

PART I

1. Categories

Definition A category \mathcal{C} is a set of *objects* $Ob(\mathcal{C})$ and for each $A, B \in Ob(\mathcal{C})$ a set of morphisms $\mathcal{C}(A, B)$ (or $Mor_{\mathcal{C}}(A, B)$) and for each A, B, C a map $\mathcal{C}(A, B) \times \mathcal{C}(B, C) \rightarrow \mathcal{C}(A, C)$ (the image of this map at (f, g) is denoted by $f \circ g$ and is called the composition of f and g) such that the following properties hold:

- (i) there exists $1_A \in \mathcal{C}(A, A)$ such that $f \circ 1_A = f, 1_A \circ f = f$ whenever the compositions are defined;
- (ii) $Mor_{\mathcal{C}}(A, B) \cap Mor_{\mathcal{C}}(A', B') \neq \emptyset$ iff $A = A', B = B'$;
- (iii) the composition is associative.

We shall denote $Mor_{\mathcal{C}}$ the union of the sets $Mor_{\mathcal{C}}(A, B)$. The set $Ob(\mathcal{C})$ can be identified with the subset of this set by assigning to A the morphism 1_A .

To avoid some logical problems in the future we shall assume that all sets $Ob(\mathcal{C})$ and $Mor_{\mathcal{C}}(A, B)$ are subsets of a fixed set (a universum). A *universum* is a set U satisfying certain natural axioms (e.g. with each subset X the set of its parts $\mathcal{P}(X)$ is also a subset of U). In this case say that \mathcal{C} is a U -category. We shall always assume that we are dealing with some U -categories taking U large enough. Another way to solve logical problems is to admit *classes* (e.g. the set of all sets is a class but not a set). Then we define $Ob(\mathcal{C})$ to be a class, and in the case when $Ob(\mathcal{C})$ and $Mor_{\mathcal{C}}$ are sets we say that \mathcal{C} is a *small category*. The problem here is that, even when $Ob(\mathcal{C})$ is a set, the union of $Mor_{\mathcal{C}}(A, B)$ may not be a set. We stick to universums.

There are some natural definitions:

a *subcategory*: $Ob(\mathcal{C}) \subset Ob(\mathcal{C}'), \mathcal{C}(A, B) \subset \mathcal{C}'(A, B)$. A subcategory is *full* if $\mathcal{C}(A, B) = \mathcal{C}'(A, B)$.

The *dual category* \mathcal{C}° : $Ob(\mathcal{C}^\circ) = Ob(\mathcal{C}), \mathcal{C}^\circ(A, B) = \mathcal{C}(B, A)$.

A morphism $f : A \rightarrow B$ is called *monomorphism* or *injective* if the map $Mor_{\mathcal{C}}(C, A) \rightarrow Mor_{\mathcal{C}}(C, B)$ defined by the composition is injective. A map is an *epimorphism* or *surjective* if it is injective when considered to be a morphism in the dual category. A morphism which is a mono and epi is called *bijective*.

One may also define a monomorphism $u : A \rightarrow B$ as a morphism admitting a left inverse morphism $v : B \rightarrow A$ such that $v \circ u = 1_A$. Dually we get the notion of a *epimorphism*. A morphism which is a mono and epi in this sense is called an *isomorphism*. However, the new notions are not equivalent to the previous ones and very rarely used in the theory of general categories.

The set of morphisms $Mor_{\mathcal{C}}(A, B)$ has a natural quasi-order $f \leq g$ if $g = h \circ f$ for some $h : B \rightarrow B$. We say that two morphisms are equivalent if $f \leq g$ and $g \leq f$. A *subobject* of an object A is an equivalence class of monomorphisms $B \rightarrow A$. Dually one defines a notion of a *factor-object*.

Examples 1. (*Sets*): objects = sets, morphisms = maps, compositions = compositions of maps, $1_A = id_A$. Note that each subobject of a set S contains a unique representative which is a subset of S . A factor-object is an equivalence class of surjections $A \rightarrow B$. There is no a natural choice of a representative for a factor-object.

2. Various subcategories of (*Sets*) defined by putting some structures on the sets and taking morphisms to be maps compatible with these structures:

Mod_R = modules over a ring R with linear maps as morphisms

In particular, we have the category $Vect_k$ of vector spaces over a field k .

$Rings$ = associative rings, Com = commutative rings.

Top = topological spaces, morphisms = continuous maps, so on.

$Diff$ = differential manifolds. The notion of a subobject has a natural meaning: a submodule, a subring, a subspace.

3. Let Γ be an oriented graph. Define the category \mathcal{C}_Γ as follows. Its objects are the vertices, its morphisms are paths. It is not a subcategory of (*Sets*). The dual category of \mathcal{C}_Γ is $\mathcal{C}_{\Gamma'}$, where Γ' is obtained from the graph Γ by inverting the orientation of arcs in Γ .

4. Let M be a semigroup with the unity e (a monoid). It defines a category $\langle M \rangle$ with one object e and morphisms are elements from M . The composition is the multiplication. For example, if M is a group, then each morphism in $\langle M \rangle$ is an isomorphism. A category with this property is called a *groupoid*.

5 For experts: the category of affine schemes is dual to the category of commutative rings.

2. Functors

A (covariant) functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ is a map of sets $F : \text{Mor}_{\mathcal{C}} \rightarrow \text{Mor}_{\mathcal{C}'}$ which preserves the compositions and maps $\text{Ob}(\mathcal{C})$ to $\text{Ob}(\mathcal{C}')$. It is called *faithfull* if it is injective and *fully faithfull* if it is surjective on each $\text{Mor}_{\mathcal{C}}(A, B)$. A functor $F : \mathcal{C}^{\circ} \rightarrow \mathcal{C}'$ is called a *contravariant functor* from \mathcal{C} to \mathcal{C}' . From now when we say a functor we mean a covariant functor. Note that when both \mathcal{C} and \mathcal{D} are U -categories the class of functors $\text{Funct}(\mathcal{C}, \mathcal{D})$ is a set belonging to U . Very often we define a functor on objects only leaving to the reader to extend it to morphisms in a natural way.

Examples 1. \mathcal{C} is from examples 2. $F : \mathcal{C} \rightarrow (\text{Sets})$ is the *forgetful functor*, it forgets about the additional structure and sends a morphism to the corresponding map with less structure. It is always faithfull but rarely fully faithfull.

2. Let Γ be a graph. A functor $F : \mathcal{C}_{\Gamma} \rightarrow \mathcal{C}$ is called a *graph in a category*. For example, a graph in Vect_k is called a *quiver*.

3. A functor $\mathcal{F} : \mathcal{C}^{\circ} \rightarrow (\text{Sets})$ is called a *presheaf* of sets on \mathcal{C} . For example, assigning to a topological space the set of continuous functions on it is a presheaf on Top . Here, for $u : X \rightarrow Y$, we have $\mathcal{F}(u) = u^*$ is defined by composition with u . If $f : \mathcal{C}' \rightarrow \mathcal{C}$ is a functor, composing \mathcal{F} with f we obtain a presheaf on \mathcal{C}' . It is denoted by $f^*(\mathcal{F})$. For example, let X be a topological space and $\text{Op}(X)$ be the category of open subsets of X with $\text{Mor}_{\text{Op}(X)}(V, Y)$ consisting of the identity map if V is a subset of Y and empty otherwise. Then $\text{Op}(X)$ is a subcategory of Top , and restricting a pre-sheaf \mathcal{F} to $\text{Op}(X)$ we get a definition of a presheaf of sets on X . If $\mathcal{F} : \mathcal{C}^{\circ} \rightarrow (\text{Sets})$ factors through a subcategory \mathcal{C}' of (Sets) we have the notion of a \mathcal{C}' -presheaf on \mathcal{C} . for example, we can speak about the category of sheaves of abelian groups, of topological space and so on.

We can make the set of functors $\text{Funct}(\mathcal{C}, \mathcal{C}')$ into a category by defining a morphism between functors $u : F \rightarrow F'$ as a collection of morphisms $u_A : F(A) \rightarrow F'(A)$ in \mathcal{C}' such that for any $f : A \rightarrow B$ in \mathcal{C} we have

$$u_B \circ F(f) = u_A \circ F'(f).$$

There is the identity functor $I_{\mathcal{C}}$ defined in the obvious way. In particular, we have the notion of isomorphic functors.

Example 4. Let $(F_s)_{s \in S}$ be a subset of the ring of polynomials $\mathbb{Z}[T_1, \dots, T_n]$ in n variables with integer coefficients. It defines a functor $\underline{X} : (\text{Com}) \rightarrow (\text{Sets})$ by assigning to a commutative ring R the set of solutions of the system of algebraic equations $F_s(T) = 0$ in R . If $\phi : R \rightarrow R'$ is a homomorphism of rings and $(r_1, \dots, r_n) \in \underline{X}(R)$, then $(\phi(r_1), \dots, \phi(r_n)) \in \underline{X}(R')$. This defines $\underline{X}(\phi)$. Two sets of polynomials define the same functors if and only if the ideals generated by the sets are equal. Let F be another functor which assigns to R the set of homomorphisms $\mathbb{Z}[T_1, \dots, T_n]/(F_s, s \in S) \rightarrow R$. For every $r \in \underline{X}(R)$ we define an element from $F(R)$ by sending the coset of T_i to r_i . This defines an isomorphism of functors $\underline{X}(R) \rightarrow F$. Two systems (in different number of variables) define isomorphic functors if and only if the quotients rings $\mathbb{Z}[T_1, \dots, T_n]/(F_s, s \in S)$ and $\mathbb{Z}[T_1, \dots, T_{n'}]/(F_{s'}, s' \in S')$ are isomorphic.

One can define the notion of isomorphism of categories in the obvious way but this is rarely used. A weaker notion is an *equivalence of categories*: $F : \mathcal{C} \rightarrow \mathcal{C}'$ is an equivalence of categories if there exists a functor $G : \mathcal{C}' \rightarrow \mathcal{C}$ and isomorphisms of functors (considered as objects of $\text{Funct}(\mathcal{C}, \mathcal{C}')$) $u : F \circ G \rightarrow I_{\mathcal{C}'}$ and $v : G \circ F \rightarrow I_{\mathcal{C}}$ such that the map $\text{Mor}_{\mathcal{C}'}(F(A), G(B)) \rightarrow \text{Mor}_{\mathcal{C}}(A, B)$ defined by applying G is the inverse to the map $F : \text{Mor}_{\mathcal{C}}(A, B) \rightarrow \text{Mor}_{\mathcal{C}'}(F(A), F(B))$.

Example 5. For any set S let \mathcal{C}_S be the category with $\#\text{Mor}_{\mathcal{C}}(A, B) = 1$ for any $A, B \in \text{Ob}(\mathcal{C})$. Then for any S, S' the categories \mathcal{C}_S and $\mathcal{C}_{S'}$ are equivalent. They are isomorphic if and only if S and S' have the same cardinality.

3. Representable functors

Let $\check{\mathcal{C}} = \text{Funct}(\mathcal{C}^{\circ}, (\text{Sets}))$. It is the category of *presheaves* on \mathcal{C} . Note that $\check{\mathcal{C}}$ is not a U -category if \mathcal{C} is a U -category because the class of sets is a set. So, to be rigorous we have to consider U -presheaves restricting ourselves by a subcategory of (Sets) formed by U -sets.

For every $X \in \text{Ob}(\mathcal{C})$ define the presheaf h_X by

$$h_X(A) = \text{Mor}_{\mathcal{C}}(A, X), \quad h_X(A \rightarrow B) = \text{Mor}_{\mathcal{C}}(B, X) \rightarrow \text{Mor}_{\mathcal{C}}(A, X),$$

where the latter map is defined by composing with $A \rightarrow B$.

If $f : X \rightarrow Y$ is a morphism in \mathcal{C} , then we have a natural map $h_X(A) = \text{Mor}_{\mathcal{C}}(A, X) \rightarrow \text{Mor}_{\mathcal{C}}(A, Y)$ defined by composition with f . This defines a functor

$$h : \mathcal{C} \rightarrow \check{\mathcal{C}}.$$

It is called the *Yoneda functor*.

Lemma (Yoneda). *The Yoneda functor is fully faithful.*

Proof. We have to show that, for any $X, Y \in \text{Ob}(\mathcal{C})$, the map

$$h_{X,Y} : \text{Mor}_{\mathcal{C}}(X, Y) \rightarrow \text{Mor}_{\check{\mathcal{C}}}(h_X, h_Y)$$

is bijective. Notice that $\text{Mor}_{\mathcal{C}}(X, Y) = h_Y(X)$. For any $\mathcal{F} \in \check{\mathcal{C}}$ let us define a bijection

$$u : \mathcal{F}(X) \rightarrow \text{Mor}_{\check{\mathcal{C}}}(h_X, \mathcal{F})$$

such that $u = h_{X,Y}$ if $\mathcal{F} = h_Y$. Then we will be done. Take $s \in \mathcal{F}(X)$, then $u(s)$ must be a morphism from h_X to \mathcal{F} . For any $A \in \text{Ob}(\mathcal{C})$ it must assign a map $u(s)(A) : h_X(A) \rightarrow \mathcal{F}(A)$. We define it by

$$u(s)(A)(A \rightarrow X) = \mathcal{F}(A \rightarrow X)(s) \in \mathcal{F}(A).$$

If $\mathcal{F} = h_Y$, $s : X \rightarrow Y$, then $h_Y(A \rightarrow X)(s) = h_{X,Y}(s)$. It remains to show that u is bijective. We define the inverse map

$$v : \text{Mor}_{\check{\mathcal{C}}}(h_X, \mathcal{F}) \rightarrow \mathcal{F}(X)$$

as follows. Let $F : h_X \rightarrow \mathcal{F}$ be a morphism of functors. It sends the set $h_X(X) = \text{Mor}_{\mathcal{C}}(X, X)$ to $\mathcal{F}(X)$. Define $v(F)$ to be equal to the image of 1_X .

Let us check that $v \circ u = \text{id}$. Let $s \in \mathcal{F}(X)$. Then $v \circ u(s) = u(s)(X)(1_X) : \mathcal{F}(\text{id} : X \rightarrow X)(s) = s$.

Let us check that $u \circ v = \text{id}$. Let $f : h_X \rightarrow \mathcal{F}$ be a morphism of functors. Consider the diagram

$$\begin{array}{ccc} h_X(X) & \rightarrow & \mathcal{F}(X) \\ \downarrow & & \downarrow \\ h_X(Y) & \rightarrow & \mathcal{F}(Y) \end{array}.$$

corresponding to $\alpha : Y \rightarrow X$ in the definition of functor morphisms. The image of 1_X under the top arrow is equal to $v(f)$. Its image under the right vertical arrow is $\mathcal{F}(\alpha)(v(f)) = u(v(f))(Y)(\alpha)$. Now the image of 1_X under the left vertical arrow is the composition $\alpha \circ \text{id}_X = \alpha$. Its image under the bottom horizontal arrow is $f(Y)(\alpha)$. This shows that $u \circ v(f) = f$. This finishes the proof.

Corollary 1. *Let $u : F \rightarrow G$ is a morphism of presheaves. Then it is a monomorphism if and only if for any $A \in \text{Ob}(\mathcal{C})$ the map of sets $F(A) \rightarrow G(A)$ is injective.*

Proof. $\mathcal{F}(A) \rightarrow \mathcal{F}'(A) = \text{Mor}_{\check{\mathcal{C}}}(h_A, \mathcal{F}) \rightarrow \text{Mor}_{\check{\mathcal{C}}}(h_A, \mathcal{F}')$. So, if $f(A)$ is not injective f is not injective, by definition of injective morphisms. This proves the sufficiency. The converse easily follows from the definitions.

Corollary 2. *A morphism $f : X \rightarrow Y$ in \mathcal{C} is an isomorphism iff $h(f) : h_X \rightarrow h_Y$ is an isomorphism in $\check{\mathcal{C}}$.*

Definition A functor $\mathcal{F} \in \check{\mathcal{C}}$ is called representable (by an object $A \in \text{Ob}(\mathcal{C})$) if it is isomorphic to the functor h_A .

Examples 1. The functor \underline{X} from section 2, Example 4 is representable by the quotient ring $k[T]/((F_s))$.

2. Let $\mathcal{C} = \text{Mod}_R^n$. Consider the functor $M \rightarrow M^{\oplus n}$. It is representable by the free module R^n . Indeed there is a natural isomorphism

$$f_M : \text{Mor}_{\mathcal{C}}(R^n, M) = \text{Hom}_R(R^n, M) \cong M^n, \phi \rightarrow (\phi(e_1), \dots, \phi(e_n)),$$

where (e_1, \dots, e_n) is the standard basis of M^n .

Let $(X_i)_{i \in I}$ be a family of objects in \mathcal{C} . Consider the functor $\mathcal{F}(Z) = \prod_{i \in I} \text{Mor}_{\mathcal{C}}(Z, X_i)$. Here we use the usual definition of cartesian products of sets. If this functor is representable by an object A , then we have, for any Z a bijection

$$\text{Mor}_{\mathcal{C}}(Z, A) = \prod_{i \in I} \text{Mor}_{\mathcal{C}}(Z, X_i).$$

Thus any element (\dots, f_i, \dots) of the RHS defines a unique morphism $f : Z \rightarrow A$. The image of 1_A defines the maps $p_i : A \rightarrow X_i$ such that $f_i = p_i \circ f$. We denote in this case A by $\prod_{i \in I} X_i$ and call it the *direct product* of the family of objects X_i . It is defined uniquely up to isomorphism. A category is called a category with products (resp. with finite products) if products exist for any families (resp. finite families).

Dually one defines the notion of the *direct sum* $\coprod_{i \in I}$ of objects.

Examples 3. The direct products in (Sets) coincide with cartesian products. The direct sum correspond to disjoint unions.

4. The finite direct products and arbitrary direct sums in Mod_R coincide with usual direct sums of modules.

5. In the category of k -algebras the tensor product is the direct sum. The finite direct products are the direct products of algebras.

6. In $\check{\mathcal{C}}$ direct products and direct sums exist and are defined by taking the direct products and direct sums of the value sets. This allows one to define the direct product of objects in any category as an object in $\check{\mathcal{C}}$ by considering \mathcal{C} as a full subcategory of $\check{\mathcal{C}}$.

7. Let S be an object of \mathcal{C} . Consider the category \mathcal{C}/S whose objects are morphisms $X \rightarrow S$ in \mathcal{C} and morphisms $(f : X \rightarrow S) \rightarrow (g : Y \rightarrow S)$ are morphisms $u : X \rightarrow Y$ such that $g \circ u = f$. The object $1_S : S \rightarrow S$ is the *final object* of \mathcal{C} , i.e. an object e of a category such that for any other object there exists a unique morphism to e . Direct products in \mathcal{C}/S are called *fibred products* over S . If $\mathcal{C} = (\text{Sets})$, the product of $f_i : X_i \rightarrow S$ is the pre-image of the diagonal $\Delta_S \subset \prod_{i \in I} S$ under the product of the maps $\prod_{i \in I} f_i : \prod_{i \in I} X_i \rightarrow \prod_{i \in I} S$.

8. In category (Top) the direct products are usual cartesian products equipped with the product topology.

5. Structures on objects of a category.

The Yoneda functor $\mathcal{C} \rightarrow \check{\mathcal{C}}$ allows one to define a structure on an object in \mathcal{C} . Let \mathcal{C}' be some subcategory of (Sets) whose objects are sets with some additional structure. We say that $X \in \text{Ob}(\mathcal{C})$ is \mathcal{C}' -object if h_X is a \mathcal{C}' -pre-sheaf. For example, we may speak about a group object in \mathcal{C} . It is an object such that all sets $h_X(A) = \text{Mor}_{\mathcal{C}}(A, X)$ are equipped with a group structure such that for any morphism $A \rightarrow B$ in \mathcal{C} the corresponding map of sets $h_X(B) \rightarrow h_X(A)$ is a homomorphism of groups. Here are some examples:

Example 1. Let $\mathcal{C} = \text{Rings}^\circ$. Consider the functor $R \rightarrow R^*$. Then it is represented by the ring $\mathbb{Z}[t, t^{-1}]$ since

$$\text{Mor}_{\mathcal{C}}(\mathbb{Z}[t, t^{-1}], R) \rightarrow R^*, \quad \phi \rightarrow \phi(t)$$

is a functorial bijection. Thus $\mathbb{Z}[t, t^{-1}]$ is a group-object in the category Rings° .

Example 2 The ring $\mathbb{Z}[t]$ is a ring-object in the category (Rings) . Indeed $\text{Mor}_{\mathcal{C}}(\mathbb{Z}[t, t^{-1}], R) = R$.

If \mathcal{C} admit finite products and contains a final object e then one can give an equivalent definition of a group-object by requiring the existence of morphisms $\mu : X \times X \rightarrow X, \beta : X \rightarrow e, \alpha : e \rightarrow X$ satisfying some natural commutative diagrams expressing the properties of associativity, the existence of inverse and the existence of the neutral element. In fact, the existence of group structure on each $h_X(A)$ defines a map $h_X(A) \times h_X(A) \rightarrow h_X(A)$. Using the Yoneda theorem, it defines a morphism $X \times X \rightarrow X$.

6. Inductive and projective limits.

Let $F : \mathcal{I} \rightarrow \mathcal{C}$ be a functor. For any $A \in \text{Ob}(\mathcal{C})$ denote by F_A the *constant functor* $F_A(i \rightarrow j) = 1_A$. Consider the functor on \mathcal{C} by $I(A) = \text{Mor}_{\text{Funct}(\mathcal{I}, \mathcal{C})}(F, F_A)$. An element of $I(A)$ is a set of maps $f_i : F(i) \rightarrow A$ defined for each $i \in \text{Ob}(\mathcal{I})$ such that $f_i = f_j \circ \phi$, where $\phi : i \rightarrow j$ is a morphism in \mathcal{I} . If $I(A)$ is representable, then the representing object is called the *inductive limit* of F . The dual notion is the *projective limit*.

In many applications, \mathcal{I} is the category defined by a partially ordered set I identified with a graph with arrows corresponding to $i \leq j$. A functor $F : \mathcal{I} \rightarrow \mathcal{C}$ is called in this case an *inductive system*. The inductive limit is denoted by

$$\lim_{i \in I} \text{ind } F_i = \lim_{i \in I} \rightarrow F_i.$$

The dual notion is a *projective system*. The projective limit is denoted by

$$\lim_{i \in I} \text{proj } F_i = \lim_{i \in I} \leftarrow F_i.$$

Here are examples:

Example 1. I with the trivial order $i \leq j$ iff $i = j$. Then F is defined by the set $X_i = F(i)$ of objects in \mathcal{C} . The inductive limit is the direct sum, the projective limit is the direct product. By definition the direct sum (resp. direct product) of the empty set I is the *cofinal* or *initial* (resp. *final*) object of the category, i.e. $\#\mathcal{C}(e, X) = 1$ for any $X \in \text{Ob}(\mathcal{C})$, (resp. $\#\mathcal{C}(X, f) = 1$ for any $X \in \text{Ob}(\mathcal{C})$).

Example 2. Take I with a minimal element i_0 and no other elements are comparable by order. Then we get the notion of fibred products and fibred sums. Another special case is when \mathcal{I} consists of two elements with two morphisms. The functor $F\mathcal{I} \rightarrow \mathcal{C}$ is defined by two arrows in \mathcal{C} : $A \rightrightarrows B$. The projective limit is called the *equalizer* of the pair of morphisms. The dual notion is the *co-equalizer*.

Example 3. Let $I = \mathbb{N}$ with the usual order. Let $\mathcal{C} = \text{Com}$. Fix a prime number p . Consider the functor $F(n) = \mathbb{Z}/p^n$ with $F(n) \rightarrow F(m)$, $m \leq n$, equal to the natural factor map $\mathbb{Z}/p^n \rightarrow \mathbb{Z}/p^m$. The direct limit is the ring \mathbb{Z}_p of p -adic number. The inductive limit is the p -torsion subgroup of \mathbb{Q}/\mathbb{Z} .

Example 4. Inductive and projective limits of inductive systems exist if $\mathcal{C} = (\text{Sets})$. To construct the projective limit we define $\lim_{\mathcal{I}} F$ as the subset of $\prod_{i \in I} F(i)$ which consists of string (\dots, a_i, \dots) such that $(F(j) \rightarrow F(i))(a_j) = a_i$ if $j \leq i$. The inductive limit is defined as the quotient of the set $\cup_{i \in I} F(i)$ by the minimal equivalence relation containing the relation $(a_i, a_j) \in R$ if the previous equality holds.

Theorem 1. *The following properties of a category \mathcal{C} are equivalent:*

- (i) \mathcal{C} has projective limits;
- (ii) \mathcal{C} has direct products and equalizers;
- (iii) \mathcal{C} has direct products and fibred products of two objects.

Proof. Assume direct products exist. First notice that equalizers of a pair $A \rightrightarrows B$ is the fibred product $A \times_{A \times B} A$, where the two morphisms $A \rightarrow A \times B$ are $1_A \times f$ and $1_A \times g$. To show that $\lim_{\text{proj}} d$ exists if (ii) holds we construct a pair of morphisms

$$\prod_{i \in I} d(i) \rightrightarrows \prod_{u \in \text{Mor}(\mathcal{C})} d(\text{end}(u)),$$

where the first arrow is defined by the projections $\prod \rightarrow d(\text{end}(u))$ and the second one by the composition of projection to $d(\text{source}(u))$ and the morphism $d(u) : d(\text{source}(u)) \rightarrow d(\text{end}(u))$. It is immediately seen that the projective limit is isomorphic to the equalizer of this pair.

We leave to the reader to state the dual statement about the inductive limits.

Definition We say a $F : \mathcal{C} \rightarrow \mathcal{C}'$ is *left exact* (resp. *right exact*) if it commutes with finite projective (resp. inductive) limits. A functor which is left and right exact is called *exact*.

If $F : \mathcal{C}^\circ \rightarrow \mathcal{C}'$ is a contravariant functor. Then it is left exact if it transforms inductive limits in \mathcal{C} into projective limits.

Corollary. A functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ is left exact if and only if it commutes with direct products and equalizers. In other words, if

$$F\left(\prod_{i \in I} X_i\right) = \prod_{i \in I} F(X_i),$$

$$F(\text{Ker}(A \rightrightarrows B)) = \text{Ker}(F(A) \rightrightarrows F(B)).$$

Example 5. Let \mathcal{F} be a presheaf on the category $\text{Open}(X)$. Let us add to $\text{Open}(X)$ direct sums (for example, by embedding it into the category of presheafs where direct sums exist). Let $U_i \rightarrow U$ be an open covering of an open set U . Then U is the co-equalizer of the pair

$$\prod_{(i,j) \in I \times I} U_i \cap U_j \rightrightarrows \prod_{i \in I} U_i \rightarrow U.$$

Here we embed $U_i \cap U_j$ in U_i and U_j to define the two arrows. A presheaf is called a sheaf if it is left exact. This means that it defines an exact sequence of sets

$$\mathcal{F}(U) \rightarrow \text{Ker}\left(\prod_{i \in I} \mathcal{F}(U_i)\right) \rightrightarrows \prod_{(i,j) \in I \times I} \mathcal{F}(U_i \cap U_j).$$

The exactness means that $\mathcal{F}(U)$ is the equalizer of the the pair of maps. For example, the presheaf $U \rightarrow \mathcal{C}(U)$ of continuous functions is a sheaf.

Remark 1. Commute with the direct sums of empty set of indices means that the functor sends the cofinal object to the cofinal object.

We leave to the reader to state the dual statement about the right exactness.

Corollary. Suppose that \mathcal{C} has inductive limits. Then $h_X : \mathcal{C} \rightarrow (\text{Sets})$ is right exact.

Proof. It is enough to check that h_X commutes with direct sums and fibred sums. We have, by definition of the direct sum,

$$h_X\left(\coprod_{i \in I} Y_i\right) = \prod_i h_X(Y_i),$$

and similarly

$$h_X\left(A \coprod_C B\right) = h_X(A) \times_{h_X(C)} h_X(B).$$

This gives a necessary condition for representability of a presheaf of sets:

Theorem 2. Suppose $\mathcal{F} \in \check{\mathcal{C}}$ is a representable functor and \mathcal{C} has inductive limits. Then \mathcal{F} is right exact.

Let us see how to reconstruct a presheaf by its values using inductive limit. For any $F \in \check{\mathcal{C}}$ consider the category \mathcal{C}/F of objects of \mathcal{C} over F (here we identify \mathcal{C} with $\check{\mathcal{C}}$ using the Yoneda functor). Let $d : \mathcal{C}/F \rightarrow \check{\mathcal{C}}$ be the forgetful of the arrow functor $\mathcal{C}/F \rightarrow \check{\mathcal{C}}$.

Proposition 1.

$$\lim_{\mathcal{C}/F} \text{ind } d \cong F.$$

(or symbolically:

$$F = \lim_{X \rightarrow F} \text{ind } h_X.)$$

Proof. By definition, for any $(u : X \rightarrow F) \in \text{Ob}(\mathcal{C}/F)$ there is a unique morphism $d(u) \rightarrow F$. The set of such morphisms defines a morphism ϕ from the inductive limit to F . Suppose G is an object of $\check{\mathcal{C}}$ such that for any u as above there is a morphism $d(u) = h_X \rightarrow G$ (of course satisfying the functoriality condition). Let

$x \in F(X)$ correspond to some $u : X \rightarrow F$ and let $y \in G(X)$ corresponds to $d(u)$. It is easy to check that the family of maps of sets $f_X : x \rightarrow y$ defines a morphism of presheaves $F \rightarrow G$. Taking G to be the inductive limit, we get a morphism f from G to the inductive limit. It is easy to see that this is the inverse of ϕ .

7. Adjoint functors.

Let $F : \mathcal{C} \rightarrow \mathcal{C}'$ be a functor. For any $B' \in \text{Ob}(\mathcal{C}')$ consider the functor $X \rightarrow \text{Mor}_{\mathcal{C}'}(F(A), B')$. Suppose this functor is representable. Denote the representing object by $G(B')$. Then we have a bijection

$$\text{Mor}_{\mathcal{C}}(A, G(B')) \rightarrow \text{Mor}_{\mathcal{C}}(F(A), B')$$

which must be "functorial" in A and B' . It is easy to see that the correspondence $B' \rightarrow G(B')$ is functor on \mathcal{C}' . It is called the *right adjoint functor* of F . The functor F is called the *left adjoint* of G . To say more symmetrically, we introduce the product of the categories $\mathcal{C} \times \mathcal{C}'$. Suppose that we have two functors $F : \mathcal{C} \rightarrow \mathcal{C}'$ and $G : \mathcal{C}' \rightarrow \mathcal{C}$. Consider the functors $\mathcal{C} \times \mathcal{C}' \rightarrow (\text{Sets})$ defined by

$$\text{Mor}_{\mathcal{C}}(?, G(?)) : (A, B') \rightarrow \text{Mor}_{\mathcal{C}}(A, G(B')), \quad \text{Mor}'_{\mathcal{C}'}(F(?), ?) : (A, B') \rightarrow \text{Mor}'_{\mathcal{C}'}(F(A), B').$$

We require that there exists an isomorphism of these functors:

$$u : \text{Mor}_{\mathcal{C}}(?, G(?)) \rightarrow \text{Mor}'_{\mathcal{C}'}(F(?), ?).$$

In particular, taking $B' = F(A)$ we have a bijective map $\text{Mor}_{\mathcal{C}}(A, G \circ F(A)) \rightarrow \text{Mor}'_{\mathcal{C}'}(F(A), F(A))$. The pre-image of $1_{F(A)}$ defines a morphism $A \rightarrow G \circ F(A)$. Similarly, we get a morphism $F \circ G(B') \rightarrow B'$. It is easy to see that, varying A and B' we get the morphisms of functors

$$G \circ F \rightarrow \text{Id}_{\mathcal{C}}, \quad F \circ G \rightarrow \text{Id}_{\mathcal{C}'}$$

The pair of such morphisms is called the *adjunction morphisms*. Here are some examples.

Examples 1. Let $\mathcal{C} = \text{Mod}_R$. The functor $A \rightarrow M \otimes A$ is adjoint to $B \rightarrow \text{Hom}(B, M)$.

2. Let HTop be the subcategory of Top which consists of Hausdorff spaces. Let F be the inclusion functor. The adjoint functor assigns to any topological space the maximal Hausdorff factor-object (see exercise).

3. Let $F : \mathcal{C} \rightarrow (\text{Sets})'$ is a forgetful functor. The adjoint functor gives a construction of a free object in the category. For example, $\mathcal{C} = (\text{Groups})$ is the category of groups. The adjoint of the forgetful functor assigns to a set S the free group with the set of generators S . If $\mathcal{C} = \text{Com}$, the adjoint functor assigns to a set S the polynomial ring in the set of variables indexed by S . If $\mathcal{C} = \text{Top}$, then the adjoint functor assigns to a set S the discrete topological space S .

4. Let $\mathcal{C} = \text{HTop}_c$ be the category of compact Hausdorff topological spaces. The forgetful functor $\text{HTop}_c \rightarrow \text{HTop}$ has a right adjoint. This is the Stone-Chech compactification. It is defined as follows. One embeds X in the space $\mathbb{R}^{C(X)}$ by sending x to the map $f \rightarrow f(x)$. Then one compactifies X by taking the closure of X in $\mathbb{R}^{C(X)}$, where the latter has the standard product topology.

