Enriques Surfaces II

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Preface

The book gives a contemporary account of the study of the class of projective algebraic surfaces known as Enriques surfaces. These surfaces were discovered more than 125 years ago in an attempt to extend the characterization of rational algebraic curves via the absence of regular (or holomorphic) differential 1-forms to the two-dimensional case.

The theory of differential forms on algebraic varieties of arbitrary dimension and their birational invariance was laid out in the works of Clebsch and Noether between 1870 – 1880. The further developments of these ideas and clarifying their geometric meaning was undertaken by the school of Italian algebraic geometry, who were probably the first to define one of the main goals of algebraic geometry, namely the classification of algebraic varieties up to birational equivalence. They also understood the significance of vector spaces of regular differential forms. One of the main achievements of their work was the classification of algebraic surfaces, which is mainly due to Castelnuovo and Enriques. Central results of this classification are achieved via the analysis of the canonical and pluri-canonical linear systems and the Albanese map. The main numerical invariants are $q$, $p_g$, and $P_n$, which are by definition the dimensions of the vector spaces of regular 1-forms, regular 2-forms, and regular $n$-pluri-canonical forms, respectively. A rational variety, that is, an algebraic variety birational to projective space, has no nonzero regular forms and the converse is true for algebraic curves. In 1894, Castelnuovo proved that vanishing of $q$, $p_g$, and $P_2$ is sufficient for the rationality of an algebraic surface. In discussions with Enriques, whether the condition $P_2 = 0$ can be eliminated, both came up with an example that shows that it cannot be done. In the example of Enriques, one has $P_{2n} = 1$, $P_{2n+1} = 0$ for all $n \geq 0$ and in the example of Castelnuovo, one has $P_n = [1 + \frac{n}{2}]$, that is, linear growth as $n$ tends to infinity. Enriques mentions this example in a letter to Castelnuovo in July 22, 1894 [217, Letter 11] and he also mentions it in his 1896 paper [213, §39]. Castelnuovo’s example is discussed in his 1896 paper [108]. In the later development of the classification of algebraic surfaces, these two examples occupy different places: Enriques’ example is of Kodaira dimension 0 and shares this class with abelian surfaces, K3 surfaces, and hyperelliptic surfaces. On the other hand, Castelnuovo’s example is a surface of Kodaira dimension 1. The Enriques construction has a birational model that is a non-normal surface of degree 6 in $\mathbb{P}^3$ that passes through the edges of the coordinate tetrahedron with multiplicity 2. It was dubbed an Enriques sextic surface and the notion of an Enriques surface as a smooth projective surface with $q = 0$ and $P_2 = 1$ occurs in Artin’s thesis from 1960 [20], in Shafarevich’s seminar in 1961 – 1963 [5], as well as in Kodaira’s 1963 paper [389, part 3, p.719].

In 1906, Enriques proved that every (general) surface with invariants $p_g = q = 0$ and $P_2 = 1$ is birationally equivalent to an Enriques sextic. He also gave other birational models of his surfaces,
for example, as double planes branched along a certain curve of degree 8, an Enriques octic. A special case of the double plane construction was known to Enriques already in 1896 [217, Letter 302].

A minimal, smooth, and projective surface satisfies $p_g = q = 0$, $P_2 = 1$ if and only if its fundamental group is of order 2 and its universal cover is isomorphic to a K3 surface, which is characterized by being a minimal, smooth, and projective projective surface with invariants $q = 0$, $p_g = P_2 = 1$. This was already understood by Enriques, who proved that the pre-image of his sextic surface under the double cover of $\mathbb{P}^3$ branched along the union of four coordinate planes is birationally equivalent to a K3 surface [215]. This leads to the modern definition of an Enriques surface as the quotient of a K3 surface by a fixed-point-free involution. This point of view suggests that the theory of Enriques surface may be understood as a part of the theory of K3 surfaces, which is widely discussed and used in the modern literature, see, for example, [47], [312], or [405]. However, most usage of K3 surfaces in the study of Enriques surfaces consists in applying transcendental methods related to the theory of periods of K3 surfaces, which has little to do with the fascinating intrinsic geometry of Enriques surfaces.

The classification of algebraic surfaces was extended to the fields of positive characteristic in the work of Bombieri and Mumford [516], [77] and [78]. In particular, they gave a characteristic-free definition of Enriques surfaces. It turns out that Enriques surfaces in characteristic 2 live in a completely different and beautiful world that has many features, which have no analogue in characteristic $\neq 2$. For example, the canonical double cover still exists, but is a torsor under one of the three finite group schemes $\mu_2$, $\mathbb{Z}/2\mathbb{Z}$, $\alpha_2$ of order 2. Accordingly, this splits Enriques surfaces in characteristic 2 into three different classes, which are called classical, $\mu_2$-surfaces (or singular surfaces), and $\alpha_2$-surfaces (or supersingular surfaces). In the case where the canonical cover is inseparable, it is never a smooth surfaces and in some cases it is a rational surface, so it is not birationally equivalent to a K3 surface. Since there are many good modern expositions of the theory of algebraic surfaces and the theory of Enriques surfaces over the complex numbers (see, for example [47]), our priority is in providing the first complete as possible treatment of Enriques surfaces over fields of arbitrary characteristic. The price that we have to pay for this noble goal is reflected in the size of our book and also in requiring many more technical tools that we use. We collect all these needed tools in Chapter 0 and in fact, more than we need, in the hope to serve as a useful reference source for the study of algebraic surfaces over fields of arbitrary characteristic.

The authors have to admit that the initial goal of providing a complete exposition of the theory of Enriques surfaces over fields of arbitrary characteristic turned out to be too ambitious. Among the important topics that had to be left out are the theory of vector bundles on Enriques surfaces, arithmetic properties, as well as the theory of special subvarieties of the moduli spaces of algebraic curves that represent curves lying on Enriques surfaces.

We are grateful to many colleagues for many valuable discussions, which allowed us to improve the exposition as well to include many results previously unknown to the authors. They include D. Allcock, W. Barth, K. Hulek, T. Katsura, J. Keum, W. Lang, E. Looijenga, V. Nikulin, G. Martin, S. Mukai, Y. Matsumoto, H. Ohashi, C. Peters, N. Shepherd-Barron, I. Shimada, M. Schütt, M. Reid, Y. Umezu, A. Verra.

Each chapter ends with a bibliographical note, where we tried our best to give the credit to the
original research discussed in this chapter. Special credit goes to François Cossec, who did not participate in the present project, but whose contribution to the theory of Enriques surfaces is hard to overestimate. Some of the results, which had not found a place in the first edition of Part I, are now included in the new editions of Part I and II and are based on his unpublished results communicated to the second author.
Chapter 6 ‘Nodal Enriques surfaces’ is an expanded revision of Chapter 3 from the first edition of Part I [134]. In section 1 we discuss canonical isotropic sequences and introduce the non-degeneracy invariant \( \text{nd}(S) \) of an Enriques surface. It is a maximal length of an sequence \((f_1, \ldots, f_k)\) of nef numerical classes of divisors with \( f_i \cdot f_j = 1 - \delta_{ij} \). The main result of this section is Corollary 7.1.6 that asserts that \( \text{nd}(S) \geq 3 \) always if the characteristic \( \neq 2 \). We give a new conceptual proof of this result which is different from the case-by-case proof sketched by F. Cossec in [131]. The situation in characteristic 2 is much more complicated. There are extra-special surfaces of three different type discussed in section 2 for which \( \text{nd}(S) \leq 2 \). One expects that \( \text{nd}(S) \geq 3 \) for all other surfaces, the first edition contains a proof of this result that consists of more than 30 pages of case-by-case of possible dual graphs of sets of smooth rational curves arising from the assumption that \( \text{nd}(S) \leq 2 \).

In section 3 we discuss among other things the results of Cossec and the first author [135] on the minimal degrees of smooth rational curves on polarized Enriques surfaces. In section 4 we introduce different invariants of an Enriques surface that controls the set of smooth rational curves. Among them are the Nikulin root invariants, the nodal and the Reye lattices. We also discuss an important class of special Enriques surfaces arising from the Hessian surfaces of cubic surfaces and compute their nodal invariants. In the last section 6 we define the notion of a general nodal Enriques surface and give their different geometric characterizations.

Chapter 7 ‘Reye congruences’ discusses a classical construction of nodal Enriques surfaces as a smooth congruences of lines in the Grassmannian of lines in \( \mathbb{P}^3 \). The canonical cover of such surfaces is birationally isomorphic to a famous Cayley quartic symmetroid. Some of the modern exposition of this theory can be found in a paper of F. Cossec [133] and the book of the first author [180]. The novel feature here is the analogous construction of Reye congruences and their relationship to Enriques surfaces in characteristic 2.

Chapter 8 ‘Automorphisms of Enriques surface’ plays the central role in Part Two. In section 1, after reminding some general facts about the group scheme of automorphisms of algebraic varieties, we prove that the automorphism group of an Enriques surface is discrete and hence it is determined by its representation in the group of isometries of the lattice \( \text{Num}(S) \) of numerical divisor classes. This opens a way to relate these groups to discrete groups of isometries of hyperbolic space \( \mathbb{H}^9 \).

In section 2, we study the finite group of automorphisms of an Enriques surface which act identically on \( \text{Num}(S) \) or on \( \text{Pic}(S) \). The surfaces for which such group is non-trivial are very rare and have been completely classified over the field of complex numbers by S. Mukai and Y. Namikawa [503], [505]. We supply the classification in all characteristics different from 2. The characteristic 2
case is the subject of paper [190] the first author and G. Martin. We give an exposition of the main results from this paper.

The group of automorphisms of a general Enriques surface over complex numbers was determined by W. Barth and C. Peters [46], and independently by V. Nikulin [533], in the early eighties. In section 3 we reprove their result by using purely geometrical methods that do not make any assumption on the characteristic of the ground field. Section 10 of Chapter 2 of the first edition of Part I contains a preparatory computer computations needed for this proof. The new proof gets rid of this by using some nice and short lattice-theoretical arguments due to D. Allcock.

Section 4 extends the results of the previous section to the case of a general nodal Enriques surface. The structure of the group of automorphisms of such surfaces over fields of arbitrary characteristic was announced by F. Cossec and the first author in a short note [136] and we give here the complete proof of the announced results. Again we replace some computer computations from [134] by another lattice-theoretical argument due to D. Allcock [8]. As application we discuss in section 5 automorphism groups of a Cayley quartic symmetroid.

The next two sections 6 and 7 are devoted to expositions of results of H. Ito and H. Ohashi [330] on cyclic groups of automorphisms of complex Enriques surfaces. We also discuss the results of S. Mukai and H. Ohashi about automorphisms of Mathieu type of Enriques surfaces [508], [509] and [510]. Again, we use different methods not relying on the theory of periods of Enriques surfaces that allow us to extend these results to the case of positive characteristic.

The question of the existence of an Enriques surface with a finite automorphisms was raised by F. Enriques. We refer to the history of this question to Bibliographical Notes to Chapter 8. In section 8 we discuss a complete classification of Enriques surfaces with finite automorphisms over a field of complex numbers due to V. Nikulin [535] in terms of periods of their K3 covers and a purely geometric classification over fields of characteristic zero due to the second author [397]. In the next section we give a brief exposition of a recent classification of such surfaces over fields of characteristic ≠ 2 due to G. Martin and characteristic 2 due to T. Katsura, the second author and G. Martin [363].

