Math. 632. Homework 2

1. Let \( k = F(t) \) be a purely transcendental extension of a field \( F \) and \( K \) be an extension of \( k \) obtained by adjoining a \( p \)-th root of \( t \). Compute the Galois group scheme of the extension \( K/k \).

2. Let \( \mathcal{F} \) be a functor from the category of algebras over a ring \( R \) to the category of sets. Assume \( \mathcal{F} \) is representable by an affine scheme of finite type over \( R \). Show that the functor \( \mathcal{F}' \) defined by \( \mathcal{F}'(K) = \mathcal{F}(K[[T]]) \) (check that this defines a functor) is representable by a scheme.

3. Show that \( X = \text{Spec} k[X,Y]/(X,Y^3) \) is contained in a closed reduced subscheme of \( \mathbb{A}^2_k \) of the form \( \text{Spec}(k[X,Y]/(f(X,Y))) \), where \( f(x,y) = 0 \) is a nonsingular curve. Show that for \( X = \text{Spec} k[X,Y]/(X^2,Y^2,XY) \) this is not true.

4. Let \( k/F \) be a finite extension of fields. Consider the functor which assigns to a \( F \)-algebra \( K \) the group of invertible elements of \( k \otimes_F K \). Show that this functor is representable by an affine group scheme. Find it explicitly in the case \( F = \mathbb{R} \) and \( k = \mathbb{C} \). Show that its real points is the group \( \mathbb{C}^* \) and its complex points is the group \( \mathbb{C}^* \times \mathbb{C}^* \). What is its Lie algebra scheme?

5. Give an example of a sheaf of ideals on a scheme \( X \) which is not a quasi-coherent sheaf.

6. Prove that assigning to a finitely generated projective module \( M \) over a domain \( A \) the vector bundle \( V(M) = \text{Spec} S(M^*) \) establishes a one-to-one correspondence between vector bundles on \( \text{Spec} A \) and finitely generated projective \( A \)-modules. Show that this defines an isomorphism of categories of projective bundles over \( \text{Spec} A \) (a subcategory of the category of schemes over \( \text{Spec} A \)) and the category of finitely generated projective modules (a full subcategory of the category of modules over \( A \)).

7. Let \( X \) be the open subscheme of \( \mathbb{A}^{n+1}_R = \text{Spec} R[T_0, \ldots, T_n] \) whose complement is the closed point \((T_1, \ldots, T_n)\). Show that \( X \) admits a morphism \( \mathbb{P}^n_R \) and an open embedding (as \( \mathbb{P}^n \)-schemes) into a line bundle over \( \mathbb{P}^n \) (may do it in the case \( n = 1 \)).

8. Let \( S \) be a scheme and \( X = \mathbb{A}^3_S \to S \) be the affine space over \( S \). Show that the canonical homomorphism \( \text{Pic} S \to \text{Pic} X \) is an isomorphism.

9. Give an example of a morphism \( X \to S \) such that it is locally isomorphic to a vector bundle over \( S \) but not a vector bundle (i.e. the transition isomorphisms are not linear).

10. Let \( K \) be a finite extension of \( \mathbb{Q} \) and \( \mathcal{O} \) be a normal subring of \( K \) with fraction field \( K \) (a maximal order in \( K \)). Show that \( \text{Pic}(\mathcal{O}) \) is a finite group and each element can be represented by a fractional ideal in \( K \) (an \( \mathcal{O} \)-submodule of \( K \)). Show that \( \text{Pic} \mathcal{O} \) is trivial when \( K = \mathbb{Q} \) or \( \mathbb{Q}(i) \) and give an example when it is not trivial.