5. Let $F : \mathcal{C} \rightarrow \mathcal{C}'$ be a functor. For any presheaf of sets $\mathcal{F} : \mathcal{C}^\circ \rightarrow (\text{Sets})$ we can define the inverse image $F^*(\mathcal{F}) \in \tilde{\mathcal{C}}$ as the composition $\mathcal{F} \circ F$. For example, if $f : X' \rightarrow X$ is a continuous map of topological spaces, it defines the functor $\text{Open}(X) \rightarrow \text{Open}(X')$ by $U \rightarrow f^{-1}(U)$. If \mathcal{F} is a presheaf on X' (i.e. a presheaf of sets on $\text{Open}(X')$), its inverse image is the presheaf $f_*(\mathcal{F})$ (called the direct image under f) defined by

$$f_*(\mathcal{F})(U) = \mathcal{F}(f^{-1}(U)).$$

This defines the functor:

$$F_* : \tilde{\mathcal{C}}' \rightarrow \tilde{\mathcal{C}} : \mathcal{F}' \rightarrow F_*(\mathcal{F}')$$

For example if $\mathcal{O}_{X'}$ is the presheaf of continuous functions (i.e. $\mathcal{O}_{X'}(U') = C(U')$), then we have a natural morphism of presheaves $\mathcal{O}_X \rightarrow f_*(\mathcal{O}_{X'})$ which is obtained by composing a continuous function $\phi : U \rightarrow \mathbb{R}$ with f .

6. A presheaf $\mathcal{F} : \mathcal{C}^\circ \rightarrow (\mathit{Sets})$ has the right adjoint G if $\mathcal{C}(S, \mathcal{F}(X)) = \mathcal{C} \circ (G(S), X)$. Taking S to be a singleton, we obtain $\mathcal{F}(X) = \mathcal{C}(X, G(S))$. This means that $G(S)$ represents \mathcal{F} . So, if \mathcal{F} is not representable, \mathcal{F} does not have left adjoint.

The left adjoint of the functor $\mathcal{F} \rightarrow u^*(\mathcal{F})$ is the functor

$$u_! : \check{\mathcal{C}} \rightarrow \check{\mathcal{C}}' : \mathcal{F}' \rightarrow u_!(\mathcal{F}')$$

In the example above, $u_!(\mathcal{F})$ is denoted by $f^{-1}(\mathcal{F})$ (the inverse image under f) and is constructed as follows. Let V be an open set on X' . Consider the family of open subsets U in X such that $V \subset f^{-1}(U)$. It is an inductive system with respect to inclusions. So, we can define the inductive limit $\lim \mathcal{F}(U)$. We set $f^{-1}(\mathcal{F})(V)$ equal to this limit. In the case of general categories, we do the same. Let $\mathcal{C} \rightarrow \mathcal{C}'$ be a functor. For any $X \in \mathcal{C}'$ we consider the following subcategory I_X^u of \mathcal{C} . Its objects are morphisms $s : X \rightarrow u(Y) \in \mathit{Ob}(\mathcal{C}')$, where $Y \in \mathit{Ob}(\mathcal{C})$. The morphisms in this category are morphisms $\phi : Y \rightarrow Y'$ in \mathcal{C}' such that $s = u \circ s'(\phi)$. Define a functor $I_X^u \rightarrow (\mathit{Sets})$ by $(X \rightarrow u(Y)) \rightarrow \mathcal{F}(Y)$. The inductive limit of this functor is taken for $u_!(\mathcal{F})(X)$. In other words,

$$u_!(\mathcal{F})(X) = \lim_{X \rightarrow u(U)} \text{ind} F(U).$$

The fact that it is a contravariant functor follows from the observation that for any morphism $X \rightarrow X'$ in \mathcal{C}' one has a natural functor $I_{X'}^u \rightarrow I_X^u$ obtained by composition ($X \rightarrow X' \rightarrow u(Y)$). Note that the inductive limit always exists, since in (Sets) inductive limits exist. Also observe that $F(U) = \emptyset$ if I_X^u is the empty category. This follows from understanding that the union of sets parametrized by the empty set is the empty set. Let \mathcal{G} be a presheaf on \mathcal{C} . Suppose we have a morphism $\mathcal{F} \rightarrow u^*\mathcal{G}$. Then for any $Y \rightarrow F(X)$, we have $\mathcal{F}(Y) \rightarrow \mathcal{G}(F(Y)) = u^*(\mathcal{G})(Y) \rightarrow \mathcal{G}(X)$. By definition of inductive limits, there is a unique morphism $u_!(\mathcal{F})(Y) \rightarrow \mathcal{G}(Y)$. This shows that

$$\check{\mathcal{C}}'(u_!(\mathcal{F}), \mathcal{G}) = \check{\mathcal{C}}(\mathcal{F}, u^*(\mathcal{G})).$$

Hence u^* is right adjoint to $u_!$.

Example 7. Let us compute $u_!(h_X)$, where $X \in \mathit{Ob}(\mathcal{C})$. Let Z be an object of \mathcal{C}' and $Y \rightarrow X$ be a morphism in \mathcal{C} . For any $f : Z \rightarrow u(Y)$ in I_Z^u we have the composition map

$$h_X(Y) \xrightarrow{u} h_{u(X)}(u(Y)) \xrightarrow{h_{u(X)}(f)} h_{u(X)}(Z).$$

This defines a map from $u_!(h_X)(Z)$ to $h_{u(X)}(Z)$. Any morphism $Z \rightarrow u(X)$ can be considered as an object of the category I_Z^u , so that we have the inverse map $h_{u(X)}(Z) \rightarrow u_!(h_X)(Z)$. Thus we have constructed a bijection $u_!(h_X)(Z) \rightarrow h_{u(X)}(Z)$ which as is easy to see defines an isomorphism of presheaves:

$$u_!(h_X) \cong h_{u(X)}.$$

Let us construct the right adjoint functor to u^* . It is denoted by u_* and must have the property

$$\check{\mathcal{C}}(u^*(\mathcal{F}), \mathcal{G}) = \check{\mathcal{C}}'(\mathcal{F}, u_*(\mathcal{G})).$$

By setting $\mathcal{F} = h_X$, we get

$$\check{\mathcal{C}}(u^*(h_X), \mathcal{G}) = \check{\mathcal{C}}'(h_X, u_*(\mathcal{G})) = u_*(\mathcal{G})(X).$$

So, we take for the definition

$$u_*(\mathcal{F})(X) = \check{\mathcal{C}}(u^*(h_X), \mathcal{F}).$$

In the example $u = f^{-1} : \mathit{Open}(Y) \rightarrow \mathit{Open}(X)$ we have, for every $U \in \mathit{Open}(X)$, and any $V \in \mathit{Open}(Y)$,

$$u^*(h_U)(V) = h_U(u(V)) = h_U(f^{-1}(V)) = \mathit{Open}(X)(f^{-1}(V), U).$$

The latter is not empty if and only if $U = f^{-1}(V')$, where $V \subset V'$. Thus, $u_*(\mathcal{F})(U) = \emptyset$ if U is not as above and $u_*(\mathcal{F})(U) = \lim_{V \subset V'} \text{ind}_V \mathcal{F}(V)$, if $U = f^{-1}(V')$.

Theorem 1. Suppose $F : \mathcal{C} \rightarrow \mathcal{C}'$ admits a right adjoint functor $G : \mathcal{C}' \rightarrow \mathcal{C}$. Then F is right exact and G is left exact.

Proof. Let $d : \mathcal{I} \rightarrow \mathcal{C}$ be a functor and $X = \lim_{\mathcal{I}} d$. By definition we have a morphism of functors $d \rightarrow X$. Applying F we get a morphism of functors $F \circ d \rightarrow F(X)$. Suppose we have another morphism of functors $F \circ d \rightarrow Y$. We have to show that there exists a morphism $F(X) \rightarrow Y$ making the diagram commutative. Applying G we get morphisms $d \rightarrow GFd \rightarrow G(Y)$. By definition of inductive limit, there is a morphism $X \rightarrow G(Y)$. Hence we have $F(X) \rightarrow FG(Y) \rightarrow Y$. Similarly we prove the second assertion.

Corollary 1. The functor $u_!$ commutes with inductive limits and the functors u^*, u_* commute with projective limits.

8. Grothendieck topologies

Let \mathcal{C} be a category. A *sieve* in \mathcal{C} is a full subcategory \mathcal{D} of \mathcal{C} such that $(X \rightarrow Y) \in \mathcal{C}(X, Y)$, where $Y \in \text{Ob}(\mathcal{C})$, implies $X \in \text{Ob}(\mathcal{D})$. A sieve of an object X of \mathcal{C} is a sieve of the category \mathcal{C}/X .

Equivalently, a sieve R of X is a subobject of h_X , we have $h_X(Y)$ is the set of morphisms $Y \rightarrow X$ from the sieve. Conversely, if R is a subobject of h_X in \mathcal{C}° we consider the subcategory \mathcal{C}/R of all arrows $Y \rightarrow R$. This is a sieve of X and the two maps from sieves of X to subobjects of h_X are inverse to each other.

If $F : \mathcal{C} \rightarrow \mathcal{C}'$ is a functor, and \mathcal{D} is a subcategory of \mathcal{C}' then its inverse image $F^*(\mathcal{D})$ is the full subcategory of \mathcal{C} which consists of objects A such that $F(A) \in \text{Ob}(\mathcal{D})$ is a morphism in \mathcal{D} . If R is a sieve in \mathcal{C}' , then its inverse image is obviously a sieve in \mathcal{C} . If $F/X : \mathcal{C}/X \rightarrow \mathcal{C}'/F(X)$ the induced functor, then the inverse image of a sieve of $F(X)$ is a sieve of X .

We shall identify objects of \mathcal{C} with presheaves on \mathcal{C} which they represent.

Definition A topology in \mathcal{C} is a set of sieves $J(X)$ of X , for each object X of $\text{Ob}(\mathcal{C})$. The following axioms must satisfy:

- (T1) If $Y \rightarrow X$ is a morphism in \mathcal{C} and $R \in J(X)$, then the set of morphisms $Z \rightarrow Y$ such that the composition $Z \rightarrow Y \rightarrow X$ belongs to R forms a sieve in $J(Y)$;
- (T2) if $R \in J(X)$ and R' is a sieve of X such that for any $Y \rightarrow X$ in R the set of morphisms $Z \rightarrow Y$ such that the composition $Z \rightarrow Y \rightarrow X$ belongs to R' forms a sieve in $J(Y)$ then R' is a sieve in $J(X)$;
- (T3) $I_X \in J(X)$.

A category with a topology is called a *site*. A functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ between two sites is called *continuous* or a *morphism of sites* if for every sieve $R \in J(X)$ in \mathcal{C} the morphism $F_!(R) \rightarrow F_!(h_X) = h_{F(X)}$ factors through an isomorphism $F_!(R) \rightarrow R'$, where $R' \in J(F(X))$.

A way to define a topology is by considering for each $X \in \text{Ob}(\mathcal{C})$, a set of coverings $U_i \rightarrow X, i \in I$. Each covering defines a sieve, the subfunctor of h_X equal to the union of the images of the subpresheaves h_{U_i} . Thus $Y \rightarrow X$ belongs the sieve if $h_Y \rightarrow h_X$ through the union of h_{U_i} . To define a topology using families of coverings $\text{Cov}(X)$ of X we must require the following properties:

- (PT1) If $Y \rightarrow X$ and $(U_i \rightarrow X)_{i \in I} \in \text{Cov}(X)$, then $U_i \times_X Y \rightarrow Y$ belongs to $\text{Cov}(Y)$;
- (PT2) if $(U_i \rightarrow X)_{i \in I} \in \text{Cov}(X)$ and for each U_i we have a covering $(V_{ij} \rightarrow U_i) \in \text{Cov}(U_i)$, then $(V_{ij} \rightarrow U_i \rightarrow X)$ belongs to $\text{Cov}(X)$;
- (PT3) the morphism 1_X belongs to $\text{Cov}(X)$.

A topology defines a family of coverings, i.e. a family of morphisms $U_i \rightarrow X$ such they span a crible from $J(X)$. If \mathcal{C} has fibred products, then a topology can be defined by a pre-topology.

Example 1. If $J(X)$ is the set of all sieves of X we get a discrete topology. If we take $J(X) = \{1_X\}$, we get the "chaotique" topologie.

To give the next example we have to say a little more about inductive limits. Let $\text{Funct}(I, \mathcal{C})$ be the category of functors. Let J be another category together with a functor $d : J \rightarrow \text{Funct}(I, \mathcal{C})$ (equivalently, we have a bifunctor $D : I \times J \rightarrow \mathcal{C}$). Suppose that projective limits of the functors $d_i : J \rightarrow \mathcal{C}, j \rightarrow D(i \times j)$ are

representable. Also assume that the functor $I \rightarrow \mathcal{C}$ defined by $i \rightarrow \lim \text{proj } d_i$ is representable. Then D has projective limit and

$$\lim_{I \times J} \text{proj } D \cong \lim_I \text{proj } \lim_J \text{proj } d_i,$$

and hence

$$\lim_I \text{proj } \lim_J \text{proj } d_i \cong \lim_J \text{proj } I \lim_I \text{proj } d_j.$$

We leave the proof to the reader. A special corollary of this is the following. Suppose $A = \text{Ker}(A_1 \rightrightarrows A_2)$, $B = \text{Ker}(B_1 \rightrightarrows B_2)$. Suppose we have two morphisms between these two pairs of arrows $A_1 \rightrightarrows B_1$ and $A_2 \rightrightarrows B_2$ (think about them as two functors from $i \rightrightarrows j$ to \mathcal{C}). Then there are two morphisms from the equalizers $A \rightrightarrows B$ and we have

$$\text{Ker}(A \rightrightarrows B) \cong \text{Ker}(\text{Ker}(A_1 \rightrightarrows B_1) \rightrightarrows \text{Ker}(A_2, B_2)).$$

Now we are ready to give our next example:

Example 2. Any category \mathcal{C} with fibred products can be equipped with a *canonical topology* TC , its covering families $(U_i \rightarrow U)$ are universal effective epimorphisms, i.e. families such that for any object X of \mathcal{C}

$$h_X(U) = \text{Ker}\left(\prod h_X(U_i) \rightrightarrows \prod (h_X(U_i \times_U U_j))\right).$$

We require that the restricted family $(U_i \times_U V \rightarrow V)$ is also universal effective epimorphism. This checks axiom (PT3) automatically. Axiom (PT2) follows from what we have said before. Axiom (PT1) is obviously checked.

One can also describe the canonical topology in terms of cibles. A canonical topology is the finest topology such that all representable presheaves are sheaves. A cible in this topology is called a strict universal epimorphism. For any covering cible of an object X in the canonical topology we must have a canonical bijection $h_Z(X) \rightarrow \check{\mathcal{C}}(R, h_Z)$. Also for any $Y \rightarrow X$ the cible $Y \times_X R$ must be a covering cible of Y so that there is a canonical bijection $h_Z(Y) \rightarrow \check{\mathcal{C}}(Y \times_X R, h_Z)$. Let us show that the set of cibles R satisfying these two properties form the canonical topology. For this we have to verify only axiom (T2) of the topology. The other axioms are satisfied by definition. By taking the fibred product $R \times_X R'$ of two cibles of X , we may assume that either $R' \subset R$ or $R \subset R'$. So we have to check that

- (i) if $R' \subset R$ and $R \in J(X)$, and for any $(Y \rightarrow X) \in R(Y)$ the cible $R' \times_X Y \in J(Y)$ then $R \in J(X)$;
- (ii) if $R' \subset R$ and $R' \in J(X)$ then $R \in J(X)$.

By Proposition 1 from section 4, R is equal to the inductive limit of representable functors h_Y which admit a morphism to R . So, R' is the inductive limit of $R' \times_X Y$. It is easy to see that in both cases, passing to the limit we obtain a bijection

$$\check{\mathcal{C}}(R, h_Z) \rightarrow \check{\mathcal{C}}(R', h_Z).$$

Since (i) and (ii) are preserved under fibres products $? \times_X Y$, we obtain that $h_Z(X) \cong \check{\mathcal{C}}(R \times_X Y, h_Z) \cong \check{\mathcal{C}}(R' \times_X Y, h_Z)$, and hence R and R' are covering cibles in both cases.

So we have shown that in the canonical topology covering cibles are characterized by the condition

$$h_Z(Y) \rightarrow \check{\mathcal{C}}(Y \times_X R, h_Z)$$

are bijective for all $Z \in \text{Ob}(\mathcal{C})$ and $(Y \rightarrow X) \in \text{Mor}_{\mathcal{C}}$.

Example 3. Let X be a topological space, $\mathcal{C} = \text{Open}(X)$. For every open set U let $\text{Cov}(U)$ be the set of open coverings of U . Clearly all the axioms of pre-topology are satisfied. Any continuous map $f : X \rightarrow Y$ of topological spaces defines a morphism of the corresponding sites $U \rightarrow f^{-1}(U)$.

Example 4. Let us define a topology on a set X . First consider a subcategory \mathcal{C} of the category of subsets of X . It must contain X and (a final and initial object). It must be closed under finite products. As usually we extend it in $\check{\mathcal{C}}$ to contain direct sums. Then we require that it contains co-equalizers. Then it is clear

that its objects can be taken as open sets in a topology on X and also as direct sums of open sets. Now if we take the canonical topology on this category, we see that the obtained topology is the same as in the previous example.

Example 5. Let X be a topological space. Assume that X is linearly connected and each point has a neighborhood with trivial fundamental group. A continuous map $Y \rightarrow X$ is called *étale* if for any point x in the image $Y \rightarrow X$ there exists a neighborhood $U(x)$ such that $f^{-1}(U(x)) = \coprod_{i \in I} V_i$, where the restriction of f to each V_i is a homeomorphism. It is easy to see that $f(Y)$ is open in X . We define the étale topology on X as a topology on the full subcategory $Et(X)$ of Top/X formed by étale maps. This category has fibred products, and we define the pre-topology by taking $Cov(U)$ to be a set of morphisms $f_i : U_i \rightarrow U$ in $Et(X)$ such that $\coprod_{i \in I} U_i \rightarrow U$ is surjective. Note that each f_i is necessary étale. We leave to the reader to check that this defines a pre-topology. It is called the étale topology of X . For any étale map $U \rightarrow V$ and any continuous map $Z \rightarrow V$ the pull-back morphism $U \times_V Z \rightarrow Z$ is étale. This easily shows that a continuous map $f : X \rightarrow Y$ defines a morphism of the étale sites.

Example 6. Let G be a group and $G - (Sets)$ be the category of G -sets. Equip it with the canonical topology. Here $Cov(U)$ are all surjective families of morphisms $U_i \rightarrow U$.

Consider the étale site $Et(X)$ of a topological space X . Define the following functor $F : Et(X) \rightarrow \pi_1(X) - (Sets)$. Here $\pi_1(X)$ is the fundamental group of X . Fix a point $x_0 \in X$. The functor F assigns to an étale cover $f : Y \rightarrow X$ the fibre $f^{-1}(x_0)$. If $\gamma : [0, 1] \rightarrow X$ is a loop with $\gamma(0) = \gamma(1) = x_0$ representing an element g of $\pi_1(X, x_0)$. By the theorem of covering homotopy, for any $y \in f^{-1}(x_0)$ there is a path $\tilde{\gamma} : [0, 1] \rightarrow Y$ such that $\tilde{\gamma}(0) = y, \tilde{\gamma}(1) = y' \in f^{-1}(x_0)$. One can show that y' depends only on y and the homotopy class of γ . This defines an action of $\pi_1(X, x_0)$ on $F(Y)$. If $g \circ f : Y' \rightarrow Y \rightarrow X$ is a composition in $Et(X)$ then $(g \circ f)^{-1}(x_0)$ is mapped π_1 -equivariantly to $g^{-1}(x_0)$. So the functor is covariant. It is easy to see that it maps covering families in $Et(X)$ to covering families in $\pi_1 - (Sets)$. The corresponding sites are in fact isomorphic. Let \tilde{X} be the universal cover of X . Given a π_1 -set S we construct an étale cover as the quotient topological space $\tilde{X} \times_{\pi_1} S = \tilde{X} \times S / \sim$, where $(x, s) \sim (x', s')$ if $x' = gx, s' = gs$ or some $g \in \pi_1$. We equip S with discrete topology and then put the quotient topology.

Example 7. Let k be a field, and $Sep(k)$ be the full subcategory of the category of commutative k -algebras which are finite separable extensions of k or their direct sums. Equip it with canonical topology. This site is isomorphic to the canonical site of the category of finite sets on which the Galois group $Aut(\bar{k}/k)$ acts with the kernel of the action of finite index.

Example 8. Let \mathcal{C} be the category dual to the category of commutative rings. A homomorphism of rings $f : A \rightarrow B$ is called *flat* if for any ideal I of A the natural homomorphism of B -modules $I \otimes_A B \rightarrow A \otimes_A B = B$ is injective. It is called strictly flat if additionally $M \otimes_A B \neq 0$ for any nonzero A -module M . Equivalently, a sequence of A -modules $N \rightarrow M \rightarrow L$ is exact in M if and only if the sequence $N \otimes_A B \rightarrow M \otimes_A B \rightarrow L \otimes_A B$ is exact in $M \otimes_A B$ (show that that)

For example any homomorphism of fields is strictly flat. I claim that a strictly flat homomorphism f defines a strict epimorphism in the category Com° . In other words, if

$$A \cong Ker(B \rightrightarrows B \otimes_A B).$$

Recall that this means that for any ring C

$$Hom(C, A) = Ker(Hom(C, B) \rightrightarrows Hom(C, B) \times_{Hom(C, A)} Hom(C, B)).$$

Consider the following sequence

$$0 \rightarrow A \rightarrow B \rightarrow B^{\otimes 2} \rightarrow \dots \rightarrow B^{\otimes r} \rightarrow \dots,$$

where $d_r : B^{\otimes r} \rightarrow B^{\otimes r+1}$ is defined as

$$d_r(b_1 \otimes \dots \otimes b_r) = \sum_{i=1}^r (-1)^{r+1} b_1 \otimes \dots \otimes b_{i-1} \otimes 1 \otimes b_{i+1} \otimes \dots \otimes b_r.$$

It is immediately checked that this is a complex, i.e. $d_{r+1} \circ d_r = 0$. Assume $A \rightarrow B$ admits a right inverse, i.e. a homomorphism $g : B \rightarrow A$ such that $g \circ f = id$. Then we can construct the homomorphisms $k_r : B^{\otimes r+1} \rightarrow B^{\otimes r}$ such that $k_{r+1}d_{r+1} + d_r k_r = 1$. Just set

$$k_r(b_1 \otimes \dots \otimes b_{r+1}) = g(b_1)b_2 \otimes \dots \otimes 1 \otimes b_i \otimes \dots \otimes b_{r+1}.$$

A standard argument (used for example in the calculus of differential forms) shows that the complex is exact in this case. Now if we extend the scalars to B (i.e. tensor the complex by B), we get the complex

$$0 \rightarrow B \rightarrow B \otimes_A B \rightarrow B^{\otimes 3} \rightarrow \dots \rightarrow B^{\otimes r} \rightarrow \dots,$$

Since $B \otimes B \rightarrow B, b \otimes b' \rightarrow bb'$ is the right inverse of $B \rightarrow B \otimes B, b \rightarrow 1 \otimes b$, we obtain an exact complex. Since $f : A \rightarrow B$ is strictly flat, the original complex is exact.

Now $A = \ker(d_1 : B \rightarrow B \otimes_A B)$ means exactly that $A \rightarrow B$ is a strict epimorphism in the category Com° . This gives a description of canonical topology in Com° . Its covering families $A \rightarrow B_i$ are flat morphisms such that $A \rightarrow \oplus_i B_i$ is strictly flat. This is called the flat topology. If A is a commutative ring we can consider the category $(Com/A)^\circ$ and equip it with the induced flat topology. Abusing the language one speaks about the flat topology of A .

9. Sheaves

Definition. A presheaf of sets \mathcal{F} on a site (\mathcal{C}, T) is called a *sheaf* if for any $X \in Ob(\mathcal{C})$ and any sieve $R \in J(X)$ the canonical map

$$\mathcal{F}(X) = \check{C}(h_X, \mathcal{F}) \rightarrow \check{C}(R, \mathcal{F})$$

is a bijection.

Recall that $\check{C}(R, \mathcal{F})$ is given by assigning to each $(Y \rightarrow X) \in R(Y) \subset h_X(Y)$ an element of $s_Y \in \mathcal{F}(Y)$ such that if $f : (Y' \rightarrow X) \rightarrow (Y \rightarrow X) \in Mor_{\mathcal{C}/X}$ the map $\mathcal{F}(f) : \mathcal{F}(Y) \rightarrow \mathcal{F}(Y')$ sends s_Y to $s_{Y'}$. So the condition of a sheaf requires that the family $\{s_Y\}$ as above originates from a unique $s \in \mathcal{F}(X)$.

Lemma 1. Let I and \mathcal{C} be two categories. Suppose \mathcal{C} has direct limits of families $X_{\alpha \in U}$ where U is the chosen universe. Assume that I has a family of objects $(i_\alpha)_{\alpha \in A}$, where A is a small set, such that the products $i_\alpha \times i_\beta$ are representable for each pair and each object of I has a morphism to some i_α . Then for any functor $F : I^\circ \rightarrow \mathcal{C}$ the projective limits exist and there is a functorial isomorphism

$$\lim \text{proj } F \cong \text{Ker} \left(\prod_{i \in I} F(i) \rightrightarrows \prod_{(i,j) \in I \times J} F(i \times j) \right).$$

Proposition. Suppose \mathcal{C} has fibred products and the topology is defined covering families $Cov(U)$. Then a presheaf is a sheaf if and only if for any $(U_i \rightarrow U) \in Cov(U)$ the sequence

$$\mathcal{F}(U) \rightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j)} \mathcal{F}(U_i \times_U U_j)$$

is bijective.

Proof. By Proposition 1 from section 4 we have

$$\mathcal{F} = \lim_{\mathcal{C}/\mathcal{F}} h_X.$$

We use the following generalization of Yoneda's lemma: for any \mathcal{F}, \mathcal{G} in \check{C}

$$\check{C}(\mathcal{F}, \mathcal{G}) \cong \lim \text{proj}_{(X,u) \in \mathcal{C}/\mathcal{F}} \mathcal{G}(X).$$

This follows from the fact

$$\check{\mathcal{C}}(\lim_{\mathcal{C}/\mathcal{F}} h_X, \mathcal{G}) = \lim_{\mathcal{C}/\mathcal{F}} \text{proj } \check{\mathcal{C}}(h_X, \mathcal{G}) = \lim_{\mathcal{C}/\mathcal{F}} \text{proj } \mathcal{G}(X).$$

Notice that \mathcal{C}/R consists of morphisms $S \rightarrow R$ which are elements of $R(S)$, i.e. morphisms $S \rightarrow X$ from R . This is our category I . It contains the objects $U_i \rightarrow X$ satisfying the assumptions of the lemma. Thus

$$\check{\mathcal{C}}(R, \mathcal{F}) \cong \lim_{U_i \rightarrow U} \text{proj } \mathcal{F}(U_i) = \text{Ker} \left(\prod_{i \in I} F(i) \rightrightarrows \prod_{(i,j) \in I \times J} F(i \times j) \right).$$

Examples 1. A representable presheaf on a canonical site is a sheaf.

2. A sheaf on a topological space is a sheaf on the site $\text{Open}(X)$ with its canonical topology. By definition of a canonical topology, $U \rightarrow C(U, Y)$ is a sheaf.

3. Since any representable pre-sheaf on the flat topology of Com° is a sheaf, the presheaf $A \rightarrow A^*$ is a sheaf. It is denoted by \mathbb{G}_m . Similarly, we have the sheaf $A \rightarrow A^+$ denoted by \mathbb{G}_a .

4. Let $\mathcal{C} = G - (\text{Sets})$ with canonical topology. We know that every representable presheaf is a sheaf. Conversely, let \mathcal{F} be a sheaf, and $A = \mathcal{F}(G)$, where G is considered as a G -set with respect to left multiplications. The set A is a G -set, the action is by functoriality. I claim that A represents \mathcal{F} in \mathcal{C} , i.e. for every G -set U we have a natural bijection $\mathcal{F}(U) = \mathcal{C}(U, A) := \text{Maps}_G(U, A)$. For every point $x \in U$ let $f_x : G \rightarrow U$ be the map $g \rightarrow gx$. Its image is the G -orbit of x . Then we have a cover $(f_x : G \rightarrow U)_{x \in U}$ in $\text{Cov}(U)$. Observe that $(f_x : G \rightarrow U) \times_U (f_y : G \rightarrow U) = \{(g, g') : gx = g'y\}$. Now $\prod_{x \in U} \mathcal{F}(G)$ can be identified with the set of functions $U \rightarrow \mathcal{F}(G)$. The group G acts on this product in two ways: $(g \circ \phi)(u) = \phi(g^{-1}x)$ and ${}^g\phi(x) = g \circ \phi(x)$. To say that ϕ commutes with the action is the same as $g \circ \phi = {}^g\phi$. But this can be also expressed by saying that ϕ belongs to the kernel K of the pair of maps

$$\prod_{x \in U} \mathcal{F}(G) \rightrightarrows \prod_{(x,y) \in U \times U} \mathcal{F}(G \times_U G).$$

By definition of a sheaf there is unique bijective map $\mathcal{F}(U) \rightarrow K$.

Theorem 1. Let \mathcal{C} be a site and $\check{\mathcal{C}}$ be the full subcategory of $\check{\mathcal{C}}$ formed by sheaves on \mathcal{C} . Let $i : \check{\mathcal{C}} \rightarrow \check{\mathcal{C}}$ be the inclusion functor. Then there exists a left adjoint functor $a : \check{\mathcal{C}} \rightarrow \check{\mathcal{C}}$. The functor a is left exact.

First we set for any $\mathcal{F}, \mathcal{G} \in \text{Ob}(\check{\mathcal{C}})$

$$\mathcal{F}(\mathcal{G}) = \check{\mathcal{C}}(\mathcal{G}, \mathcal{F}).$$

By Yoneda's lemma, there is a natural bijection:

$$\mathcal{F}(h_X) \rightarrow \mathcal{F}(X).$$

Lemma 1. Let \mathcal{C}/\mathcal{G} be the full subcategory of $\check{\mathcal{C}}/\mathcal{F}$ which consists of arrows $X = h_X \rightarrow \mathcal{G}$, where $X \in \text{Ob}(\mathcal{C})$. For any $\mathcal{F} \in \check{\mathcal{C}}$ consider \mathcal{F} as a functor on \mathcal{C}/\mathcal{G} defined by $\mathcal{F}(X \rightarrow \mathcal{G}) = \mathcal{F}(X)$. Then

$$\mathcal{F}(\mathcal{G}) = \lim_{\mathcal{C}/\mathcal{G}} \text{proj } \mathcal{F} = \lim_{(X \rightarrow \mathcal{G})} \text{proj } \mathcal{F}(X).$$

Proof. Let us show first that

$$\mathcal{G} = \lim_{X \rightarrow \mathcal{G}} \text{ind } h_X.$$

We have to construct a functorial bijection from $\mathcal{G}(A)$ and $\lim_{X \rightarrow \mathcal{G}} \text{ind } h_X(A)$ for any $A \in \text{Ob}(\mathcal{C})$. An element $u \in \mathcal{G}(A)$ defines a morphism $A \rightarrow \mathcal{G}$, and hence an element in the inductive limit of the sets $h_X(A)$, $X \in \text{Ob}(\mathcal{C}/\mathcal{G})$ corresponding to the equivalence class of the element $1_A \in h_A(A)$. Conversely, any

element from the inductive limit represented by some $a \in h_X(A)$ defines a morphism $A \rightarrow X \rightarrow \mathcal{G}$. It is easy to see that the two constructed maps are inverse to each other. Now, by definition of projective limit,

$$\mathcal{F}(\mathcal{G}) = \check{\mathcal{C}}(\mathcal{G}, \mathcal{F}) = \check{\mathcal{C}}(\lim_{(X \rightarrow \mathcal{G})} \text{ind } h_X, \mathcal{F}) = \lim_{(X \rightarrow \mathcal{G})} \text{proj } \check{\mathcal{C}}(h_X, \mathcal{F}) = \lim_{(X \rightarrow \mathcal{G})} \text{proj } \mathcal{F}(X).$$

The lemma is proven.

Corollary. *Let R be a sieve of X considered as a subcategory of \mathcal{C}/X . Then for any presheaf \mathcal{F} ,*

$$\mathcal{F}(R) = \lim_{(Y \rightarrow X) \in \text{Ob}(R)} \text{proj } \mathcal{F}(Y).$$

Proof. Consider R as a presheaf, a subpresheaf of h_X . Then $(Y \rightarrow X) \in \text{Ob}(R)$ if and only if there exists a morphism $h_Y \rightarrow R$. Then we apply the previous Lemma to $\mathcal{G} = R$.