Chapter 9 ‘Rational Coble surfaces’ is devoted to close cousins of Enriques surfaces which are smooth rational surfaces that contain an isolated curve in its anti-bicanonical linear system. They appear as nonsingular models of type II degenerations of Enriques surfaces and share many common properties with Enriques surfaces. Over a field of characteristic different from 2 they appear as quotients of K3 surfaces by an involution with the disjoint union of smooth rational curves fixed pointwise. They also arise as the blow-up of the projective plane at ≥ 10 double points of a plane curve of degree 6. Such curves and their Cremona equivalence classes were intensively studied by A. Coble in the first half of the last century. In section 1 we discuss the relationship between Coble surfaces and Enriques and K3 surfaces. In section 2 we introduce the Coble-Mukai lattice which for a general Coble surface coincides with the Enriques lattice.

In sections 2 and 3 we discuss the work of R. Winger [704] and [705] on the classification of irreducible rational plane sextics with non-trivial projective group of automorphisms.

In section 5, following Coble’s work we find the automorphism group of a general Coble surface which has the same structure as the automorphism group of a general Enriques surface.
The group of automorphisms of a Coble surface could be a finite group. We classify Coble surfaces with finite automorphism group over fields of characteristic 0 and give many examples of such surfaces in positive characteristic that arise as either a reduction to positive characteristic of Enriques surfaces with finite automorphism group or as the limits in the families of such surfaces in characteristic 0 or 2.

Chapter 10 ‘Enriques surfaces and supersingular K3 surfaces’ deals with classical or \(\alpha_2\)-Enriques surfaces in characteristic 2 whose canonical cover is birationally isomorphic to a K3 surface \(X\). It turns out that \(X\) is always a supersingular K3 surface and conversely a general such surface can be obtained in this way by considering its quotient by a rational vector field. We tried to be self-contained by giving an introduction to the theory of supersingular K3 surfaces over fields of arbitrary positive characteristic. In particular, we discuss the periods of such surfaces and Ogus’s Global Torelli Theorem. Each supersingular K3 surface comes with the Artin invariant \(\sigma\) determined by the discriminant of the Néron-Severi lattice of \(X\). The moduli space of supersingular surfaces is of dimension \(\sigma - 1\) and the surface with \(\sigma = 1\) is unique up to an isomorphism. In the last section we discuss Enriques and Coble surfaces whose canonical cover is birationally isomorphic to a supersingular K3 surface with Artin invariant equal to 1.
Chapter 6

Nodal Enriques surfaces

6.1 Canonical isotropic sequences

Recall from Chapter 1, Section 1.5 that the Enriques lattice $E_{10}$ has a root basis $(\alpha_0, \ldots, \alpha_9)$ with the Dynkin diagram

\[ \alpha_1 \bullet \alpha_2 \bullet \alpha_3 \bullet \alpha_4 \bullet \alpha_5 \bullet \alpha_6 \bullet \alpha_8 \bullet \alpha_9 \bullet \alpha_0 \] (6.1.1)

It contains 10 isotropic vectors $f_1, \ldots, f_{10}$ such that $\alpha_i = f_i - f_{i+1}, i \geq 1$, and $\alpha_0 = h - f_1 - f_2 - f_3$ for some $h \in E_{10}$. We have $f_i \cdot f_j = 1, i \neq j$, and

\[ 3h = f_1 + \cdots + f_{10}. \]

An ordered set $(f_1, \ldots, f_k)$ of $k \geq 2$ isotropic vectors with $f_i \cdot f_j = 1, i \neq j$, will be called an isotropic $k$-sequence. If $k = 1$, we assume that $f_1$ is primitive. It follows from above that an isotropic $k$ sequences exists for any $1 \leq k \leq 10$ and does not exist for $k > 10$.

Consider the standard embedding of the lattice $E_{10}$ in the standard hyperbolic lattice $I_{10}$ as described in loc.cit.. Let $k_{10} = 3e_0 - e_1 - \cdots - e_{10}$ generate the orthogonal complement of $E_{10}$ in $I_{10}$. Let $(f_1, \ldots, f_k)$ be an isotropic $k$-sequence. Then the vectors $v_i = f_i - k_{10}$ satisfy $v_i^2 = -1, v_i \cdot v_j = 0, i \neq j$. A sequence $(v_1, \ldots, v_k)$ of vectors in $I_{10}$ with $v_i^2 = -v_i \cdot k_{10} = -1, v_i \cdot v_j = 0$, is called an exceptional $k$-sequence.

We use the notation from Section 1.5.

Proposition 6.1.1. The Weyl group $W(E_{10})$ acts transitively on the set of isotropic $k$-sequences with $1 \leq k \neq 9$ and has two orbits of isotropic 9-sequences represented by $(k_{10} + e_0 - e_1 - e_2, f_3, \ldots, f_{10})$ and $(f_2, \ldots, f_{10})$.

Proof. Suppose $k = 2$. An isotropic 2-sequence $(f_1, f_2)$ (resp. $(f'_1, f'_2)$) generates a unimodular hyperbolic sublattice $U$ (resp. $U'$) of $E_{10}$. Let $E_{10} = U \oplus M = U' \oplus M'$. By Witt’s theorem, there
exists an element \( \sigma \in O(E_{10}) \) such that \( \sigma(U) = U' \). This gives \( (\sigma(f_1), \sigma(f_2)) = (\pm f'_1, \pm f'_2) \), or \((\pm f'_2, \pm f'_1)\). Composing \( \sigma \) with \(-id_{E_{10}}\), we may assume that \( \sigma \in W(E_{10}) \). Composing \( \sigma \) with an isometry of \( U \) which acts identically on \( M \), we may assume that \( (\sigma(f_1), \sigma(f_2)) = (f'_1, f'_2) \). Thus the assertion is true for \( k = 2 \). It also shows that \( W(E_{10}) \) has one orbit on the set exceptional 2-sequences. In particular, we may assume that the first two vectors in an exceptional \( k \)-sequence \((v_1, \ldots, v_k)\) coincide with the vectors \( e_1, e_2 \) from the standard basis \( (e_0, e_1, \ldots, e_{10}) \) of \([1,10]\). The orthogonal complement of the sublattice spanned by \((e_1, e_2)\) is the lattice \([1,8]\) and its Weyl group is \( W(E_8) \) that embeds naturally in \( W(E_{10}) \) by acting identically on the orthogonal complement. It is known that \( W(E_8) \) acts transitively on exceptional \( r \)-sequences with \( r \neq 7 \) and has two orbits on the set of exceptional \( 7 \)-sequences (see [157], [467]). The two orbits are represented by the exceptional \( 7 \)-sequences \((e_0 - e_1 - e_2, e_3, \ldots, e_7)\) and \((e_2, e_3, \ldots, e_7)\). This corresponds to the isotropic sequences \((f, f_3, \ldots, f_{10})\) and \((f_2, \ldots, f_{10})\), where \( f = k_{10} + e_0 - e_1 - e_2 \). This proves the assertion.

Corollary 6.1.2. If \( k \neq 9 \), any isotropic \( k \)-sequence \((g_1, \ldots, g_k)\) in \( E_{10} \) can be extended to a canonical isotropic 10-sequence \((g_1, \ldots, g_{k}, g_{k+1}, \ldots, g_{10})\).

Proof. Consider an isotropic 10-sequence \((f_1, \ldots, f_{10})\) and find \( w \in W(E_{10}) \) such that \( w(g_i) = f_i, i = 1, \ldots, k \). Then set \( g_{k+i} = w^{-1}(f_{k+i}), i = 1, \ldots, 10 - k \).

Recall that we denoted by \( U \) an abstract quadratic lattice with a basis \((f_1, f_2)\) formed by two isotropic vectors with \( f_1 \cdot f_2 = 1 \). Its generalization is the lattice \( U_{[k]} \) with a basis formed by isotropic vectors \((f_1, \ldots, f_k)\) such that \( f_i \cdot f_j = 1, i \neq j \).

Proposition 6.1.3. The lattice \( U_{[k]} \) is an even hyperbolic lattice with cyclic discriminant group of order \( k - 1 \). Every isotropic \( k \)-sequence contained in \( U_{[k]} \) is a basis. The vector \( s = f_1 + \cdots + f_k \) is a unique vector in \( U_{[k]} \) such that \( s^2 = k(k - 1) \) and \( |s \cdot f| \geq k - 1 \) for any isotropic vector \( f \in U_{[k]} \).

Proof. The explicit computation of the Gram matrix of the basis \((f_1, \ldots, f_k)\) checks the assertion about the discriminant group. It also shows that any isotropic \( k \)-sequences forms a basis of \( U_{[k]} \).

We continue to use \( S \) to denote an Enriques surface and continue to denote the reflection group generated by reflections in classes of \((-2)\)-curves on \( S \) by \( W_{S}^{\text{nod}} \).

Definition 6.1.4. A nef isotropic \( k \)-sequence in \( \text{Num}(S) \) is an isotropic \( k \)-sequence that consists of nef vectors.

Proposition 6.1.5. Let \((f_1, \ldots, f_k)\) be an isotropic \( k \)-sequence in \( \text{Num}(S) \) of effective isotropic classes. There exists a unique \( w \in W_{S}^{\text{nod}} \) such that, after reindexing, the sequence \((f'_1, \ldots, f'_k) := (w(f_1), \ldots, w(f_k))\) contains a nef isotropic subsequence \( f'_{i_1}, f'_{i_2}, \ldots, f'_{i_c} \) with \( 1 = i_1 < i_2 < \ldots < i_c \), such that, for any \( i_s < i < i_{s+1} \),

\[
f'_i = f'_{i_s} + R_{i_s,i} + \cdots + R_{i_s,i-i_s} \in W_{S}^{\text{nod}} \cdot f_{i_s},
\]

where \( R_{i_s,1} + \cdots + R_{i_s,i-i_s} \) is a nodal cycle of type \( A_{i-i_s} \).
6.1. CANONICAL ISOTROPIC SEQUENCES

**Proof.** Let \( f = f_1 + \cdots + f_k \). Let \( w' \in W_S^{\text{nod}} \) such that \( h = w'(f) = f'_1 + \cdots + f'_k \) is nef. Since \( W_S^{\text{nod}} \) sends effective divisor with non-negative self-intersection to effective divisors, the classes \( f'_i \) are effective. Then \( h^2 = k(k-1) \) and \( \Phi(h) \leq h_10 \cdot f'_1 = k-1 \). Let \( \Phi(h) = h_10 \cdot f_0 \) for some isotropic vector \( f_0 \). By Proposition 2.3.3, we can write \( f_0 = g_0 + \sum m_i R_i \), where \( g_0 \) is a nef isotropic class and \( R_i \) are \((-2)\)-curves. Since \( h \) is nef, we get \( h_10 \cdot f_0 \geq h_10 \cdot g_0 \), hence we may assume that \( f_0 \) is nef. Since \( f_0 \cdot f'_i \geq 0 \), we obtain that \( f_0 \) is one of the \( f'_i \). After reindexing, we may assume that there exists a sequence \( 1 = i_1 < \cdots < i_c \) such that \( f'_i \) are nef, and \( f'_i, i_s < i < i_{s+1} \) belong to the \( W_S^{\text{nod}} \)-orbit of \( f'_i \). If \( c = 10 \), there is nothing to prove. Assume \( c < 10 \), and \( i_{s+1} - i_s > 1 \). For any \( i_s < i < i_{s+1} \), we can write

\[
 f'_i = w(f_{i_s}) = f'_{i_s} + \sum m_\alpha R_\alpha,
\]

where \( R_\alpha \) are different \((-2)\)-curves and \( m_\alpha > 0 \). Intersecting with \( f'_{i_s} \), we find a unique \( R_\alpha \) such that \( f'_{i_s} \cdot R_\alpha = 1 \) and \( m_\alpha = 1 \). The class \( f'_{i_s} + R_\alpha \) is isotropic and

\[
 h_10 \cdot (f'_{i_s} + R_\alpha) = h_10 \cdot (f'_{i_s} + 1 - \sum m_\beta R_\beta) \leq h_10 \cdot f'_{i_s+1} \leq k-1.
\]

