Math. 632. Homework 3

1, 2. Hartshorne, Ex. 5.10, 5. 11.

3. Let $A = R[T_0, \ldots, T_n]$ be the polynomial algebra graded by the condition $\deg(T_i) = q_i > 0$ and $X = \text{Proj}A$. Show by an example, that $\mathcal{O}_X(1)$ is not locally trivial in general and prove that $\mathcal{O}_X(n)$ is locally trivial on the open subset equal to the union of $D^+(T_i)$'s such that $q_i|n$.

4. Let $i : Y \to X$ be the canonical morphism of a closed subscheme Y of X defined by an Ideal $\mathcal{I}_Y$ in $\mathcal{O}_X$. Show that for any quasi-coherent $\mathcal{O}_X$-Module $\mathcal{F}$, the inverse image $i^*(\mathcal{F})$ is isomorphic to the $\mathcal{O}_Y$-Module $\mathcal{F}/\mathcal{I}_Y\mathcal{F}$ (make sense of this quotient sheaf and its structure of a quasi-coherent sheaf on Y).

5. Let $\phi : R[T_0, \ldots, T_n] \to A$ be a surjective homomorphism of graded R-algebras and $I = Ker(\phi)$. Show that there exists an isomorophism $\Phi : X = \text{Proj}(A) \to Y$, where Y is a closed subscheme of $\mathbb{P}^n_R$ with ideal sheaf isomorphic to $\tilde{I}$. Show that $\Phi^*(\mathcal{O}_{\mathbb{P}^n_R}(1)) \cong \mathcal{O}_X(1)$ (here we identify Φ with its composition with the natural morphism $Y \to \mathbb{P}^n_R$).

6. Let I be a homogeneous ideal in $R[T_0, \ldots, T_n]$. Let K be an R-algebra. For any line L in K^{n+1} (i.e. a direct summand of R^{n+1} of rank 1) make sense of the equality $F(L) = 0$, where $F \in I$. Show that there is a natural (make sense of this too) bijection between the set of K-points of $\text{Proj}(R[T_0, \ldots, T_n]/I)$ and the set of lines L in K^{n+1} such that $F(L) = 0$, for any $F \in I$.

7. Show that the canonical projection $p : \mathbb{P}^n_R \to \text{Spec}R$ defines a homomorphism of groups p^* : $\text{Pic}(\text{Spec}R) \to \text{Pic}(\mathbb{P}^n_R)$ with quotient isomorphic to $\mathbb{Z}$ (prove this first in the case when R is a field to get the idea).

8. Let S be a scheme and $X = \mathbb{A}^n_S \to S$ be the affine space over S. Show that the canonical homomorphism $\text{Pic } S \to \text{Pic } X$ is an isomorphism.

9. Let $X = \text{Spec}(\mathbb{C}[X, Y, Z](XY + Z^2)$. Prove that $\text{Pic}(X)$ is a group of order 2. What is the class group of X?