Let us consider the set of sieves $J(X)$ as an ordered set, the order is the inclusion of pre-sheaves. Thus we can consider inductive limits over the cofiltering category $J(X)$ (the fact that it is cofiltering, in sense of HW2, Problem 1, follows from the fact that the set of covering sieves is closed under finite products). For any presheaf \mathcal{F} , set

$$\mathcal{F}^\#(X) = \lim_{R \in J(X)} \text{ind } \check{\mathcal{C}}(R, \mathcal{F}) = \lim_{R \in J(X)} \text{ind } \mathcal{F}(R). \quad (*)$$

If $Y \rightarrow X$ is a morphism in \mathcal{C} , we have $R \rightarrow R \times_X Y$ maps $J(X)$ to $J(Y)$, hence we have a canonical map of sets $\mathcal{F}^\#(X) \rightarrow \mathcal{F}^\#(Y)$. This makes $\mathcal{F}^\#$ a presheaf.

Let $R \in J(X)$. Consider the set of sieves R' of X such that each $Z \rightarrow X$ from R' factors $Z \rightarrow Y \rightarrow X$ through some $(Y \rightarrow X) \in R$ and the set of such $Z \rightarrow Y$ forms a covering sieve of Y . By axiom (T2) of a site, each such R' is covering sieve of X . We denote the set of such covering sieves of X by $J(X; R)$. Note that $R \in J(X; R)$ and any $R' \in J(X; R)$ is a subsieve of R . The following lemma generalizes the definition (*):

Lemma 2. *For any $R \in J(X)$*

$$\mathcal{F}^\#(R) = \lim_{R' \in J(X; R)} \text{ind } \mathcal{F}(R').$$

Proof. First we check that

$$\lim_{Y \rightarrow R} \text{ind } \lim_{R' \in J(Y)} \text{proj } R' = \lim_{P \in J(X; R)} \text{proj } P.$$

Since each $P \in J(X; R)$ defines a sieve $R' = P \times_Y X \in J_Y$, the projection to R' defines a map from $\lim_{R' \in J(Y)} \text{proj } R'$ to P . This gives us a map from $\lim_{R' \in J(Y)} \text{proj } R'$ to $\lim_{P \in J(X; R)} \text{proj } P$, and hence the map from the LHS to the RHS. By definition of $J(X, P)$ this map is surjective. By Lemma 1 and (*),

$$\mathcal{F}^\#(R) = \lim_{Y \rightarrow R} \text{proj } \mathcal{F}^\#(Y) = \lim_{Y \rightarrow R} \text{proj } \lim_{R' \in J(Y)} \text{ind } \mathcal{F}(R').$$

On the other hand, since $\mathcal{C}(?, T)$ transforms inductive limits to projective and projective to inductive, we have

$$\begin{aligned} \mathcal{F}(\lim_{P \in J(X; R)} \text{proj } P) &= \mathcal{F}(\lim_{Y \rightarrow R} \text{ind } \lim_{R' \in J(Y)} \text{proj } R') = \lim_{Y \rightarrow R} \text{proj } \mathcal{F}(\lim_{R' \in J(Y)} \text{proj } R') = \\ &= \lim_{Y \rightarrow R} \text{proj } \lim_{R' \in J(Y)} \text{ind } \mathcal{F}(R') = \mathcal{F}^\#(R). \end{aligned}$$

It is easy to see that

$$L : \mathcal{F} \rightarrow \mathcal{F}^\#$$

is a functor. By definition of inductive limit, we have morphisms, for every sieve $R \in J(X)$,

$$Z_R : \mathcal{F}(R) \rightarrow \mathcal{F}^\#(X).$$

Taking the sieve $h_X \in J(X)$, we get a map $\mathcal{F}(X) \rightarrow \mathcal{F}^\#(X)$. It is easy to see that the set of such maps defines a morphism of functors

$$\ell_{\mathcal{F}} : \mathcal{F} \rightarrow \mathcal{F}^\# = L(\mathcal{F}).$$

Note that

$$\ell_{\mathcal{F}}(X) = Z_{h_X}.$$

Since any $Y \rightarrow X$ defines a functor $J(X) \rightarrow J(Y), R \rightarrow R \times_X Y$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{F}(R) & \longrightarrow & \mathcal{F}^\#(X) \\ \downarrow & & \downarrow \\ \mathcal{F}(R \times_X Y) & \longrightarrow & \mathcal{F}^\#(Y) \end{array} \quad (**)$$

Lemma 3. *Let $u \in \mathcal{F}(R)$ be considered as a morphism of presheaves $R \rightarrow \mathcal{F}$ and $Z_R(u)$ as a morphism $h_X \rightarrow \mathcal{F}^\#$. The following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\ell_{\mathcal{F}}} & \mathcal{F}^\# \\ u \uparrow & & \uparrow Z_R(u) \\ R & \xrightarrow{i} & h_X \end{array}$$

Proof. We have to check that two morphisms $\ell_{\mathcal{F}} \circ u$ and $Z_R \circ i$ from R to $\mathcal{F}^\#$ are equal. By Lemma 1, it is enough to check that for any $g : Y \rightarrow R$ we have $\ell_{\mathcal{F}} \circ u \circ g = Z_R \circ i \circ g$. Let $f : Y \rightarrow X$ be the composition $i \circ g$. We have $(R \times_X Y)(Z)$ is the set of morphisms $Z \rightarrow X$ in $R(Z)$ which factor through $Z \rightarrow Y \rightarrow X$. The map $R \times_X Y(Z) \rightarrow h_Y(Z), (Z \rightarrow X) \rightarrow (Z \rightarrow Y)$ is injective and has a section $Z \rightarrow Y \rightarrow X$. Thus it is bijective, and we have an isomorphism of functors $i' : R \times_X Y \rightarrow h_Y$. Consider the diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\ell_{\mathcal{F}}} & \mathcal{F}^\# \\ u \uparrow & & \uparrow Z_R(u) \\ R & \xrightarrow{i} & h_X \\ g' \uparrow & \nearrow g & \uparrow f \\ R \times_X Y & \xrightarrow{i'} & h_Y \end{array} .$$

The bottom arrow is an isomorphism. We have

$$Z_{R \times_X Y}(u \circ g') = \ell_{\mathcal{F}} \circ u \circ g.$$

By commutativity of the diagram (**), we have

$$Z_{R \times_X Y}(u \circ g') = Z_R(u) \circ f = Z_R(u) \circ i \circ g = \ell_{\mathcal{F}} \circ u \circ g.$$

This checks what we wanted.

Lemma 4. *The sheaf $\mathcal{F}^\#$ is separated.*

Proof. Suppose there are two different $a, b \in \mathcal{F}^\#(X)$ which define the same element in $\mathcal{F}^\#(R) := \check{C}(R, \mathcal{F}^\#)$. By definition of inductive limit, we can find $a' \in \mathcal{F}(R')$ and $b' \in \mathcal{F}(R'')$ such that $Z_{R'}(a') = a, Z_{R''}(b') = b$. Since the set of covering sieves is cofiltering, we may assume that $R' = R'' \subset R$. By the previous lemma (applied to $R = R'$), the composition

$$\mathcal{F}(R') \xrightarrow{Z_{R'}} \mathcal{F}^\#(X) \xrightarrow{\mathcal{F}^\#(i)} \mathcal{F}^\#(R) \rightarrow \mathcal{F}^\#(R')$$

is equal to the value of the morphism of the functors $\mathcal{F} \rightarrow \mathcal{F}^\#$ at R' . It sends a' and b' to the same element $c \in \mathcal{F}^\#(R')$. By Lemma 2, the image of a' in b' in $\lim \text{ind}_{R'' \in J(X, R')} \mathcal{F}(R'')$ is equal. This means that there

exists some $R'' \in J(X; R')$ such that the images of a', b' in $\mathcal{F}(R'')$ are equal. But then $a = b$ in the inductive limit $\mathcal{F}(X) = \lim \text{ind}_{R \in J(X)} \mathcal{F}(R)$. This proves that $\mathcal{F}^\#$ is separated.

Lemma 5. *Assume \mathcal{F} is separated. Then $\mathcal{F}^\#$ is a sheaf.*

Proof. First of all, for any $X \in \text{Ob}(\mathcal{C})$, the canonical map $\ell_{\mathcal{F}}(X) : \mathcal{F}(X) \rightarrow \mathcal{F}^\#(X)$ is always injective. From this it follows that for any $\mathcal{G} \in \text{Ob}(\tilde{\mathcal{C}})$ the map $\mathcal{F}(\mathcal{G}) \rightarrow \mathcal{F}^\#(\mathcal{G})$ is injective. In fact, each map $\mathcal{F}(X) \rightarrow \mathcal{F}(R)$ is injective, so $\mathcal{F}(X)$ is mapped injectively into the inductive limit $\mathcal{F}^\#(X)$. We want to show that $\mathcal{F}^\#(X) \rightarrow \mathcal{F}^\#(R)$ is surjective for any $R \in J(X)$. By Lemma 2, $\mathcal{F}^\#(R) = \lim \text{ind}_{R' \in J(X; R)} \mathcal{F}(R')$. For each $R' \in J(X; R)$ the composition map $\mathcal{F}(R') \rightarrow \mathcal{F}^\#(R) \rightarrow \mathcal{F}^\#(R')$ is injective. Since $\mathcal{F}^\#$ is separated, the map $\mathcal{F}^\#(R) \rightarrow \mathcal{F}^\#(R')$ is injective. Hence the map $\mathcal{F}(R') \rightarrow \mathcal{F}^\#(R)$ is injective. Let $s \in \mathcal{F}^\#(R)$. Pick $u \in \mathcal{F}(R')$ representing s . By the above, the pair (u, R') is defined uniquely. We set $s' = Z_{R'}(u) \in \mathcal{F}^\#(X)$. It is immediately checked that s' is a lift of s .

Proof Theorem 1. By Lemmas 4 and 5, if we apply L twice, we get a sheaf $LL(\mathcal{F})$. If \mathcal{F} is a sheaf, then $\mathcal{F}^\# = \mathcal{F}$. In fact, $\mathcal{F}(R) = \mathcal{F}(X)$ and the definition (*) implies that $\mathcal{F}^\#(X) = \mathcal{F}(X)$. If $\mathcal{F} \rightarrow \mathcal{G}$, where \mathcal{G} is a sheaf, then $LL(\mathcal{F}) \rightarrow LL(\mathcal{G}) = G$. This defines a morphism from $LL(\mathcal{F})$ to G . Applying L to $LL(\mathcal{F}) \rightarrow G$ we obtain $\mathcal{F} \rightarrow \mathcal{G}$. This easily checks the adjunction. The last assertion follows from the fact that, for any sieve R , the functor $F \rightarrow \tilde{\mathcal{C}}(R, F)$ commutes with projective limits (this follows from the definition) and the functor $F \rightarrow \lim \text{ind}_{R \in J(X)} F(R)$ commutes with projective limits because the category $J(X)$ is filtering. Thus L is left exact and $a = LL$ is left exact.

Lemma 6. *Suppose a site \mathcal{C} has an initial object e . Then $a(h_e)$ is the initial object in the category $\tilde{\mathcal{C}}$.*

Proof.

Lemma 6. *Let $(s_i : U_i \rightarrow U)_{i \in I}$ be a covering family in a site \mathcal{C} . Suppose each $s_i : U_i \rightarrow U$ is a monomorphism and $U_i \times_U U_j$ is isomorphic to an initial object in \mathcal{C} . Then*

$$a(h_U) \cong a(h_{U_i}).$$

Proof. Let $s = (h(s_i)_{i \in I} : \prod_{i \in I} h_{X_i} \rightarrow h_X)$ be the canonical morphism. Since a commutes with inductive limits, we have to show that $a(s)$ is an isomorphism of sheaves.

Definition An initial object of a category \mathcal{C} is called a *strict initial object* if any morphism $X \rightarrow e$ is an isomorphism.

Proposition 1. *Suppose a site \mathcal{C} has a strict initial object e and the Grothendieck topology is finer than the canonical topology. Then for any sheaf F the set $F(e)$ is a singleton.*

Proof. We claim that the empty cribble $\emptyset \subset h_e$ of e belongs to $J(e)$. By example 2 from section 8, we have to check that, for any object Z and any $Y \rightarrow e$ we have a canonical bijection

$$h_Z(Y) \rightarrow \check{\mathcal{C}}(\emptyset \times_e Y, h_Z).(*)$$

But $Y \rightarrow e$ is an isomorphism, so that $\emptyset \times_e Y \cong \emptyset$ and $\check{\mathcal{C}}(\emptyset \times_e Y, h_Z) \cong \check{\mathcal{C}}(\emptyset, h_Z)$. If Z is not isomorphic to e , then $h_Z(e) = \emptyset$ so $\check{\mathcal{C}}(\emptyset, h_Z) = \emptyset$. So both sides of (*) are emptysets. If $Z \cong e$, then both sides are singletons. Thus, we have checked that $\emptyset \in J(e)$. But, then for any sheaf F we must have a bijection $F(e) \rightarrow F(\emptyset)$, the latter is obviously a singleton.

Corollary. *In the condition of the previous proposition assume that $(U_i \rightarrow U)$ is a covering family such that $U_i \times_U U_j$ is isomorphic to a strict initial object in \mathcal{C} . Then for any sheaf F there is a natural bijection $F(U) \rightarrow \prod_{i \in I} F(U_i)$.*

Example 5. Let c_S be a constant pre-sheaf $X \rightarrow S$, where S is a fixed set S . If S is not a singleton, and \mathcal{C} has an initial object then c_S is not sheaf. In fact, if \mathcal{C} is a canonical site with a strict initial object e , then for any sheaf F we have $F(e)$ must be a singleton. Let $\tilde{c}_S = a(c_S)$. Assume that \mathcal{C} has an initial object e

and the empty crible \emptyset of e is a covering crible. Then $\check{C}(\emptyset, c_S)$ is a singleton, so, by definition of the functor $L : \mathcal{F} \rightarrow \mathcal{F}^\#$ we must have $L(c_S)(e)$ is a singleton. This shows that c_S is not even separated if S is not a singleton. Now

$$L(c_S)(X) = \lim_{\text{ind}} \check{C}(R, c_S)_{R \in J(X)}$$

If R is generated by $(U_i \rightarrow X)_{i \in I}$ such that $U_i \cup U_j = e$ for any $i, j \in I$, then, for any Y the value of R on Y consists of the set of morphisms $Y \rightarrow X$ which factor through one of $U_i \rightarrow X$. By the assumption there is only one i such that it could happen (unless $Y = e$). This shows that $\check{C}(R, c_S)$ is defined by an element of S^I . Assume that R be the minimal covering sieve of X with the property as above (if it exists). Then we obtain $L(c_S)(X) = S^I$.

Let $u : \mathcal{C} \rightarrow \mathcal{C}'$ be a continuous functor of sites. For every covering site $R \in J(X)$ in \mathcal{C} the morphism $u_!(R) \rightarrow u(X)$ is isomorphic to a covering sieve in \mathcal{C}' . This implies that for every sheaf \mathcal{F} in \mathcal{C}' , the natural map of sets $\mathcal{F}(u(X)) \rightarrow \check{C}(u_!(R), \mathcal{F})$ is bijective. By adjunction, this implies that the map $\check{C}(a(u_!(R)), \mathcal{F}) \rightarrow \check{C}(a(h_{u(X)}), \mathcal{F})$ is bijective. This implies that the associated sheaves $a(u_!(R))$ and $a(h_{u(X)})$ are isomorphic. Since $u^*(\mathcal{F})(X) = \mathcal{F}(u(X))$, we see that for any covering sieve R of X , the natural map $u^*(\mathcal{F})(X) \rightarrow \check{C}(R, u^*(\mathcal{F})) = \check{C}'(u_!(R), \mathcal{F})$ is bijective. This implies that $u^*(\mathcal{F})$ is a sheaf. The functor u^* restricted to $\check{\mathcal{C}}'$ is denoted by u_s .

Remark One can define a continuous functor by requiring that for any sheaf \mathcal{F} on \mathcal{C}' its inverse image $u^*(\mathcal{F})$ is a sheaf on \mathcal{C} . It can be shown that this is equivalent to the condition that for any sieve $R \in J(X)$ the associated sheaves $a(u_!(R))$ and $a(h_{u(X)})$ are isomorphic. This condition is weaker than the one we used to define a continuous functor. If the topology in \mathcal{C} and \mathcal{C}' is defined by a pre-topology and $u : \mathcal{C} \rightarrow \mathcal{C}'$ commutes with fibred products the latter definition is equivalent to requiring that for any covering family $(U_i \rightarrow X)$ in \mathcal{C} the family $(u(U_i) \rightarrow u(X))$ is a covering family in \mathcal{C}' .

It is not true, in general, that $u_!(\mathcal{F})$ is a sheaf if \mathcal{F} is a sheaf. So one defines the functor

$$u^s = a \circ u_! : \check{\mathcal{C}} \rightarrow \check{\mathcal{C}}'$$

Once can show that it is a left adjoint to u_s .

Example 6. Let $\mathcal{C} = \text{Open}(Y), \mathcal{C}' = \text{Open}(X)$ and $u : \mathcal{C} \rightarrow \mathcal{C}'$ corresponds to a continuous map $f : X \rightarrow Y$ of topological spaces. Recall that for any sheaf \mathcal{F} on Y the presheaf $u_!(\mathcal{F})$ is defined by $u_!(\mathcal{F})(V) = \lim_{V \rightarrow f^{-1}(U)} \mathcal{F}(U)$. Take \mathcal{F} to be the sheaf associated to the constant pre-sheaf S . Assume U is connected by $f^{-1}(U) = V_1 \amalg V_2$ where V_1, V_2 are not-empty sets, then

$$u_!(\mathcal{F})(V_1 \cup V_2) = u_!(\mathcal{F})(V_1) = u_!(\mathcal{F})(V_2) = \mathcal{F}(U) = S.$$

However, if $u_!(\mathcal{F})$ were a sheaf, we would have $u_!(\mathcal{F})(V_1 \cap V_2) = S \times S$.

In the situation of this example, the sheaf $u^s(\mathcal{F})$ is denoted by $f^{-1}(\mathcal{F})$ and is called the inverse image of the sheaf \mathcal{F} .

10. Simplicial objects

Let ST be the *category of simplicial types*: its objects are natural numbers and morphisms $n \rightarrow m$ are non-decreasing maps from the set $[n] = \{0, 1, \dots, n\}$ to $[m]$. The category ST/k is called the category of k -truncated simplicial types.

The following are special morphisms from this category:

- $\partial_n^i : [n-1] \rightarrow [n]$: an injective increasing map with the image $[n] \setminus \{i\}$;
- $\sigma_n^i : [n+1] \rightarrow [n]$: a non-decreasing surjective map with $\#(\sigma_n^i)^{-1}(i) = 2$.

Definition A simplicial object in a category \mathcal{C} is a functor $ST^\circ \rightarrow \mathcal{C}$. The category of simplicial objects is denoted by $ST(\mathcal{C})$. For every $k \in \text{Ob}(ST)$, the category of contravariant functors from ST/k to \mathcal{C} is denoted by $ST(\mathcal{C})/k$. It is called the category of truncated simplicial objects in \mathcal{C} .

Example 1. Take $\mathcal{C} = (\mathcal{S}ets)$. Then a simplicial object is a presheaf on ST . The representable presheaf $h_n : k \rightarrow ST(k, n)$ is denoted by $\Delta[n]$. Its value at k is the set of all non-decreasing maps $[k] \rightarrow [n]$. Its subfunctor of injective maps can be identified with ordered subsets of $[n]$. If X is a simplicial set, we denote by X_n its value at n . Its elements are called n -simplices (0-simplices are *vertices*, 1-simplices are *edges*). In particular, $\Delta[n]_n$ is the fundamental n -simplex corresponding to the identity map $[n] \rightarrow [n]$. It is denoted by I_n . By Yoneda's Lemma,

$$X_n = \tilde{S}T(\Delta[n], X). \quad (1)$$

In particular, for every $x \in X_n$ there exists a unique morphism which sends I_n to x .

Let $\Delta : ST \rightarrow \tilde{S}T$ be the functor $[n] \rightarrow \Delta[n]$.

We know that in the category $\tilde{S}T$ all limits exist. For example, we can define the direct product of simplicial sets, direct sums and so on. For example,

$$Coker(\Delta(\partial_1^0), \Delta(\partial_1^1)) : \Delta[0] \rightarrow \Delta[1]$$

is called a *simplicial circle*. Let us explain why. We have $\Delta[0]_k$ are constant maps, $\Delta[1]_k$ are defined by $s \leq k$ such that $f(i) = 0$ if $i \leq s$, $f(i) = 1$ if $i > s$. The map $\Delta(\partial_1^i)$ sends $\Delta[0]_k$ to $\Delta[1]_k$ by composing with the map $\partial_1^i : [0] \rightarrow [1]$. Thus the first map sends the constant map to the constant map with value 0, and the second sends the constant map to the constant map with value 1. Thus the value of the cokernel at k is equal to the set of numbers $s \in [0, k]$ where 0 is identified with k .

2. Let $\mathcal{C} = Top$. For each $n \in \mathbb{N}$ let I^n be the direct sum of the unit interval $[0, 1] \subset \mathbb{R}$ equipped with the topology induced from the Euclidean topology of \mathbb{R}^n . We view I^n as the set of maps $I^{[n]}$ so that $n \rightarrow I^n$ defines a contravariant functor from ST to Top . This is a simplicial object of Top .

More generally, take any category \mathcal{C} with finite direct products, and its object X . Then the functor $n \rightarrow X^n$ is a simplicial object of the category \mathcal{C} .

3. For each $n \in \mathbb{N}$ let

$$\Delta_n \{ (x_0, \dots, x_n) \in \mathbb{R} : 0 \leq x_i, \sum_{i=0}^n x_i = 1. \}.$$

For any $X \in Ob(Top)$ let

$$X_n = Top(\Delta_n, X).$$

Elements of this set are called *singular simplices* of X . If $f : [n] \rightarrow [m]$ is a morphism in ST , we define $\Delta(f) : \Delta_n \rightarrow \Delta_m$ as the unique affine linear map sending the vertex $e_i \in \Delta_n$ to the vertex $e_{f(i)} \in \Delta_m$. This makes (Δ_n) a *cosimplicial topological space*. The composition map

$$X_m \rightarrow X_n, \alpha \rightarrow \Delta(f) \circ \alpha$$

defines the functor $X : ST^\circ \rightarrow (\mathcal{S}ets)$. It is easy to see that we get ourselves a functor from (Top) to $ST((\mathcal{S}ets))$.

Let us construct the left adjoint functor $ST((\mathcal{S}ets)) \rightarrow (Top)$. It is called the *geometric realization* of a simplicial set. We define the geometric realization as follows. Let $X = (X_n)$ be a simplicial set. Consider X_n as a topological space with discrete topology. The topological space

$$\prod_{n=0}^{\infty} \Delta_n \times X_n / R,$$

where R is the minimal equivalence relation which identifies the points $(s, x) \in \Delta_n \times X_n$ and $(t, y) \in \Delta \times X_m$ if $y = X(f)(x)$, $s = \Delta_f(t)$ for some morphism $f : m \rightarrow n$. The topology is the factor-topology. In other words,

$$|X| = \lim_{ST \times ST^\circ} \text{ind} \Delta \times X,$$

where $\Delta \times X$ is considered as a functor on the category $ST \times ST^\circ$ with values in (Top) which assigns to $[n] \times [m]$ the topological space $\Delta_n \times X_n$. The geometric realization provides a functor $ST((\mathcal{S}ets)) \rightarrow (Top)$.

Theorem 1. *The functor of geometric realization is left adjoint of the functor $S : (Top) \rightarrow ST((Sets)), X \rightarrow S(X) = (Top(\Delta_n, X))$. In particular, we have natural bijection*

$$Top(|X|, Y) \rightarrow \check{S}T(X, S(Y)).$$

Example 4 The geometric realization of $\Delta[n]$ is homeomorphic to Δ_n . It follows from the previous theorem that the functor of geometric realization commutes with inductive limits. This implies that

$$\begin{aligned} |Coker(\Delta(\partial_1^0), \Delta(\partial_1^1) : \Delta[0] \rightarrow \Delta[1])| &= Coker(|\Delta(\partial_1^0)|, |\Delta(\partial_1^1)| : |\Delta[0]| \rightarrow |\Delta[1]|) = \\ Coker(|\Delta(\partial_1^0)|, |\Delta(\partial_1^1)| : \Delta_0 \rightarrow \Delta_1) &= S^1. \end{aligned}$$

It is convenient to think about the set $\Delta_n \times X_n$ as the set of n -simplices Δ_n indexed by the set X_n , $(\Delta_n, x), x \in X_n$. Then $|X|$ is obtained by gluing $\#X_0$ 0-simplices, $\#X_1$ 1-simplices, and so on together with respect to the gluing defined by the morphisms $X(f) : [n] \rightarrow [m]$. A topological space together with such gluing is called a *triangulated space*. The gluing is the *triangulation*.

Example 5 Let us find a simplicial set $X = (X_n)$ such that $|X| \cong \Delta_p \times \Delta_q$. An element of X_n is a sequence of $n + 1$ distinct pairs of integers $\{(i_0, j_0), \dots, (i_n, j_n)\}$, where

$$0 \leq i_0 \leq \dots \leq i_n \leq p, \quad 0 \leq j_0 \leq \dots \leq j_n \leq q.$$

If $f : [m] \rightarrow [n]$ is a morphism in (ST) , then we define $\check{f}(\{(i_0, j_0), \dots, (i_n, j_n)\}) = \{(i'_0, j'_0), \dots, (i'_m, j'_m)\}$ where $i'_k = i_{f(k)}, j'_k = j_{f(k)}$. Now we have a map

$$\phi_n : \Delta_n \times X_n \rightarrow \Delta_p \times \Delta_q$$

It assigns to $(\Delta_n, \{(i_0, j_0), \dots, (i_n, j_n)\})$ the subset of the RHS equal to the convex hull of the points $(e_{i_s}, e_{j_s}) \in \mathbb{R}^{p+1} \times \mathbb{R}^{q+1}$. It is clear that this defines a map from $|X|$ to $\Delta_p \times \Delta_q$. We refer to [GM] to verify that the map is bijective. Let us check in the case $p = q = 1$. Then $\Delta_1 \times \Delta_1$ is the square. It is easy to see that $\Delta_0 \times X_0$ consists of 4 vertices, $\Delta_1 \times X_1$ consists of 5 copies of Δ_1 . They are mapped to the four sides and one of the diagonals (joining (e_0, e_0) with (e_1, e_1)). The set $\Delta_2 \times X_2$ consists of 2 2-dimensional simplices corresponding to the sequences $\{(0, 0), (0, 1), (1, 0)\}$ and $\{(0, 0), (1, 0), (1, 1)\}$. These are mapped to the two triangles divided by the diagonal. The sets X_n with $n > 2$ are empty.

Let ST_k be the full subcategory of ST whose objects are $[n]$ with $n \leq k$. We have a natural inclusion functor $i_k : ST_k \rightarrow ST$. If $X \in ST(\mathcal{C})$, the composition functor $X \circ i_k^* : ST_k^\circ \rightarrow \mathcal{C}$ is called the *k-truncated simplicial object*. We have the truncation functor

$$i_k^* : ST(\mathcal{C}) \rightarrow (ST/k)(\mathcal{C}), \quad X \rightarrow i_k(X).$$

The left (resp. the right) adjoint is denoted by sk_k (resp. $cosk_k$). We also consider the compositions

$$\begin{aligned} Cosk_k : ST(\mathcal{C}) &\rightarrow (ST/k)(\mathcal{C}) \xrightarrow{cosk_k} ST(\mathcal{C}), \\ Sk_k : ST(\mathcal{C}) &\rightarrow (ST/k)(\mathcal{C}) \xrightarrow{sk_k} ST(\mathcal{C}), \end{aligned}$$

the adjointness properties give functorial morphisms:

$$sk_k X \rightarrow X, \quad X \rightarrow cosk_k X.$$

If $\mathcal{C} = (Sets)$, then $sk_k = (F_k)!$ (resp. $cosk_k = (F_k)^!$) and is defined by

$$\begin{aligned} Sk_k(X)_n &= \lim_{[n] \rightarrow [m], m \leq k} \text{ind} X_m, \\ Cosk_k(X)_n &= \check{S}T_k(i_k^*(\Delta[n]), i_k^*(X)) = \\ \check{S}T_k(sk_k(i_k^*(\Delta[n])), X) &= \check{S}T(sk_k(\Delta[n]), X). \end{aligned}$$

Here we used that the functor i_k^* is right adjoint to sk_k .

We have

$$i_k(sk_k(X)) = i_k(X),$$

So that $sk_k(X)$ has the same simplices as X up to dimension k and all simplices in the above dimension are degenerations of n -simplices with $n \leq k$ (i.e. each $x \in X_m$ with $m \geq n$ is equal to $X(f)(y)$ for some $y \in X_n, n \leq k$ and some surjective morphism $m \geq n$).

More precisely, we call an n -simplex $x \in X_n$ *degenerate* if there exists a surjection $f : [n] \rightarrow [m]$ from ST such that $x = X(f)(y)$ for some $y \in X_m$. We have the following:

Lemma (Eilenberg-Zilber). For any $x \in X_n$ as above, there exists a non-degenerate m -simplex $y \in X_m$ and a surjective morphism $f : [n] \rightarrow [m]$ such that $X(f)(y) = x$. The pair (f, y) with this property is defined uniquely.

Proof. The existence of (f, y) is obvious. Suppose we have another pair (f', y') so that $x = X(f)(y) = X(f')(y')$. Let σ, σ' be sections of f and f' , respectively. Then

$$X(f' \circ \sigma)(y') = X(\sigma)(X(f')(y')) = X(\sigma)(x) = X(\sigma)(X(f)(y)) = y.$$

. Since y is non-degenerate, this is possible only if $f' \circ \sigma$ is injective. This implies $m' \leq m$. Similarly, we show that $m \leq m'$ so that $m = m'$ and $f' \circ \sigma$ is an increasing bijection, i.e. the identity. Thus $y = y'$ and also it follows that $f = f'$.

Corollary. Let $|X|$ be the geometric realization of a simplicial set $X = (X_n)$. Then each point of $|X|$ can be uniquely represented by $(s, x) \in \Delta_n \times X_n$, where x is a non-degenerate simplex. In particular, each element of $|sk_k(X)|$ is uniquely represented by $(s, x) \in \coprod_{n=0}^k \Delta_n \times X_n$, where $n \leq k$.

By eq(1), we have

$$Cosk_k(X)_n = Mor(i_k^*(\Delta[n]), i_k^*(X)).$$

Thus the morphism

$$X_n = \check{S}T(\Delta[n], X) \rightarrow (cosk_k X)_n = \check{S}T(sk_k(\Delta[n]), X)$$

can be interpreted as follows: any simplicial map $sk_k(\Delta[n]) \rightarrow X$ can be canonically extended to a simplicial map $\Delta[n] \rightarrow X$.

Example 6 Let A be an object of \mathcal{C} considered as a 0-truncated simplicial object $ST_0^\circ \rightarrow \mathcal{C}$. Then

$$cosk_0(A)_n = \check{S}T(sk_0 \Delta[n], A) = A^{n+1}.$$

So, we obtain the example of a simplicial object in \mathcal{C} from Example 3.

Generalizing the construction of geometric realization of a simplicial set, we can define a geometric realization of a simplicial object in (Top) . It is defined in the same way, only now the topology in X_n is not necessary discrete.

Let us see what is a simplicial object in the category $ST((Sets))$. By definition its set of m -simplices is a simplicial sets $(X_n)_m$. They can be interpreted as a contravariant functor on the category $ST \times ST$. Its morphisms are direct products of non-decreasing pairs of maps $(f, g) : [n] \times [n'] \rightarrow [m] \times [m']$. So, we can view an object of $ST(ST((Sets)))$ as *bisimplicial set* (X_{nm}) . Any two simplicial sets (X_n) and (Y_n) define a bisimplicial set $(X_n \times Y_m)$ with obvious morphisms. It is called the direct product, although the terminology is a little confusing, since the direct product in the category $ST((Sets))$ is $(X_n \times Y_n)$. This is called the *diagonal* of the bisimplicial set $X = (X_{nm})$ and is denoted by DX . One can extend the notion of geometric realization to bisimplicial sets. It can be done in three different ways:

- I. Define $|X|^D$ as $|DX|$.
- II. For each n we can define the geometric realization of the simplicial set (X_{nm}) . This is a simplicial object in (Top) . Then we can take the geometric realization of this object. Denote it by $|X|^I$.
- III. Using the second index we define $|X|^{II}$.

Theorem (Eilenberg-Zilber). There is a homeomorphism

$$|X|^D \cong |X|^I \cong |X|^{II}.$$

Proof. We refer for the proof to [GM]. Let us see the assertion in the case $X_{nm} = \Delta[n] \times \Delta[m]$. First we check that $|X|^D$ is homeomorphic to $\Delta_n \times \Delta_m$. For any $(f, g) \in X_{n,n}$ set $\{(i_0, j_0), \dots, (i_k, j_k)\}$, where $i_s = f(s), j_s = g(s)$. The simplex (f, g) is non-degenerate if and only if all pairs (i_s, j_s) are distinct. This shows that $|X|$ is homeomorphic to $\Delta_p \times \Delta_q$.

On the other hand, we know that $|\Delta_n| = \Delta_n$. This implies that $|X|^I$ is the geometric realization of the simplicial set $\Delta_n \times \Delta_m$. It is easy to see that it is homeomorphic to $\Delta_n \times \Delta_m$.