Since \( \Phi(h) = k-1 \), the class \( f'_{i_s} + R_\alpha \) must be one of the classes \( f'_{i_s}, i_s < i < i_{s+1} \). After reindexing, we may assume that \( f'_{i_{s+1}} = f'_{i_s} + R_{i_{s+1}} \), where \( R_\alpha = R_{i_{s+1}} \). Assume \( i > i_s \). Then

\[
 1 = f'_{i_{s+1}} \cdot f'_i = (f'_{i_{s+1}} + R_{i_{s+1}}) \cdot (f'_{i_s} + R_{i_{s+1}} + \sum \beta \neq \alpha m_\beta R_\beta) = R_{i_{s+1}} \cdot (\sum \beta \neq \alpha m_\beta R_\beta).
\]

This shows that there exists a unique \( \beta \) such that \( R_{i_{s+1}} \cdot R_\beta = 1 \) and \( m_\beta = 1 \). As above we show that \( f'_{i_{s+2}} + R_{i_{s+1}} + R_\beta \) is equal to one of \( f'_{i_s}, i_s < i < i_{s+1} \). After reindexing, we may assume that \( f'_{i_{s+2}} = f'_{i_s} + R_{i_{s+1}} + R_{i_{s+2}} \), where \( R_{i_{s+2}} = R_\beta \). Continuing in this way we show that, after reindexing \( f'_{i} = f'_{i_s} + R_{i_{s+1}} + \cdots + R_{i_{s,i-i_{s}-1}}, i_s < i < i_{s+1} \), where \( R_{i_{s+1}} + \cdots + R_{i_{s,i-i_{s}}} \) is a nodal cycle of type \( A_{i-i_{s}} \).

An isotropic \( k \)-sequence \( (f_1, \ldots, f_k) \) which, after reindexing, is equal to the sequence \( (f'_1, \ldots, f'_k) \) described in the previous lemma, is called canonical.

It also follows from the lemma that for any isotropic \( k \)-sequence there exists a unique \( w \in W_S^{\text{nod}} \) such that \( (w(f_1), \ldots, w(f_k)) \) is a canonical isotropic \( k \)-sequence.

The number \( c \) of nef members in a canonical isotropic \( k \)-sequence is called the degeneracy invariant of the sequence. We say that a canonical isotropic \( k \)-sequence is \( c \)-degenerate if its degeneracy invariant is equal to \( c \). A canonical isotropic \( k \)-sequence with the degeneracy invariant equal to \( k \) is called non-degenerate.

**Proposition 6.1.6.** A canonical isotropic sequence \( (f_1, \ldots, f_{10}) \) is non-degenerate if and only if \( f = f_1 + \cdots + f_{10} \) is an ample numerical divisor class.

**Proof.** Assume that \( (f_1, \ldots, f_{10}) \) is non-degenerate. Then \( f \) is nef and \( f^2 = 90 > 0 \). Suppose \( f \cdot R = 0 \) for some \((-2)\)-curve \( R \). Then \( R \cdot f_i = 0 \) for all \( i \). Since \( f_1, \ldots, f_{10} \) generate \( \text{Num}(S) \otimes \mathbb{Q} \), we get a contradiction. Conversely, suppose \( f \) is ample but \( (f_1, \ldots, f_{10}) \) is degenerate. Then we can find some \( f_i \) which is equal to \( f_{i-1} + R \), where \( f_{i-1} \) is nef and \( R \cdot f_{i-1} = 1 \). It follows from the definition of a canonical isotropic sequence that \( (f_1 + \cdots + f_{10}) \cdot R = 0 \). □
Proposition 6.1.7. A canonical isotropic sequence \((f_1, \ldots, f_k)\) with \(k \neq 9\) with degeneracy invariant \(c\) can be extended to a canonical isotropic 10-sequence with degeneracy invariant \(c' \geq c\).

Proof. Applying Corollary 6.1.2, we can extend \((f_1, \ldots, f_k)\) to a maximal isotropic 10-sequence \((f_1, \ldots, f_k, f_{k+1}, \ldots, f_{10})\). Applying some \(w \in W_{S}^{\text{nod}}\) we obtain a canonical isotropic sequence \((f'_1, \ldots, f'_k)\). Let \(f'_1, \ldots, f'_c\) be the nef vectors in this sequence. Then each vectors \(f'_i\) belongs to the \(W_{S}^{\text{nod}}\)-orbits of one of these vectors. In particular, the nef vectors in \((f_1, \ldots, f_k)\) belong to the orbits of \(f'_1, \ldots, f'_c\). Since two different nef vectors cannot belong to the same orbit, we may assume that \(f'_i = f_i, i = 1, \ldots, c\). The vectors \(f_{c+1}, \ldots, f_k\) belong to the \(W_{S}^{\text{nod}}\)-orbits of \(f_1, \ldots, f_c\). The vectors \(f'_{c+1}, \ldots, f'_k\) belong to the \(W_{S}^{\text{nod}}\)-orbits of \(f'_1, \ldots, f'_c\). Since the orbits of different isotropic vectors are disjoint, we obtain that \(f_{c+i} = f'_{c+i}, i = c+1, \ldots, k\). \(\square\)

Definition 6.1.8. A primitive lattice embedding \(j : U_{[k]} \hookrightarrow \text{Num}(S)\) is called a \(U_{[k]}\)-marking of \(S\). Two \(U_{[k]}\)-markings \(j\) and \(j'\) are called equivalent if there exists an isometry \(\sigma\) of \(U_{[k]}\) and an element \(w \in W_{S}^{\text{nod}}\) such that \(j' = w \circ j \circ \sigma\).

We say that a \(U_{[k]}\)-marking is canonical (resp. non-degenerate) if the image \((f_1, \ldots, f_k)\) of its canonical basis \((f_1, \ldots, f_k)\) is a canonical (resp. non-degenerate) isotropic \(k\)-sequence. It follows from Proposition 6.1.5 that any \(U_{[k]}\)-marking is equivalent to a canonical \(U_{[k]}\)-marking.

When \(k = 2\), the lattice \(U_{[2]}\) coincides with the standard hyperbolic plane lattice \(U\). So, in this case we just say a U-marking. A canonical U-marking is called a \(U\)-pair. A non-degenerate \(U\)-pair is uniquely defined by an ordered pair of nef isotropic vectors \(f_1, f_2\) with \(f_1 \cdot f_2 = 1\). A canonical degenerate \(U\)-pair is uniquely defined by a choice of a nef isotropic vector \(f\) and the class \(r\) of a smooth rational curve such that \(f \cdot r = 1\).

Definition 6.1.9. The non-degeneracy invariant of an Enriques surface is the maximal length \(\text{nd}(S)\) of a non-degenerate isotropic sequence.

Of course, if \(S\) has no \((-2)\)-curves, then any isotropic \(k\)-sequence is non-degenerate and the non-degeneracy invariant is equal to 10.

In characteristic 2 it may happen that \(\text{nd}(S) = 1\) if \(S\) is extra \(\tilde{E}_8\)-special in the following sense. It contains a half-fiber \(F\) of type \(\tilde{E}_8\) and a \((-2)\)-curve intersecting \(F\) with multiplicity 1. The dual graph of the ten \((-2)\)-curves is a \(T_{2,3,7}\)-diagram 6.1.

\[
\begin{array}{ccccccccccc}
R_2 & R_3 & R_4 & R_5 & R_6 & R_7 & R_8 & R_9 & R_{10} \\
\cdot & R_1 & & & & & & & &
\end{array}
\]

The classes of the curves \(R_i\) form a crystallographic root basis in \(\text{Num}(S)\). Recall that this means that the reflection subgroup of \(W_{S}^{\text{nod}}\) generated by the reflections \(s_{R_i}\) is of finite index in \(O(\text{Num}(S))\). It follows from Proposition 0.8.19 that it coincides with \(W_{S}^{\text{nod}}\), hence the set of \(W_{S}^{\text{nod}}\)-orbits of isotropic vectors coincides with the set of primitive nef isotropic classes. It follows from the theory of reflection groups that the set of \(W_{S}^{\text{nod}}\)-orbits is equal to the number of parabolic subdiagrams of maximal rank. In our case it is equal to 8. Looking at the diagram we find that there is only one such subdiagram, hence there is only one genus one fibration on \(S\).
Theorem 6.1.10. Suppose $p \neq 2$ or $p = 2$ and $S$ is not extra-$\tilde{E}_8$-special. Then any isotropic nef class $f$ can be extended to a non-degenerate canonical 2-sequence $(f_1, f_2)$. In particular, the non-degeneracy invariant $\text{nd}(S)$ of a not extra-$\tilde{E}_8$-special Enriques surface is greater than or equal to 2.

Proof. It is enough to show that one can find a canonical 10-sequence with the non-degeneracy invariant $\geq 2$. Suppose it does not exist. Starting from any isotropic class $f = f_1$ we extend it to an isotropic 10-sequence and then apply an element of $W^\text{nod}_S$ to transform the latter to a canonical 10-sequence $(f_1, f_2, \ldots, f_{10})$ with the non-degeneracy invariant $c$ equal to 1. By Proposition 6.1.5, we may assume that $f_i = f_1 + R_1 + \cdots + R_{i-1}$, where $R_1 + \cdots + R_9$ is a nodal cycle of type $A_9$ with $(R_1 + \cdots + R_9) \cdot f_1 = R_1 \cdot f_1 = 1$. Let $f_1 = [F]$ for some genus one curve $F$ and $|2F|$ be the corresponding genus one pencil. Since $R_i \cdot f_1 = 0$, the nodal cycle $R_2 + \cdots + R_9$ is contained in some member $D$ of $|2F|$. The classification of genus one curves on $S$ shows that $D$ is of type $\tilde{A}_8$ or $\tilde{E}_8$.

Case 1: $D$ is of type $\tilde{A}_8$ and $D = D_{\text{red}}$.

Let $D = R_2 + \cdots + R_9 + R_{10}$. Since $R_1 \cdot D = 2$ and $R_1$ intersects $R_2 + \cdots + R_9$ with multiplicity 1, we must have $R_1 \cdot R_{10} = 1$. This is pictured on the following diagram:

```
F   R_1  R_2  R_3  R_4  R_5  R_6  R_7  R_8  R_9
    \downarrow R_{10}
```

Consider the divisor $F' = R_1 + R_2 + R_{10}$. It is a genus one curve of type $\tilde{A}_2$ intersecting $F$ with multiplicity 1. The pair $[F], [F']$ is a non-degenerate isotropic 2-sequence.

