Math. 632. Homework 1 (Part I)

1. Let A be a finitely generated algebra over a field k. Prove that the set of maximal ideals of A is dense in the Zariski topology on Spec$A$. Give an example of a ring A where this is not true.

2. Give an example of a sheaf of ring on a topological space X such that for any point $x \in X$ the ring of germs at x is not a local ring.

3. Let $X = \mathbb{R}^n$ with usual topology defined by the Euclidean distance. For any open subset U define $\mathcal{O}_X(U)$ to be ring of functions of class C^k.

(i) Show that this defines a sheaf of rings and the pair $(X, \mathcal{O}_X)$ is a geometric space.

(ii) Show that one can define an n-dimensional manifold of class C^k as a geometric space locally isomorphic to $(X, \mathcal{O}_X)$.

(iii) Show that maps of manifolds of class C^k are exactly morphisms of the corresponding geometric spaces.

4. Describe explicitly the fibres of the morphism $f : \text{Spec}B \to \text{Spec}A$, where $A \to B$ is one of the following homomorphisms of rings $\phi : A \to B$:

(i) $A = \mathbb{Z}$, $B = \mathbb{Z}[\sqrt{-1}]$, the ring of Gaussian integers, ϕ is the natural inclusion of rings.

(ii) $A = \mathbb{C}[u, v]$, $B = \mathbb{C}[x, y, z, u, v]/(x^2 + uy^2 + v)$ and ϕ is defined by sending u, v to the cosets of u, v modulo the ideal $(x^2 + uy^2 + v)$.

5. Let $\mathbb{G}_a = \text{Spec } k[t]$ (resp. $\mathbb{G}_m = \text{Spec } k[t, t^{-1}]$) be the additive group (resp. the multiplicative group scheme) over a field k of characteristic $p > 0$. Show that the homomorphism of rings $k[t] \to k[t]$ (resp. $k[t, t^{-1}] \to k[t, t^{-1}]$), defined by the formula $t \mapsto t^p$ is a homomorhism of group schemes. Describe its kernel.

6. Prove that a scheme is separated if and only if the intersection of any affine subsets is affine. Show that any affine scheme is separated.