Let \mathcal{C} be a category. Define a simplicial set $N(\mathcal{C}) = (N(\mathcal{C})_n)$ as follows: $S(\mathcal{C})_0 = Ob(\mathcal{C})$, $S(\mathcal{C})_1 = Mor(\mathcal{C})$, $S(\mathcal{C})_n$ are sequences of morphisms of length n :

$$A_0 \xrightarrow{\phi_0} A_1 \xrightarrow{\phi_1} \dots A_n. \quad (*)$$

If $f = \{i_0 \leq i_1 \dots i_m\} : [m] \rightarrow [n]$ then $X(f)$ sends (*) to

$$A_{i_0} \xrightarrow{\psi_0} A_{i_1} \xrightarrow{\psi_{i_1}} \dots A_{i_m},$$

where $\psi_s : A_{i_s} \rightarrow A_{i_{s+1}}$ is the composition of $\phi_{i_{s+1}-1} \circ \dots \circ \phi_{i_s}$. If f is degenerate we insert the identity morphisms. It is clear that non-degenerate simplices correspond to sequences of morphisms (*) such that no morphism ϕ_i is the identity.

We call $N(\mathcal{C})$ the *nerv* of \mathcal{C} . Its geometric realization of $B(\mathcal{C}) = |N(\mathcal{C})|$ is called the *classifying space* of the category \mathcal{C} .

Example 6 Consider the category defined by the graph

$$B \rightrightarrows C \leftarrow A.$$

There are 3 non-degenerate simplices in dimension 0 and 3 in dimension 1. The classifying space of the category looks like a loop with an arm: On the other hand if \mathcal{C} is

$$A \rightarrow B \rightrightarrows C$$

we have additionally two 1-simplices and 2 2-simplices. The classifying space looks as follows.

If A is equal to the kernel of $B \rightrightarrows C$, we have only one additional 1-simplex.

Example 7 Let \mathcal{C} be the groupoid category defined by a group G . Since \mathcal{C} has only one object, its nerv has only one vertex. All 1-dimensional simplices are loops corresponding to elements of G . Nondegenerate n -simplices correspond to ordered sets (g_1, \dots, g_n) of elements $g_i \neq 1$ from G . The fundamental group of the nerv is equal to G and its universal covering is contractible. Thus the classifying space of \mathcal{C} is $K(G, 1)$.

Here are some properties of the nerv of a category:

1) If $\#\mathcal{C}(A, B) \leq 1$ for any pair of objects, then the classifying space is contractible. For example, the classifying space of $Open(X)$ is a simplicial complex. Recall that a simplicial complex is a topological space X together with a homeomorphism $\phi : X \rightarrow |Y|$, where $|Y|$ is a geometric realization of a topological space such that the intersection of the images of two simplices belongs to the triangulation, two simplices intersect along their common face, and the set is closed if and only if its intersection with each simplex is closed. For example, the nerv of the category $B \rightrightarrows C \leftarrow A$ is not a simplicial complex.

2) If \mathcal{C} has a final object f , then classifying space is contractible. This is because the nerv of \mathcal{C} is homeomorphic to the cone over the nerv of $\mathcal{C} \setminus f$.

3) The classifying spaces of the category and the dual category coincide.

Since in many interesting cases the classifying space of a category is contractible, one proceeds differently, using Grothendieck topology and the nerves of hypercoverings (see below).

11. Homotopy theory

Let $f, g : X \rightarrow Y$ be two morphisms of simplicial objects. A *homotopy* between f and g is a morphism of functors $h : \Delta[1] \times X \rightarrow Y$ (from ST° to $(Sets) \times \mathcal{C}$ and from ST° to \mathcal{C} such that

$$f = h \circ (\Delta(\delta_1^0) \times 1 : \Delta[0] \times X \rightarrow \Delta[1] \times X \rightarrow Y,$$

$$g = h \circ (\Delta(\partial_1^1) \times 1 : \Delta[0] \times X \rightarrow \Delta[1] \times X \rightarrow Y.$$

Here we identify $\Delta[0] \times X$ with X . In general the homotopy is not an equivalence relation. To define the equivalence relation we consider n -steps homotopies. Here we replace $\Delta[1]$ with the simplicial set

$$\begin{array}{ccccccc} \Delta[1] & & \Delta[1] & & \Delta[1] & & \Delta[1] \\ & \swarrow & \nearrow & & \swarrow & & \nearrow \\ & \Delta[0] & & \Delta[0] & & \Delta[0] & \\ & & & & & & \end{array}.$$

Thus we can define the homotopy category of simplicial objects by taking for morphisms the step-homotopy equivalence classes of morphisms of simplicial objects. They form the set $[X, Y]$.

Example 1 Consider the category $\{0 < 1\}$ of two objects and one morphism between them. A functor $\mathcal{C} \times \{0 < 1\} \rightarrow \mathcal{D}$ is just a morphism between two functors $F \rightrightarrows G$ from \mathcal{C} to \mathcal{D} . By Milnor's theorem, for any two simplicial sets

$$|X \times Y| \cong |X| \times |Y|.$$

We have $|N(\{0 < 1\})| = |\Delta[1]| = |\Delta_1|$. $B(\mathcal{C}) \times B(\{0 < 1\}) \rightarrow B(\mathcal{D})$ defines a homotopy between two maps of the classifying spaces of the categories $B(F), B(G) : B(\mathcal{C}) \rightarrow B(\mathcal{D})$.

In particular, suppose that $F : \mathcal{C} \rightarrow \mathcal{D}$ has a left or right adjoint. Then $B(F) : B(\mathcal{C}) \rightarrow B(\mathcal{D})$ is a homotopy equivalence.

We shall consider only simplicial sets. Let $\Delta_k^n, k = 0, \dots, n+1$, denotes the smallest simplicial subset of $\Delta[n+1]$ which contains the simplices $(\partial_n^i : [n] \rightarrow [n+1]) \in \Delta[n+1]_n$, where $i \neq k$. For example, Δ_k^1 corresponds to the union of 1-faces of the 2-simplex containing the vertex k . It is easy to see that $(\Delta_k^n)_m$ consists of all non-decreasing maps $[m] \rightarrow [n+1]$ whose image does not contain the set $[n] \setminus \{k\}$. The geometric realization of Δ_k^n is the union of n -dimensional faces of Δ_n which contain the vertex e_k .

A (k, n) -horn of X is a simplicial map $\Delta_k^n \rightarrow X$. A simplicial set X is called *complete* or a *Kan complex* if each (k, n) -horn extends to a simplicial map $\Delta[n+1] \rightarrow X$. Obviously, $\Delta[n]$ is a complete simplicial set.

Example 1 The singular simplicial set $[n] \rightarrow Top(\Delta_n, X)$ associated of a topological space X is always complete.

Lemma. *Let $\check{S}T_c$ be the full subcategory of the category of simplicial sets whose objects are complete simplicial sets. Then the relation of homotopy on the set of morphisms is an equivalence relation.*

The inclusion functor $\check{S}T_c \rightarrow \check{S}T$ has the left adjoint in the homotopy categories $Ex^\infty : \check{S}T \rightarrow \check{S}T_c$. By definition, any map of a simplicial set X to a complete simplicial set Y there exists a unique (up to homotopy map) $Ex^\infty(X) \rightarrow Y$.

Now we define the simplicial category by defining morphisms

$$[X, Y] = [X, Ex^\infty Y] = [Ex^\infty X, Ex^\infty Y].$$

The notion of a Kan complex admits a relativization. One defines the notion of a Kan fibration $p : X \rightarrow Y$. This is a simplicial map satisfying a certain property of homotopy lifting. Namely we require that for any morphism $f : \Delta_k^n \rightarrow X$ such that $f \circ p : \Delta_k^n \rightarrow Y$ extends to a morphism $g : \Delta[n+1] \rightarrow Y$ there exists an extension $\bar{f} : \Delta[n+1] \rightarrow X$ such that $p \circ \bar{f} = g$. It is clear that the fibres of a Kan fibration are Kan complexes. Here by a fibre we mean the following. A point of a simplicial set Y is a morphism of simplicial sets $\Delta[0] \rightarrow Y$. The fibre is the fibred product $X \times_Y \Delta[0]$. Each simplicial map is homotopy equivalent to a Kan fibration.

It follows from the definition that the k -coskeleton $Cos_k(X)$ of a complete simplicial set can be defined as follows. We say that $x, y \in X_n$ are equivalent and write $x \sim y$, if, considered as morphisms $\Delta[n] \rightarrow X$, their composition with $Sk_k(\Delta) \rightarrow \Delta[n]$ are equal. Then $Cos_k(X)_n = X_n / \sim$.

One can introduce homotopy groups of simplicial sets. Two simplices $x, x' \in X_n$ are called *comparable* if $X(\partial_n^i)(x) = X(\partial_n^i)(x')$ for all $i \in [n]$. Two comparable simplices are called *homotopy equivalent* if there exists $y \in X_{n+1}$ such that $X(\partial_{n+1}^n)(y) = x, X(\partial_{n+1}^{n+1})(y) = x'$ and

$$X(\partial_{n+1}^i)(y) = X(\sigma_n^i)(X(\partial_n^{n-1})(x)) = X(\sigma_n^i)(X(\partial_n^{n-1})(x')), \quad 0 \leq i \leq n.$$

When X is complete this is an equivalence relation.

We say that X is *pointed* if there is given a simplicial map $e : \Delta[0] \rightarrow X$. If $n \geq 1$, the set of equivalence classes of n -simplexes of X congruent to $e_n = e(\Delta[0]_n) \in X_n$ forms a group, the *n-homotopy group* of X , and is denoted by $\pi_n(X)$. The operation of multiplication is defined by assigning to $x, x' \in X_n$ the simplex $X(\partial_n^n(z))$, where $z \in X_{n+1}$ extends the functik $\Delta_n^n \rightarrow X$ defined by sending the map $\partial_n^i : [n] \rightarrow [n+1] \in \Delta[n+1]_n$ to $e_n \in X_n$ for $i = 0, \dots, n-2$, sending ∂_n^{n-1} to x and sending ∂_n^{n+1} to x' . One can show that

$$\pi_n(X) \cong \pi_n(|X|).$$

Lemma.

$$\pi_n(\text{Cosk}_k(X)) = 0, \quad n > k.$$

For each $m \geq n$, we have a natural surjective simplicial map

$$p_m^n : \text{Cosk}_m(X) = \check{S}T(Sk_n \Delta[?], X) \rightarrow \check{S}T(Sk_m \Delta[?], X) = \text{Cosk}_n(X)$$

corresponding to inclusion of functors $Sk_m(Y) \rightarrow Sk_n(Y)$. It induces an isomorphism of homotopy groups in dimension $\leq m$. In particular, the geometric realization fibre of $p_{n-1}^n : \text{Cosk}_n(X) \rightarrow \text{Cosk}_{n-1}(X)$ is $K(\pi_n(|X|), n)$. Here we use the homotopy exact sequence for Kan fibrations.

The family of maps (p_m^n) defines a projective system functor $n \rightarrow \text{Cosk}_n(X)$. This projective limit is not representable in the homotopy category of simplicial sets. One has to enlarge the category by adding projective limits. This leads to the definition of pro-simplicial objects and their morphisms using the extended notion of homotopy. Then

$$\hat{X} = \lim \text{proj } \text{Cosk}_n(X).$$

exists in the enlarged category and is called the *profinite completion* of X . The sequence

$$X \rightarrow \dots \rightarrow \text{Cosk}_{n+1}(X) \rightarrow \text{Cosk}_n(X) \rightarrow \text{Cosk}_{n-1}(X) \rightarrow \dots \rightarrow \text{Cosk}_0(X)$$

defines a canonical morphism (in the extended category) $X \rightarrow \hat{X}$. We can also get the sequence of Kan fibrations:

$$X \rightarrow \dots \rightarrow \text{Cosk}_{n+1}(X)' \rightarrow \text{Cosk}_n(X)' \rightarrow \text{Cosk}_{n-1}(X) \rightarrow \dots \rightarrow \text{Cosk}_0(X)'$$

Here we replace $Y \rightarrow \text{Cosk}_n(X)$ with $Y' \rightarrow \text{Cosk}_n(X)'$ to apply the relative Ex^∞ transforming a map to a Kan fibration. We use that $f : Y \rightarrow \text{Cosk}_n$ gives

$$\text{Cosk}_{n+1}(f) : \text{Cosk}_{n+1}(Y) \rightarrow \text{Cosk}_{n+1}(\text{Cosk}_n(X)) = \text{Cosk}_n(X).$$

This replaces X with homotopy equivalent X' admitting a *Postnikov* decomposition. Each fibre of the morphism $\text{Cosk}_{n+1}(X)' \rightarrow \text{Cosk}_n(X)'$ is an Eilenberg-Maclane space $K(\pi_n(X), n)$.

12. Hypercoverings

Let \mathcal{C} be a site. We assume that \mathcal{C} has finite fibred products, has an initial object \emptyset and the following *distributivity* property is satisfied: For every family of objects $X, Y_i, i \in I$ in \mathcal{C}/S such that $\coprod_i Y_i$ is representable, the canonical morphism

$$\coprod_i X \times_S Y_i \rightarrow X \times (\coprod_i Y_i)$$

is an isomorphism. This allows to deduce that the decomposition of an object as a direct sum of connected objects is unique. Indeed, write $A = \coprod A_i = \coprod B_j$. Then $A \times_A B_j = B_j = \coprod (A_i \times_A B_j)$ implies that for one i $A_i \times_A B_j = B_j$ and

For example, every topos satisfies this condition.

An object X is called *connected* if $X \neq \emptyset$ and X has no non-trivial coproduct decompositions. A category is locally connected if every object is a coproduct of contractible objects. It is *connected* if it is

locally connected and its final object is connected. We have the functor which assigns to \mathcal{C} the set of its connected components:

$$\pi_0 : \mathcal{C} \rightarrow (\mathit{Sets}).$$

If K_\bullet is a simplicial object in \mathcal{C} we can extend π_0 by defining

$$\pi_0(K_\bullet) = (\pi_0(K_i)).$$

This is a simplicial set.

Every covering $(U_i \rightarrow X)$ of an object X can be viewed as a morphism $U = \coprod U_i \rightarrow X$. We can define the simplicial object $\mathit{cosk}_0(U) = (U \times_X \dots \times_X U)$. Then we can apply π_0 to get a simplicial set $\pi_0(\mathit{cosk}_0(U))$ and then take its geometric realization as an object of the homotopy category. Denote it by $|U|$. If W is a subcovering, i.e. $W \rightarrow X$ factors $W \rightarrow U \rightarrow X$, we get $|W| \rightarrow |U|$, and we can take the projective limit object $\{|U|\}_{U \in \mathit{Cov}(X)}$. This could be a definition of the homotopy type of X . If \mathcal{C} has a final object, we can define the homotopy type of \mathcal{C} as the homotopy type of its final object. Unfortunately the definition is too rude and does not give what we expect when taking for example \mathcal{C} the category of CW-complexes or algebraic varieties. We have to generate from $U \rightarrow X$ more refined coverings. They are called hypercoverings.

Definition A morphism $X \rightarrow Y$ is called a *covering morphism* if one of the equivalent conditions is satisfied

- (i) for any $A \rightarrow Y \in \mathit{Mor}(\mathcal{C})$ the image of $X \times_Y A \rightarrow A$ is a covering sieve of A ;
- (ii) the morphism of associated sheaves $a(h_X) \rightarrow a(h_Y)$ is an epimorphism in the category of sheaves;
- (iii) for any sheaf \mathcal{F} the canonical map $\mathcal{F}(Y) \rightarrow \mathcal{F}(X)$ is injective.

For example, if $f : X \rightarrow Y \in \mathit{Cov}(Y)$, we know that, for any sheaf \mathcal{F} , we have

$$\mathcal{F}(Y) = \mathit{Ker}(\mathcal{F}(X) \rightrightarrows \mathcal{F}(X \times_Y X)),$$

and hence f is a covering morphism.

Let \mathcal{C} be a category with finite products. Let $p > 0$.

Definition A simplicial object $K \in \mathit{ST}(\mathcal{C})$ is called a *hypercovering* in \mathcal{C} of type p if the following properties are satisfied:

- (HR1) the canonical morphisms $K_n \rightarrow \mathit{Cosk}_p(K)_n$ are isomorphisms;
- (HR2) for each $n \geq 0$, the canonical morphism $K_{n+1} \rightarrow \mathit{Cosk}_n(K)_{n+1}$ is a covering morphism;
- (HR3) the canonical morphism $K_0 \rightarrow e$, where e is the final object of \mathcal{C} is a covering morphism. A hypercovering is a hypercovering of some type p . A hypercovering of an object X is a hypercovering in \mathcal{C}/X .

Notice that in the case when the site is canonical, (HR2) says that $K_{n+1} \rightarrow \mathit{Cosk}_n(K)_{n+1}$ is an epimorphism in the category of sheaves \mathcal{C}^\sim .

Let us make some comments about the definition of a hypercovering.

1) It follows from the definition of a coskeleton that

$$\mathit{cosk}_n(X)_{n+1} = \{(x_0, \dots, x_{n+1}) \in \mathit{cosk}_n(X)_{n+2} : X(\partial_n^j)(x_j) = X(\partial_n^{j-1})(x_i), i < j\}.$$

For example, when $n = 1$ this means that (x_0, x_1, x_2) are three 1-simplices which form a triangle with vertices in X_1 . This can be interpreted as $\mathit{cosk}_n(X)_{n+1}$ fills all the simplicial holes in $\mathit{cosk}_n(X)_n$.

Thus the map $K_{n+1} \rightarrow (\mathit{cosk}_n(K_\bullet))_{n+1}$ 2) Condition (HR2) means that $\pi_n(K_\bullet) = 0$ when $n > p$. For example, if $p = 0$, $|K_\bullet|$ is contractible.

3) The geometric realization of a hypercovering in (Sets) is always contractible. In fact $K_{n+1} \rightarrow \mathit{cosk}_n(K)_{n+1} = \mathit{ST}(\mathit{sk}_n(\Delta_{n+1}, X))$ is surjective implies that any map from the boundary of Δ_{n+1} to X can be extended to a map from Δ_{n+1} to X . In fact, hypercoverings in (Sets) are characterized by this property.

We can define the category of hypercoverings of X .

Example 2 When $\mathcal{C} = (\mathit{Sets})$ with canonical topology, the condition (HR2) implies that K is complete and contractible. Indeed the natural map

$$K_{n+1} = \mathit{ST}(\Delta[n+1], K) \rightarrow \mathit{Cosk}_n(K)_{n+1} = \mathit{ST}(\mathit{Sk}_n(\Delta[n+1]), K)$$

is surjective.

For example, $\text{Cosk}_0(X) = X$ with $X_n = X^{n+1}$ is a hypercovering of X . Starting from a covering $K_0 : U \rightarrow X$, we can take $K_1 = \text{cosk}_0(K_0)_1 = U \times_X U$, then $K_2 = \text{cosk}_1(U \times_Y U \rightrightarrows U)_2 = U \times_Y U_Y U$ and so on. A general hypercovering allows one to refine such hypercoverings.

Definition A *pointed site* is a site together with a continuous morphism $p : \mathcal{C}(\text{Sets})$ from a site \mathcal{C} to the canonical site (Sets) . Recall that this means that it carries covering morphisms to surjective maps. We shall also assume that p is exact, i.e. commutes with coproducts, finite fibred products and arbitrary direct products. If X is a simplicial object in a pointed site, then we call its a pointed simplicial object if we choose a point in $p(X_0)$.

An example of a pointed set is the functor from $\text{Et}(X)$ to (Sets) which takes any étale covering $\pi : U \rightarrow X$ to the fibre $\pi^{-1}(x_0)$, where x_0 is a fixed points of X .

Now we take the category of hypercoverings in \mathcal{C} and apply π_0 to obtain

$$\Pi(\mathcal{C}) = \{\pi_0(K_\bullet)\}_{K_\bullet \in \text{HR}(\mathcal{C})}.$$

This is an object in the homotopy category $\text{pro} - \mathcal{H}$ of pro-simplicial sets. The functor $\Pi : \mathcal{C} \rightarrow \text{pro} - \mathcal{H}$ is called the *Verdier functor*.

If \mathcal{C} is a pointed site we can consider pointed hypercoverings (i.e. they become pointed simplicial objects). Then $\Pi(\mathcal{C})$ becomes an object of the homotopy theory of pointed simplicial sets. So, one can define homology of \mathcal{C} with coefficients in an abelian group

$$H_q(\mathcal{C}, A) = H_q(\Pi(\mathcal{C}), A)$$

as well as the homotopy groups

$$\pi_q(\mathcal{C}) = \pi_q(\Pi(\mathcal{C})).$$

Of course, here we have to develop the theory of homotopy groups and homology in the homotopy theory of pro-objects. We refer for this to [Artin-Mazur].

There is another approach due to Saul Lubkin. Let $\mathcal{U} = (U_i)_{i \in I}$ be a covering of a topological space by non-empty open sets. We define the category $\mathcal{C}(\mathcal{U})$ as a full subcategory generated by U_i and then take its nerv and the classifying space. Let N be the nerv. We have $N_0 = I, N_1 = \{i, j \in I : U_i \subset U_j\}$ and so on. For each $n \in \mathbb{N}$ consider the set \mathcal{N}_n of functions $f : [n] \rightarrow I$ such that

$$U_f = U_{f(0)} \cap \dots \cap U_{f(n)} \neq \emptyset.$$

Obviously $[n] \rightarrow \mathcal{N}_n$ forms a simplicial set. It is (or its geometric realization) is often called the *nerv* of the covering. Assume (U_i) is closed under the non-empty intersections. Each U_f defines an n -simplex in the nerv N of the covering by assigning to it the chain of morphisms

$$U_{f(0)} \cap \dots \cap U_{f(n)} \rightarrow U_{f(1)} \cap \dots \cap U_{f(n)} \rightarrow \dots \rightarrow U_{f(n)}.$$

Conversely, for any chain of inclusions among the members of the covering

$$U_{i_0} \subset U_{i_1} \subset \dots \subset U_{i_n}$$

we define $f : [n] \rightarrow I$ by $f(k) = i_k$, so that

$$U_{i_n} = U_{f(n)}, \quad U_{i_{n-1}} = U_{f(n)} \cap U_{f(n-1)}, \dots, U_{i_0} = U_{f(0)} \cap \dots \cap U_{f(n)}.$$

get an isomorphism of simplicial sets $\mathcal{N} \cong N(\mathcal{U})$.

It is easy to see that if we start with a covering $U = \cup_{i \in I} U_i$, and define the hypercovering $K_\bullet = \text{cosk}_0(\coprod_{i \in I} U_i)$, then the simplicial set $\pi_0(K_\bullet)$ is isomorphic to the nerv of the covering $(U_i)_{i \in I}$.

When X is an algebraic variety with Zariski topology this simplicial set is contractible since $K_\bullet = \Delta_n$, where $n = \#I$. This leads to consider Grothendieck topologies different from the Zariski topology.

Consider the following example:

Example 1. Let $X = \mathbb{P}^1(\mathbb{C})$, $U_1 = X \setminus \{0\}$, $U_2 = X \setminus \{1\}$, U_2 is the universal cover of $U_1 \cap U_2$. We have $\pi_1(U_2) = \mathbb{Z}$ and so any deck transformation $U_2 \rightarrow U_2$ is a morphism in $Et(X)$. So our category can be described as

$$e \leftarrow \mathbb{Z} \rightarrow e'$$

Here \mathbb{Z} denotes the object U_2 . The nerv of the subcategory formed by U_2 is equivalent to S^1 . The nerv of $\{e, \mathbb{Z}\}$ is a cone over a circle. So the nerv of the whole category is the gluing of two cones, hence S^2 .

More generally, let $\mathcal{U} = (U_i)_{i \in I}$ be a finite covering of X . Assume that for any $x \in U_i \cap U_j$ there exists U_k such that $x \in U_k \subset U_i \cap U_j$. Let $J \subset I$ be the subset of $j \in I$ such that there exists a point $x \in U_j$ such that U_j is a smallest open subset containing x . Let \mathcal{U}' be the subcover formed by such sets. Suppose that all sets U_j are spaces $K(\pi, 1)$. Then a theorem of Lubkin asserts that covering in $Et(X)$ formed by universal coverings of U_j is defines the nerv whose geometric realization is homotopy equivalent to X .

Note that any algebraic variety over \mathbb{C} admits a cover by such sets. The proof is by induction using fibrations in curves. Use that if a base and a fibre are $K(\pi, 1)$'s, then the space is $K(\pi, 1)$.

PART 2

1. Abelian categories

A category is called *additive* if for any object A , the set $\mathcal{C}(B, A)$ has a structure of an abelian group such that the composition map is bilinear. Moreover one assumes that finite products exist in \mathcal{C} . Since we always have morphisms $A \rightarrow A \times B, B \rightarrow A \times B$ defined by $i_A = (1_A, 0), i_B = (0, 1_B)$. If $f : A \rightarrow Z$ and $g : B \rightarrow Z$ we define $A \times B \rightarrow Z$ by the formula $h = f \circ p_A + g \circ p_B$. Clearly,

$$h \circ i_A = (f \circ p_A + g \circ p_B) \circ i_A = (f \circ p_A) \circ i_A = f.$$

Similarly we check that $h \circ i_B = g$.

We shall denote the direct sum by $A \oplus B$.

Finally we require the existence of an object with $\mathcal{C}(A, A) = \{0\}$. Obviously, a final or cofinal object satisfies this property. We call such object a zero object. All of them are isomorphic and we choose one of them and denote it by 0.

We define the kernel of a morphism $f : A \rightarrow B$ as the kernel $ker(f)$ (if it exists) of the pair $A \rightrightarrows B$, where one arrow is f and another arrow is the zero morphism. By definition, for any object Z we have the exact sequence of abelian groups

$$0 \rightarrow Hom(Z, ker(f)) \rightarrow Hom(Z, A) \rightarrow Hom(Z, B).$$

We define the cokernel of f as the cokernel $coker(f)$ of the pair $A \rightrightarrows B$. By definition, there is an exact sequence of abelian groups

$$0 \rightarrow Hom(coker(f), Z) \rightarrow Hom(B, Z) \rightarrow Hom(A, Z).$$

It follows from the definition that $ker(f) \rightarrow A$ is injective and $B \rightarrow coker(f)$ is a surjective morphism.

Note that the canonical morphism $coim(f) \rightarrow im(f)$ is always bijective but not necessary isomorphism.

Remark. Note that h_A is an abelian presheaf, so the Yoneda functor embeds \mathcal{C} into the category $\check{\mathcal{C}}^{ab}$ of abelian presheaves on \mathcal{C} . In particular we can speak about kernels and cokernels of morphisms $h_A \rightarrow h_B$ in $\check{\mathcal{C}}^{ab}$. It follows from the definition that the kernel $ker(f)$ represents $ker(h_A \rightarrow h_B)$. However, it is not true that the cokernel $coker(f)$ represents $coker(h_A \rightarrow h_B)$. For example, we may take $\mathcal{C} = (Ab)$ and $f : \mathbb{Z} \rightarrow \mathbb{Z}$ to be the multiplication by $d > 1$. Then $coker(f) \cong \mathbb{Z}/d$ but $T(\mathbb{Z}/d) = coker(h_A \rightarrow h_B)(\mathbb{Z}/d) = 0$. If T were representable, we would have $T(\mathbb{Z}) = \mathbb{Z}/d \cong Hom(\mathbb{Z}, A) = A$ and hence $T(\mathbb{Z}/d) = Hom(\mathbb{Z}/d, \mathbb{Z}/d) \neq 0$. This shows that T is not representable.

Example 1. Let A be an abelian group. Consider the additive category which consists of two objects 0 and e and $\mathcal{C}(e, a) = \text{End}(A)$, where $\text{End}(A)$ denotes the group of endomorphisms of A . It is easy to see that, in general, this category does not contain kernels or cokernels.

We can define the image $\text{im}(f)$ as the kernel of $B \rightarrow \text{coker}(A \rightarrow B)$ and the coimage as the cokernel of $\text{ker}(f) \rightarrow A$. There is a canonical morphism $s : \text{coim}(f) \rightarrow \text{im}(f)$ such that the morphism f decomposes

$$A \xrightarrow{p} \text{coim}(f) \rightarrow \text{im}(f) \xrightarrow{i} B.$$

The morphism s is not necessary an isomorphism, as the following example shows:

Example 2. Let \mathcal{C} be the category of filtered vector spaces. Its object are vector spaces together with a filtration $V = F_0(V) \supset F_1(V) \supset \dots \supset F_n(V) \supset \dots \supset \{0\}$. A morphism is a linear map which preserves the filtrations. For example, suppose V has two filtrations $(F_i(V))$ and $(F'_i(V))$ such that $F_i(V) \subset F'_i(V)$ and with strict inclusion for some i . Then the identity linear map is bijective but not an isomorphism.

Definition. An additive category is called *preabelian* if kernels and cokernels exist.

Example 3. The category from Example 1 is preabelian. We take the kernel K of a linear map f and define the filtration by $F_i(K) = K \cap F_i(V)$. We take the cokernel C and define the filtration by $F_i(C) = F^i/F^i \cap \text{im}(f)$.

Lemma 1. Assume \mathcal{C} is preabelian. Then finite projective and inductive limits exist.

Proof. It is enough to show that $A \times_C B$ and $A \coprod_C B$ exist, where $a : A \rightarrow C, b : B \rightarrow C$ (resp. $C \rightarrow A, C \rightarrow B$) are morphisms. We define

$$A \times_C B = \text{Ker}(A \times B \rightrightarrows C) = \text{Ker}(A \oplus B \rightarrow C),$$

where $A \oplus B \rightarrow C$ is the difference of $(a, 0)$ and $(0, b)$ and similarly

$$A \coprod_C B = \text{Coker}(C \rightarrow A \oplus B).$$

Lemma 2. Assume \mathcal{C} is preabelian. Let $f : A \rightarrow B$. The canonical morphism $s : I = \text{coim}(f) \rightarrow I' = \text{im}(f)$ is bijective.

Proof. Let $Z = \text{ker}(s)$. Consider $T = A \times_B Z$, where $Z \rightarrow B$ is the composition $Z \rightarrow I \rightarrow I' \rightarrow B$. There is a projection $T \rightarrow A$ such that its composition with f is zero. For any $a : X \rightarrow A$ such that the composition $X \rightarrow A \rightarrow B$ is zero, there is a morphism $X \rightarrow A \times_B Z$ defined by $(a, 0)$. Its composition with the projection $T \rightarrow A$ is equal to a . Thus T satisfies the definition of the kernel of f and, by its uniqueness, the projection $T \rightarrow A$ factors through an isomorphism $p : T \rightarrow \text{ker}(f)$. But obviously, the composition of the morphism $(0, 1_Z) : Z \rightarrow T$ with p is zero. Since p is injective, the morphism $1_Z : Z \rightarrow Z$ must be zero, i.e. $Z = 0$.

Similarly, if $Z = \text{coker}(s)$, we consider $B \oplus_A Z$.

A category is called *abelian* if it is additive and two additional properties are satisfied:

(A1) kernel and cokernel exist,

(A2) the canonical morphism from coimage to the image is an isomorphism.

Proposition 1. A bijective morphism in an abelian category is an isomorphism.

Proof. By definition, a morphism $f : A \rightarrow B$ is bijective if for any Z the maps $\mathcal{C}(B, Z) \rightarrow \mathcal{C}(A, Z)$ and $\mathcal{C}(Z, A) \rightarrow \mathcal{C}(Z, B)$ are injective. This implies that the kernel and cokernel of f are the zero objects. Thus $\text{im}(f) = A$ and $\text{coim}(f) = B$.

Examples 4. The category of Mod_R of modules over any ring R .

5. Let \mathcal{C} be an abelian category, for any category \mathcal{C}' , the category of functors $\text{Funct}(\mathcal{C}', \mathcal{C})$ is an abelian category. In fact, let $F, G : \mathcal{C}' \rightarrow \mathcal{C}$ be two functors and $u, v : F \rightarrow G$ be two natural transformation. We can

define $(u+v)_X = u_X + v_X : F(X) \rightarrow G(X)$. Since $F \oplus G$ can be easily defined by $(F \oplus G)(X) = F(X) \oplus G(X)$, this makes $\text{Funct}(\mathcal{C}', \mathcal{C})$ an additive category. Now, if $u : F \rightarrow G$ we can define $\ker(u)(X) = \ker(u_X : F(X) \rightarrow G(X))$ and $\text{coker}(u)(X) = \text{coker}(u_X : F(X) \rightarrow G(X))$. Let $\alpha : \text{coim}(u) \rightarrow \text{im}(v)$. Then $\alpha_X : \text{coim}(u_X) \rightarrow \text{im}(u_X)$ is bijective, so α is an isomorphism.

In particular, we see that the category $\tilde{\mathcal{C}}^{ab}$ of *abelian presheaves* (= presheaves $F : \mathcal{C} \rightarrow (\text{Sets})$ which factor through the category of abelian groups) is abelian.