Case 2: $D$ is of type $\tilde{A}_8$ and $D = 2F$. We have the following picture:

```
F   R_1  R_2  R_3  R_4  R_5  R_6  R_7  R_8  R_9
    \downarrow R_{10}
```

We see that the divisor $D = R_6 + R_7 + R_8 + R_9 + R_{10} + R_1 + R_2 + R_3 + R_4$ is of type $\tilde{E}_8$ and since $D \cdot R_5 = 3$, it must be a half-fiber. Therefore $p = 2$ and the surface is not a $\mu_2$-surface. On the other hand, we see that the divisor $R_2 + \cdots + R_{10}$ is of type $\tilde{A}_8$ and $R_1 \cdot D = 1$. Thus the surface admits an elliptic fibration with a double fiber of type $\tilde{A}_8$, and hence it is a $\mu_2$-surface. This contradiction excludes this case.

Case 3: $D$ is of type $\tilde{E}_8$. One of the two possible cases is when $S$ is extra $\tilde{E}_8$-special. In other case, we have the following picture:

```
F   R_1  R_2  R_3  R_4  R_5  R_6  R_7  R_8  R_9
    \downarrow R_{10}
```

Since $R_{10}$ comes with multiplicity 3, the curves $R_1$ and $R_{10}$ do not intersect. We see that a divisor $D' = R_1 + D - R_8 - R_9$ is the support of a divisor of type $\tilde{E}_7$ such that $F \cdot D' = 1$. Thus $(F, D')$ is a non-degenerate $U$-pair.

\qed
Corollary 6.1.11. Suppose $S$ is not $\tilde{E}_8$-special. Then there exists a degree 2 cover $f : S \to D$, where $D$ is an anti-canonical quartic del Pezzo surface $D_1, D_2$ or $D_3$. In particular, if $p \neq 2$, $D$ is a 4-nodal anti-canonical quartic del Pezzo surface.

Proof. Let $|2F|$ and $|2F'|$ be two genus one fibrations such that $([F], [F'])$ is a non-degenerate $U$-pair. Then the linear system $|2F + 2F'|$ is a non-special bielliptic linear system and the assertion follows from Theorem 6.1.10.

Lemma 6.1.12. Let $(f_1, f_2, f_3)$ be a non-degenerate isotropic 3-sequence and let $(F_1, F_2, F_3)$ be its lift in $\text{Pic}(S)$. Let $\phi : S \to D$ be a bielliptic map defined by the linear system $|2F_1 + 2F_2|$. Then the image of $F_3$ is a hyperplane section of $D$ and its pre-image is equal to the union $F_3 + G_3$, where $G_3$ is the unique genus one curve from $|2F_1 + 2F_2 - F_3|$.

Proof. Since $(2F_1 + 2F_2 - F_3)^2 = 0$, $(2F_1 + 2F_2 - F_3) \cdot F_1 = 1$, we get $h^0(2F_1 + 2F_2 - F_3) = h^1(2F_1 + 2F_2 - F_3) = 1$ and the exact sequence

$$0 \to \mathcal{O}_S(2F_1 + 2F_2 - F_3) \to \mathcal{O}_S(2F_1 + 2F_2) \to \mathcal{O}_{F_3}(2F_1 + 2F_2) \to 0$$

shows that the image $C$ of $F_3$ spans a hyperplane. The pre-image of $C$ is the union of $F_3$ and the unique genus one curve $G_3 \in |2F_1 + 2F_2 - F_3|$.

Theorem 6.1.13. Assume that $p \neq 2$. Then any non-degenerate canonical isotropic 2-sequence $(f_1, f_2)$ can be extended to a non-degenerate canonical isotropic 3-sequence $(f_1, f_2, f_3)$.

Proof. In view of the previous Lemma it is enough to find a hyperplane section $C$ of $D$ that splits under the cover.

Let $S \xrightarrow{\sigma} S' \xrightarrow{\sigma'} D$ be the Stein factorization and $\tilde{D} \xrightarrow{\tilde{\sigma}} \tilde{S}' \xrightarrow{\tilde{\sigma}'} \tilde{D}$ its lift to the minimal resolution of singularities $\tilde{S} \to S$ as in diagram (3.3.2) from Volume 1.

We denote the preimage of $\mathcal{O}_D(n)$ on $\tilde{D}$ by $\mathcal{O}_{\tilde{D}}(n)$ and identify $C$ and $W$ with their pre-images on $\tilde{D}$. Let $\mathcal{L} \cong \mathcal{O}_D(1)$ be the invertible sheaf that defines the finite cover $\tilde{\sigma}' : \tilde{S}' \to \tilde{D}$. The branch curve of this map is a curve $W \in |\mathcal{O}_{\tilde{D}}(2)|$ and the sum of the four $(-2)$ curves from $|2A|$. The exact sequence

$$0 \to \mathcal{O}_{\tilde{D}}(A - C) \to \mathcal{O}_{\tilde{D}}(A) \to \mathcal{O}_C(A) \to 0$$

easily shows that $h^0(\mathcal{O}_C(A)) = 0$, hence the restriction of $A$ to $C$ defines a non-trivial 2-torsion Cartier divisor class $\epsilon$. Suppose we show that there exists $C$ such that $\mathcal{M}_C := \mathcal{O}_C(1)(\epsilon)$ has a section with the zero divisor $D$ such that $2D$ is cut out by $W$. Then, as we explained in section 0.2 in Volume 1, the curve $C$ splits under the cover.

Consider the variety $M$ of pairs $(C, D)$, where $C$ is a hyperplane section of $D$ and $D \in |\mathcal{O}_D(1)(\epsilon)|$. Its projection to $|\mathcal{O}_D(1)| \cong \mathbb{P}^4$ is a $\mathbb{P}^3$ projective bundle with fiber $|\mathcal{O}_C(1)(\epsilon)|$. For any $(C, D) \in M$, we have the invertible sheaf $\mathcal{M}_C$ isomorphic to the restriction of $\mathcal{L}$ to $C$. Let

$$m_C : H^0(\mathcal{O}_D(1)) \to H^0(\mathcal{O}_D(2)), s' \mapsto s_C \otimes s'$$

where $s_C$ is a section of $\mathcal{O}_D(1)$ with zero subscheme equal to $C$. The quotient space is a linear space $V$ of dimension $13 - 5 = 8$. For any $D \in |\mathcal{C}(1)(\epsilon)|$ the divisor $2D$ is cut out by unique $W' \in \mathbb{P}^3$. 

|\mathcal{O}_D(2)|. This defines a map \( r : M \to \mathbb{P}(V) \cong \mathbb{P}^7 \) of varieties of dimension 7. Let \( r(C, D) = W' \). Then the double cover of \( D \) branched along \( W' \) and \( 2A \) corresponding to the sheaf \( \mathcal{O}_D(1)(A) \) is a surface birationally isomorphic an Enriques surface \( S' \) with a non-degenerate isotropic sequence \((G_1, G_2, G_3)\). Let \( S' \to D \) be the bielliptic map defined by the linear system \([2G_1 + 2G_2]\) defines the bielliptic map \( S' \to D \) with branch curve \( W' \) and \( G_3 \) arises from splitting \( C \). Since for a fixed \( W' \) there is only finitely many primitive isotropic classes \( g_3 \) such that \( ([G_1], [G_2], g_3) \) is an isotropic sequence, the map \( r \) is of finite degree and hence surjective. In particular, we find a pair \((C, D)\) lying over our original branch curve \( W \), and we are done.

\[ \mathbf{Corollary 6.1.14.} \text{ Let } nd(S) \text{ be the non-degeneracy invariant of an Enriques surface over a field of characteristic } p \neq 2. \text{ Then} \]

\[ nd(S) \geq 3. \]

\[ \mathbf{Remark 6.1.15.} \text{ It is claimed in [134, Theorem 3.5.1] that the assertion of the theorem remains true in characteristic 2 except for explicitly classified extra-special surfaces with } nd(S) \leq 2. \text{ There were many more cases to consider in this case and the proof occupies 32 pages of the book. Since we are not sure that the authors have not missed some cases, to be on the safe side, we have stated the theorem only in the case } p \neq 2. \]

We refer to Sections 8.8 and 8.10 where we compute \( nd(S) \) for Enriques surfaces with finite automorphism group.

\[ \mathbf{Remark 6.1.16.} \text{ Let } (v_1, v_2) \text{ be an isotropic 2-sequence in } E_{10}. \text{ Let us count how many isotropic vectors } v \text{ extend it to an isotropic 3-sequence } (v_1, v_2, v). \text{ Let } U = \langle v_1, v_2 \rangle \text{ then } v \in (v_1 - v_2)^\perp \cong \langle v_1 + v_2 \rangle \oplus U^\perp \cong (2) \oplus E_8. \text{ Write } v = a_1(v_1 + v_2) + w, w \in U^\perp. \text{ Intersecting with } v_1 + v_2 \text{ we find that } a_1 = 1. \text{ Since } v^2 = 0, \text{ we get } w \in (U^\perp)_{-2}. \text{ The number of vectors of square-norm } -2 \text{ in } E_8 \text{ is equal to 240. However, our vector } v \text{ comes from the positive cone in } E_{10} \otimes \mathbb{R}, \text{ hence we have } 120 = 2^4 \cdot (2^4 - 1) \text{ possibilities for } v. \text{ This is equal to the cardinality of a Steiner complex of odd theta characteristics on a genus 5 the number of pairs of odd theta characteristic } \{\theta_1, \theta_2\} \text{ with fixed difference } \epsilon = \theta_1 - \theta_2 \text{ (see [180, Proposition 5.4.7]). If we assume that } W \text{ is a nonsingular curve of genus 5, then a choice of } C \text{ as above defines an odd theta-characteristic } \theta \text{ on } W \text{ such that } \mathcal{O}_W(\theta - \theta_0) \cong \mathcal{O}_W(A), \text{ where } 2\theta_0 = C \cap W. \text{ Thus we see that in this case there are 120 ways to extend } (f_1, f_2) \text{ to } (f_1, f_2, f_3). \]

Further note that the branch curve \( W \) always admits two even vanishing characteristics different by \( \mathcal{O}_W(A) \). They are defined by ruling by planes in two quadrics of rank 3 containing \( D_1 \). However they define hyperplane sections \( C \) that pass through two opposite vertices of the quadrangle of lines on \( D_1 \). These curve lift to \( \hat{D} \) and intersect two of the exceptional curves from the branch locus. They do not split under the cover.

### 6.2 Extra-special and exceptional Enriques surfaces

We have already encountered a surface for which \( nd(S) = 1 \). It contains \((-2)\)-curves forming the following intersection graph of type \( T_{2,3,7} \).
It follows from the definition of a crystallographic root basis that the classes of the curves corresponding to the vertices of this diagram form a crystallographic root basis in $\text{Num}(S)$.

**Definition 6.2.1.** An Enriques surface is called nodal reflective surface if it contains a crystallographic root basis formed by the divisor classes of $(-2)$-curves.

We will later give a classification of such surfaces. It coincides with the classification of Enriques surfaces with finite automorphism group. It follows from this classification that in the case when $\text{char}(k) \neq 2$, the cardinality $c$ of such root bases is larger than or equal to 12. However, in characteristic 2 we have several possibilities with $c < 12$ given in the following theorem.

First let us prove the following lemma of multiple use.

