-abelian group of order $p^2q$. It is isomorphic to the semi-direct product $\mathbb{Z}/p^2 \rtimes \mathbb{Z}/q$, where $\alpha : \mathbb{Z}/q \to Aut(\mathbb{Z}/p^2)$ is defined by sending $1 \mod q$ to the automorphism $x \mapsto a^c x$, where $(a) = (\mathbb{Z}/p^2)^*$. Changing $a^c$ by any power prime to $q$ does not change the group (up to isomorphism). The group depends only on the choice of a cyclic subgroup of $(\mathbb{Z}/p^2)^*$ of order $q$, which is unique.

Now consider case b) $H \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$. In this case the group is isomorphic to the semi-direct product $(\mathbb{Z}/p)^2 \rtimes \mathbb{Z}/q$ with respect to a homomorphism

$$\alpha : \mathbb{Z}/q \to Aut((\mathbb{Z}/p)^2) = GL_2(\mathbb{F}_p).$$

The group $GL_2(\mathbb{F}_p)$ is of order $(p-1)(p^2-1)p$. Thus in order that $\alpha$ be non-trivial the number $q$ must divide $(p-1)(p^2-1)p$, or equivalently, $(p^2-1)$. Thus, by Lemma 0.1, the number of non-isomorphic non-abelian groups is equal to the number of conjugacy classes of cyclic subgroups of order $q$ in the group $GL_2(\mathbb{F}_p)$.

Examples 1. Let $|G| = 245$. We have $p = 7, q = 5$. Since $q$ does not divide $p^2 - 1$, all groups of order 245 are abelian. Notice that this is the smallest number of the form $p^2q$, where $p > q$ and $q$ does not divide $p^2 - 1$. Indeed, for any prime number $p > 2$, the number $p^2 - 1$ is always divisible by 3.

2. Let $G = 75$. We have $p = 5, q = 3$. Since 3 does not divide $p - 1$, the $p$-subgroup is isomorphic to $(\mathbb{Z}/p)^2$ (we consider only non-abelian groups). Since 3 divides $p + 1$, we have a non-abelian group of this order. We have to find the conjugacy classes of cyclic subgroups of order 3 in $GL_2(\mathbb{F}_5)$. The

image of such group in $\mathbb{F}_5^*$ under the determinant homomorhism is trivial. Thus the group is contained in $SL_2(\mathbb{F}_5$. The group $PS_2(\mathbb{F}_5)$ is isomorphic to the icoshaedron group of order 60. We know that there exists only one conjugacy class of order 3 in the icosahedron group. The pre-image of this class in $SL_2(F_5)$ consists of elements $(\pm g)$, where $g$ is of order 3. Thus we have only one class of cyclic groups $(g)$. So, there is a unique non-abelian group of order 75.

CASE 2: $p < q$. Similarly to the above we obtain that $\mid Syl_q(G) \mid= 1, p$, or $p^2$. Since $p < q$, $p$ cannot be congruent to 1 mod $q$. If $p^2 \equiv 1 \mod q$, then $q \mid (p-1)(p+1)$. This is possible only if $q = 2, p = 3$.

Assume $(p,q) \neq (2,3)$, i.e. $|G| \neq 12$. Thus there exists a unique Sylow group $K$ of order $q$. It is isomorphic to $\mathbb{Z}/q$. We also have $\mid Syl_p(G) \mid= 1$ or $q$. In the latter case we must have $q \equiv 1 \mod p$. If $\mid Syl_p(G) \mid= 1$, then $K$ acts, by conjugation, on the unique $p$-Sylow subgroup $H$ of order $p^2$. As above $|Aut(H)| = p(p-1)$ or $p(p-1)^2(p+1)$. But, since $p < q$, $q$ can divide this number only in the case $p = 2, q = 3$ and $H = (\mathbb{Z}/2)^2$. In all other cases $K$ acts trivially on $H$, hence elemenst of $K$ and $H$ commute, and since both groups are abelian, the group $G$ must be abelian, and equal to the direct sum of $H$ and $K$.

So we assume $\mid Syl_p(G) \mid= q$ and $q \equiv 1 \mod p$. By Lemma 0.1, $G$ must be isomorphic to the semi-direct product $K \rtimes H$ with respect to $\alpha : H \to Aut(K) \cong \mathbb{Z}/(q-1)$ Since we are interested in non-abelian groups we assume that $\alpha$ is not trivial.

Assume $H \cong \mathbb{Z}/p^2$. We have $q - 1 = pk$. If $k$ is not divisible by $p$, then $\alpha(H)$ is the unique cyclic group of order $p$. The subgroup $Ker(\alpha)$ is isomorphic to $\mathbb{Z}/p$ and is equal to the center of the group $G$. Thus $\bar{G} = G/Z(G)$ is the unique non-abelian group of order $pq$ isomorphic to $\mathbb{Z}/q \rtimes \mathbb{Z}/p$. The pre-image of any subgroup of $\bar{G}$ of order $p$ in $G$ is a cyclic subgroup of order $p^2$ containing $Z(G)$. This gives $q$ subgroups of order $p^2$.