Let \mathcal{C} be a site and $\tilde{\mathcal{C}}^{ab}$ be the full subcategory of $\tilde{\mathcal{C}}^{ab}$ which consists of sheaves. If we restrict the sheafification functor $a : \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{C}}$, we obtain a functor $a : \tilde{\mathcal{C}}^{ab} \rightarrow \tilde{\mathcal{C}}^{ab}$ which is left adjoint to the inclusion functor $i : \tilde{\mathcal{C}}^{ab} \rightarrow \tilde{\mathcal{C}}^{ab}$.

Lemma 3. *Let $u : F \rightarrow G$ be a morphism in $\tilde{\mathcal{C}}^{ab}$. Then*

$$\ker(u) = \ker(i(u)), \quad \text{coker}(u) = a(\text{coker}(i(u))).$$

Proof. Let $K = \ker(i(u))$. For any covering sieve R of $X \in \text{Ob}(\mathcal{C})$, we have the following commutative diagram:

$$\begin{array}{ccccc} K(X) & \rightarrow & F(X) & \rightarrow & G(X) \\ \downarrow & & \downarrow & & \downarrow \\ K(R) & \rightarrow & F(R) & \rightarrow & G(R) \end{array}$$

Since the vertical arrows $F(X) \rightarrow F(R)$ and $G(X) \rightarrow G(R)$ are bijective, this implies that the vertical arrow $K(X) \rightarrow K(R)$ is bijective. Thus K is a sheaf. Then it is immediately checked that $K = \ker(u)$.

Let $C = \text{coker}(i(u))$. The composition $G \rightarrow C \rightarrow a(C)$ defines a morphism $G \rightarrow a(C)$ such that the composition $F \rightarrow G \rightarrow a(C)$ is the zero morphism. By adjunction, for any sheaf Z , there is a canonical bijection $\tilde{\mathcal{C}}^{ab}(a(C), Z) \cong \tilde{\mathcal{C}}^{ab}(C, i(Z))$. Thus if $F \rightarrow G \rightarrow Z$ is the zero morphism of sheaves, the natural morphism of presheaves $C \rightarrow i(Z)$ defines a morphism of sheaves $a(C) \rightarrow Z$. This shows that $a(C) = \text{coker}(u)$.

Proposition 2. *Let \mathcal{C} be a site. Then the category of abelian sheaves is abelian.*

Proof. By the previous lemma, we have kernels and cokernels. Let $u : F \rightarrow G$ be a morphism in $\tilde{\mathcal{C}}^{ab}$. Let $C = \text{coker}(i(u))$. By the same lemma, $\text{coim}(u) = a(\text{coim}(u(i)))$ and $\text{im}(u) = \ker(G \rightarrow a(C)) = \ker(a(G) \rightarrow a(C))$. Since a is left exact (see Theorem 1 in section 9), we obtain $\text{im}(u) = a(\text{im}(u(i)))$ and thus canonical morphism $\text{coim}(u) \rightarrow \text{im}(u)$ is equal to $a(\alpha)$, where α is the canonical morphism $\text{coim}(i(u)) \rightarrow \text{im}(i(u))$. Since $\tilde{\mathcal{C}}^{ab}$ is an abelian category, α is an isomorphism. Hence $a(\alpha)$ is an isomorphism.

We shall consider the following additional axioms which are often satisfied:

(A3) Arbitrary direct sums exist in \mathcal{A} .

For example this implies that for any family of monomorphisms $U_i : A_i \rightarrow A$ there exists the supremum $\sup u_i : B \rightarrow A$. It is enough to take $\coprod A_i$ and consider the canonical monomorphism of the image of $\coprod A_i \rightarrow A$. We will always identify a subobject with a choice of a corresponding monomorphism. So we can speak about the supremum of subobjects.

(A4) = (A3) and the direct sum of any family of monomorphisms is a monomorphism.

(A5) = (A3) and for any directed increasing family of subobjects (A_i)

$$(\sup A_i) \cap B \cong \sup(A_i \cap B).$$

Here we define for two subobjects of an object X

$$A \cap B = \ker(X \rightarrow (X/A) \times (X/B)).$$

Here X/A denotes $\text{coker}(A \rightarrow X)$. Clearly $A \cap B$ serves as $\inf(A, B)$.

The axiom (AB5) is equivalent to the following:

(AB5)' = (AB3) and for any increasing set of subobjects A_i of A

$$\sup_{i \in I} A_i \rightarrow B \cong \lim_{i \in I} \text{ind } A_i.$$

Note that since for each A_i there is a morphism to sup we have always a canonical morphism $\text{lim ind} \rightarrow \text{sup}$.

A set of objects G_i of a category \mathcal{C} is called a family of *generators* if for any object X and its subobject Y not equal to X there exists a morphism $G_i \rightarrow X$ which does not factor through the canonical morphism $Y \rightarrow X$. If \mathcal{C} is abelian and has direct sums we can replace (G_i) with $G = \coprod_i G_i$.

Definition. An abelian category is called a *Grothendieck category* if (A3) is satisfied and it contains a generator.

Examples 6. Consider the category of $\tilde{\mathcal{C}}$ where \mathcal{C} is a small category, i.e. the class of objects is a set. Then the set of representable functors is a set of generators. Note that the objects of $\tilde{\mathcal{C}}$ do not form a set. For this we have to restrict ourselves with presheaves with values in a fixed universe.

7. Let $\mathcal{A} = (Ab)$ be the category of abelian groups. Take $X = \mathbb{Z}$. Then for any object A we have a homomorphism $\mathbb{Z} \rightarrow A$.

8. Let $\mathcal{A} = Mod_R$, then we take $M = R$.

Lemma 4. Let

$$B \xrightarrow{\alpha} A \xrightarrow{\alpha'} B' \rightarrow 0, \quad C \xrightarrow{\gamma} A \xrightarrow{\gamma'} C' \rightarrow 0$$

be two exact sequences in an abelian category \mathcal{A} . Suppose that the composition $\gamma' \circ \alpha$ is an epimorphism. Then $\alpha' \circ \gamma$ is an epimorphism.

Proof. The exact sequences are equivalent to the exact sequences of abelian groups

$$0 \rightarrow \mathcal{A}(B', X) \rightarrow \mathcal{A}(A, X) \rightarrow \mathcal{A}(B, X),$$

$$0 \rightarrow \mathcal{A}(C', X) \rightarrow \mathcal{A}(A, X) \rightarrow \mathcal{A}(C, X),$$

where X is any object in \mathcal{A} . The property that $\alpha' \circ \gamma$ is an epimorphism is equivalent to the injectivity of the composition map

$$\mathcal{A}(C', X) \rightarrow \mathcal{A}(A, X) \rightarrow \mathcal{A}(B, X)$$

and the property that $\alpha' \circ \gamma$ is an epimorphism is equivalent to the injectivity of the composition map

$$\mathcal{A}(B', X) \rightarrow \mathcal{A}(A, X) \rightarrow \mathcal{A}(C, X).$$

Thus the assertion follows from the following easy fact: Let

$$0 \rightarrow N \rightarrow M \rightarrow N' \quad 0 \rightarrow L \rightarrow M \rightarrow L',$$

be exact sequences of abelian groups such that the composition $N \rightarrow M \rightarrow L'$ is injective. Then the composition $L \rightarrow M \rightarrow N'$ is injective.

Lemma 5. An epimorphism in an abelian category is a universal epimorphism.

Proof. Let $f : A \rightarrow B$ be an epimorphism and $v : C \rightarrow B$ be a morphism. We have an exact sequence

$$0 \rightarrow A \times_B C \rightarrow A \oplus C \rightarrow B \rightarrow 0.$$

where $A \oplus C \rightarrow B$ is defined as the difference $f - v$. Also we have an obvious exact sequence

$$0 \rightarrow A \rightarrow A \oplus C \rightarrow C.$$

Now the composition $A \rightarrow A \oplus C \rightarrow B$ is obviously epimorphism, hence the composition $A \times_B C \rightarrow A \oplus C \rightarrow C$ is an epimorphism as follows from the previous lemma.

We can consider presheaves with values in any category. They are just contravariant functors $F : \mathcal{C}^\circ \rightarrow \mathcal{D}$. If \mathcal{C} is a site, we say that F is a sheaf with values in \mathcal{D} if for for any $A \in \text{Ob}(\mathcal{D})$ the presheaf of sets

$$F_A : X \rightarrow \mathcal{D}(A, F(X))$$

is a sheaf.

Lemma 6. Let $u : \mathcal{D}_1 \rightarrow \mathcal{C}_2, v : \mathcal{D}_2 \rightarrow \mathcal{C}_1$ a pair of adjoint functors. Let $F : \mathcal{C} \rightarrow \mathcal{D}_2$ be a presheaf with values in \mathcal{D}_2 on a site \mathcal{C} . If F is a sheaf, then the presheaf $G = v \circ F : X \rightarrow v(F(X))$ is a sheaf with values in \mathcal{D}_1 .

Proof. By adjunction, for any $A \in \mathcal{D}_1$ we have $\mathcal{D}_1(A, v(F(X))) = \mathcal{D}_2(u(A), F(X))$. Thus $G_A = F_{u(A)}$ is a sheaf.

Example 9. Take $\mathcal{D}_2 = (Ab), \mathcal{D}_1 = (Sets)$, let v be the forgetful functor. Then an abelian sheaf is a sheaf of sets.

Now if we take $\mathcal{D}_2 = (Sets), \mathcal{D}_1 = (Ab)$ and v the adjoint of the forgetful functor, we obtain for any sheaf of sets that the presheaf $X \rightarrow \mathbb{Z}^F(X)$ is a sheaf of abelian groups.

Proposition 3. The category $\tilde{\mathcal{C}}^{ab}$ is an abelian category satisfying properties (A5) and has generators.

Proof. We already know that $\tilde{\mathcal{C}}^{ab}$ is an abelian category. Let us check (A3). Let $(F_i)_{i \in I}$ be an arbitrary family of sheaves. Their direct sum $\oplus_{i \in I} F_i$ obviously exists as a presheaf. Since the functor of sheafification commutes with inductive limits (since it has the right adjoint) we obtain

$$a(\oplus_{i \in I} F_i) \cong \oplus_{i \in I} a(F_i) \cong \oplus_{i \in I} F_i.$$

Thus for any sheaf G we have

$$\tilde{\mathcal{C}}^{ab}(\oplus_{i \in I} F_i, G) = \check{\mathcal{C}}(\oplus_{i \in I} F_i, i(G)) = \prod_{i \in I} \check{\mathcal{C}}(F_i, i(G)) = \prod_{i \in I} \tilde{\mathcal{C}}^{ab}(a(F_i), G) = \prod_{i \in I} \tilde{\mathcal{C}}^{ab}(F_i, G).$$

This proves that $\oplus_{i \in I} F_i$ is a sheaf and is a direct sum in $\tilde{\mathcal{C}}^{ab}$.

Let (F_i) be an increasing family of subobjects of F in $\tilde{\mathcal{C}}^{ab}$. Set $A = \cup_{i \in I} i(F_i)$. This is a subpresheaf of F . Clearly, $A = im(\oplus_{i \in I} F_i) \rightarrow F$ and since a is exact we obtain that $a(A) = \sup_{i \in I} F_i$ in the category of sheaves. But obviously $A = \lim ind i(F_i)$. Since a is right exact, we are done.

Finally to get generators, we take the set $\mathbb{Z}_X, X \in Ob(\mathcal{C})$. Here the sheaf \mathbb{Z}_X is defined as follows. Let $v : (Sets) \rightarrow (Ab)$ be the functor "free abelian group generated by a set". Then for any sheaf $F : \mathcal{C} \rightarrow (Sets)$ the composition $v \circ F : \mathcal{C} \rightarrow (Ab)$ is an abelian sheaf. Applying this construction to $F = a(h_X)$ we obtain the sheaf \mathbb{Z}_X . We have, for any abelian sheaf F

$$\tilde{\mathcal{C}}^{ab}(\mathbb{Z}_X, F) = \tilde{\mathcal{C}}(a(h_X), F) = \check{\mathcal{C}}(h_X, F) = F(X).$$

This easily implies that the set of sheaves \mathbb{Z}_X is a set of generators.

Theorem 1 (Gabriel). For each abelian category \mathcal{C} there exists a fully faithfully additive exact functor to a Grothendieck category.

Proof. We consider $\tilde{\mathcal{C}}^{ab}$ with respect to the canonical Grothendieck topology described in Example 6. Let $R = a \circ h : \mathcal{C} \rightarrow \tilde{\mathcal{C}}^{ab}$ be the composition of the Yoneda functor h and the sheafification functor a . Since h and a are left exact, R is left exact. To prove that R is right exact, it suffices to show that it transforms epimorphisms to epimorphisms. Let $f : A \rightarrow B$ be an epimorphism in \mathcal{C} and let $H = coker(h(f))$. For any $X \in Ob(\mathcal{C})$ we have an exact sequence

$$h_A(X) \rightarrow h_B(X) \rightarrow H(X) \rightarrow 0.$$

Let $\bar{a} \in H(X)$ and $a : X \rightarrow B$ be its pre-image in $h_B(X)$. Consider the fibred product $Y = A \times_B X$ corresponding to the morphisms f and a :

$$\begin{array}{ccc} Y & \xrightarrow{p} & X \\ \downarrow q & & \downarrow a \\ A & \xrightarrow{f} & B. \end{array}$$

By Lemma 5, the projection $p : Y = A \times_B X \rightarrow X$ is an epimorphism. We want to show that the image of \bar{a} in $H(Y)$ under the map $H(p) : H(X) \rightarrow H(Y)$ is zero. Since p is a covering morphism in the canonical site, this will imply that the associated sheaf $a(H)$ is zero. Consider the diagram

$$\begin{array}{ccccccc}
h_A(X) & \longrightarrow & h_B(X) & \longrightarrow & H(X) & \longrightarrow & 0 \\
\downarrow h_A(p) & & \downarrow h_B(p) & & \downarrow H(p) & & \\
h_A(Y) & \longrightarrow & h_B(Y) & \longrightarrow & H(Y) & \longrightarrow & 0.
\end{array}$$

Let $q : Y \rightarrow A$ be the first projection. Since $a \circ p = f \circ q$, we have $h_B(p)(a) = a \circ p = f \circ q = h(f)(Y)(q)$. Since the bottom row is exact, the image of $h_B(p)(a)$ in $H(Y)$ is equal to zero. But it is equal to $H(p)(\bar{a})$.

Example 10. Consider the example from Remark 1. Then $\text{coker}(\mathbb{Z} \rightarrow \mathbb{Z}) \cong \mathbb{Z}/d$ and hence the sheaf associated to the presheaf $\text{coker}(h_{\mathbb{Z}} \rightarrow h_{\mathbb{Z}})$ is isomorphic to $a(h_{\mathbb{Z}/d}) = h_{\mathbb{Z}/d}$. In particular, we see that $\text{coker}(h_{\mathbb{Z}} \rightarrow h_{\mathbb{Z}})$ in the category of presheaves is not isomorphic to a sheaf.

2. Cohomology

Let \mathcal{A} be an additive category. A *chain complex* is a sequence of morphisms

$$K : \dots \xrightarrow{d_n} K_n \xrightarrow{d_{n-1}} K_{n-1} \rightarrow \dots$$

such that $d_{n-1} \circ d_n = 0$. We can define the *homology* by

$$H_i(K) = \text{coker}(\text{im}(d_n) \rightarrow \ker(d_{n-1})).$$

One can show that

$$\text{coker}(\text{im}(d_n) \rightarrow \ker(d_{n-1})) \cong \text{coker}(K_n \rightarrow \ker(d_{n-1})) \cong \ker(\text{coker}(d_n) \rightarrow K_{n-2}).$$

More generally, if $ST(\mathcal{A})$ is the category of simplicial objects in \mathcal{A} . For any object K of this category, the boundary maps $d_n^i = \Delta_n^i : K_n \rightarrow K_{n-1}$ belong to the abelian group $\mathcal{A}(K_n, K_{n-1})$ hence we can form their alternating sum

$$d_n = \sum_{i=0}^n (-1)^i d_n^i.$$

Since $d_{n-1}^i \circ d_n^j = d_{n-1}^j \circ d_n^i$ (both of the compositions correspond to the maps $[n-2] \rightarrow [n]$ such that the image does not contain i, j) and they occur in the sum $d_{n-1} \circ d_n$ with opposite sign, we obtain

$$d_{n-1} \circ d_n = 0.$$

Now we can define the *homology groups* of a simplicial object

$$H_i(K_\bullet) = \text{coker}(\text{im}(d_n) \rightarrow \ker(d_{n-1})).$$

When we replace \mathcal{A} with the dual category we obtain the notions of *cochain complexes* and *cohomology groups*.

Let \mathcal{A} be a category of abelian sheaves on a site \mathcal{C} . Let K be a simplicial object in \mathcal{C} . Then, for any abelian sheaf from \mathcal{A} we can define a cosimplicial abelian group $(\mathcal{F}(K_n))$. Then its cohomology are called the *Čech cohomology* of \mathcal{F} with respect to K_\bullet and denoted by

$$\check{H}^n(K, \mathcal{F}).$$

For example, we may take $\mathcal{C} = \text{Open}(X)$ (as always with disjoint sum added). A simplicial object $K = \text{cosk}_0(U)$, where $U = \coprod_{i \in I} U_i$ corresponds to an open cover of X . Then we obtain the Čech cohomology of an open cover with coefficients in a sheaf \mathcal{F} .

Assume that $(U_i \rightarrow U)_{i \in I}$ is a covering family. We can define a hypercovering $\text{cosk}_0(\coprod U_i)$ with

$$K_0 = \coprod U_i, \quad K_1 = \coprod_{(i,j) \in I \times I} U_i \times_U U_j, \dots$$

It follows from the definition of a sheaf

$$H^0(K_\bullet, \mathcal{F}) = \mathcal{F}(U).$$

More generally K_\bullet is a hypercovering in \mathcal{C} . Then $K_1 \rightarrow \text{cosk}_0(K_0)_1 = K_0 \times K_0$ is a covering morphism so $\mathcal{F}(K_0 \times K_0) \rightarrow \mathcal{F}(K_1)$ is injective. Since $K_0 \rightarrow f$ is a covering morphism,

$$\mathcal{F}(f) = \ker(\mathcal{F}(K_0) \rightrightarrows \mathcal{F}(K_0 \times K_0)) \cong \ker(\mathcal{F}(K_0) \rightrightarrows \mathcal{F}(K_1)) = H^0(K_\bullet, \mathcal{F}).$$

One defines Čech cohomology of a sheaf \mathcal{F} by

$$\check{H}^i(\mathcal{F}) = \lim_{K_\bullet \in \text{HR}(\mathcal{C})} \text{ind} H^i(K_\bullet, \mathcal{F}).$$

3. Injective objects

A functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ of additive categories is called *additive* if it defines linear maps on the groups of morphisms.

For example, the functor $\mathcal{C}(?, Y)$ and $\mathcal{C}(X, ?)$ are additive functors from \mathcal{C} to (Ab) .

Another example $\tilde{\mathcal{C}} \rightarrow (Ab), \mathcal{F} \rightarrow \mathcal{F}(X)$.

An additive functor is called *left exact* if $F(\text{Ker}(A \rightarrow B)) = \text{Ker}(F(A) \rightarrow F(B))$. Dual definition for right exactness. An additive functor which is left and right exact is called *exact*.

For example, $\mathcal{C}(?, A)$ is left exact and $\mathcal{C}(X, ?)$ is right exact. The functor $\mathcal{F} \rightarrow \mathcal{F}(X)$ is left exact but not right exact.

Example 1. Consider the Grothendieck topology in an additive category \mathcal{A} with $\text{Cov}(X)$ equal to the set of *universal epimorphisms* $Y \rightarrow X$ (i.e. epimorphisms such that for any $Z \rightarrow X$ the projection $Y \times_X Z \rightarrow Z$ is an epimorphism). Let $f : A \rightarrow B$ be an epimorphism. Consider the fibred product $A \times_B A \rightarrow A$ and let p_1, p_2 be the projections. The morphism $p_1 - p_2 : A \times_B A \rightarrow A$ defines a morphism $u : A \times_B A \rightarrow \text{Ker}(A \rightarrow B)$. Since it admits a section $s : B \rightarrow A$ defined by $(i_A, 0)$, where $i : \text{ker} \rightarrow A$, we see that u is an epimorphism. It is also an universal epimorphism. In fact, if $\phi : C \rightarrow B$ is a morphism, we can define a section of $A \times_B C \rightarrow C$ by $s' = (s \circ \phi, 1_C)$. By definition of a sheaf, the morphism $F(u) : F(\text{ker}(f)) \rightarrow F(A \times_B A)$ is injective. Thus

$$\begin{aligned} \text{ker}(F(A) \rightarrow F(\text{ker}(f))) &= \text{ker}(F(A) \rightarrow F(\text{ker}(f))) \xrightarrow{F(u)} F(A \times_B A) = \\ & \text{ker}(F(A) \xrightarrow{F(p_1 - p_2)} A \times_B A) = \text{ker}(F(A) \rightrightarrows A \times_B A). \end{aligned}$$

This shows that F is a sheaf if and only if it is left exact.

Since h_A is left exact, we obtain that all representable presheaves are sheaves. Conversely, if $(A \rightarrow B) \in \text{Cov}(B)$ and not epimorphism, we find X such that h_X is not a sheaf.

Example 2. Let \mathcal{C} be an abelian category and $R : \mathcal{C} \rightarrow \tilde{\mathcal{C}}^{ab}$ be the additive functor $a \circ h$. Then R is exact. This follows from the proof of Gabriel's theorem.

Definition. An object I of an abelian category \mathcal{A} is called *injective* if the functor h_I is exact. An injective object in the dual category \mathcal{A}° is called *projective*.

It follows from the definition that X is injective if and only if for any monomorphism $A \rightarrow B$ and any morphism $A \rightarrow X$ there exists an extension $B \rightarrow X$.

Dually, X is projective if and only if for any epimorphism $A \rightarrow B$ and a morphism $X \rightarrow B$ there exists an extension $X \rightarrow A$. For example, a free R -module is a projective object in Mod_R .

The next result describes injective objects in any "good" abelian category.

Theorem 1. *Assume that \mathcal{A} is a Grothendieck abelian category. Then an object I is injective if and only if for any subobject V of a generator U in \mathcal{A} any morphism $V \rightarrow I$ extends to a morphism $U \rightarrow I$.*

Proof. The condition is obviously necessary. Let us prove its sufficiency. Let $u : B \rightarrow A$ be a monomorphism and $f : B \rightarrow X$ be a morphism. Consider the set of subobjects of (B', u') which dominate such that f can be extended to B' . We can take the supremum and try to prove that it must be A . So, we may assume that $B \neq A$ is the supremum and f does not extend to A . Let U be a generator such that there exists a morphism $j : U \rightarrow A$ with image not contained in B . Take $V = U \times_A B \rightarrow B$. The projection $j' : V \rightarrow U$ makes it a subobject of U . Let $B' = j(U) + B$. By assumption on j , $B \neq B'$. Consider the exact sequence $V \rightarrow U \times B \rightarrow B' \rightarrow 0$, where the first morphism $\phi' : V \rightarrow U \times B$ is equal to $(i, -j')$ and the second one $\phi : U \times B \rightarrow B'$ is equal to $j \oplus u$. We claim that any morphism $f : B \rightarrow X$ can be extended to $f' : B' \rightarrow X$. This will contradict the maximality of B . To define f' , it suffices to define $v : U \times B \rightarrow X$ such that its composition with ϕ' is equal to 0 and the composition with $(0, 1_B) : B \rightarrow U \times B$ is equal to f . Let s be an extension of $f \circ j' : V \rightarrow X$ to U . Define $v = (s, f) : U \times B \rightarrow X$. It is easy to see that it is what we want.

Example 3. Let $\mathcal{C} = (Ab)$. A generator is \mathbb{Z} . For any subobject $n\mathbb{Z}$ a morphism $f : n\mathbb{Z} \rightarrow A$ is given by the image $a = f(n) \in A$. Let $f' : \mathbb{Z} \rightarrow A$ be its extension and $b = f'(1)$. Then $f'(n) = nf'(1) = f(n) = b$. This shows that A is *divisible group*, i.e. any equation $nx = a$ in the group has a non-zero solution. Since this condition is sufficient, we see that an abelian group is injective if and only if it is a divisible group.

Example 4. Let \mathcal{A} be the category of abelian sheaves on a canonical site \mathcal{C} , where \mathcal{C} is an abelian category. We know that $U = \bigoplus_{X \in \mathcal{A}} h_X$ is a generator in \mathcal{A} . In fact, it is a projective object in \mathcal{C} . Let us prove it. Let $v : A \rightarrow B$ be an epimorphism of sheaves and $f : U \rightarrow B$ be a morphism. The composition of f with the canonical morphism $h_X \rightarrow U$ is a morphism $h_X \rightarrow B$. By Yoneda's Lemma it is defined by an element of $b_X \in B(X)$. In fact, we see that f is described by the family $(b_X)_{X \in Ob(\mathcal{C})}$. Since v is an epimorphism of sheaves, there is an extension $f' : U \rightarrow A$. In fact, let $u_X \in U(X)$ and $f(X)(u_X) = b_X \in B(X)$. Since v is an epimorphism in the category of sheaves, we can find an epimorphism $Y \rightarrow X$ such that the image b_Y of b_X in $B(Y)$ is equal to $v_Y(a_Y)$ for some $a_Y \in A(Y)$.

Proposition 1. *Let $F : \mathcal{A} \rightarrow \mathcal{A}'$ be an additive functor of abelian categories and $G : \mathcal{A}' \rightarrow \mathcal{A}$ be its right adjoint additive functor. Assume that F is exact. Then for any injective object I of \mathcal{C}' the object $G(I)$ is injective.*

Proof. Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence in \mathcal{A} . We have to show that the sequence

$$0 \rightarrow Hom(A, G(I)) \rightarrow Hom(B, G(I)) \rightarrow Hom(C, G(I)) \rightarrow 0$$

is exact. By definition of the adjointness this is equivalent to the exactness of the sequence

$$0 \rightarrow Hom(F(A), I) \rightarrow Hom(F(B), I) \rightarrow Hom(F(C), I) \rightarrow 0.$$

But it is exact since F is exact.

Example 5. Let \mathcal{C} be a site and $X \in Ob(\mathcal{C})$. Consider the subcategory $\{X\}$ of \mathcal{C} which consists of one object X and the set of morphisms $\{1_X\}$. Let \mathcal{A} be the category of abelian presheaves on $\{X\}$. It is isomorphic to the category of abelian groups. Let \mathcal{A}' be the category of abelian sheaves on \mathcal{C} . Let $i : \{X\} \rightarrow \mathcal{C}$ be the inclusion. Consider the functor $i^* : \mathcal{A}' \rightarrow \mathcal{A}$. We have $i^*(F) = F(A) \in Ob(Ab)$. On the other hand, for any sheaf \underline{A} on $\{X\}$ defined by an abelian group A we have

$$i_!(\underline{A})(Y) = \lim_{Y \rightarrow X} \text{ind} \underline{A}(X) = A^{h_X(Y)}.$$

Here we use that the inductive system $Y \rightarrow X$ is discrete, i.e. no morphisms between different objects. The functor i_i is obviously exact (since for any set S the functor $A \rightarrow A^S$ in (Ab) is exact). Thus we obtain that the functor $F \rightarrow F(X)$ sends injective to injectives, in other words, the value of an injective abelian presheaf are injective abelian groups.

Similarly we get that the values of an injective sheaf are injective groups.

Definition. We say that \mathcal{A} has enough injective objects if for any object X there exists a monomorphism to an injective object $u : X \rightarrow I$.

Let us remind some definitions from the set theory which we shall use in the proof of the next theorem. Let (E, Γ) be a pair consisting of a set and an order relation $\Gamma \subset E \times E$ on it. We say that (E, Γ) is isomorphic to (E', Γ') if there exists a bijection $f : E \rightarrow E'$ such that $f(i) \leq f(j)$ if and only if $i \leq j$. The class of isomorphisms is called an *ordinal number* or just an *ordinal*. Using the axiom of choice we pick up a representative (E, Γ) in each ordinal and denote it $Ord(E)$. The cardinality of an ordinal is the cardinality of the set E . There is a natural definition of sum and product of ordinals (using the definition of an order on the sum or product of two sets). Also there is an order on the set of ordinals compatible with sums and products. The supremum of a set of ordinals exists and is an ordinal.

Theorem 2. *Every Grothendieck category \mathcal{A} has enough injective objects.*

Proof. Let A be an object of \mathcal{A} . Let $I(A)$ be the set of monomorphisms $u_i : V_i \rightarrow A$, where V_i is a subobject of a generator U . Consider $A \times U^{\oplus I(A)}$ and define $f : \oplus_{i \in I(A)} V_i \rightarrow M_1(A)$ such that its restriction to each V_i is the morphism (u_i, α_i) , where $\alpha_i : V_i \rightarrow U$ is the canonical embedding of V_i into a copy of U corresponding to i . Set $M_1(A) = \text{coker}(f)$. Then the composition $f(A)$ of $(1_A, 0) : A \rightarrow A \times U^{\oplus I(A)}$ and $A \times U^{\oplus I(A)} \rightarrow M_1(A)$ is a monomorphism. This follows from the fact that the morphism $\oplus_{i \in I(A)} V_i \rightarrow U^{\oplus I(A)}$ is a monomorphism (AB4). Also, any morphism $u_i : V_i \rightarrow A$ extends to a morphism $U \rightarrow A$ (take the restriction of $U^{\oplus I(A)} \rightarrow M_1(A)$ to the summand corresponding to i). Next we repeat the construction by replacing A with $M_1(A)$. Let us use now the transfinite induction. For any ordinal number i let us construct an object $M_i(A)$ and, for any two ordinals $i \leq j$, a monomorphism $M_i(A) \rightarrow M_j(A)$ such that $M_i(A)$ form an inductive system for all $i < i_0$ (i_0 is a fixed ordinal). For $i = 0$ we take $M_0(A) = A$ and for $i = 1$ we take $M_1(A)$ and the monomorphism $M_0(A) \rightarrow M_1(A)$ as constructed in above. Suppose we have such a construction for all ordinals $j < i$. If $i = j + 1$ for some j , we set $M_i(A) = M_1(M_j(A))$ and $M_j(A) \rightarrow M_{j+1}(A)$ equal to $f(M_j(A))$, where f is as above (this also defines all morphisms $M_k(A) \rightarrow M_i(A)$ for all $k \leq i$). If i is the limit ordinal, we set $M_i(A) = \lim \text{ind}_{j < i} M_j(A)$ and take the morphisms $M_j(A) \rightarrow M_i(A)$ to be the inductive limit of the constructed monomorphisms (the inductive limit of monomorphisms is a monomorphism). Let k be the smallest ordinal whose cardinality is strictly greater than the cardinality of the set of subobjects of U . Set $M(A) = M_k(A)$. It suffices to prove that $M(A)$ is injective. By Theorem 1, it is enough to show that for any subobject V of U and a morphism $v : V \rightarrow M(A)$ the image $v(V)$ is contained in some $M_i(A)$ where $i < k$. Since $M_k(A) = \sup M_i(A)$, we have $V = \sup_{i < k} v^{-1}(M_i(A))$ (here we use the axiom (AB5)). The set of subobjects of V has smaller cardinality than the cardinality of k . Also any set of ordinals $< k$ with k as its limit has cardinality equal to the cardinality of k (because the cardinality of k is not equal to the limit ordinal number). This implies that $v^{-1}(M_i(A))$ becomes constant starting from some $i_0 < k$. Thus $V = v^{-1}(M_{i_0})$ and we are done.

Remark 1. Here is a direct proof of Theorem 2 in the case when \mathcal{A} is the category of left R -modules. Let

$$E_R = \text{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z}).$$

It has a natural structure of an R -module. Let us see that it is an injective module. Let $u : M \rightarrow M'$ be an injective homomorphism of R -modules. We have the following commutative diagram:

$$\begin{array}{ccccc} \text{Hom}_R(M, E_R) & \xrightarrow{\text{Hom}_R(u, 1)} & \text{Hom}_R(M', E_R) & & \\ \downarrow & & \downarrow & & \\ \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\text{Hom}_{\mathbb{Z}}(u, 1)} & \text{Hom}_{\mathbb{Z}}(M', \mathbb{Q}/\mathbb{Z}) & & \end{array} .$$

Here the vertical arrows are defined naturally: if $f : M \rightarrow \text{Hom}_{\mathbb{Z}}(R, N)$ then we define $M \rightarrow N$ by $m \rightarrow f(m)(1)$. It is clear that it is an isomorphism. Since \mathbb{Q}/\mathbb{Z} is a divisible abelian group, it is an injective module over \mathbb{Z} . Hence the bottom arrow is surjective. Thus the top arrow is surjective.

For each R -module M we set

$$I^0(M) = E_R^{\text{Hom}_R(M, E_R)}.$$

As a direct sum of injective modules it is an injective module. Let $e_M : M \rightarrow I^0(M)$ be the homomorphism defined by

$$e(m) = (\phi(m))_{\phi \in \text{Hom}_R(M, E_R)}.$$

I claim that e_M is injective. Suppose $e_M(m) = 0$. Then for any $\phi : M \rightarrow E_A$ we have $\phi(m) = 0$. Notice that for any abelian group G there exists a quotient isomorphic to \mathbb{Z}/n , where $n \neq 1$. Hence there exists a non-trivial homomorphism $G \rightarrow \mathbb{Q}/\mathbb{Z}$. This easily implies that for any module M there exists a non-trivial homomorphism from the submodule $Rm \subset M$ to $\text{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ (since $\text{Hom}_R(Rm, \text{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})) = \text{Hom}_{\mathbb{Z}}(Rm, \mathbb{Q}/\mathbb{Z})$).