**Lemma 6.2.2.** Let $F_1, F_2$ form a non-degenerate $U$-pair. Then, $F_1$ and $F_2$ have no common irreducible components.

**Proof.** By Proposition 2.5.2, a fiber $F_1$ is numerically 2-connected, i.e. if we write $F_1$ as a sum of two proper effective divisors $F_1 = D_1 + D_2$, then $D_1 \cdot D_2 \geq 2$. Now, if $D_1$ is the maximal effective divisor with $D_1 \leq F_1$ and $D_1 \leq F_2$ and if we let $F_1 = D_1 + D_2$ and $F_2 = D_1 + D'_2$ be decompositions into effective divisors, we have $D_2, D'_2 \geq 0$. Therefore $1 = F_1 \cdot F_2 = (D_1 + D_2) \cdot F_2 = D_2 \cdot D_1 + D_2 \cdot D'_2 \geq D_2 \cdot D_1$, hence $D_1 = 0$. 

**Theorem 6.2.3.** Let $B$ be a crystallographic root basis in $\text{Num}(S)$ formed by $k$ classes of $(-2)$-curves such that, for any $\alpha, \beta \in B$, $\alpha \cdot \beta \leq 2$. Assume that $k \leq 11$. Then the intersection graph of the curves is one of the following.

Every such crystallographic basis of $(-2)$-curves is realized.
Proof. We will show later in Section 8.10 that such crystallographic bases are realized in characteristic 2. Now let us show that the five diagrams are the only ones that can be realized under the assumption of the theorem.

Let $\Gamma$ be the Coxeter diagram of the reflection group $W_B$ of the crystallographic basis formed by the curves $R_i$. Since the fundamental polytope $P(B)$ is of finite volume and is not compact (since $W(E_{10})$ is not cocompact), $\Gamma$ contains a maximal rank parabolic subdiagram $P$ of rank 8. Let $P_1, \ldots, P_m$ be its connected components, and let $n_i$ be the number of vertices in $P_i$. Then

$$8 = \sum_{i=1}^{m} (n_i - 1) = -m + \sum_{i=1}^{m} n_i \leq k - 1 - m$$

gives $m \leq k - 9$.

Assume $k = 10$. Then $m = 1$. A connected parabolic diagram with 9 vertices must be of type $\tilde{A}_8$, $\tilde{E}_8$, or $\tilde{D}_8$. We must have an additional vertex $v_{10}$. By assumption, $\alpha \cdot \beta \leq 2$, hence the edge between $(v_i, v_{10})$ is simple or double. In the latter case, it defines a parabolic subdiagram that is not contained in a unique maximal rank parabolic subdiagram $P$. Thus the edge $(v_i, v_{10})$ is simple. In the case $\tilde{A}_8$, we get the following diagram

Here the red vertex corresponds to $v_{10}$. It is clear that the graph contains a parabolic subdiagram of type $\tilde{E}_7$. It is not of maximal rank and it is not contained in a maximal rank parabolic subdiagram. This contradicts the assumption $k = 10$.

A similar argument shows that in the case $\tilde{E}_8$ and $\tilde{D}_8$, we get the following diagrams:

Let us assume that $k = 11$. We will apply Vinberg’s Theorem 0.8.22. The assumption on $B$ is equivalent to that the fundamental polytope $P(B)$ has no divergent faces.

Suppose there is a non-connected maximal rank parabolic subdiagram $P$. Then there exists a unique vertex $v \notin P$. Suppose we have another non-connected maximal rank parabolic subdiagram $P'$. One of its connected components must contain $v$. Another must be contained in $P$, a contradiction. We conclude that any other maximal rank parabolic subdiagram must be connected. In particular, $v$ is connected to only one vertex of each connected component of $P$ unless it happens to be of type $\tilde{A}_1$. Thus there are two possibilities: either all maximal rank parabolic subdiagrams are connected, or there exists a unique non-connected maximal rank parabolic subdiagram.

Case 1: All maximal rank parabolic subdiagrams are connected.

In particular, the diagram has no multiple edges and any $v \notin P$ is connected to at most one other vertex.
**Case 1a:** Γ contains a maximal rank parabolic subdiagram $P$ of type $\tilde{A}_8$.

The diagrams are of the following pattern:

Let $r$ be the smallest number of vertices between the vertices connected to the red vertices. Then, if $r = 0, 1, 2, 3$, we find a parabolic subdiagram $\tilde{D}_5, \tilde{D}_6, \tilde{D}_7, \tilde{E}_7$, respectively. This obviously contradicts the assumption that all maximal rank parabolic subdiagrams are connected.

**Case 1b:** Γ contains a maximal rank parabolic subdiagram $P$ of type $\tilde{D}_8$.

Adding only one new vertex, we get the following possible diagrams:

(a) ![Diagram](a.png)  (b) ![Diagram](b.png)

(c) ![Diagram](c.png)  (d) ![Diagram](d.png)

In cases (a), (b), (c) we have a parabolic subdiagram of type $\tilde{D}_4, \tilde{D}_5, \tilde{D}_6$. For the same reason as in the previous case, it cannot be contained in a connected parabolic subdiagram of maximal rank.

In case (d), adding the second red vertex connected to any vertex except the last extreme vertex, we obtain a parabolic subdiagram of one of the types $\tilde{D}_5, \tilde{D}_4, \tilde{E}_6, \tilde{D}_6, \tilde{D}_7$ which is not contained in a maximal rank parabolic subdiagram. It remains to consider the case of the following possible diagram:

![Diagram](e.png)

In this case, the diagram contains a root lattice $E_6 + A_4$ of rank 10 which is impossible by the fact that $\text{NS}(S)$ has signature $(1, 9)$.

**Case 2:** There exists a maximal rank parabolic subdiagrams $P$ that consists of two connected components $P_1, P_2$.

In this case there is only one additional vertex $v$. Any parabolic subdiagram different from $P_1$ and $P_2$ must be a connected maximal rank parabolic subdiagram. In particular, there are no double edges unless it describes $P_1$ or $P_2$. Also, $v$ is connected to only one vertex of $P_1$ and $P_2$ unless $P_1$ or $P_2$ is defined by a double edge.

**Case 2a:** $P_1$ and $P_2$ are of types $\tilde{A}_s, \tilde{A}_t$ with $s + t = 8$.

If $s, t > 2$, the diagram contains a parabolic subdiagram of type $\tilde{D}_6$:
It cannot be a part of any connected maximal rank parabolic subdiagram.

If \( s = 2, t = 6 \), we have the following subdiagram

In this case we see a subdiagram of type \( \tilde{E}_6 \) not contained in any maximal rank parabolic subdiagram.

If \( s = 1 \), we have the following possible subdiagrams:

In both cases we have a parabolic subdiagram of type \( \tilde{E}_7 \) which is not a part of a maximal rank parabolic subdiagram.

**Case 2b**: \( P_1 \) is of type \( \tilde{D}_s \) and \( P_2 \) is of type \( \tilde{A}_t \) with \( s + t = 8 \).

If \( t > 2 \), one immediately see that there exists a parabolic subdiagram of type \( \tilde{D}_k \) which is not a part of a maximal rank parabolic subdiagram. Assume \( t = 1 \). Then \( s = 7 \). If the new vertex is joined to one of the two tri-valent vertices of the subdiagram of type \( \tilde{D}_7 \), we obtain a parabolic subdiagram of type \( \tilde{D}_4 \) which is not contained in a maximal rank parabolic subdiagram. If it is joined to one of the two extreme vertices then we get a parabolic subdiagram of type \( \tilde{E}_7 \) with the same property. In the remaining cases we get a parabolic subdiagram of type \( \tilde{E}_6 \) with the same property.

**Case 2c**: \( P_1 \) and \( P_2 \) are of type \( \tilde{D}_4 \). If the new vertex is connected to a 4-valent vertex of the subdiagrams of type \( \tilde{D}_4 \), we obtain a parabolic subdiagram of type \( \tilde{D}_6 \) or \( \tilde{D}_7 \) which is not contained in a maximal rank parabolic subdiagram. This gives the diagram

Since the fibers of type \( \tilde{D}_4 \) are double fibers, we see that \( K_S \neq 0 \) and \( p = 2 \). We know that the jacobian fibration with these types of reducible fibers must be a quasi-elliptic fibration.

**Case 2d**: \( P_1 \) is of type \( \tilde{E}_6 \) and \( P_2 \) is of type \( \tilde{A}_2 \).

If the new vertex joins a vertex of \( P_1 \) different from the extreme one, we find a parabolic subdiagram of type \( \tilde{E}_6 \) which is not contained in a maximal rank parabolic subdiagram. The only possibility is the following:
Let us index the vertices by the corresponding \( \tilde{\text{E}}_8 \) defined by the parabolic diagrams of type \( \tilde{\text{E}}_8 \) with reducible fibers of type \( M \mod R_f \). This is excluded.

**Case 2d:** \( P_1 \) is of type \( \tilde{\text{E}}_7 \) and \( P_2 \) is of type \( \tilde{A}_1 \).

The only possibility here is given in cases \((E_7^7)\) and \((E_7^2)\) of the Theorem.

**Definition 6.2.4.** An Enriques surface is called extra-special if \( \text{nd}(S) \leq 2 \).

It follows from Theorems 6.1.10 and 6.1.13 that there are no extra-special surfaces in characteristic different from two.

**Proposition 6.2.5.** Let \( S \) be an Enriques surface that admits a crystallographic basis of one of the types \( \tilde{\text{E}}_8, \tilde{\text{E}}_7, \tilde{D}_8 \) from the assertion of Theorem 6.2.3. Then \( S \) is extra-special. The surface of type \( \tilde{E}_7^2 \) has \( \text{nd}(S) = 3 \) and the surface of type \( \tilde{D}_4 + \tilde{D}_4 \) has \( \text{nd}(S) = 4 \).

**Proof.** Let \( S \) have a crystallographic basis of type \( \tilde{E}_8 \). It has only one genus one fibration, hence \( \text{nd}(S) = 1 \).