If $q \equiv 1 \mod p^2$, then there is also the case when $\alpha$ is injective. Then $Z(G)$ is trivial, and $G \cong \mathbb{Z}/q \rtimes \mathbb{Z}/p^2$ is defined uniquely by $\alpha$.

Assume $H \cong (\mathbb{Z}/p)^2$. Since any two subgroups of order $p$ of $H$ differ by an automorphism of $H$, there is a unique non-abelian group. Its center is again $Ker(\alpha) \cong \mathbb{Z}/p$ and the quotient $G/Z(G)$ is the unique non-abelian group of order $pq$. The pre-image of any subgroup of $\bar{G}$ of order $p$ in $G$ is a non-cyclic subgroup of order $p^2$ containing $Z(G)$. This gives $q$ subgroups of order $p^2$.

It remains to consider the case $|G| = 12$.

Case a): $|Syl_3(G)| = 4, |Syl_2(G)| = 1$. Let $H$ be a Sylow 2-group. If $H = \mathbb{Z}/4$, then $Aut(H) \cong \mathbb{Z}/2$ and $\alpha : \mathbb{Z}/3 \to \mathbb{Z}/2$ is trivial. So this implies that $G$ abelian but this is impossible since $|Syl_3(G)| = 4$. Thus $H = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $\mathbb{Z}/3 \to Aut((\mathbb{Z}/2)^2) = SL_2(\mathbb{F}_2) \cong S_3$ is a nontrivial homomorphism. Its image is the unique subgroup of order 3 of $S_3$. This case is realized by the group $A_4 \cong (\mathbb{Z}/2)^2 \rtimes \mathbb{Z}/3$.

Case b): $|Syl_3(G)| = 1$, a Sylow 2-subgroup is isomorphic to $(\mathbb{Z}/2)^2$. In this case there exists a non-trivial homomorphism $(\mathbb{Z}/2)^2 \to Aut(\mathbb{Z}/3)$. It has kernel of order 2 whose generator commutes with the unique subgroup $K$ of order 3. Thus $G$ contains a normal cyclic group $G_1$ of order 6. $G$ must be the dihedral group $D_{12}$. We have three Sylow 2-subgroups, each is generated by a pair of reflections with respect to orthogonal symmetry axes.

Case c) $|Syl_3(G)| = 1$, a Sylow 2-subgroup is isomorphic to $\mathbb{Z}/4$. In this case there exists a non-trivial homomorphism $\mathbb{Z}/4 \to Aut(\mathbb{Z}/3)$ with kernel of order 2. This kernel is the center of the group. Thus $G/Z(G) \cong \mathbb{Z}/3 \rtimes \mathbb{Z}/2 \cong S_3$. Again, we have 3 Sylow 2-subgroups obtained by taking the pre-images of 3 cyclic groups of order 2 in $S_3$.

Case d) $|Syl_3(G)| = 3, |Syl_2(G)| = 3$. It is easy to see, by listing all elements, that this case is not realized.

Notice that the case when both $H$ and $K$ are normal is not realized for a non-abelian group. In fact it is true that the semi-direct product of two normal subgroups is the direct product. This is easy to check. We have

$$(h, 1) \cdot (1, k)(h, 1)^{-1} = (h, k) \cdot (h^{-1}, 1) = (h \cdot {}^k h^{-1}, k).$$

This shows that the subgroup $K$ is normal if and only if ${}^k h = h$ for all $k \in K, h \in H$. This implies that the homomorphism $\alpha : K \to Aut(H)$ is trivial, and the product is semi-direct product.

Example 1. Any group of order 45 is abelian (because $q = 5$ is not congruent to 1 modulo $p = 3$.

Example 2. $|G| = 63, p = 3, q = 7$. Since $q \equiv 1 \mod p$ there are two non-isomorphic non-abelian groups. One is isomorphic to the semi-direct product $(\mathbb{Z}/3)^2 \rtimes \mathbb{Z}/7$, another one to $\mathbb{Z}/9 \rtimes \mathbb{Z}/7$. Both have nontrivial center of order 3.

Example 3. $|G| = 20, p = 2, q = 5$. Since $q \equiv 1 \mod p^2$ there are three non-isomorphic non-abelian groups. Two with non-trivial center $(\mathbb{Z}/2)^2 \rtimes \mathbb{Z}/5$ and $\mathbb{Z}/4 \rtimes \mathbb{Z}/5$. One with trivial center $\mathbb{Z}/4 \rtimes \mathbb{Z}/5$.