Theorem 3. *For every abelian category \mathcal{A} there is a fully faithful exact functor to a category of left modules over an associative ring.*

Proof. First of all, using Gabriel's theorem we embed \mathcal{A} into a Grothendieck category by using an exact functor. Let $U = \bigoplus_{X \in \text{Ob}(\mathcal{C})} h_X$ be its generator. Note that U is a projective object of \mathcal{A} . Let $I = \bigcup_{A \in \text{Ob}(\mathcal{A})} \text{Mor}_{\mathcal{A}}(U, A)$ and $P = \bigoplus_I U$. For any $A \in \text{Ob}(\mathcal{A})$ there is an epimorphism

$$P_A = \bigoplus_{\text{Mor}_{\mathcal{A}}(U, A)} U \rightarrow A.$$

For any morphism $P_A \rightarrow A$ there exists a unique morphism, and hence an epimorphism

$$s_A : P \rightarrow A.$$

Note that the functor $A \rightarrow \text{Mor}_{\mathcal{A}}(P, A)$ is exact. Let

$$R = \text{Mor}_{\mathcal{A}}(P, P).$$

This is an associative ring. For every $X \in \text{Ob}(\mathcal{A})$ the group $M(X) = \text{Mor}_{\mathcal{A}}(P, X)$ is a left module over R . Consider the functor $M : X \rightarrow M(X)$ from \mathcal{A} to Mod_R . By Example 4 from the previous section, P is a projective object. Hence the functor M is exact. Also, since P is a generator, M is faithful.

Let us show that it is fully faithful. Suppose $\phi : M(A) \rightarrow M(B)$ is a homomorphism of R -modules. We want to find $f : A \rightarrow B$ such that $M(f) = \phi$. Let $u : P \rightarrow A$ be an epimorphism and $K = \ker(u)$. Let $v : P \rightarrow B$ be an epimorphism. Applying M we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \rightarrow & M(K) & \rightarrow & R & \rightarrow & M(A) & \rightarrow & 0 \\ & & & & s \downarrow & & \downarrow \phi & & \\ & & & & R & \rightarrow & M(B) & \rightarrow & 0 \end{array}.$$

Here the homomorphism s is defined by the projectivity of R as an object of Mod_R . Since any endomorphism of a ring R is of the form $x \rightarrow x \circ r$, where $p : P \rightarrow P$ is an endomorphism of P , we obtain a diagram

$$\begin{array}{ccccccccc} 0 & \rightarrow & K & \rightarrow & P & \rightarrow & A & \rightarrow & 0 \\ & & & & r \downarrow & & \downarrow f & & \\ & & & & P & \rightarrow & B & \rightarrow & 0 \end{array}.$$

Since the composition $M(K) \rightarrow R \rightarrow R \rightarrow M(B)$ is zero, the composition $K \rightarrow P \rightarrow P \rightarrow B$ is zero (because M is exact). Thus there exists a morphism $f : A \rightarrow B$ such that the diagram

$$\begin{array}{ccc} P & \rightarrow & A \\ r \downarrow & & f \downarrow \\ P & \rightarrow & B \end{array}.$$

This implies the diagram

$$\begin{array}{ccc} R & \rightarrow & M(A) \\ s \downarrow & & \downarrow M(f) \\ P & \rightarrow & B \end{array}$$

is commutative. Since $R \rightarrow M(A)$ is epimorphic, $M(f) = \phi$.

4. Derived category

Let $K(\mathcal{A})$ denote the category of cochain complexes

$$K^\bullet = (K^n, d_K^n : K^n \rightarrow K^{n+1})_{n \in \mathbb{Z}}$$

in an abelian category \mathcal{A} . It is a preabelian category. Its objects are cochain complexes.

We shall denote by $K(\mathcal{A})^+$ the full subcategory formed by complexes such that there exists N such that $K^i = 0$ for $i < N$. Similarly we define $K(\mathcal{A})^-$ and $K(\mathcal{A})^b = K(\mathcal{A})^+ \cap K(\mathcal{A})^-$.

Let

$$H^\bullet : K(\mathcal{A}) \rightarrow K(\mathcal{A}),$$

be the functor which is defined by taking the cohomology of a cochain complex. We consider $(H^\bullet(K^\bullet), d_K)$ as a cochain complex with zero coboundary morphisms. We shall use the following notations:

$$\begin{aligned} \tau_{\leq n}(K^\bullet)_i &= \begin{cases} K^i & \text{if } i \leq n \\ 0 & \text{if } i > n \end{cases}, & d_{\tau_{\leq n}(K^\bullet)}^i &= d_K^i, i \leq n. \\ \tau_{\geq n}(K^\bullet)_i &= \begin{cases} K^i & \text{if } i \geq n \\ 0 & \text{if } i < n \end{cases}, & d_{\tau_{\geq n}(K^\bullet)}^i &= d_K^i, i \geq n. \end{aligned}$$

In particular, we set

$$\begin{aligned} \tau_+(K^\bullet) &= \tau_{\geq 0}(K^\bullet), & \tau_-(K^\bullet) &= \tau_{\leq 0}(K^\bullet). \\ K^\bullet[n]^i &= K^{n+i}, & d_{K^\bullet[n]}^i &= (-1)^n d_K^{n+i}. \end{aligned}$$

We define the dual complex \check{K}^\bullet as a complex in $K(\mathcal{A}^\circ)$ defined by

$$(\check{K}^\bullet)^i = K^{-i}, \quad d_{\check{K}^\bullet}^i = d_K^{-i}.$$

Definition. A morphism $f : (K^\bullet, d_K) \rightarrow (L^\bullet, d_L)$ is called *homotopy to zero* if there exist morphisms $k^i : K^i \rightarrow L^{i-1}$ such that

$$f^i = k^{i+1} d_K^i + d_L^{i-1} k^i.$$

Two morphisms are homotopy equivalent $f \sim g$ if the difference is homotopy equivalent to zero. It is clear that the homotopy to zero morphisms form a subgroup in the group $Hom_{K(\mathcal{A})}(K^\bullet, L^\bullet)$ (the corresponding homotopy maps are the sums). Thus the homotopy equivalence is an equivalence relation. Let $Hom_{HK(\mathcal{A})}(K^\bullet, L^\bullet)$ be the quotient group by the subgroup of morphisms which are homotopy to zero.

Lemma 1. *Let $f : K^\bullet \rightarrow L^\bullet, g : L^\bullet \rightarrow M^\bullet$ be two morphisms of complexes. Assume that f is homotopy to 0. Then $g \circ f$ is homotopy to zero.*

Proof. Let (k^i) define a homotopy for f . Then $g(k^i)$ will define a homotopy for $g \circ f$.

Let $HK(\mathcal{A})$ denote the category whose objects are complexes in \mathcal{A} and morphisms are equivalence classes of morphisms in $K(\mathcal{A})$ modulo the homotopy equivalence relation. The previous lemma shows that this is well-defined.

Lemma 2. *If $f \sim g$ then*

$$H^\bullet(g) = H^\bullet(f).$$

Proof. It is enough to check that $H^\bullet(f) = 0$ if $f \sim 0$. This is immediate.

Corollary. The functor $H^\bullet : K(\mathcal{A}) \rightarrow K(\mathcal{A})$ extends to a functor $H^\bullet : HK(\mathcal{A}) \rightarrow HK(\mathcal{A})$.

We shall identify objects of \mathcal{C} with complexes X^\bullet such that $X^n = 0, n \neq 0$ and $X^0 = X$ (i.e. $\tau_{\leq -1} = \tau_{\geq 1} = 0$).

Definition. Let $A \in Ob(\mathcal{A})$. A complex $I^\bullet = (I^n)_{n \geq 0}$ together with a morphism $e_A : A \rightarrow I^\bullet$ is called an *injective resolution* of an object A if

- (i) $I^\bullet = \tau_+(I^\bullet)$;
- (ii) each I^n is injective;
- (iii) $H^n(I^\bullet) = 0, n > 0$;
- (iv) $e : A \rightarrow I^0$ is a kernel of $d^0 : I^0 \rightarrow I^1$ (in particular, $A \cong H^0(I^\bullet)$.)

A *projective resolution* is a complex such that its dual complex is an injective resolution in \mathcal{A}° .

Theorem 1. Let I^\bullet be an injective resolution of A and J^\bullet be an injective resolution of B . Then any morphism $f : A \rightarrow B$ can be extended to a morphism of complexes $I^\bullet \rightarrow J^\bullet$. Two different extensions are homotopy equivalent.

Proof. We have

$$\begin{array}{ccccccccccc} A & \rightarrow & I^0 & \rightarrow & I^1 & \rightarrow & \dots & \rightarrow & I^{n-1} & \rightarrow & I^n & \rightarrow & \dots \\ \downarrow & & & & & & & & & & & & \\ B & \rightarrow & J^0 & \rightarrow & J^1 & \rightarrow & \dots & \rightarrow & J^{n-1} & \rightarrow & J^n & \rightarrow & \dots \end{array}$$

Since $e_A : A \rightarrow I^0$ is a monomorphism, and J^0 is injective, the composition $e_B \circ f : A \rightarrow J^0$ extends to a morphism $f^0 : I^0 \rightarrow J^0$ such that $f^0 \circ d_A = d_B \circ f$. Assume that we can extend f to a morphism of the truncated complexes $\tau_{\leq n}(I^\bullet) \rightarrow \tau_{\leq n}(J^\bullet)$. Let $f^i : I^i \rightarrow J^i$ be the corresponding morphisms. Since $d_J^n \circ f^n \circ d_I^{n-1} = d_J^n \circ d_J^{n-1} \circ f^{n-1} = 0$ we see that f^n defines a morphism from $im(d_I^n)$ to J^{n+1} . Since I^{n+1} is injective we can extend it to a morphism $f^{n+1} : I^{n+1} \rightarrow J^{n+1}$. This extends f to a morphism of the truncated complexes $\tau_{\leq n+1}(I^\bullet) \rightarrow \tau_{\leq n+1}(J^\bullet)$. This proves the existence of an extension.

Let us prove the second assertion. Obviously it is enough to show that an extension of the zero morphism $A \rightarrow B$ is homotopy equivalent to zero. Since $f^0 \circ e_A = 0$, the morphism f^0 factors through $coker(e_A) = im(d_K^0)$. Since I^1 is injective, the latter extends to a morphism $k^1 : I^1 \rightarrow J^0$. Clearly, $f^0 = k^1 \circ d^0 + k^0 \circ d^{-1}$, where $k^0 = 0, d^{-1} = 0$. Assume we can construct a homotopy $(k^i), i \leq n+1$, for the truncated morphism $\tau_{\leq n} : \tau_{\leq n}(I^\bullet) \rightarrow \tau_{\leq n}(J^\bullet)$. Thus $f^n = k^{n+1} \circ d_K^n + d_J^{n-1} \circ k^n$. Let $\alpha = f^{n+1} - k^{n+1} \circ d_K^n : I^{n+1} \rightarrow J^{n+1}$. It is easy to see that $d_J^n \circ \alpha = 0$. Thus α factors through $im(d_K^{n+1})$ and then extends to $k^{n+2} : I^{n+2} \rightarrow J^{n+1}$. We have $f^{n+2} = k^{n+2} \circ d_K^{n+1} + d_J^n \circ k^{n+1}$ and, by induction, we are done.

Corollary. For any object X there exists an injective resolution. For any two injective resolutions of X there exists a morphism between the resolution which is defined uniquely up to homotopy.

We would like to identify objects with their injective resolutions. This would be possible if we build a category where all injective resolutions of an object define the same object. This is an idea of the derived category.

Theorem 2. Let \mathcal{C} be a small category and S be a set of its morphisms. There exists a category $\mathcal{C}[S^{-1}]$ and a functor $Q : \mathcal{C} \rightarrow \mathcal{C}[S^{-1}]$ satisfying the following properties:

- (i) for any $f \in S$, $Q(f)$ is an isomorphism;
- (ii) if $F : \mathcal{C} \rightarrow \mathcal{C}'$ is a functor satisfying property (i), then there exists a unique functor $G : \mathcal{C}[S^{-1}] \rightarrow \mathcal{C}'$ such that $F = G \circ Q$.

Proof. The idea is simple we have to add formally inverses of all $s \in S$. Let us consider an oriented graph Γ' whose vertices are objects of \mathcal{C} and morphisms as edges. For each $s \in S$ add the edge from $end(s)$ to $tail(s)$. Let Γ be the new graph. Now define the category $\mathcal{C}[S^{-1}]$ as follows. Its objects are vertices of Γ . Its morphisms correspond to paths with the same origin and the end modulo the following equivalence relation: two paths are equivalent if they obtained from each other by the following elementary operations:

- (a) two edges can be replaced by the edge corresponding to the composition with respect to
- (b) the two path corresponding to the edges s and its inverse is equivalent to the loops corresponding to the identity morphisms (of the origin and the end of the path).

Remark 1. It is easy to see that the functor $\mathcal{C} \rightarrow \mathcal{C}[S^{-1}]$ is universal with respect to morphisms $F : \mathcal{C} \rightarrow \mathcal{D}$ such that $F(s)$ is invertible for any $s \in S$.

Definition. The category $\mathcal{C}[S^{-1}]$ is called the *localization* of \mathcal{C} with respect to the set of morphisms S .

Now we take $\mathcal{C} = HK(\mathcal{A})$ and S is a set of morphisms $f : K^\bullet \rightarrow L^\bullet$ such that $H^\bullet(f)$ is an isomorphism (such morphisms are called *quasi-isomorphisms*). The obtained category is called the *derived category* of \mathcal{A} and is denoted by $D(\mathcal{A})$. Replacing $HK(\mathcal{A})$ with $HK(\mathcal{A})^\pm$ or $HK(\mathcal{A})^b$ we obtain the derived categories $D^\pm(\mathcal{A})$ (of complexes bounded from below or above) and $D(\mathcal{A})^b$ of bounded complexes.

Theorem 3. *The inclusion $\mathcal{A} \subset K(\mathcal{A})$ extends to a morphism*

$$i : \mathcal{A} \rightarrow D(\mathcal{A}).$$

The image \bar{I}^\bullet of any injective resolution I^\bullet of X in $D(\mathcal{A})$ is isomorphic to $i(X)$. For each $X \in Ob(\mathcal{A})$ pick one injective resolution $\tau(X)$ of X . Then $X \rightarrow \tau(X)$ is an isomorphism from $i(\mathcal{A})$ and the subcategory of $D(\mathcal{A})$ generated by $\tau(X)$.

Proof. This follows from the definitions.

Example 1. Let \mathcal{A} is semi-simple, t.e. all short exact sequences split. Then $D(\mathcal{A})$ is isomorphic to the category of cyclic complexes, i.e. complexes with zero differentials.

Definition. A set S of morphisms in a category \mathcal{C} is called *localizable* if it satisfies the following properties:

(FR1) S is closed under compositions;

(FR2) for any $s : X \rightarrow Y$ from S and a morphism $f : Z \rightarrow Y$ there exists $g : W \rightarrow X$ and $t : W \rightarrow Z$ from S such that $s \circ g = f \circ t$. Also the similar property holds when we reverse the arrows;

(FR3) for any $f, g : A \rightrightarrows B$ the existence of $s \in S$ such that $s \circ f = s \circ g$ is equivalent to the existence of $t \in S$ such that $f \circ t = g \circ t$.

Condition (i) means that we can write each $s^{-1}f$ in $\mathcal{C}[S^{-1}]$ in the form gt^{-1} or can write each fs^{-1} in the form $t^{-1}g$. Let $f : X' \rightarrow Y$ be a morphism in \mathcal{A} and $s : X' \rightarrow X$ belongs to S . If we denote the morphism fs^{-1} in $\mathcal{C}[S^{-1}]$ as a "roof"

$$\begin{array}{ccc} X' & & \\ s \downarrow & \searrow f & \\ X & & Y \end{array}$$

Two roofs define the same morphism if they can be extended to a common roof

$$\begin{array}{ccccc} & & X' & & \\ & \nearrow & s \downarrow & \searrow f & \\ \tilde{X} & & X & & Y \\ & \searrow & s' \uparrow & \nearrow f' & \\ & & X'' & & \end{array}$$

To check that this is indeed an equivalence relation one has to use the third property of localizing sets. If $(X', s, t) : X \rightarrow Y$ is equivalent to (X'', t, g) by means of (Z', r, h) and (X'', t, g) is equivalent to (X''', u, e) by means of (Z'', p, i) , then we first define $sr : Z' \rightarrow X, tp : Z'' \rightarrow X$, then take $v : W \rightarrow Z', k : W \rightarrow Z''$ such that $srv = tpk$. Then $f_1 = hv, f_2 = pk$ satisfy $tf_1 = tf_2$. Thus we find $w : Z''' \rightarrow W$ such that $f_1w = f_2w$. If we take $q = rvw : Z''' \rightarrow X'$ and $j = ikw : Z''' \rightarrow X''$ we get the equivalence $(X', s, t) \sim (X''', u, e)$.

The composition is defined by composing the roofs:

$$\begin{array}{ccccc} & & \tilde{X} & & \\ & \swarrow & & \searrow & \\ & X' & & X'' & \\ s \swarrow & & \searrow f & & \searrow f' \\ X & & Y & & Z \end{array}$$

Proposition 1. *Let S be a localizing set of morphisms in an additive category \mathcal{C} . Then $\mathcal{C}[S^{-1}]$ is an additive category.*

Proof. The idea is to reduce to common denominator. Suppose we have two morphisms $X \rightarrow Y$ represented by two roofs $fs^{-1} = (s, f)$ and $f's'^{-1} = (s', f')$. Let $U = Z \times_X Z'$ with respect to $s : Z \rightarrow X, s' : Z' \rightarrow X$. Then the two projections $r : U \rightarrow Z, r' : U \rightarrow Z'$ are quasi-isomorphisms. Then we have $t = sr = s'r' : U \rightarrow X$. Let $g = fr, g' = f'r'$; they are morphisms from U to Y . Then $gt^{-1} = fs^{-1}, g't^{-1} = f's'^{-1}$ and we define $fs^{-1} + f's'^{-1}$ to be equal to $t^{-1}(g + g')$.

The notion of localization is closely related to the notion of a *quotient-category*.

A subcategory \mathcal{B} of an abelian category \mathcal{A} is called *thick* if any zero object of \mathcal{A} belongs to \mathcal{B} , \mathcal{B} is closed under taking a subobject or a quotient-object, and if $u : B \rightarrow A$ is a subobject of A and B and $\text{coker}(u)$ belong to \mathcal{B} , then B belong to \mathcal{B} .

Now the factor-category \mathcal{C}/\mathcal{B} is defined as follows. First of all its objects are objects of \mathcal{A} . For any $A \in \text{Ob}(\mathcal{A})$ let \mathcal{C}_A denote the full subcategory of \mathcal{A}/A which consists of monomorphisms $A' \rightarrow A$ such that $A/A' \in \mathcal{B}$. Let \mathcal{C}_A° denotes the same thing for the dual category \mathcal{A}° . We have a functor $\text{hom} : \mathcal{C}_A \times \mathcal{C}_B^\circ \rightarrow (\text{Sets})$ defined by $\text{hom}(A', B') = \mathcal{A}(A', B')$. For any A, B in \mathcal{A} We set

$$\text{Mor}_{\mathcal{C}/\mathcal{B}}(A, B) = \lim_{\mathcal{C}_A \times \mathcal{C}_B^\circ} \text{ind hom}.$$

If we assume that \mathcal{A} is small (or the class of subobjects is a set for any object) then this limit exists. For example, if A, B belong to \mathcal{B} , then \mathcal{C}_A and \mathcal{C}_A° contain 0 and hence any two elements in $\text{hom}(A', B')$ are equivalent to the zero map. We define the composition of new morphisms as follows. Let $\bar{u} : A' \rightarrow B'$ be a representative of $u : A \rightarrow B$ and $\bar{v} : B'' \rightarrow C'$ is a representative of $v : B \rightarrow C$. Let $X = \text{im}(B'' \rightarrow B \rightarrow B')$. It is a subobject of B' and a factor-object of B'' . Let $C'' = C' \amalg_{B''} C'$. This is a quotient-object of C' . Let $A'' = A' \times_{B'} X \rightarrow A'$. This is a subobject of A' . We can replace A' with A'' and C' with C'' and take for the composition the morphism $A'' \rightarrow X \rightarrow C''$. Notice that \mathcal{C}_A is closed under taking an overobject of a subobject of A . Similarly \mathcal{C}_A° is closed under passing from B' to B'' where $A \rightarrow B'' \rightarrow B'$ is a composition of epimorphisms.

Notice that $\mathcal{A}(A, B)$ is mapped to the inductive limit (since A/A and $\ker(B \rightarrow B)$ belong to \mathcal{B}). The identity functor defines a functor $F : \mathcal{A} \rightarrow \mathcal{C}/\mathcal{B}$. It is easy to see that it is exact and $F(A) = 0$ iff $A \in \mathcal{B}$. Also each object in the factor-category is of the form $F(A)$.

5. Exact triangles

Before we show that the set of quasi-isomorphisms in $HK(\mathcal{A})$ is a localizing set we have to introduce some constructions familiar from homotopy theory.

Recall the *cone construction* from algebraic topology. Let $f : X \rightarrow Y$ be a continuous map of topological spaces. We define the cone C_f of f as the topological space

$$C_f = Y \amalg X \times [0, 1] / \sim,$$

where $(x, 1) \sim f(x)$ and $(x, 0) \sim (x_0, 0)$ for some fixed $x_0 \in X$. In the case Y is a point, $C_f = \Sigma X$ is the suspension of X . In the case when $X \rightarrow Y$ is an inclusion, C_f is homotopy equivalent to Y/X . So in this case C_f is an analog of a cokernel. Also note that there is an inclusion $Y \rightarrow C_f$ and if we apply the cone construction to this we get that C_f/Y is homotopy equivalent to ΣX . Thus we have a sequence of morphisms in the homotopy category:

$$X \rightarrow Y \rightarrow C_f \rightarrow \Sigma X.$$

Recall that we have the suspension isomorphism:

$$\tilde{H}_{n+1}(\Sigma X) \cong \tilde{H}_{n+1}(X).$$

Also recall that $\Sigma X = X \times [0, 1]$ with $X \times \{0\}$ and $X \times \{1\}$ identified with a point (a double cone). Since $X \times (0, 1)$ and $X \times [0, 1)$ are contractible and $X \times (0, 1)$ is homotopy equivalent to X , we can use the Mayer-Vietoris exact sequence:

$$\tilde{H}_{n+1}(X_1 \cup X_2) \rightarrow \tilde{H}_n(X_1 \cap X_2) \rightarrow \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) \rightarrow \tilde{H}_n(X_1 \cup X_2) \rightarrow \dots$$

Thus, using the suspension isomorphism this gives, a long exact sequence

$$H_i(X) \rightarrow H_i(Y) \rightarrow H_i(C_f) \rightarrow H_{i-1}(X) \rightarrow H_{i-1}(Y) \rightarrow \dots \quad (*)$$

In the case when $Y \rightarrow X$ is an inclusion, $H_i(C_f) \cong H_i(Y, X)$ and we get the exact sequence of a pair (X, Y) . Also, if f induces the isomorphism on homology and X, Y are simply-connected, we get $H_*(C_f) = 0$. By Hurevich's theorem, $\pi_*(C_f) = 0$, and by Whitehead theorem C_f is homotopy to a point. This implies that f is a homotopy equivalence.

There is an analogous construction for simplicial sets. Let $f : (X_n) \rightarrow (Y_n)$ be a morphism of simplicial sets. Then C_f is a simplicial set with

$$(C_f)_i = Y_i \cup \bar{X}_{i-1},$$

where $\bar{X}_{i-1} = X_{i-1} \times *$ is the cone over X_{i-1} . Passing to the cochain complexes this leads to the cochain complex

$$C_X^\bullet[1] \oplus C_Y^\bullet, \quad d = \begin{pmatrix} d_X[1] & 0 \\ f & d_Y \end{pmatrix}.$$

This can be taken for the definition of the cone C_f for any morphism of cochain complexes $f : X^\bullet \rightarrow Y^\bullet$ in $K(\mathcal{A})$.

Definition. Let $f : X^\bullet \rightarrow Y^\bullet$ be a morphism in $K(\mathcal{A})$. Define the *cone* of f as the complex

$$C_f = X^\bullet[1] \oplus Y^\bullet, \quad d = \begin{pmatrix} d_X[1] & 0 \\ f & d_Y \end{pmatrix}.$$

Define the *cylinder* $Cyl(f)$ as the complex

$$Cyl(f) = X^\bullet \oplus X^\bullet[1] \oplus Y^\bullet, \quad d = \begin{pmatrix} d_X & -1 & 0 \\ 0 & d_X[1] & 0 \\ 0 & f & d_Y \end{pmatrix}.$$

Example 1. Let $f : X \rightarrow Y$ considered as a morphism of complexes. Then $(C_f)^{-1} = X, C_f^0 = Y$ and $d = f$. Thus $H^0(C_f) = Ker(f)$ and $H^1(C_f) = Coker(f)$.

Example 2. Take $f = id : X^\bullet \rightarrow X^\bullet$. Let $k^i : X^{i+1} \oplus X^i \rightarrow X^i \oplus X^{i-1}$ defined by $(x^{i+1}, x^i) \rightarrow (x^i, 0)$. This defines the homotopy between 1_{C_f} and 0_{C_f} .

Lemma 1. Let $f : X^\bullet \rightarrow Y^\bullet$ be a morphism of complexes. There is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \rightarrow & Y^\bullet & \rightarrow & C_f & \rightarrow & X^\bullet[1] & \rightarrow & 0 \\ & & a \downarrow & & \parallel & & & & \\ 0 & \rightarrow & X^\bullet & \rightarrow & Cyl(f) & \rightarrow & C_f & \rightarrow & 0 \end{array} \quad (*)$$

The first vertical arrow is a quasi-isomorphism.

Proof. We define a by $(k^i, k^{i+1}, l^i) \rightarrow (0, 0, l^i)$ and its inverse $b : (k^i, k^{i+1}, l^i) \rightarrow f(k^i) + l^i$. Now $b \circ a = id$. We define the homotopy $h^i : Cyl_i \rightarrow Cyl_{i-1}$ by $(k^i, k^{i+1}, l^i) = (0, k^i, 0)$. We have

$$\begin{aligned} a \circ b(k^i, k^{i+1}, l^i) &= (0, 0, f(k^{i+1}) + l^i), \\ (d_{Cyl(f)}^{i-1} h^i + h^i d_{Cyl(f)}^i)(k^i, k^{i+1}, l^i) &= (-k^i, -d_K^i(k^i), f(k^i)) + (0, d_K^i(k^i) - k^{i+1}, 0) = \\ &= (-k^i, -k^{i+1}, f(k^{i+1})) = (a \circ b - 1_{Cyl(f)})(k^i, k^{i+1}, l^i). \end{aligned}$$

This checks that $a \circ b \sim id$.

Corollary. Let $f : X^\bullet \rightarrow Y^\bullet$ be a morphism in $HK(\mathcal{A})$. Then it can be extended to a sequence

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} C_f \xrightarrow{h} X^\bullet[1],$$

where the composition of any two morphisms is zero.

Proof. We define $g : Y \rightarrow C_f = X^\bullet[1] \oplus Y^\bullet$ as the natural inclusion and the morphism $C_f^\bullet \rightarrow X[1]$ as the natural projection. By the proof of the previous lemma, the composition $g \circ f : X^\bullet \rightarrow Y^\bullet \rightarrow C_f$ is homotopy equivalent to the composition $X^\bullet \rightarrow Cyl(f) \rightarrow C_f$ which is zero. The composition $h \circ g : Y^\bullet \rightarrow C_f \rightarrow X^\bullet[1]$ is zero because the top row in Lemma 1 is an exact sequence.

Definition. A triangle in $K(\mathcal{A})$ is a diagram of the form

$$K^\bullet \rightarrow L^\bullet \rightarrow M^\bullet \rightarrow K^\bullet[1].$$

A *distinguished triangle* is a triangle which is quasi-isomorphic to the part of the diagram from the previous lemma

$$K^\bullet \rightarrow Cyl(f)^\bullet \xrightarrow{\pi} C_f^\bullet \rightarrow K^\bullet[1].$$

Lemma 2. Any short exact sequence of complexes is quasi-isomorphic to the middle row of the diagram (*).

Proof. Let

$$0 \rightarrow K^\bullet \xrightarrow{f} L^\bullet \xrightarrow{g} M^\bullet \rightarrow 0$$

be an exact sequence in $K(\mathcal{A})$. We define $\beta : Cyl(f) \rightarrow L^\bullet$ to be equal to b from the previous lemma and $\gamma : C_f \rightarrow M^\bullet$ by composing the natural projection $C_f \rightarrow L^\bullet$ with g . We have $\ker(\gamma) = K^\bullet[1] \oplus Imf = K^\bullet[1] \oplus K^\bullet = C_{1_K}^\bullet$. By Example 2, the latter complex has trivial cohomology. Thus, using the exact sequence $0 \rightarrow Ker(\gamma) \rightarrow C_f \rightarrow M^\bullet \rightarrow 0$ we obtain that γ is a quasi-isomorphism.

Theorem 1. Any distinguished triangle

$$K^\bullet \rightarrow L^\bullet \rightarrow M^\bullet \rightarrow K^\bullet[1]$$

defines an exact sequence of cohomology:

$$\dots \rightarrow H^i(K^\bullet) \rightarrow H^i(L^\bullet) \rightarrow H^i(M^\bullet) \rightarrow H^i(K^\bullet[1]) = H^{i+1}(K^\bullet) \rightarrow \dots$$

Proof. It is enough to prove it for the distinguished triangle:

$$K^\bullet \rightarrow Cyl(f)^\bullet \rightarrow C_f^\bullet \xrightarrow{g} K^\bullet[1].$$

We have the exact sequence

$$0 \rightarrow K^\bullet \rightarrow Cyl(f)^\bullet \rightarrow C_f \rightarrow 0.$$

It gives the exact sequence of cohomology

$$\dots \rightarrow H^i(K^\bullet) \rightarrow H^i(Cyl(f)) \rightarrow H^i(C_f) \xrightarrow{\delta} H^{i+1}(K^\bullet) \rightarrow \dots$$

It remains to identify the coboundary morphism δ with $H^i(g)$. We use that $Cyl(f)^i = K^i \oplus C_f^i$ and check the definitions.

Theorem 2. *In the category $HK(\mathcal{A})$ quasi-isomorphisms form a localizing set of morphisms.*

Proof. Let $s : K^\bullet \rightarrow L^\bullet$ and $g : M^\bullet \rightarrow L^\bullet$, where s is a q.i.. We have to find a q.i. $t : N^\bullet \rightarrow M^\bullet$, and a morphism of complexes $g : N^\bullet \rightarrow K^\bullet$ making the square

$$\begin{array}{ccc} N^\bullet & \xrightarrow{t} & M^\bullet \\ \downarrow & & \downarrow g \\ K^\bullet & \xrightarrow{s} & L^\bullet \end{array} \quad (**)$$

commutative. Let $\pi = (0, 1_L) : L^\bullet \rightarrow C_s = K^\bullet[1] \oplus L^\bullet$, Consider the following diagram in $HK(\mathcal{A})$:

$$\begin{array}{ccccccc} C_{\pi \circ g}[-1] & \xrightarrow{t} & M^\bullet & \xrightarrow{\pi \circ g} & C_s & \rightarrow & C_{\pi \circ g} \\ h \downarrow & & \downarrow g & & \parallel & & \downarrow \\ K^\bullet & \xrightarrow{s} & L^\bullet & \xrightarrow{\pi} & C_s & \rightarrow & K^\bullet[1] \end{array}$$

where $t = (1_M, 0, 0) : M^\bullet \rightarrow C_{\pi \circ g}[-1] = M^\bullet \oplus K^\bullet[-1] \oplus L^\bullet[-1]$ and $h(m^i, k^i, l^{i-1}) = -k^i$. We claim that the left square in the diagram is quasi-isomorphic to the square (**). We have

$$\begin{aligned} t &: (m^i, k^i, l^{i-1}) \rightarrow -m^i, \\ g \circ t - s \circ h &: (m^i, k^i, l^{i-1}) \rightarrow g(m^i) + s(k^i). \end{aligned}$$

Let $p^i : C_{\pi \circ g}[-1]^i \rightarrow L^{i-1}$ defined by the natural projection. We have

$$(p^{i+1} d_{C_{\pi \circ g}[-1]}^{i-1} + d_L^i p^i)(m^i, k^i, l^{i-1}) = g(m^i) + s(k^i) - d_L^{i-1}(l^{i-1}) + d_{L^\bullet[-1]}^i(l^i) = g(m^i) + s(k^i).$$

This shows that $g \circ t \sim s \circ h$. Thus we see that $g \circ t \sim s \circ h$ so that the square is indeed commutative in $HK(\mathcal{A})$

Since s is a q.i., the complex C_f is *acyclic* (i.e. all cohomology are zero). Since the top row is a distinguished triangle, t must be a q.i..