Suppose \( S \) contains a crystallographic basis of \((-2)\)-curves of type \( \tilde{E}_7^2 \). Then \( S \) has one genus one fibration \( |2F_1| \) with reducible fibers \( F_1 \) of type \( \tilde{E}_7 \) and \( \tilde{A}_1 \) and two genus one fibrations \( |2F_2|, |2F_3| \) with reducible fibers of type \( \tilde{E}_8 \). We may assume that \( F_1 \) is of type \( \tilde{E}_7 \). Let \( D_2, D_3 \) be the divisors defined by the parabolic diagrams of type \( \tilde{E}_8 \). We have to decide whether \( D_i \sim 2F_i \) or \( D_i \sim F_i \). Let us index the vertices by the corresponding \((-2)\)-curves as follows

\[
\begin{array}{cccccccccc}
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1 & & & & & & & & 11
\end{array}
\]

and denote the corresponding curves by \( R_i \). Consider the genus one pencil \( |D_1| \) with two fibers of types \( \tilde{E}_7 \) and \( \tilde{A}_1 \) (or \( \tilde{A}_1^* \)). The curve \( R_9 \) is its bisection and we see that the first fiber is double and the second is not. We see that \( R_{10} + R_{11} \) is a double fiber, hence \( v = \frac{1}{2}[R_{10} - R_{11}] \in \text{Num}(S) \). Let \( M \) be the sublattice of \( \text{Num}(S) \) generated by the curves \( R_i \). It is easy to see that \( M \) is isomorphic to the lattice \( E_7 \oplus A_1 \oplus U \), where \( A_1 \) is generated by \( v \). Its discriminant group is generated by \( r_1 = \frac{1}{2}v \mod M \) and \( r_2 = \frac{1}{2}[R_2 + R_4 + R_1] \mod M \) with \( r_1^2 = -\frac{1}{2}, r_2^2 = -\frac{3}{2}, r_1 \cdot r_2 = 0 \) computed in the discriminant group of \( M \). We see that \( r = r_1 + r_2 \) is a unique isotropic vector in the discriminant group. Adding it to \( M \) we obtain a unimodular lattice that must coincide with \( \text{Num}(S) \). Since \( D_2 \cdot r \) and \( D_3 \cdot r \) are even, we see that \( D_2 \in |2F_2|, D_3 \in |2F_3| \) are simple fibers. Now, it is easy to see that \( F_1 \cdot F_2 = F_1 \cdot F_3 = F_2 \cdot F_3 = 1 \). This shows that \( \text{nd}(S) = 3 \).
Suppose $S$ contains a crystallographic basis of $(-2)$-curves of type $\tilde{E}_7$
\[\begin{array}{cccccccccccc}
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\end{array}\] (6.2.3)

Then $S$ has two genus one pencils $|2F_1|$ and $|2F_2|$. The first one has two half-fibers of type $\tilde{E}_7$ and $A_1$ (or $A_1^*$) and the second one has a fiber (or half-fiber) $D_2$ of type $\tilde{E}_8$. We observe that $F_1 \cdot D_2 = 2$. Consider the sublattice $M$ of $\text{Num}(S)$ spanned by the classes of $(-2)$-curves represented by the vertices of the diagram. It is easy to see that it is isomorphic to the lattice $A_1 \oplus E_7 \oplus U$. It is a sublattice of index 2 in $\text{Num}(S)$. The vectors $v_1 = [R_1 + R_2 + R_4]$ and $v_2 = [R_1 + R_6 + R_8 + R_{10}]$ have even intersection with all curves in the diagram, hence, $\frac{1}{2}v_1$ and $\frac{1}{2}v_2$ belong to the dual lattice $M^\vee$. The residues of $r_1, r_2$ modulo $M$ generate the discriminant group of $M$. We have $r_2^2 = 0$, hence $r_2$ mod $M$ is an isotropic vector in $\text{discr}(M)$ and thus joining $\frac{1}{2}v_2$ to $M$ we obtain the unimodular lattice that must coincide with $\text{Num}(S)$. Since $[D_2]$ intersects any vector in $M$ evenly and $[D_2] \cdot v_2 = 0$, we obtain that $D_2$ intersects all classes in $\text{Num}(S)$ evenly, and hence $D_2$ is a double fiber. Thus $F_1 \cdot F_2 = 1$ and $\text{nd}(S) = 2$.

Suppose $S$ contains a crystallographic basis of $(-2)$-curves of type $\tilde{D}_8$.
\[\begin{array}{cccccccc}
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 10 \\
\end{array}\] (6.2.4)

Then $S$ has one genus one fibration $|2F_1|$ with reducible half-fiber $F_1$ of type $\tilde{D}_8$ and two genus one fibrations $|2F_2|, |2F_3|$ with reducible fibers of type $\tilde{E}_8$. Let $D_2, D_3$ be the divisor defined by the parabolic diagrams of type $\tilde{E}_8$. We have to decide whether $D_i \sim 2F_i$ or $D_i \sim F_i$.

Consider the sublattice $M$ of $\text{Num}(S)$ generated by the $(-2)$-curves represented by the vertices of the diagram. It is easy to see that it is isomorphic to $D_8 \oplus U$, a sublattice of index 2 in $E_{10}$. The curve $R_1 + R_5 + R_7 + R_{10}$ has even intersection with all curves in the diagram. As in the previous cases, we show that the discriminant group of $M$ is generated by vectors $r_1 = \frac{1}{2}[R_1 + R_5 + R_7 + R_{10}]$ mod $M$ and $r_2 = \frac{1}{2}[R_1 + R_5 + R_7 + R_{10}]$ mod $M$ with $r_1 + r_2 = \frac{1}{2}[R_6 + R_{10}]$ mod $M$. We have $r_1^2 \equiv r_2^2 \equiv 0$, $(r_1 + r_2)^2 \equiv 1$. Adding one of the vectors $r_1, r_2$ to $M$ generates a sublattice isomorphic to $E_{10}$. Since both of $r_1, r_2$ cannot belong to $\text{Num}(S)$, only one of them belongs to $\text{Num}(S)$. This shows that one of the fibers of type $\tilde{E}_8$ is non-double, let it be $D_2 \sim 2F_2$ and let $D_3 = F_3$ be a half-fiber. Now, we find that $F_1 \cdot F_2 = 1, F_1 \cdot F_3 = 2, F_2 \cdot F_3 = 1$. This shows that $([F_1], [F_2])$ is the only non-degenerate isotropic 2-sequences. This gives $\text{nd}(S) = 2$.

Suppose $S$ is of type $\tilde{D}_4 + \tilde{D}_4$, we index the vertices as follows
\[\begin{array}{cccccccc}
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
0 & 1 & 10 \\
\end{array}\] (6.2.5)

and denote by $R_i$ the corresponding $(-2)$-curves. We have one maximal parabolic subdiagram of type $\tilde{D}_4 + \tilde{D}_4$ and nine maximal parabolic subdiagrams of type $\tilde{D}_8$. Using the classification of extremal jacobian genus one fibrations on rational surfaces we find that the genus one fibration corresponding to the diagram of type $\tilde{D}_4 + \tilde{D}_4$ is quasi-elliptic. The curve $R_5$ is the curve of cusps
and we have two half-fibers $F_1 = 2R_0 + R_1 + R_2 + R_3 + R_4$ and $F_2 = 2R_{10} + R_6 + R_7 + R_8 + R_9$ of type $\tilde{D}_4$.

Let $M \cong U \oplus D_4 \oplus D_4$ be the sublattice $\text{Num}(S)$ of index 4 spanned by the numerical divisor classes of the curves $R_i$. The numerical classes $r_1 = \frac{1}{2}[R_2 + R_3 + R_8 + R_9]$ and $r_2 = \frac{1}{2}[R_1 + R_2 + R_7 + R_9]$ generate a maximal isotropic subspace in the discriminant group of $M$, and hence, as above, we obtain that $\text{Num}(S)$ is generated by $M$ and the classes $r_1, r_2$. Let $F_{a,b}, a = 1, 2, 3, b = 7, 8, 9$ be the parabolic subdiagram spanned by the curves $R_i$ except $R_a$ and $R_b$. We check that $F_{1,7}, F_{3,8}$ and $F_{2,9}$ intersect both $r_1$ and $r_2$ with multiplicity 2. All other $F_{a,b}$ intersect one of the $r_i$ with multiplicity 1. This implies that the $F_{1,7}, F_{3,8}$ and $F_{2,9}$ are simple fibers of genus one fibrations, all other $F_{a,b}$ are half-fibers. We also check that $F_{1,7}, F_{3,8}$ and $F_{2,9}$ intersect each other with multiplicity 4 and intersect $F_1$ with multiplicity 2, hence the half-fibers of these fibrations together with $F_1$ form a non-degenerate canonical 4-sequence. It is a maximal such sequence, hence $\text{nd}(S) = 4$.

From now, referring to an extra-special surface of type $\tilde{E}_7$ we say that it is an extra-special surface of type $\tilde{E}_7$.

The following theorem was claimed in [134, Theorem 3.5.2], however its proof was based on lengthy case-by-case considerations and there is no guarantee that it is correct.

**Theorem 6.2.6.** An extra-special surface must be one of the three types $\tilde{E}_8, \tilde{E}_7, \tilde{D}_8$.

**Proposition 6.2.7.** Let $S$ be an extra-special Enriques surface.

- An extra-special surface of type $\tilde{E}_8$ has only one genus one fibration. It is quasi-elliptic and has a half-fiber of type $\tilde{E}_8$.

- An extra-special surface of type $\tilde{E}_7$ has two genus one fibrations $|2F_1|$ and $|2F_2|$. Both of them are quasi-elliptic. One of them has two half-fibers of types $\tilde{E}_7$ and $\tilde{A}_1^*$, hence $S$ is a classical Enriques surface. Another one has a simple fiber of type $\tilde{E}_8$. We have $F_1 \cdot F_2 = 1$.

- An extra-special surface of type $\tilde{D}_8$ has three genus one fibrations $|2F_1|, |2F_2|, |2F_3|$. The first fibration has a half-fiber of type $\tilde{D}_8$. The fibration $|2F_2|$ has a simple fiber of type $\tilde{E}_8$ and the fibration $|2F_3|$ has a half-fiber of type $\tilde{E}_8$. We have $F_1 \cdot F_2 = F_2 \cdot F_3 = 1, F_1 \cdot F_3 = 2$.

**Proof.** We have to prove only the statements about quasi-elliptic fibrations. The rest follows from the proof of the previous Proposition. Suppose $S$ is an extra-special surface of type $\tilde{E}_8$. Let us see that its unique genus one fibration $f : S \to \mathbb{P}^1$ must be quasi-elliptic. Suppose it is elliptic. We know that the $(-2)$-curves $R_i$ generate $\text{Num}(S)$ and only one of them is not contained in fibers. It is a special bisection $R$. It follows that any bisection of the fibration must be numerically equivalent to $R + A$, where $A$ is contained in fibers. Thus the generic fiber has a unique degree two point. However, by Riemann-Roch, a divisor of degree 2 on an elliptic curve moves in a pencil. Thus $f$ is a quasi-elliptic fibration and $R$ is its curve of cusps. Recall from Table 4.9 that the jacobian fibration of $f$ has a unique singular fiber.
Suppose $S$ is an extra-special surface of type $\tilde{E}_7$. It has two genus one fibrations of type $\tilde{E}_7 + \tilde{A}_1$ and of type $\tilde{E}_8$. The first fibration has two reducible half-fibers, one of them is of additive type $\tilde{A}_1$. This implies that the second half-fiber is of type $\tilde{A}_1$. The classification of extremal genus one fibrations on a rational surface in Tables ?? and 4.9 shows that the extremal elliptic fibration with a reducible fiber of type $\tilde{E}_7$ has the second reducible fiber of type $\tilde{A}_1$. Thus our fibration must be quasi-elliptic. Also we must have $K_S \neq 0$. It follows that the other half-fiber of the other genus one fibration must be of additive type. The classification of extremal rational elliptic surfaces in characteristic 2 shows that the second half-fiber must be smooth and its absolute invariant is 0. Thus it is a supersingular curve, but then the fiber must be wild. Since $K_S \neq 0$, we get a contradiction.

Suppose $S$ is an extra-special surface of type $\tilde{D}_8$. It has one genus one pencil $|2F_1|$ with a half-fiber $F_1$ of type $\tilde{D}_8$ and two genus one pencils $|2F_2|, |2F_3|$ of type $\tilde{E}_8$. We have seen in the proof of Proposition 6.2.5 that only one of the latter fibrations has a half-fiber of type $\tilde{E}_8$, so we may assume that $F_2$ is of this type and $F_3$ is irreducible. We also have $F_1 \cdot F_2 = 1, F_1 \cdot F_3 = 2, F_2 \cdot F_3 = 1$.

