Math. 632. Homework 2

1. Let $k = F(t)$ be a purely transcendental extension of a field F and K be an extension of k obtained by adjoining a p-th root of t. Compute the Galois group scheme of the extension K/k.

2. Let $\mathcal{F}$ be a functor from the category of algebras over a ring R to the category of sets. Assume $\mathcal{F}$ is representable by an affine scheme of finite type over R. Show that the functor $\mathcal{F}'$ defined by $\mathcal{F}'(K) = \mathcal{F}(K[[T]])$ (check that this defines a functor) is representable by a scheme.

3. Show that $X = \text{Spec } k[X,Y]/(X,Y^3)$ is contained in a closed reduced subscheme of $\mathbb{A}^2_k$ of the form $\text{Spec}(k[X,Y]/(f(X,Y))$, where $f(x,y) = 0$ is a nonsingular curve. Show that for $X = \text{Spec } k[X,Y]/(X^2,Y^2,XY)$ this is not true.

4. Let k/F be a finite extension of fields. consider the functor which assigns to a F-algebra K the group of invertible elements of $k \otimes_F K$. Show that this functor is representable by an affine group scheme. Find it explicitly in the case $F = \mathbb{R}$ and $k = \mathbb{C}$. Show that its real points is the group $\mathbb{C}^*$ and its complex points is the group $\mathbb{C}^* \times \mathbb{C}^*$. What is its Lie algebra scheme?

5. Give an example of a sheaf of ideals on a scheme X which is not a quasi-coherent sheaf.

6. Prove that assigning to a finitely generated projective module M over a domain A the vector bundle $\mathbb{V}(M) = \text{Spec } S(M^*))$ establishes a one-to-one correspondence between vector bundles on Spec$A$ and finitely generated projective A-modules. Show that this defines an isomorphism of categories of projective bundles over Spec A (a subcategory of the category of schemes over Spec$A$) and the category of finitely generated projective modules (a full subcategory of the category of modules over A).

7. Let X be the open subscheme of $\mathbb{A}^{n+1}_R = \text{Spec } R[T_0,\dots,T_n]$ whose complement is the closed point $(T_1,\dots,T_n)$. Show that X admits a morphism $\mathbb{P}^n_R$ and an open embedding (as $\mathbb{P}^n$-schemes) into a line bundle over $\mathbb{P}^n$ (may do it in the case $n = 1$).

8. Let S be a scheme and $X = \mathbb{A}^n_S \to S$ be the affine space over S. Show that the canonical homomorphism Pic $S \to$ Pic X is an isomorphism.

9. Give an example of a morphism $X \to S$ such that it is locally isomorphic to a vector bundle over S but not a vector bundle (i.e the transition isomorhisms are not linear).

10. Let K be a finite extension of $\mathbb{Q}$ and $\mathcal{O}$ be a normal subring of K with fraction field K (a maximal order in K). Show that Pic$(\mathcal{O})$ is a finite group and each element can be represented by a fractional ideal in K (an $\mathcal{O}$-submodule of K). Show that Pic $\mathcal{O}$ is trivial when $K = \mathbb{Q}$ or $\mathbb{Q}(i)$ and give an example when it is not trivial.