Similarly,

$$\begin{array}{ccccccc} C_f[-1] & \xrightarrow{\tau} & K^\bullet & \xrightarrow{s} & L^\bullet & \rightarrow & C_f \\ \parallel & & \downarrow g & & \downarrow h & & \parallel \\ C_f[-1] & \xrightarrow{g\tau} & M^\bullet & \xrightarrow{t} & C_{g \circ \tau} & \rightarrow & C_f \end{array}$$

is used to check the property about fibred products.

Finally we have to check the third property of localizing sets. Let $f : K^\bullet \rightarrow L^\bullet$ be a morphism in $HK(\mathcal{A})$. Assume $sf = 0$ for some q.i. $s : L^\bullet \rightarrow M^\bullet$. We have to show that $ft = 0$ for some q.i. $t : N^\bullet \rightarrow K^\bullet$. Consider the following commutative diagram:

$$\begin{array}{ccccccc} C_s[-1] & \xrightarrow{\delta} & L^\bullet & \xrightarrow{s} & M^\bullet & & \\ \parallel & & \uparrow f & & & & \\ C_s[-1] & \xrightarrow{g} & K^\bullet & \xrightarrow{t} & C_g[-1] & & \end{array}$$

The morphism g is defined as follows

$$g^i : K^i \rightarrow C_s[-1]^i = L^i \oplus M^{i-1}, \quad k^i \rightarrow (f^i(k^i), -h^i(k^i)),$$

where h defines the homotopy $sf \sim 0$. We have $tf = tg\delta = 0$ (because $tg = 0$ in $HK(\mathcal{A})$ as follows from Corollary to Lemma 2). Since s is a q.i., t is a q.i..

6. Triangulated categories

Definition. An additive category \mathcal{C} is called *triangulated* if it is equipped with the following data:

- (i) An additive automorphism $T : \mathcal{C} \rightarrow \mathcal{C}$ (the *shift functor*).

(ii) A class of distinguished triangles (closed under a naturally defined isomorphism of triangles)

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

(we also write it as

$$\begin{array}{ccc} & Z & \\ [1] \swarrow & & \nwarrow \\ X & \rightarrow & Y \end{array}$$

to justify the name). The following axioms must be satisfied:

(TR1) $X \xrightarrow{1_X} X \rightarrow 0 \rightarrow T(X)$ is distinguished;

(TR2) (*extension axiom*) any morphism $f : X \rightarrow Y$ can be completed to a distinguished triangle;

(TR3) (*shift axiom*) a triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ is distinguished if and only if

$$Y \xrightarrow{v} Z \xrightarrow{w} T(X) \xrightarrow{-T(u)} T(Y)$$

is distinguished;

(TR4) Any commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ f \downarrow & & \downarrow g \\ X' & \xrightarrow{u'} & Y' \end{array} .$$

extends to a morphism of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & T(X) \\ f \downarrow & & \downarrow g & & \downarrow h & & \downarrow T(f) \\ X' & \xrightarrow{u'} & Y & \xrightarrow{v'} & Z & \xrightarrow{w'} & T(X') \end{array}$$

(TR5) (*octaedron axiom*) Given three triangles

$$X \xrightarrow{u} Y \xrightarrow{j} Z' \rightarrow T(X), \quad Y \xrightarrow{v} Z \rightarrow X' \xrightarrow{i} T(Y), \quad X \xrightarrow{vu} Z \rightarrow Y' \rightarrow T(X)$$

there exist two morphisms $f : Z' \rightarrow Y', g : Y' \rightarrow X'$ such that $(1_X, v, f), (u, 1_Z, g)$ are morphisms of triangles and $(Z', Y', X', f, g, T(j) \circ i)$ is a distinguished triangle. One can illustrate it by the following diagrams:

$$\begin{array}{ccccccc} X' & \longleftarrow & \text{---} & Z & X' & \longleftarrow & \text{---} & Z \\ \downarrow & [1] \searrow & \nearrow & \uparrow & \downarrow & \nwarrow & \swarrow & \uparrow \\ & Y & & & & Y' & & \\ \downarrow & \swarrow & \nwarrow & \downarrow & \downarrow & \nearrow & [1] \searrow & \downarrow \\ Z' & \text{---} [1] \text{---} & \longrightarrow & X & Z' & \text{---} [1] \text{---} & \longrightarrow & X \end{array} .$$

Here the upper and the bottom triangle in the left diagram are distinguished triangles and the other two triangles are commutative. In the right diagram, the upper and the bottom triangles are commutative and

the other two are distinguished triangles. It is also required that the two possible morphisms $Y \rightarrow Y'$ (factored through Z and Z') are equal. The axiom says that the left diagram can be completed to the right diagram.

Remark 1. It follows from Lemma 2 that the cone of a morphism $A \rightarrow B$ represented by an embedding of morphisms of complexes $0 \rightarrow K^\bullet \rightarrow L^\bullet$ is isomorphic to an object represented by L^\bullet/K^\bullet . We denote it by B/A . The octahedron axiom shows that if $A \rightarrow B$ and $B \rightarrow C$ are represented by monomorphisms of complexes, then

$$C/B \cong (C/A)/(B/A).$$

Indeed, let $X = C/B, Y = B/A$. We have

$$\begin{array}{ccccccc} C/B & \longleftarrow & \text{---} & \text{---} & C & C/B & \longleftarrow & \text{---} & \text{---} & C \\ \downarrow & \searrow & & \nearrow & \uparrow & \downarrow & \swarrow & & \swarrow & \uparrow \\ & & B & & A & & & C/A & & \\ \downarrow & \swarrow & & \swarrow & \downarrow & \downarrow & \nearrow & & \searrow & \downarrow \\ B/A & \text{---} & \text{---} & \longrightarrow & A & B/A & \text{---} & \text{---} & \longrightarrow & A \end{array} .$$

Remark 2. Using the shift axiom (TR3) allows one to extend the commutative diagram formed by g and h to a morphism of triangles by constructing a morphism f (we use the notation of (TR4)). Also it allows one to extend a morphism $f : X \rightarrow Y$ to a distinguished triangle $Z \rightarrow X \rightarrow Y \rightarrow Z[1]$.

Proposition 1. Let

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

be a distinguished triangle.

- (i) The composition of any two consecutive morphisms is zero.
- (ii) For any object M the sequences

$$\begin{aligned} \dots &\rightarrow \text{Hom}_C(M, T^i(X)) \rightarrow \text{Hom}_C(M, T^i(X)) \rightarrow \text{Hom}_C(M, T^i(X)) \rightarrow \text{Hom}_C(M, T^{i+1}(X)) \rightarrow \dots, \\ \dots &\rightarrow \text{Hom}_C(T^{i+1}(X), M) \rightarrow \text{Hom}_C(T^i(Z), M) \rightarrow \text{Hom}_C(T^i(Y), M) \rightarrow \text{Hom}_C(T^i(X), M) \rightarrow \dots, \end{aligned}$$

are exact.

- (iii) In axiom (TR4), if f and g are isomorphisms then h is also an isomorphism.
- (iv) If u is zero, then there exists $\gamma : Z \rightarrow Y$ such that $\gamma \circ w = 1_{T(X)}$.

Proof. (i) and (ii) By axiom (TR2) it is enough to check the exactness at Y . Extend $(1_X : X \rightarrow X) \rightarrow (u : X \rightarrow Y)$ to a morphism of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{1_X} & X & \xrightarrow{0} & 0 & \xrightarrow{0} & T(X) \\ \downarrow 1_X & & \downarrow u & & \downarrow h & & \downarrow 1_{T(X)} \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & T(X) \end{array} .$$

Since $h : 0 \rightarrow Z$ is the zero morphism we get $vu = 0$. Consider

$$\begin{array}{ccccccc} U & \xrightarrow{0} & 0 & \xrightarrow{0} & T(U) & \xrightarrow{-1} & T(U) \\ \downarrow f & & \downarrow 0 & & \downarrow T(f) & & \downarrow \\ Y & \xrightarrow{v} & Z & \xrightarrow{w} & T(X) & \xrightarrow{-T(u)} & T(Y) \end{array} .$$

Each arises from the triangles

$$\begin{array}{ccccccc}
 U & \xrightarrow{1_U} & U & \xrightarrow{0} & 0 & \xrightarrow{0} & T(U) \\
 \downarrow g & & \downarrow f & & \downarrow 0 & & \downarrow T(g) \\
 X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & T(X)
 \end{array}
 .$$

By (TR2) we can find the third arrow in the second diagram $T(g) : T(U) \rightarrow T(X)$ and then reconstruct the first arrow $g : U \rightarrow X$ in the first diagram (by shifting and applying T^{-1}). From the commutativity of the first square we get $f = ug$.

(iii) Apply (ii) to $\text{Hom}_{\mathcal{C}}(Z', ?)$ to both sequences, use the five-lemma to deduce that the homomorphism $\text{Hom}(h) : \text{Hom}(Z', Z) \rightarrow \text{Hom}(Z', Z')$ is an isomorphism. This implies that there exists $\phi \in \text{Hom}(Z', Z)$ such that $h \circ \phi = 1_{Z'}$. Similarly, using $\text{Hom}_{\mathcal{C}}(? , Z)$ we find the left inverse of h .

(iv) Assume $u = 0$. Apply (TR4) to

$$\begin{array}{ccccccc}
 X & \xrightarrow{0} & 0 & \xrightarrow{0} & T(X) & \xrightarrow{1_{T(X)}} & T(X) \\
 \downarrow 1_X & & \downarrow 0 & & \downarrow \gamma & & \downarrow 1_{T(X)} \\
 X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & T(X)
 \end{array}
 .$$

Here the top triangle is constructed by using (TR1) and (TR3).

Corollary 1. *Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ extends $u : X \rightarrow Y$. Then Z is defined uniquely modulo isomorphism.*

Definition. We shall call Z a *cone* over u and denote it by C_u .

Remark 3. In the cone notation the octahedron axiom shows that

$$C_{v \circ u} = T^{-1}(C_w),$$

where $w : C_v \rightarrow T(C_u)$ is the composition $C_v \rightarrow T(Y) \rightarrow T(C_u)$.

Corollary 2. *A morphism in a triangulated category is an isomorphism if and only if its cone is a zero object.*

Proof. Assume $u : X \rightarrow Y$ is an isomorphism. Applying (TR1) and (TR4) we get that $X \rightarrow Y \rightarrow C_u \rightarrow T(X)$ is isomorphic to $X \rightarrow X \rightarrow 0 \rightarrow T(X)$. Conversely, if $C_u = 0$ we apply (TR3) to get a distinguished triangle $0 \rightarrow X \rightarrow Y \rightarrow 0$. By Proposition 1 (ii), we get that u is an isomorphism.

Corollary 3. *Let \mathcal{C} be a triangulated abelian category. Then any monomorphism $u : X \rightarrow Y$ splits, i.e. u is an isomorphism to a direct summand of Y .*

Proof. Extend u to a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

Let $\alpha = T^{-1}(w) : T^{-1}(Z) \rightarrow X$. Applying (TR3) we get a distinguished triangle $T^{-1}(Z) \xrightarrow{\alpha} X \rightarrow Y \rightarrow Z$. By Proposition 1 (i), we obtain that $f \circ \alpha = 0$ and hence $\alpha = 0$ since u is a monomorphism. By Proposition 1 (iv) there exists $\gamma : Z \rightarrow Y$ such that $v \circ \gamma = 1_Z$. It remains to prove that $s = u \oplus \gamma : X \oplus Z \rightarrow Y$ is an isomorphism. By Corollary 2 this is equivalent to $C_f = 0$. We shall apply the octahedron axiom. Let $i : X \rightarrow X \oplus Z$ be the natural inclusion and $p : X \oplus Z \rightarrow Z$ be the natural projection. The triangle

$$X \xrightarrow{i} X \oplus Z \xrightarrow{p} Z \rightarrow T(X)$$

is distinguished. This follows from considering the triangle $0 \rightarrow Z \rightarrow Z \rightarrow 0$ and applying (TR3) to $f : X \rightarrow 0, g = p : X \oplus Z \rightarrow X$ to obtain a morphism $C_i \rightarrow Z$, and then applying (TR3) again to $f : 0 \rightarrow X, g : Z \rightarrow X \oplus Z$ to obtain its inverse $Z \rightarrow C_i$. Now apply (TR5). We have

$$\begin{array}{ccccccc} C_s & \longleftarrow & \text{---} & \text{---} & Y & C_s & \longleftarrow & \text{---} & \text{---} & Y \\ \downarrow & \searrow & & \nearrow & \uparrow & \downarrow & \swarrow & & \swarrow & \uparrow \\ & & X \oplus Z & & & & & Z & & \\ \downarrow & \swarrow & & \swarrow & \downarrow & \downarrow & \nearrow & & \searrow & \downarrow \\ Z & \text{---} & \text{---} & \longrightarrow & X & Z & \text{---} & \text{---} & \longrightarrow & X \end{array}$$

Here the upper and the bottom triangle in the left diagram are distinguished triangles (with vertical arrows are of degree 1) and the other two triangles are commutative. In the right diagram, the upper and the bottom triangles are commutative and the other two are distinguished triangles. It follows that $C_s = C_{1_Z}$ and hence is zero.

Remark 4. The previous corollary says that an abelian triangulated category is semi-simple. In particular the category of abelian groups is not triangulated. Let \mathcal{C} be a semi-simple abelian category. Then it can be triangulated and each distinguished triangle is isomorphic to the triangle

$$X \rightarrow Y \rightarrow \text{Ker}(f) \oplus \text{Coker}(f) \rightarrow X[1].$$

Definition. A functor $F : \mathcal{C} \rightarrow \mathcal{A}$ from triangulated category \mathcal{C} to an abelian category \mathcal{A} is called *cohomological* if for any distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

the sequence

$$\dots \rightarrow F(T^i(X)) \rightarrow F(T^i(Y)) \rightarrow F(T^i(Z)) \rightarrow F(T^{i+1}(X)) \rightarrow \dots$$

is exact in \mathcal{A} .

Example 1. The functors $X \rightarrow \text{Hom}_{\mathcal{C}}(U, X)$ and $X \rightarrow \text{Hom}(X, U)$ for a fixed U are cohomological. This follows from Proposition 1.

It follows from Remark 2 that the category $K(\mathcal{A})$ cannot be triangulated unless \mathcal{A} is semi-simple. Instead we consider the category $HK(\mathcal{A})$. We use the same shift functor $X^\bullet \rightarrow X^\bullet[1]$ and consider the set of distinguished triangles equal to the set of triangles isomorphic in $HK(\mathcal{A})$ to a triangle $X \xrightarrow{f} Y \rightarrow C_f \rightarrow X[1]$ in $K(\mathcal{A})$.

Theorem 1. *The category $HK(\mathcal{A})$ is triangulated with respect to the functor $T = ?[1]$ and the set of distinguished triangles is equal to the set of triangles isomorphic to a triangle $X \xrightarrow{f} Y \rightarrow C_f \rightarrow X[1]$ in $K(\mathcal{A})$.*

Proof. We shall check each axiom.

(TR1) This follows from Example 3 since the identity morphism of C_{1_X} is homotopy to the zero morphism so that $C_{1_X} \cong 0$ in $HK(\mathcal{A})$.

(TR2) We know that any morphism in $HK(\mathcal{A})$ can be represented by a morphism in $K(\mathcal{A})$. We extend it to a distinguished triangle in $K(\mathcal{A})$ and then consider the corresponding triangle in $HK(\mathcal{A})$.

(TR3) Let $X \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \xrightarrow{w} X^\bullet[1]$ be a distinguished triangle in $HK(\mathcal{A})$. We may assume that $Z^\bullet \cong C_u$. It suffices to check that $C_v \cong X[1]$ and $-T(u) : X[1] \rightarrow Y[1]$ is isomorphic to the canonical morphism $C_v \rightarrow Y[1]$ such that the triangle

$$Y^\bullet \rightarrow C_u \xrightarrow{s} C_v \rightarrow Y^\bullet[1]$$

is isomorphic to the triangle

$$Y^\bullet \xrightarrow{v} Z^\bullet \xrightarrow{w} X^\bullet[1] \xrightarrow{-u[1]} Y^\bullet[1]$$

Define $\alpha : X^\bullet[1] \rightarrow C_v$ by the formula $\alpha(x^{i+1}) = (-u^{i+1}(x^{i+1}), x^{i+1}, 0)$. To prove that $(1, 1, \alpha, 1)$ defines a morphism of triangles, we have to check that $s \sim \alpha \circ w$. The homotopy is defined by $h^i(x^{i+1}, y^i) = (y^i, 0, 0)$. Finally we need to show that the morphism we have just constructed is an isomorphism. Define $\beta : C_v \rightarrow X^\bullet[1]$ as the projection. Then $\beta \circ \alpha = 1_X$ and $\alpha \circ \beta$ is given by the formula $(y^{i+1}, x^{i+1}, y^i) \rightarrow (-f^{i+1}(x^{i+1}), x^{i+1}, 0)$. The homotopy is given by $h^i : (y^{i+1}, x^{i+1}, y^i) \rightarrow (y^i, 0, 0)$.

(TR4) We may assume that $Z = C_u$, $Z' = C_{u'}$. We take $h = f[1] \oplus g$.

(TR5) It suffices to find a distinguished triangle isomorphic to

$$C_u \xrightarrow{f} C_{vu} \xrightarrow{g} C_v \xrightarrow{h} C_u[1],$$

where

$$\begin{aligned} f : C_u &= X^\bullet[1] \oplus Y^\bullet \xrightarrow{(1, v)} C_{vu} = X[1] \oplus Z, \\ g : C_{vu} &= X^\bullet[1] \oplus Z^\bullet \xrightarrow{(T(u), 1)} C_v = Y^\bullet[1] \oplus Z^\bullet, \\ h : C_v &= Y^\bullet[1] \oplus Z^\bullet \xrightarrow{(1, 0)} C_u[1] = X^\bullet[2] \oplus Y^\bullet[1]. \end{aligned}$$

For this we have to find an isomorphism

$$a : C_f = C_u[1] \oplus C_{vu} = X^\bullet[2] \oplus Y^\bullet[1] \oplus X[1] \oplus Z^\bullet \cong C_v = Y^\bullet[1] \oplus Z^\bullet.$$

To check that this and the above morphisms are morphisms of complexes.

We have

$$\begin{aligned} (d_{C_{vu}}^i f^i)(x^{i+1}, y^i) &= d_{C_{vu}}^i(x^{i+1}, v(y^i)) = (-d_X^{i+1}(x^{i+1}), (vu)^{i+1}(x^{i+1}) + d_Z^i(v(y^i))) \\ f^{i+1}(d_{C_u}(x^{i+1}, y^i)) &= f^{i+1}(-d_X^{i+1}(x^{i+1}), u^{i+1}(x^{i+1}) + d_Y^i(y^i)) = (-d_X^{i+1}(x^{i+1}), (vu)^{i+1}(x^{i+1}) + v^i(d_Y^i(y^i))). \end{aligned}$$

This checks that f is a morphism of complexes. We have

$$\begin{aligned} (d_{C_v}^i g^i)(x^{i+1}, z^i) &= d_{C_v}^i(u^{i+1}(x^{i+1}), z^i) = -d_Y^{i+1}(u^{i+1}(x^{i+1}), (vu)^{i+1}(x^{i+1}) + d_Z^i(z^i)), \\ g^i(d_{C_v}^i(x^{i+1}, z^i)) &= g^i(-d_X^{i+1}(x^{i+1}), (vu)^{i+1}(x^{i+1}) + d_Z^i(z^i)) = (u^{i+1}(x^{i+1}), (vu)^{i+1}(x^{i+1}) + d_Z^i(z^i)). \end{aligned}$$

This checks that g is a morphism of complexes. We have

$$(d_{C_u[1]}^i h^i)(y^{i+1}, z^i) = d_{C_u[1]}^i(0, y^{i+1}) = (0, d_Y^{i+1}(y^{i+1})),$$

$$h^i(d_{C_v^i}(y^{i+1}, z^i)) = h^i(d_Y^{i+1}(y^{i+1}), v^{i+1} + d_Z(z^i)) = (0, d_Y^{i+1}(y^{i+1})).$$

This checks that h is a morphism of complexes. To construct an isomorphism $a : C_f \rightarrow C_v$ we observe that $C_f = \text{Cyl}(\phi)$, where $\phi : X[1] \rightarrow Y[1] \oplus Z$, where $\phi(x^{i+1}) = (u^{i+1}(x^{i+1}), 0)$. It follows from Lemma 1 that $\text{Cyl}(\phi)$ is isomorphic to $C_v = Y[1] \oplus Z$ in $HK(\mathcal{C})$.

Definition. Let \mathcal{A} be a triangulated category and S be a localizing set. We say that S is compatible with triangulation if

- (FR4) $s \in S$ iff $T(s) \in S$;
- (FR5) in (TR4) if $f, g \in S$ we can find $h \in S$.

Definition. A morphism between triangulated categories is called a ∂ -functor if it transforms distinguished triangles to distinguished triangles and commutes with the shift functor.

Example 2. Any additive functor between abelian categories $F : \mathcal{A} \rightarrow \mathcal{B}$ defines a ∂ -functor of the corresponding categories $HK(\mathcal{A}) \rightarrow HK(\mathcal{B})$. This follows from the fact that, for any morphism $f : X \rightarrow Y$,

$$F(C_f) = C_{F(f)}, \quad F(\text{Cyl}(f)) = \text{Cyl}(F(f))$$

so that it transforms the standard triangles $X \rightarrow \text{Cyl}(f) \rightarrow C_f \rightarrow X[1]$ to a standard triangle. Since an additive functor transforms isomorphisms to isomorphisms, it is a ∂ -functor.

Theorem 2. Let \mathcal{C} be a triangulated category and S be a localizing set compatible with the triangulation. The localized category $\mathcal{C}[S^{-1}]$ is a triangulated category such that the canonical functor $Q : \mathcal{C} \rightarrow \mathcal{C}[S^{-1}]$ is a ∂ -functor.

Proof. Since objects do not change we take for T the shift in \mathcal{C} . We define a distinguished triangle in $\mathcal{D} = \mathcal{C}[S^{-1}]$ as a triangle isomorphic to the image of a distinguished triangle under the functor Q . We shall check the axioms one by one. First recall that

$$\text{Hom}_{\mathcal{D}}(X, Y) = \lim_{s: X' \rightarrow X} \text{ind} \text{Hom}_{\mathcal{C}}(X', Y).$$

(TR1) Obvious.

(TR2) We choose a representative $f' : X' \rightarrow Y$ of $f : X \rightarrow Y$ for some $s : X' \rightarrow X$ from S . Extend f' to a distinguished triangle $X' \rightarrow Y \rightarrow Z \xrightarrow{w} X'[1]$ in \mathcal{C} . Then it is isomorphic in \mathcal{D} to the triangle

$$X \rightarrow Y \rightarrow Z \xrightarrow{T(s)w} X[1].$$

(TR3) Obvious.

(TR4)

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & T(X) \\
\uparrow s & & \uparrow t & & \uparrow r & & \uparrow T(s) \\
X'' & \xrightarrow{u''} & Y'' & \xrightarrow{v''} & Z'' & \xrightarrow{w''} & T(X'') \\
\downarrow \tilde{f} & & \downarrow \tilde{g} & & \downarrow \tilde{h} & & \downarrow T(\tilde{f}) \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & T(X')
\end{array}$$

Here the morphism $X \rightarrow Y$ in the localized category is represented by a pair (s, \tilde{f}) and similar for the morphism $G : Y \rightarrow Z$, $w : Z \rightarrow T(X)$. We would like to construct the morphisms u'', v'', w'' . We do it by first replacing f and g with equivalent roofs.

First, by property (FR2) of localizing sets, we find a morphism $\tilde{t} : \tilde{X} \rightarrow X''$ from S and a morphism $\tilde{u} : \tilde{X} \rightarrow Y''$ making the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{u}} & Y'' \\ \tilde{t} \downarrow & & \downarrow t \\ X'' & \xrightarrow{us} & Y. \end{array}$$

commutative. It is easy to check that the compositions $u' \tilde{f} \tilde{t}$ and $\tilde{g} \tilde{u}$ from \tilde{X} to Y' are equal in the localized category but not in the original one. This easily implies that there exists a morphism $q : \tilde{X}' \rightarrow \tilde{X}$ from S such that the compositions $u' \tilde{f} \tilde{t} q$ and $\tilde{g} \tilde{u} q$ define the same morphism $\tilde{X}' \rightarrow Y'$ in \mathcal{C} . Now replace X'' with \tilde{X}' and $f = (s, u)$ with $(s \tilde{t} q, \tilde{f}, \tilde{t} q)$. We have a morphism $u'' : X'' \rightarrow Y''$ defined by the roof $(\tilde{q}, \tilde{u} q)$ which makes the square

$$\begin{array}{ccc} X'' & \xrightarrow{u''} & Y'' \\ s \downarrow & & \downarrow \tilde{u} q \\ X & \xrightarrow{u} & Y. \end{array}$$

commutative. Extend u'' to a distinguished triangle $X'' \xrightarrow{u''} Y'' \xrightarrow{v''} Z'' \xrightarrow{w''} X''[1]$ in \mathcal{C} . By Property (TR4) we can find a morphism $r : Z'' \rightarrow Z$. By property (FR4), $r \in S$. Again we apply (TR4) to the commutative square

$$\begin{array}{ccc} X'' & \xrightarrow{u''} & Y'' \\ q \tilde{t} \tilde{f} \downarrow & & \downarrow \tilde{g} \tilde{u} q \\ X' & \xrightarrow{u'} & Y'. \end{array}$$

to obtain a morphism $\tilde{h} : Y'' \rightarrow Y'$. The pair $h = (r, \tilde{h})$ defines a morphism $Z \rightarrow Z'$ in the localized category such that the triple (f, g, h) is a morphism of the distinguished triangles.

(TR5) We leave it as a homework exercise.

To apply the previous lemma to our situation $\mathcal{C} = HK(\mathcal{A})$ we have to check that the localizing set of quasi-isomorphisms satisfies (FR4) and (FR5). This follows from the following more general statement (we apply it by taking the functor $F = H^\bullet$).

Lemma 1. *Let $F : \mathcal{C} \rightarrow \mathcal{A}$ be a cohomological functor from a triangulated category to an abelian category. Let S be the set of morphisms in \mathcal{C} such that $F(T^i(s))$ is an isomorphism for all $i \in \mathbb{Z}$. Then S is a localizing set compatible with triangulations.*

Proof. This is similar to the proof of Theorem 1 from the previous section. The first property is trivial. To check the second one, assume that $s : X \rightarrow Y$ belongs to S and $g : Y' \rightarrow Y$ is a morphism in \mathcal{C} . Extend s

to a distinguished triangle $X \xrightarrow{s} Y \xrightarrow{v} Z \rightarrow T(X)$ and extend $v' = g \circ v : Y' \rightarrow Z$ to a distinguished triangle $W \xrightarrow{t} Y' \rightarrow Z \rightarrow T(W)$. Applying (TR4) (and Remark 1) we find a morphism of triangles

$$\begin{array}{ccccccc}
W & \xrightarrow{t} & Y' & \xrightarrow{v'} & Z & \xrightarrow{w'} & T(W) \\
\downarrow f & & \downarrow g & & \downarrow 1_Z & & \downarrow T(f) \\
X & \xrightarrow{s} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & T(X)
\end{array}$$

It remains to show that $t \in S$. Applying the long exact sequence to the image of the bottom triangle under F we obtain that $F(T^i(Z)) = 0$ for all i . Applying the same to the top triangle, we see that $F(T^i(t)) = 0$ for all i . Thus $t \in S$. We leave the proof of the dual statement in (FR2) to the reader.

Let us prove (FR3). Suppose $f : X \rightarrow Y$ is such that there exists a morphism $s : Y \rightarrow Y'$ from S with $s \circ f = 0$. Let $Z \xrightarrow{u} Y \xrightarrow{s} Y' \rightarrow T(Z)$ be a distinguished triangle obtained by applying (TR2) and (TR3). Under the canonical map $Hom(X, Y) \rightarrow Hom(X, Y')$ the map f goes to zero. Since Hom is a cohomological functor, there exists $g \in Hom(X, Z)$ such that $f = v \circ g$. Again we find a triangle $W \xrightarrow{t} X \xrightarrow{g} Z \rightarrow T(W)$. By Proposition 1 (i), $g \circ t = 0$, hence $f \circ t = v \circ (g \circ t) = 0$. It remains to show that $t \in S$. Since $s \in S$, $F(T^i(Z)) = 0$ for all i . From the second triangle we find that $F(T^i(t)) = 0$ for all i .

Property (FR4) is trivial and property (FR5) follows from the exact sequence of a cohomological functor and the five-lemma.

Corollary 1. $D(\mathcal{A})$ is a triangulated category.

Definition. A subcategory of a triangulated category is called a *triangulated subcategory* if two sides of a triangle belongs to it implies that the third side belongs to it.

Corollary 2. The set of quasi-isomorphisms in any triangulated subcategory of $HK(\mathcal{A})$ is a localizing set compatible with triangulations.

Examples. 1. Let \mathcal{B} be a thick subcategory of an abelian category \mathcal{A} . Consider the subcategory $HK_{\mathcal{B}}(\mathcal{A})$ of $HK(\mathcal{A})$ formed by complexes whose cohomology belong to \mathcal{B} . Then $HK_{\mathcal{B}}(\mathcal{A})$ is a triangulated subcategory.

2. Let $HK(\mathcal{I})$ be the subcategory of $HK(\mathcal{A})$ which consists of complexes whose objects are injective objects of \mathcal{A} . Then it is triangulated. This follows from the proof of Theorem 2 (since a direct sum of injective complexes is injective).

7. Injective complexes

The main goal is to prove the following

Theorem 1. Assume \mathcal{A} has enough injective objects. Then $D^+(\mathcal{A})$ is equivalent to the subcategory $I^+(\mathcal{A})$ of $HK(\mathcal{A})^+$ generated by complexes of injective objects bounded from below.

Lemma 1. Any morphism from an acyclic complex to an injective bounded from below complex is homotopic to zero.

Proof. This proved it while proving Theorem 1 in section 6.

Lemma 2. Any quasi-isomorphism of an injective bounded from below complex to another complex has a homotopy inverse.

Proof. Extend $f : I^\bullet \rightarrow X^\bullet$ to a distinguished triangle $I^\bullet \rightarrow X^\bullet \rightarrow Z^\bullet \rightarrow I^\bullet[1]$. Since f and $f[1]$ are a q.i., we see that Z^\bullet is acyclic. By the previous lemma, $Z \rightarrow I^\bullet[1]$ is homotopic to zero. Thus, by Prop. 1 from §6, f has the inverse in $H(\mathcal{A})$.

Lemma 3. Let \mathcal{B} be a full subcategory of a category \mathcal{C} and S is a localizing set in \mathcal{C} . Assume that for any morphism $s : X \rightarrow X'$ from S with $X \in \text{Ob}(\mathcal{B})$ there exists a morphism $f : X' \rightarrow Z$ with $Z \in \text{Ob}(\mathcal{B})$ such that $f \circ s \in S$. Then $S' = S \cap \text{Mor}(\mathcal{B})$ is a localizing set and the natural functor $\mathcal{B}[S'^{-1}] \rightarrow \mathcal{B}[S^{-1}]$ is fully faithful.

Proof. A morphism of $B \rightarrow B'$ of two objects in $\mathcal{C}[S^{-1}]$ is defined by morphisms $s : X \rightarrow X', g : Y \rightarrow X'$ in \mathcal{C} , where $s \in S$. The morphisms $f \circ s : X' \rightarrow Z$ and $f \circ g : Y \rightarrow Z$ will represent this morphism by a morphism from in $\mathcal{B}[S'^{-1}]$.

Corollary. Let $I(\mathcal{A})^+$ be the full subcategory of $HK(\mathcal{A})^+$ formed by injective complexes and $D^+(I)$ its derived category. Then the canonical functor $D^+(I) \rightarrow D^+(\mathcal{A})$ is fully faithful.

Proof. Apply Lemmas 2 and 3.

Lemma 4. Let \mathcal{A} be an abelian category with enough injective objects. Then every $X^\bullet \in HK^+(\mathcal{A})$ admits a quasi-isomorphism into an injective complex.