Remark 6.2.8. It follows from Remark [363, Remark 12.4] that in the third case of Proposition 6.2.7, if $|2F_1|$ is quasi-elliptic fibration, then the remaining two fibrations are elliptic.

We have also a partial converse statement.

**Proposition 6.2.9.** Let $S$ be a classical or $\alpha_2$-Enriques surface in characteristic 2. Then

- $S$ is extra-special of type $\tilde{E}_8$ if and only if it admits a quasi-elliptic fibration with a half-fiber of type $\tilde{E}_8$.

- $S$ is extra-special of type $\tilde{E}_7$ if and only if it admits a quasi-elliptic fibration with a simple fiber of type $\tilde{E}_8$ and a half-fiber of type $\tilde{E}_7$.

**Proof.** We have proved already that the properties are necessary. Suppose $S$ has a quasi-elliptic fibration with a half-fiber of type $\tilde{E}_8$. Then the curve of cusps $R$ is its special bisection. Together with irreducible component of the reducible half-fiber they form a crystallographic basis of type $\tilde{E}_{10}$. Thus the surface is extra-special of type $\tilde{E}_8$.

Suppose $S$ has a quasi-elliptic fibration with simple fiber $F$ of type $\tilde{E}_8$. Let $C$ be the curve of cusps. Then it intersects $F$ at its irreducible component of multiplicity 2. Thus we have two possible diagrams of $(-2)$-curves

\[
\begin{array}{c}
\bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\
C \\
\bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet
\end{array}
\]

The first diagram shows that $S$ admits fibration with a half-fiber of type $\tilde{E}_7$. The second diagram shows that $S$ does not contain a fiber of type $\tilde{E}_7$. In fact, it describes an extra-special surface of type $\tilde{D}_8$.

\[\square\]
Let us give a construction of a classical extra-special Enriques surface.

**Example 6.2.10.** Assume $p = 2$. Let $D = D'_1$. Its minimal resolution $X$ is obtained from $\mathbb{P}^2$ by blowing up 5 points $p_5 \succ p_4$ and $p_3 \succ p_2 \succ p_1$ such that there exists a line $\ell_1$ through the points $p_1, p_2, p_3$ and a line $\ell_2$ through the points $p_1, p_4, p_5$. Choose projective coordinates $t_0, t_1, t_2$ to assume that $p_2 = [0, 0, 1], p_1 = [1, 0, 0]$ and $\ell_1 = V(t_2), \ell_2 = V(t_1)$. We know from (3.3.15) that the linear system $[6e_0 - 2(e_1 + \cdots + e_5)]$ is represented by plane sextics

$$\Phi = t_0^4t_2^2 + \alpha t_0^3t_1t_2^2 + t_0^2t_1t_2A(t_1, t_2) + t_0t_1^2t_2B(t_1, t_2) + t_1^2C(t_1, t_2) = 0$$

In the affine coordinates $x = t_1/t_0, y = t_2/t_0$ at $p_1$, the equation becomes

$$G = y^2 + ay^2x + yxA(x, y) + yx^2B(x, y) + x^2C(x, y) = 0,$$

Consider the inseparable $\mu_2$-cover of $D$ given by affine equation

$$z^2 + xy(y^2 + ay^2x + yxA(x, y) + yx^2B(x, y) + x^2C(x, y)) = 0. \quad (6.2.6)$$

First, by change of the variables $z \mapsto z' + P(x, y)$, we may assume that $A(x, y) = bxy$ and $C = C'(x^2, y^2)$. Let us resolve the singular point $(x, y, z) = (0, 0, 0)$. After the first blow-up given by $z = z', y = ux$ and the normalization, the equation at the point $p_2 \succ p_1$ becomes

$$z'^2 + u(u^2 + axv^2 + bx^2v^2 + x^3uB(1, u) + x^4C'(1, u^2)) = 0.$$ 

After the second blow-up given by $z' = z''$, $u = vx$ and the normalization, the equation at $p_2$ becomes

$$z''^2 + xv(u^2 + axv^2 + bx^2v^2 + x^2vB_2(1, xv) + x^2C'(1, x^2v^2)) = 0.$$ 

After the third blow-up given by $z'' = wx, v = tx$ and the normalization, the equation at $p_3$ becomes

$$w^2 + t(t^2 + axt^2 + bx^2t^2 + xB(1, x^2t) + C'(1, x^4t^2)) = 0. \quad (6.2.7)$$

We are trying to see that the surface has a singular point of type $E_8^{(0)}$ at the exceptional curve $E$ over $p_3$. Plugging in $x = 0$, we see that the singular point has the coordinate $t = \alpha \neq 0$, where $C'(1, x^4t^2) = \alpha^2 + \beta x^4t^2 + \gamma x^8t^4$. After we replace $t$ with $t' = t + \alpha$, we see that we have to kill the coefficients at monomials of degree $\leq 2$. So, we may rewrite equation (6.2.7) in the form

$$w^2 + (t + \alpha)(t^2 + dx^3(t + \alpha)^2 + ex^5(t + \alpha)^3 + \beta x^4(t + \alpha)^2 + \gamma x^8(t + \alpha)^4)) = 0. \quad (6.2.8)$$

Replacing $w$ with $w + \sqrt{\alpha}(t + \sqrt{\beta}x^2(t + \alpha) + \sqrt{\gamma}x^4(t + \alpha)^2)$, we rewrite the equation in the form

$$w^2 + t(t^2 + dx^3(t + \alpha)^2 + ex^5(t + \alpha)^3 + \beta x^4(t + \alpha)^2 + \gamma x^8(t + \alpha)^4))$$

$$+ \alpha(dx^3(t + \alpha)^2 + ex^5(t + \alpha)^3) = 0. \quad (6.2.9)$$

This forces us to make $d = 0$, so we can write the equation in the form

$$w^2 + t^3A + x^5B = 0,$$

where $A = 1 + A_0(x, t)$ and $B = e\alpha^4 + B_0(x, t)$, where $A_0(0, 0) = B_0(0, 0) = 0$. It is easy to see that the coefficients $e, \alpha$ are not zeros since otherwise the curve $W'$ degenerates so that the cover $S \to \mathbb{P}^2$ is not normal. Thus we obtain a singularity of type $E_8^{(0)}$.

Returning to the original equation of the surface, we can rewrite it in the form

$$t_3^2 + t_1t_2(t_0^2t_2^3 + et_0^2t_1^2 + \alpha t_0^2 + \beta t_1^2t_2^2 + \gamma t_1^2t_2) = 0. \quad (6.2.10)$$

Replacing $x_0$ with $x_0 + \sqrt{\beta}x_1$ and $w$ with $w + \sqrt{\alpha}x_2$, we may assume that $\beta = 0$. After scaling
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We define the equation
\[ t_3^2 + t_1t_2(t_0^6t_2^2 + t_0t_1^2t_2^3 + t_0^6 + \gamma t_1^2t_2) = 0, \quad \gamma \neq 0. \tag{6.2.11} \]

This shows that our family of extra-special classical Enriques surfaces of type $\tilde{E}_8$ depends on one parameter.

**Example 6.2.11.** In this example we construct an extra-special classical Enriques surface of type $\tilde{E}_7$. We use the inseparable bielliptic map $\phi : S \to D_1$ defined by the unique non-degenerate $U$-pair of half-fibres $(F_1, F_2)$. In our usual coordinates $(t_0, t_1, t_2)$ used to write down the equation (3.3.9) of the branch curve, we take $W$ to be the union of a line $\ell = V(x_2 + x_1)$ and a plane quintic $W_0 \in \{5e_0 - e_1 - 2(e_2 + \cdots + e_9)\}$ that intersects $\ell$ at the point $[0, 1, 1]$ on the line $x_0 = 0$ with multiplicity 4. The local equation of $W$ at this point is $x(x + y^4) = 0$.

We take for the sextic part $W'$ of the branch curve the union of a line $\ell$ passing through the point $p_1 = [1, 0, 0]$ and a point $q = [0, 1, 1]$ on the line $x_0 = 0$ a curve of degree 5 which is tangent to $\ell$ at $q$ with multiplicity 2 and is tangent to $\ell$ at $p_1$ with multiplicity 3. According to (3.3.9), the equation of $W'$ must be of the form
\[
(t_1 + t_2)(t_0^4A_1 + a_1t_0^3t_1t_2 + t_0^2t_1t_2A_3 + t_0t_1^2t_2^2A_4 + t_1^2t_2A_5) = 0,
\]
where $A_1, A_3, A_4, A_5$ are linear forms in variables $t_1, t_2$. Plugging in $t_1 = t_2 = 1$, we see that, to satisfy our conditions, we must have $A_1 = t_1 + t_2, a_1 = 0, A_4 = 0$, and $A_5 = a_4(t_1 + t_2), a_4 \neq 0$.

By scaling $t_1$ and $t_2$, we may assume that $a_4 = 1$. By scaling $t_0$, we may assume that $a_1 = 1$. Thus the equation becomes
\[
G = (t_1 + t_2)(t_0^4A_1(t_1 + t_2) + t_0^3t_1t_2(c_1t_1 + c_2t_2) + t_1^2t_2(t_1 + t_2)) = 0.
\]

The equation depends on two parameters.

We check that the double plane $z^2 + t_1t_2G(t_0, t_1, t_2) = 0$ has a simple singular point of type $A_2$ over the point $q$ and a simple point of type $E_7$ over $p_1$. The preimage of the line $\ell$ gives us a simple fiber of type $\tilde{E}_8^{(0)}$, the preimage of the exceptional curve over $p_1$ gives a half-fiber of type $\tilde{E}_7$ and the pre-image of the line $x_0 = 0$ gives a half-fiber of type $\tilde{A}_1^*$. In Section 8.9 we will give another construction of extra-special surfaces.

In the remaining part of this section we will follow the work of T. Ekedahl and N. Shepherd-Barron [208] and provide more details for their proof of Theorem 1.4.10. Following the authors, we introduce the following definition.

**Definition 6.2.12.** A classical Enriques surface $S$ in characteristic 2 is called exceptional if it admits a non-zero regular vector field.

As we shall see that not every extra-special surface is exceptional and vice versa not every exceptional surface is extra-special.

Assume that the K3-cover $X$ of $S$ is not normal and let $A$ be the conductrix of $S$. Recall from section 1.3 that $2A$ is equal to the divisorial part of a 1-form $\omega$ spanning $H^0(S, \Omega^1_{S/k})$. We have proved that $A^2 = -2, h^0(A) = 1$, each irreducible component of $A$ is a $(-2)$-curve, and $A$ is numerically connected. Also, we have an exact sequence
\[
0 \to \mathcal{O}_S \to \tilde{\pi}_*\mathcal{O}_Y \to \omega_S(A) \to 0,
\]
where $\tilde{\pi} : Y \to S$ is the composition of the K3-cover $\pi : X \to S$ and the normalization map $\sigma : Y \to X$. We also have the exact sequence

$$0 \to \mathcal{O}_S(2A) \to \Omega^1_{S/k} \to \mathcal{J}_Z(K_S - 2A) \to 0,$$

where $Z$ is the 0-dimensional part of the scheme $\langle \omega \rangle$ of zeros of $\omega$ with $h^0(\mathcal{O}_Z) = 4$. For any point $x \in S$, we denote the length of $\mathcal{O}_Z$ at $x$ by $\langle \omega \rangle_x$.