Proof. Without loss of generality we may assume that $X^p = 0, p < 0$. Let $f^0 : X^0 \rightarrow I^0$ be an embedding of X^0 into an injective object. We have $0 = \text{Im}(d_X^{-1}) \cong \text{Im}(d_I^{-1}) = 0$, where we consider f^0 as a morphism of truncated complexes $\tau_+ X^\bullet \rightarrow \tau_+ I^\bullet$. Assume by induction that we have constructed I^0, \dots, I^p and morphisms $f^k : X^k \rightarrow I^k$ such that the morphism f^\bullet of the truncated complexes $\tau_{\leq p-1} X^\bullet \rightarrow \tau_{\leq p-1} I^\bullet$ is a quasi-isomorphism and $f^p : \text{Im}(d_X^{p-1}) \rightarrow \text{Im}(d_I^{p-1})$ is an isomorphism. Embed

$$(I^p / \text{Im}(I^{p-1} \rightarrow I^p)) \oplus_{X^p} X^{p+1}$$

in I^{p+1} and define the morphisms $d_I^p : I^p \rightarrow I^{p+1}$ and $f^{p+1} : X^{p+1} \rightarrow I^{p+1}$ in the obvious way. It is clear that $I^{p-1} \rightarrow I^p \rightarrow I^{p+1}$ is zero. Note that

$$\text{Ker}(I^p \rightarrow I^{p+1}) = \text{Ker}(I^p \rightarrow (I^p / \text{Im}(I^{p-1} \rightarrow I^p)) \oplus_{X^p} X^{p+1}).$$

The restriction of the morphism f^p to $\text{Ker}(X^p \rightarrow X^{p+1})$ defines a morphism to $\text{Ker}(I^p \rightarrow I^p \oplus_{X^p} X^{p+1})$. This defines a morphism

$$\text{Ker}(X^p \rightarrow X^{p+1}) / \text{Im}(X^{p-1} \rightarrow X^p) \rightarrow \text{Ker}(I^p \rightarrow I^{p+1}) / \text{Im}(I^{p-1} \rightarrow I^p)$$

which as is easy to see must be an isomorphism.

Proof of Theorem 1: Since any quasi-isomorphism in $I^+(\mathcal{A})$ is an isomorphism (Lemma 4), $I^+(\mathcal{A}) = D^+(I)$. By the previous Corollary the functor $I^+(\mathcal{A}) \rightarrow D^+(\mathcal{A})$ is fully faithful. Now we apply the previous lemma to show that each complex is isomorphic in $D^+(\mathcal{A})$ to an injective complex.

PART 3

1. Derived functors

Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor. It extends to a ∂ -functor $F : HK(\mathcal{A}) \rightarrow HK(\mathcal{B})$. If it extends to a functor from the derived category it should transform quasi-isomorphisms to quasi-isomorphisms. We are seeking for the closest functor which does this job.

For any triangulated subcategory $HK(\mathcal{A})^*$ of $HK(\mathcal{A})$ we shall denote by $HK(\mathcal{A})_{qis}^*$ the localization of $HK(\mathcal{A})^*$ with respect to the set of isomorphisms.

Definition. A subcategory $HK(\mathcal{A})^*$ of $HK(\mathcal{A})$ is called localizing if the canonical functor $HK(\mathcal{A})_{qis}^* \rightarrow D(\mathcal{A})$ is fully faithful.

Definition Let $HK(\mathcal{A})^*$ be a localizing subcategory of $HK(\mathcal{A})$. Let $F : HK(\mathcal{A})^* \rightarrow HK(\mathcal{B})$ be a ∂ -functor. Let $Q : HK(\mathcal{A})^* \rightarrow D^*(\mathcal{A})$ be the localizing functor. The right derived functor of F is a ∂ -functor

$$R^*F : D^*(\mathcal{A}) \rightarrow D(\mathcal{B})$$

together with a morphism of functors

$$\xi : Q \circ F \rightarrow R^*F \circ Q$$

from $HK^*(\mathcal{A})$ to $D(\mathcal{B})$. It must satisfy the following universal property: If $G : D^*(\mathcal{A}) \rightarrow D(\mathcal{B})$ is another functor with $\zeta : Q \circ F \rightarrow G \circ Q$ then there is a morphism of functors $\eta : R^*F \rightarrow G$ such that $\zeta = (\eta \circ Q) \circ \xi$.

Theorem 1. Let F as above. Suppose there is a triangulated subcategory L of $HK^*(\mathcal{A})$ such that

- (i) every object of $HK^*(\mathcal{A})$ admits a q.i. into an object of L , and
- (ii) if $I^\bullet \in Ob(L)$ is acyclic, then $F(I^\bullet)$ is also acyclic. Then F has a right derived functor (R^*F, ξ) . Furthermore, for any $I^\bullet \in Ob(L)$, the morphism

$$\xi(I^\bullet) : Q \circ F(I^\bullet) \rightarrow R^*F \circ Q(I^\bullet)$$

is an isomorphism in $D(\mathcal{B})$.

Proof. First of all F takes q.i. in L into q.i.. Indeed if $s : I_1^\bullet \rightarrow I_2^\bullet$ is a q.i. then we extend it to a distinguished triangle with the third side $J^\bullet \in Ob(L)$. Since S is a q.i., we get $F(J^\bullet)$ is acyclic. Then $F(s)$ is a q.i.. Hence F passes to the quotient to give a functor $\bar{F} : \mathcal{L}_{Qis} \rightarrow D(\mathcal{B})$ with the property $\bar{F} \circ Q = Q \circ F$ on L . Now we use that the natural functor $V : \mathcal{L}_{Qis} \rightarrow D^*(\mathcal{A})$ is an equivalence of categories (here we use property (i)). Let $U : D^*(\mathcal{A}) \rightarrow \mathcal{L}_{Qis}$ be its quasi-inverse together with functorial isomorphisms

$$\alpha : 1_{\mathcal{L}_{Qis}} \rightarrow U \circ V, \quad 1_\beta : 1_{D^*(\mathcal{A})} \rightarrow V \circ U.$$

Then define

$$R^*F \rightarrow \bar{F} \circ U.$$

We define a morphism of functors

$$\xi : Q \circ F \rightarrow R^*F \circ Q = \bar{F} \circ U \circ Q$$

as follows.

Let $X^\bullet \in Ob(HK(\mathcal{A})^*)$ and $I^\bullet \in Ob(L)$ such that $Q(X^\bullet) = U(Q(I^\bullet))$. There is an isomorphism

$$\beta(Q(X^\bullet)) : Q(X^\bullet) \rightarrow V \circ U(Q(X^\bullet)) = V(Q(I^\bullet)).$$

We represent it by a roof (t, s) , where $t : X^\bullet \rightarrow Y^\bullet$ and $s : I^\bullet \rightarrow Y^\bullet$ are quasi-isomorphisms. Using property (i), we can find an equivalent roof such that $Y^\bullet \in Ob(L)$. Applying F , we get a diagram in $HK(\mathcal{B})$:

$$\begin{array}{ccc} F(X^\bullet) & & F(I^\bullet) \\ F(t) \searrow & & \swarrow F(s) \\ & F(Y^\bullet) & . \end{array}$$

Since $F(I^\bullet)$ is acyclic, the argument from above shows that $F(s)$ is an isomorphism. This gives a morphism in $D(\mathcal{B})$

$$\xi(X^\bullet) : Q \circ F(X^\bullet) \rightarrow Q \circ F(I^\bullet) = \bar{F} \circ Q(I^\bullet) = \bar{F} \circ U \circ Q(X^\bullet) = R^*F \circ Q(X^\bullet).$$

One checks that this morphism does not depend on the choice of a roof defining a morphism $\beta(Q(X^\bullet))$. Finally, if $X^\bullet \in Ob(L)$ then $F(t)$ is a quasi-isomorphism and $\xi(X^\bullet)$ is an isomorphism in $D(\mathcal{B})$.

Corollary. Let \mathcal{A}, \mathcal{B} be abelian categories and $F : HK(\mathcal{A})^+ \rightarrow HK(\mathcal{B})$ be a ∂ -functor (e.g. defined by an additive functor $\mathcal{A} \rightarrow \mathcal{B}$). Assume that \mathcal{A} has enough injective objects, then the functor $R^*F : D(\mathcal{A})^+ \rightarrow D(\mathcal{B})$ exists.

Proof. We have to check only the second assumption in the theorem. For every acyclic injective complex I^\bullet the identity morphism is homotopy equivalent to zero. Thus it is a zero object in $HK(\mathcal{A})$ and hence its image in $HK(\mathcal{B})$ is zero. Thus $F(I^\bullet)$ is acyclic.

Definition Let $R^*F : \mathcal{A} \rightarrow D(\mathcal{B})$ be composition of R^*F and the natural functors $\mathcal{A} \rightarrow D(\mathcal{A})$. Then the functor

$$R^i F : \mathcal{A} \rightarrow H^i(R^*F(\mathcal{A}))$$

is called the i -th *derived functor* of F .

Examples. 1. Consider the additive functor $F = Hom_{\mathcal{A}}(X, ?) : \mathcal{A} \rightarrow (Ab)$. It extends to a functor $HK(\mathcal{A})^+ \rightarrow HK(Ab)$. Let $Y \rightarrow I^\bullet$ be an injective resolution of an object Y of \mathcal{A} . Then, by definition, $R^*F(Q(X)) = \bar{F}(I^\bullet) = (Hom_{\mathcal{A}}(X, I^n))_{n \geq 0}$. Composing it with the cohomology functor we obtain an abelian group

$$R^n F(Q(X)) = Ker(Hom Hom_{\mathcal{A}}(X, I^n)) \rightarrow Hom_{\mathcal{A}}(X, I^{n-1}) / Im(Hom Hom_{\mathcal{A}}(X, I^n)) \rightarrow Hom_{\mathcal{A}}(X, I^n).$$

It is denoted by $Ext_{\mathcal{A}}^i(X, Y)$. More generally, we can define $Ext_{K(\mathcal{A})}^i(X^\bullet, Y^\bullet)$ (this is called the *hyperext*). For this we consider the functor from $K(\mathcal{A}) \rightarrow K((Ab))$ by setting $Hom^\bullet(X^\bullet, Y^\bullet) = (Hom^n(X^\bullet, Y^\bullet))$, where

$$Hom^n(X^\bullet, Y^\bullet) = \prod_{p \in \mathbb{Z}} Hom_{\mathcal{A}}(X^p, Y^{p+n})$$

and

$$d^n = \prod (d_X^{p-1} + (-1)^{n+1} d_Y^{p+n}).$$

Here we fix X^\bullet and consider Y^\bullet as an argument. One can prove (see Exercises) that

$$H^i(R^*(Hom^\bullet(X^\bullet, Y^\bullet)) \cong Hom_{D(\mathcal{A})}(Q(X^\bullet), Q(Y^\bullet[n])).$$

From this it follows that we can write long exact sequences of exts.

2. Let $\mathcal{A} = Mod_R$ and $F(M) = M \otimes_R N$, where N is a fixed R -module. Then the derived functors $R^i F(M)$ are denoted by $Tor_R^i(M, N)$. Note that we may take L in the proof of Theorem 1 to be the category of complexes consisting of flat R -modules. So, we can compute the derived functor using flat resolutions.

3. Let $\tilde{\mathcal{C}}^{ab}$ be a category of abelian sheaves on a site \mathcal{C} . We know that it has enough injective objects so the derived functor exists for any additive functor $F : \tilde{\mathcal{C}}^{ab} \rightarrow \mathcal{A}$ to an abelian category. For example, for any object X of \mathcal{C} we may take the “global section functor” $F : \tilde{\mathcal{C}}^{ab} \rightarrow (Ab)$ defined by

$$F(\mathcal{F}) = \mathcal{F}(X).$$

By definition, the i -th cohomology of X with coefficients in \mathcal{F} are defined by

$$H^i(R^*F(X)) = H^i(X, \mathcal{F}).$$

Note that, since F is left exact, we get

$$H^0(X, \mathcal{F}) \cong F(X).$$

Let $\mathbb{Z}_X = i_!(\mathbb{Z})$, where \mathbb{Z} considered as a sheaf on the category defined by the object X and $i_!$ is the direct image functor. Then

$$F(X) = Hom_{\tilde{\mathcal{C}}^{ab}}(\mathbb{Z}_X, \mathcal{F})$$

and hence

$$H^i(X, \mathcal{F}) = Ext_{\mathcal{Z}_{ab}}^i(\mathbb{Z}_X, \mathcal{F}).$$

In particular, we can write long exact sequences associated with any short exact sequence of sheaves.

9. Spectral sequences.

They arise from considering the composition of derived functors.

Let $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{C}$ be two additive functors. We want to compare $R^*(G \circ F)$ and $R^*(G) \circ R^*(F)$.

Theorem 1. *Let $HK(\mathcal{A})^*$ be a localizing subcategory of $HK(\mathcal{A})$ and $HK(\mathcal{B})^\dagger$ be a localizing subcategory of $HK(\mathcal{B})$. Let $F : HK(\mathcal{A})^* \rightarrow HK(\mathcal{B})$ and $G : HK(\mathcal{B})^\dagger \rightarrow HK(\mathcal{C})$ be ∂ -functors. Assume that $F(HK(\mathcal{A})^*) \subset HK(\mathcal{B})^\dagger$ and $R^*(G \circ F), R^*(G)$ and $R^*(F)$ exist. Then there is a unique morphism of functors*

$$\zeta : R^*(G \circ F) \rightarrow R^*(G) \circ R^*(F)$$

such that the diagram

$$\begin{array}{ccc} Q \circ G \circ F & \xrightarrow{\xi_G} & R^*G \circ Q \circ F \\ \xi_{G \circ F} \downarrow & & \downarrow \xi_F \\ R^*(G \circ F) \circ Q & \xrightarrow{\zeta \circ Q} & R^*(G) \circ R^*(F) \end{array}$$

commutative. Moreover, if $\mathcal{L} \subset HK(\mathcal{A})^*$ and $\mathcal{M} \subset HK(\mathcal{B})^\dagger$ subcategories satisfying the assumptions of Theorem 1, and $F(\mathcal{L}) \subset \mathcal{M}$ then ζ is an isomorphism.

Proof. The existence of ζ follows immediately from the universal property of the derived functor. The second property follows obviously from the construction of the derived functors (since the restriction of $R^*(G \circ F)$ to \mathcal{L}_{Qis} is equal to the composition of the functors $\bar{G} \circ \bar{F}$).

Recall the definition of a spectral sequence. Let \mathcal{A} be an abelian category. A *spectral sequence* in \mathcal{A} is a collection $(E_r, H^n), r, n \in \mathbb{Z}, r \geq 1$ of complexes $(E_r = (E_r^n, d_r)$ and a collection of objects H^n (called the *limit* of the spectral sequence) with filtration of subobjects (i.e. a family of subobjects $F^p(H^n)$ such that $F^{k'}(H^n)$ is a subobject of $F^k(H^n)$ if $k' \geq k$). The following properties must be satisfied:

- (SS1) each $E_r^n = \bigoplus_{p,q \in \mathbb{Z}, p+q=n} E_r^{p,q}$;
- (SS2) $d_r(E_r^{p,q}) \subset E_r^{p+r, q-r+1}$;
- (SS3) there are isomorphisms $\alpha_r : H(E_r) \cong E_{r+1}$ of bigraded complexes. In other words

$$\alpha_r^{p,q} : ker(d_r^{p,q}) / im(d_r^{p-r, q+r-1}) \cong E_{r+1}^{p,q}.$$

(SS4) there exists r_0 such that $d_r^{p,q}$ and $d_r^{p-r, q+r-1}$ are equal to zero for $r \geq r_0$, thus $E_r^{p,q} \cong E_{r_0}^{p,q}$ for $r \geq r_0$. The objects are denoted by $E_\infty^{p,q}$

(SS5) for each $p, q \in \mathbb{Z}$, there is an isomorphism $\beta_{p,q} : E_\infty^{p,q} \rightarrow Gr^p(H^{p+q}) = F^p(H^{p+q}) / F^{p+1}(H^{p+q})$.

The spectral sequences form an additive category. Its morphisms are morphisms $f_r^{p,q} : E_r^{p,q} \rightarrow' E_r^{p,q}$ which commute with differentials $d_r, 'd_r$ and isomorphisms $\beta_r^{p,q}$.

Often one uses the following notation for a spectral sequence

$$E_r^{p,q} \implies H^n.$$

Before we give examples of spectral sequence, let us observe some corollaries.

Proposition 1. Assume $\chi : Ob(\mathcal{A}) \rightarrow \mathbb{Z}$ is a function satisfying the following properties: for any short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in \mathcal{A} we have

$$\chi(B) = \chi(A) + \chi(C).$$

Then

$$\sum_{p,q} (-1)^{p+q} \chi(E_r^{p,q}) = \sum_n \chi(E^n).$$

provided all the sums contain only finitely many non-zero terms.

Remark. More generally for any additive category one can define the *Grothendieck group* $K(\mathcal{A})$ as the free abelian group generated by objects from \mathcal{A} modulo the subgroup generated by elements $A + C - B$, where A, B, C as in the statement of Proposition 2. Let us denote $[A]$ the element of $K(\mathcal{A})$ represented by an object A . Then

$$\sum_{p,q} (-1)^{p+q} [E_r^{p,q}] = \sum_{n \in \mathbb{Z}} (-1)^n [E^n].$$

provided all the sums contain only finitely many non-zero terms.

Proposition 2. Assume that $E_2^{p,q} = 0$ unless $p \geq 0, q = 0$. Then

$$E_2^{p,0} \cong G^p(H^p).$$

Proof. This follows from the fact that all differentials d_2 are equal to zero.

Proposition 3. Assume that $E_2^{p,q} = 0$ when $p, q < 0$ and $F^i(H^n) = 0$ for $i > n$, $H^n = F^0(H^n)$. Then the exists the following five-term exact sequence:

$$0 \rightarrow E_2^{1,0} \xrightarrow{\bar{\beta}^{1,0}} H^1 \rightarrow E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0} \xrightarrow{\bar{\beta}^{2,0}} H^2.$$

Proof. Since $d_r^{1,0} : E_r^{1,0} \rightarrow E_r^{1+r, -r+1} = 0$ and $E_r^{-r, r-1} = 0$ for $r \geq 2$, we obtain $\alpha_r^{1,0} : E_r^{1,0} \rightarrow E_{r+1}^{1,0}$ is an isomorphism for $r \geq 2$. Thus,

$$E_r^{1,0} \cong E_\infty^{1,0}.$$

Similarly, we get

$$\begin{aligned} E_\infty^{0,1} &\cong \text{Ker}(d_2^{0,1} : E_2^{0,1} \rightarrow E_2^{2,0}). \\ E_\infty^{2,0} &\cong \text{Coker}(d_2^{0,1} : E_2^{0,1} \rightarrow E_2^{2,0}). \end{aligned}$$

Now $\beta^{p,q} : E_\infty^{1,0} \rightarrow F^1(H^1)/F^2(H^1) = F^1(H^1)$ is an isomorphism, so there is an injective homomorphism $\bar{\beta}^{1,0} : E_2^{1,0} \rightarrow H^1$. Also, $E_\infty^{0,1} \cong G^0(H^1) = F^0(H^1)/F^1(H^1) = H^1/Im(\beta^{1,0})$. Similarly, $E_\infty^{2,0} \cong G^2(H^2) = F^2(H^2)/F^3(H^2) = F^2(H^2)$ and hence there is an injective homomorphism $\bar{\beta}^{2,0} : E_\infty^{2,0} \rightarrow H^2$. this proves the proposition.

Example 1. Let (K, d) be a *differential object* in an abelian category, i.e. $K \in Ob(\mathcal{A})$ and $d \in End(K)$ such that $d^2 = 0$. We also assume that K has a filtration (K_p) such that

$$d(K_p) \subset K_p$$

We set

$$H(K) = \text{Ker}(d)/\text{Im}(d).$$

Note that the differential d induces the differential on

$$Gr(K) = \bigoplus_{p \in \mathbb{Z}} K^p / K^{p+1}.$$

The object $Gr(K)$ is a differential graded object, i.e. $K = \bigoplus_{n \in \mathbb{Z}} K^n$ with $d(K^n) \subset K^{n+1}$. Note that $G^\bullet = (K^p/K^{p+1})$ is a cochain complex. Conversely, given a complex it defines a graded For any integer r set

$$\begin{aligned} Z_r^p &= Ker(d : K_p \rightarrow K_p/K_{p+r}), \\ B_r^p &= K_p \cap d(K_{p-r}). \end{aligned}$$

Clearly, $Z_r^p \supset dZ_{r-1}^{p-r+1}$ and $Z_r^p \supset Z_{r-1}^{p+1}$. Set

$$E_r^p = Z_r^p / (dZ_{r-1}^{p-r+1} + Z_{r-1}^{p+1}) = Z_r^p / (B_{r-1}^p + Z_{r-1}^{p+1}), \quad E_r = \sum_p E_r^p.$$

Let us see what the differential does. Obviously it sends Z_r^p to Z_r^{p+r} and $dZ_{r-1}^{p-r+1} + Z_{r-1}^{p+1}$ to dZ_{r-1}^{p+1} . Thus it induces a morphism

$$d_r : E_r^p \rightarrow E_r^{p+r} = Z_r^{p+r} / B_{r-1}^{p+r} + Z_{r-1}^{p+1+r}.$$

Let us compute the cohomology

$$H^p(E_r) = ker(d_r : E_r^p \rightarrow E_r^{p+r}) / im(d_r : E_r^{p-r} \rightarrow E_r^p).$$

Let us assume that our category is a subcategory of the category of modules, so that we could operate with element of objects. If $x \in ker(d_r : E_r^p \rightarrow E_r^{p+r})$ then $dx \in dZ_{r-1}^{p+1} + Z_{r-1}^{p+1+r}$ so $dx = dy + z$, where $y \in Z_{r-1}^{p+1}$ and $z \in Z_{r-1}^{p+1+r}$. Setting $u = x - y$ we obtain $du \in Z_{r-1}^{p+1+r} \subset K_{p+r}$ and $u \in K_{p+1}$. Thus $u \in Z_{r+1}^p$. This shows that $x = y + u$, where $y \in Z_{r-1}^{p+1}$, $u \in Z_{r+1}^p$. This gives

$$ker(E_r^p \rightarrow E_r^{p+r}) = (Z_{r+1}^p + Z_{r-1}^{p+1}) / (dZ_{r-1}^{p-r+1} + Z_{r-1}^{p+1}).$$

Now

$$B^p(E_r) = im(d_r : E_r^{p-r} \rightarrow E_r^p) = (dZ_{r-1}^{p-r} + Z_{r-1}^{p+1}) / (dZ_{r-1}^{p-r+1} + Z_{r-1}^{p+1}).$$

This gives us

$$H^p(E_r) = (Z_{r+1}^p + Z_{r-1}^{p+1}) / (dZ_{r-1}^{p-r} + Z_{r-1}^{p+1}) = Z_{r+1}^p / (Z_{r+1}^p \cap (dZ_{r-1}^{p-r} + Z_{r-1}^{p+1})).$$

Since $dZ_{r-1}^{p-r} \subset Z_{r+1}^p$ and $Z_{r+1}^p \cap Z_{r-1}^{p+1} = Z_{r-1}^p$, we obtain

$$H^p(E_r) = Z_{r+1}^p / (dZ_{r-1}^{p-r} + Z_{r-1}^{p+1}) = E_{r+1}^p.$$

Define a filtration on $H(K)$ by

$$F^p H(K) = im(H(K_p) \rightarrow H(K)).$$

Now assume additionally that K is a graded differential module, i.e.

$$K = \bigoplus_{n \in \mathbb{Z}} K^n$$

with $d(K^n) \subset K^{n+1}$. Moreover, we assume that the grading is compatible with the filtration in the sense

$$K_p = \bigoplus_{q \in \mathbb{Z}} K_p \cap K^q.$$

Set

$$\begin{aligned} Z_r^{p,q} &= Z_r^p \cap K^{p+q}, \quad B_r^{p,q} = B_r^p \cap K^{p+q}, \\ E_r^{p,q} &= Z_r^{p,q} / (B_{r-1}^{p,q} + Z_{r-1}^{p+1,q-1}). \end{aligned}$$

Then d_r induces a morphism

$$d_r^{p,q} : E_r^{p,q} \rightarrow E_r^{p-r,q-r+1},$$

and

$$E_{r+1}^{p,q} = \ker(d_r^{p,q} / \text{im}(d_r^{p-r, q+r-1})).$$

Set

$$F^p H^q(K) = \ker(d : K_p \cap K^q \rightarrow K_p \cap K^{q+1}) / \text{Im}(d : K_p \cap K^{q-1} \rightarrow K_p \cap K^q).$$

This defines a filtration in $H^n(K)$.

Assume now that there exist $p_+(n)$ and $p_-(n)$ such that

$$F^{p_+(n)} K^n = K^n, \quad F^{p_-(n)} K^n = 0.$$

Then for any (p, q) and $r_0 = \max\{p_+(p+q+1) - p_-(p+q) + 1, p_+(p+q) - p_-(p+q-1) + 1\}$

$$d_r^{p,q} = d_r^{p-r, q+r-1} = 0, r \geq r_0.$$

Thus we can define the groups $E_\infty^{p,q}$. We have

$$E_\infty^{p,q} = Z_r^{p,q} / Z_{r-1}^{p+1, q-1}, r \geq r_0,$$

Since $F^{p+r} K^{p+q+1} = 0, r \geq r_0$, we have

$$Z_r^{p,q} = \ker(F^p K^{p+q} \rightarrow F^p K^{p+q+1} / F^{p+r} K^{p+q+1}) = \ker(F^p K^{p+q} \rightarrow F^p K^{p+q+1}) = F^p H^{p+q}.$$

Similarly we get

$$Z_{r-1}^{p+1, q-1} = \ker(F^{p+1} K^{p+q} \rightarrow F^{p+1} K^{p+q+1}) = F^{p+1} H^{p+q}.$$

Thus

$$E_\infty^{p,q} \cong G^p H^{p+q}.$$

Example 2. Now let us use some special filtrations in the complex. For example, let us consider a *double complex* $K^{\bullet, \bullet} = (K^{p,q})$. It is equipped with two differentials:

$$d_1 : K^{p,q} \rightarrow K^{p+1, q}, \quad d_2 : K^{p,q} \rightarrow K^{p, q+1}.$$

Let

$$K = \bigoplus_{p,q} K^{p,q} = \bigoplus_{n \in \mathbb{Z}} K^n,$$

where

$$K^n = \bigoplus_{p,q=n} K^{p,q}.$$

We equip K with the differential

$$d = d_1 + d_2.$$

Clearly,

$$\begin{aligned} d(K^{p,q}) &\subset K^{p+1, q} + K^{p, q+1}, \\ d(K^n) &\subset K^{n+1}. \end{aligned}$$

Thus (K, d) has a structure of a graded differential module.

Let us introduce the following two filtrations in K :

$$F_I^r(K) = \bigoplus_{p \geq r} K^{p,q}, \quad F_{II}^r(K) = \bigoplus_{q \geq r} K^{p,q}.$$

This defines two spectral sequences

$$I_r^{p,q} \implies H^n, \quad II_r^{p,q} \implies H^n.$$

Let us compute the first two "pages" of these spectral sequences. First notice that for any differential module K with filtration, we have

$$Z_0^p = K_p, \quad Z_{-1}^p = K_{p+1}, \quad dZ_{-1}^{p+1} \subset K_{p+1}.$$

Thus $E_0^p = K_p/K_{p+1}$, and

$$E_1^p = H(K_p/K_{p+1}) = d^{-1}(K_{p+1}) \cap K_p/(dK_p + K_{p+1}).$$

To compute the differential $E_1^p \rightarrow E_2^p$ we use the exact sequence

$$0 \rightarrow K_{p+1}/K_{p+2} \rightarrow K_p/K_{p+2} \rightarrow K_p/K_{p+1} \rightarrow 0.$$

It gives the exact sequence of cohomology with coboundary morphism

$$\delta^p : E_1^p = H(K_p/K_{p+1}) \rightarrow E_1^{p+1} = H(K_{p+1}/K_{p+2}).$$

It is easy to see that this coincides with the differential d^1 . Thus

$$E_2^p = \text{Ker}(\delta_p)/\text{Im}(\delta^{p+1}) = H(\oplus_{p \in \mathbb{Z}} H(K_p/K_{p+1})).$$

Now assume that $K = \oplus_n K^n$ is graded. Then

$$E_1^{p,q} = K_p \cap K^{p+q}/K_{p+1} \cap K^{p+q}.$$

Consider the first spectral sequence. Then

$$E_0^p = K_p/K_{p+1} = \oplus_q K^{p,q}.$$

We see that the first differential d_1 defines the zero morphism on E_0^p , hence

$$E_1^p = H(\oplus_q K^{p,q} \xrightarrow{d_2} K^{p,q+1}).$$

So

$$E_1^{p,q} = \text{Ker}(K^{p,q} \xrightarrow{d_2} K^{p,q+1})/\text{Im}(K^{p,q-1} \xrightarrow{d_2} K^{p,q}).$$

Let " $H(K)$ " denote the cohomology of the graded object $\oplus_p K^{p,\bullet}$ with respect to the differential d_2 . It has the grading by p and the differential induced by d_1 . We get

$$E_2^p = {}'H^p({}''H(K)) = \oplus_{q \in \mathbb{Z}} E_2^{p,q},$$

$$E_2^{p,q} = \text{Ker}(d_1 : E_1^{p,q} \rightarrow E_1^{p+1,q})/\text{Im}(d_1 : E_1^{p-1,q} \rightarrow E_1^{p,q}).$$

Similarly, we get for the second spectral sequence

$$II_2^q = {}''H^q({}'H(K)) = \oplus_{p \in \mathbb{Z}} I_2^{p,q},$$

Both spectral sequences converge to $H(K^\bullet)$ but the filtrations are different.

Definition A double complex $I^{\bullet,\bullet}$ in an abelian category \mathcal{A} is called a *Cartan-Eilenberg resolution* of a complex K^\bullet if it satisfies the following conditions

- (CE1) Each term $I^{p,q}$ is an injective object in \mathcal{A} ;
- (CE2) $I^{p,q} = 0$ if $j < 0$ or $i \leq 0$;
- (CE3) there is a morphism of complexes $\epsilon : K^\bullet \rightarrow I^{\bullet,0}$ such that

$$K^\bullet \xrightarrow{\epsilon} I^{\bullet,0} \xrightarrow{d_2} I^{\bullet,1} \xrightarrow{d_2} \dots$$

is a resolution of K^\bullet in $K(\mathcal{A})$;

(CE4) the resolution from (CE3) induces the resolutions

$$\begin{aligned} B^i(K^\bullet) &\xrightarrow{\epsilon} B_I^i(I^{\bullet,0}) \xrightarrow{d_2} B_I^i(I^{\bullet,1}) \xrightarrow{d_2} \dots, \\ Z^i(K^\bullet) &\xrightarrow{\epsilon} Z_I^i(I^{\bullet,0}) \xrightarrow{d_2} Z_I^i(I^{\bullet,1}) \xrightarrow{d_2} \dots, \\ H^i(K^\bullet) &\xrightarrow{\epsilon} H_I^i(I^{\bullet,0}) \xrightarrow{d_2} H_I^i(I^{\bullet,1}) \xrightarrow{d_2} \dots; \end{aligned}$$

(CE5) the short exact sequences

$$\begin{aligned} 0 \rightarrow B_I^i(I^{\bullet,j}) \rightarrow Z_I^i(I^{\bullet,j}) \rightarrow H_I^i(I^{\bullet,j}) \rightarrow 0, \\ 0 \rightarrow Z_I^i(I^{\bullet,j}) \rightarrow K^{ij} \rightarrow B_I^{i+1}(I^{\bullet,j}) \rightarrow 0 \end{aligned}$$

split.

Note that the property (CE5) shows that the complexes

$$B_I^i(I^{\bullet,\bullet}), \quad Z_I^i(I^{\bullet,\bullet}), \quad H_I^i(I^{\bullet,\bullet})$$

are injective and provide injective resolutions of the objects $B^i(K^\bullet), Z^i(K^\bullet), H^i(K^\bullet)$. One can prove that the complex $(K^{\bullet,i})_i$ in $K(\mathcal{A})$ is an injective object in the category of complexes.

Lemma 1. *Consider the abelian category $K(\mathcal{A})^+$ of bounded from below complexes in an abelian category \mathcal{A} . Assume \mathcal{A} has enough injective objects. Then $K(\mathcal{A})^+$ has enough injective objects, and any injective resolution of an object in $K(\mathcal{A})^+$ is a Cartan-Eilenberg resolution.*

Proof.

Lemma 2. *Let $\epsilon : K^\bullet \rightarrow L^{\bullet,\bullet}$ be a resolution of a complex K^\bullet in $K(\mathcal{A})$. Then the natural morphism of complexes $K^\bullet \rightarrow L^\bullet$ is a quasi-isomorphism.*

Proof. We apply the spectral sequence of the double complex $L^{\bullet,\bullet}$. Since $H(L^{p,q-1} \rightarrow L^{p,q} \rightarrow L^{p,q+1}) = 0$ for $q > 0$ the spectral sequence degenerates, and we get that the canonical morphism

$$K^\bullet \rightarrow L^\bullet.$$

is a quasi-isomorphism

Theorem 2. *Keep the notation of Theorem 1. Assume that \mathcal{L} and \mathcal{M} are the categories of injective complexes, and \mathcal{A}, \mathcal{B} have enough injective objects. Then for any object X in \mathcal{A} there exists a spectral sequence functorial in X with*

$$E_2^{p,q} = R^p G(R^q F(X))$$

which converges to $R^n(G \circ F)(X)$.

Proof. Recall that we replace X with its injective resolution I^\bullet quasi-isomorphic to X and then consider the complex $F(I^\bullet)$. Now we have to replace it with an injective complex quasi-isomorphic to $F(I^\bullet)$. For this we choose its Cartan-Eilenberg resolution $L^{\bullet,\bullet}$ and then obtain that $R^p G(R^q F(X)) = H^p(G(H_I^q(L^{\bullet,\bullet})))$. On the other hand, $R^n(G \circ F)(X) = H^n(G(F(I^\bullet)))$.