Suppose $R$ is an irreducible component of multiplicity $m$ of a non-multiple fiber $F$ of $f$. Let $t$ be a local parameter at $f(F)$. If $m$ is odd, then we can write $f^*(t) = \epsilon u^m$, where $u = 0$ is a local equation of $R$ at its general (not generic) point $x$ and $\epsilon$ is a unit at $x$. Passing to formal completion, and applying Hensel’s Lemma, we can replace $u$ with $\epsilon^{1/m} u$ and obtain $f^*(t) = u^m$. This gives $\omega = df^*(t) = mu^{m-1} du$, hence $R$ enters in $2A$ with multiplicity $m - 1$. On the other hand, if $m$ is even, we obtain $\omega = u^m \epsilon du$, so that $R$ enters in $2A$ with multiplicity $\geq m$. If $f$ is quasi-elliptic, then at a general point of the curve of cusps $\mathcal{C}$, we have $f^*(t) = y^2 + x^3$, so that $\omega = x^2 dx$ vanishes on $C$ with multiplicity 2. Thus $\mathcal{C}$ enters in $A$ with multiplicity 1.

Suppose $f$ is smooth at a closed point $x$. We can choose a local parameter $u$ at $x$ such that the equation of the fiber $F$ through $x$ is given by local equation $u = 0$. Then $f^*(t) = \epsilon u$, where $u$ is a unit in $\mathcal{O}_{S,x}$. Replacing $u$ with $\epsilon u$, we may assume that $\epsilon = 1$. Thus $\omega = du$ and hence does not vanish at $x$. Suppose $f$ is an elliptic fibration. Then any curve $C$ not contained in a fiber intersects some fiber at its smooth point. This implies that $\omega$ does not vanish at a general point of $C$.

To summarize, we obtain that $A$ is a combination of irreducible components of fibers if $f$ is elliptic and $2A = 2\mathcal{C} + 2A'$, where $A'$ is a combination of irreducible components of fibers, if $f$ is quasi-elliptic.

Suppose $x$ is an isolated singular point of a non-multiple fiber $F$ of an elliptic fibration. We have

$$\langle \omega \rangle_x = \dim_k \text{Ext}^1(\Omega^1_{S/k,x}, \mathcal{O}_{S,x})$$

(see [150]). The number $\langle \omega \rangle_x$ is called the **Milnor number** of $f$ at $x$.

If $x$ is an ordinary double point, then, in some local coordinates $u, v$ at $x$, we can write $f^*(t) = uv$. This implies that $\langle \omega \rangle_x = 1$. Thus, if the fiber is of type $I_n$, then

$$\sum_{x \in F} \langle \omega \rangle_x = v_x(\Delta), \quad (6.2.12)$$

where $\Delta$ is the discriminant of the associated jacobian fibration.

If $x$ is an ordinary cusp (resp. a triple point) of the fiber, then $f^*(t) = \epsilon (u^2 + v^2)$ (resp. $\epsilon (u^2 v + v^2 u)$), and $\langle \omega \rangle_x$ depends on the unit $\epsilon$. It follows from [149], Théorème 2.6, that (6.2.12) still holds. This time $\langle \omega \rangle_x = \epsilon(F) + \alpha$, where $\alpha$ is the invariant of wild ramification. In our case $\langle \omega \rangle_x = 4$ in each case, hence $Z = \{x\}$ and the singularity of the K3-cover over $x$ is a rational double point of type $D_4^{(0)}$. If $F$ has a cusp at $x$, then the invariant of the wild ramification is equal to 2, otherwise it is equal to zero.

Let us derive some immediate corollaries of our previous discussion.

**Proposition 6.2.13.** Let $p = 2$ and let $S$ be a classical or $\alpha_2$-surface. Let $\omega$ be a generator of $H^0(S, \Omega^1_{S/k})$. Let $f : S \to \mathbb{P}^1$ be an elliptic fibration with a non-multiple fiber of type $D_n$ or $E_n$. Then the divisorial part $D$ of the scheme of zeros of $\omega$ is equal to $2A$, where $A$ is defined by the
following weighted graph with weights indicating the multiplicities of the irreducible components.

\[
\begin{array}{ccc}
\tilde{D}_n & \tilde{E}_6 & \tilde{E}_7 \\
1 & 1 & 1 \\
\tilde{D}_4 & \tilde{D}_6 & \tilde{D}_8 \\
1 & 1 & 1 \\
\tilde{E}_7 & \tilde{E}_8 & \\
1 & 1 & 1 \\
\end{array}
\]

(in case \(\tilde{D}_n\) the components are all multiple components of the fiber). If \(f : S \to \mathbb{P}^1\) is a quasi-elliptic fibration then \(D\) is described by the following diagrams, where the star indicates the curve of cusps.

\[
\begin{array}{ccc}
\tilde{D}_4 & \tilde{D}_6 & \tilde{D}_8 \\
1 & 1 & 1 \\
\tilde{E}_7 & \tilde{E}_8 & \\
1 & 1 & 1 \\
\end{array}
\]

**Proof.** If \(R\) is a component of a non-multiple fiber of a genus one fibration of even multiplicity \(m\) with local equation \(u = 0\) at its general point, then \(d\pi^*(t) = u^m dt\). It is easy to see that \(dt\) can vanish on \(R\) only with even multiplicity. Suppose \(f\) is an elliptic fibration, then \(A\) is equal to \(A' + B\), where \(A'\) is as in the assertion of the proposition and \(B\) is a combination of components \(R_i\) entering in the fiber containing \(A\). One easily computes \(A^2 = (A' + B)^2\) and obtain that \(A^2 < -2\) if \(B \neq 0\) contradicting (1.3.7). If \(f\) is quasi-elliptic, then we know that the curve of cusps \(C\) enters in \(A\) with multiplicity 1. We also know which component of \(F\) the curve \(C\) intersects (see Remark 4.9.9). The rest of the argument is the same. \(\square\)

**Corollary 6.2.14.** Assume \(p = 2\) and let \(\pi : X \to S\) be the K3-cover of an Enriques surface. Suppose \(X\) is birationally isomorphic to a K3 surface. Then \(S\) does not admit a quasi-elliptic fibration.

**Proof.** We know that a \(\mu_2\)-surface does not admit a quasi-elliptic fibration. In other cases the canonical cover is inseparable. Moreover we know that \(H^0(S, \Omega^1_{S/k}) \neq \{0\}\) and, by Proposition 0.2.17, \(X\) is singular over zeros of a nonzero regular 1-form \(\omega\). However, we know that the curve of cusps \(C\) enters in the scheme of zeros of \(\omega\). Thus \(X\) is not normal, and hence not birationally isomorphic to a K3 surface. This contradicts the assumption of the Corollary. \(\square\)

One can analyze the conductrix \(A\) for surfaces with \(H^0(S, \Theta_{S/k}) \neq 0\) not assuming that it is supported at non-multiple fibers. The classification of possible configurations of \(A\) is more delicate and we refer for this to [208]. As we noted before in the proof of Theorem 1.4.10, this analysis allows the authors to classify exceptional Enriques surfaces.

Recall from the proof of Theorem 1.4.10 that \(H^0(S, \Theta_{S/k}) \neq 0\) if and only if \(h^0(2A + K_S) \neq \{0\}\). Suppose \(S\) is exceptional and the dual graph of the support of \(A\) is a part of a Dynkin diagram of finite type. Let \(D \in |2A + K_S|\). We have \(4A \sim 2D\), hence \(4A\) and \(2D\) span a pencil. Since its moving part has non-negative self-intersection, we get a contradiction. Thus \(h^0(2A + K_S) = 0\) and \(S\) cannot be exceptional.
Suppose \( S \) is exceptional. Then the dual graph of the support of \( A \) is not a part of a Dynkin diagram of finite type.

Suppose \( f \) is an elliptic fibration. By inspection of the graphs from Proposition 6.2.13, we see that the support of \( A \) is equal to the support of a half-fiber. The further classification of possible conductrices shows that the only possibilities for the dual graphs of \( A \) are the following.

\[
\begin{align*}
\tilde{E}_6 & : 1 & 1 & 2 & 1 & 1 & 1 & 1 \\
\tilde{E}_7 & : 1 & 1 \\
\tilde{E}_8 & : 2 & 3 & 5 & 4 & 4 & 3 & 2 & 1
\end{align*}
\]

Assume \( K_S \neq 0 \) and let \( F \) be a half-fiber containing the support of \( A \) and let \( F' \) be another half-fiber. In all three cases \( A = F - B \), where \( 0 < 2B \leq F \). Thus \( 2A + K_S \sim 2F - 2B + K_S \sim F' + (F - 2B) > 0 \). Thus in all these cases we have a nonzero regular vector field.

Assume now that \( f \) is a quasi-elliptic fibration. Let \( A = A' + C \), where \( C \) is the curve of cusps. Suppose the support of \( A' \) is contained in a non-multiple fiber \( F \). By inspection of the list in Proposition 6.2.13, we find that condition (\( \ast \)) is satisfied in the last two cases. In the case \( \tilde{E}_7 \), we observe that \( A \) is contained in a half-fiber \( F \) of type \( \tilde{E}_6 \) in some genus one fibration on \( S \). So, this case has been already considered and we have concluded that in this case \( 2A + K_S \) is effective.

In the case \( \tilde{E}_8 \), let \( A'' \) is obtained from \( A \) by deleting the component \( R \) which is extreme on the right side of the diagram. Then \( A'' + C \) is a part of half-fiber \( G \) of type \( \tilde{E}_7 \) of some genus one fibration with \( R \) being a special 2-section. Moreover, \( A'' + C = G - B \), where \( 0 < 2B < G \). Thus \( 2A + K_S \sim 2R + 2A'' + 2C \sim 2R + 2G - 2B + K_S \sim 2R + G' + (G - 2B) > 0 \), where \( G' \) is another half-fiber of \( [2G] \).

Next we assume that \( f \) is a quasi-elliptic fibration, \( A \) is contained in a half-fiber \( F \) of \( f \) and condition (\( \ast \)) is satisfied. It follows from [208] that there are two possible cases for \( A \):

\[
\begin{align*}
\tilde{E}_7 & : 1 & 2 & 2 & 3 & 2 & 2 & 1 & 1 \\
\tilde{E}_8 & : 2 & 3 & 5 & 4 & 4 & 3 & 3 & 2 & 1
\end{align*}
\]

In both cases \( 2A \) contains \( F \), hence \( 2A + K_S \) is linearly equivalent to an effective divisor containing the second half-fiber. Note that in the second case \( S \) is an \( \tilde{E}_8 \)-special Enriques surface.

The above discussion provides more details for the proof of Theorem 1.4.10. Let us summarize what we have found.

**Theorem 6.2.15.** Let \( S \) be an exceptional Enriques surface and \( \omega \) is a basis of \( H^0(S, \Theta_{S/k}) \). Then its divisorial part \( D \) is equal to \( 2A \), where \( A \) is a curve with dual weighted graphs from (6.2.13), (6.2.14) and the last diagram from Proposition 6.2.13. The surface contains a genus one fibration...
with a half-fiber of type $\tilde{E}_6$, or $\tilde{E}_7$, or $\tilde{E}_8$. If the fibration is quasi-elliptic with a half-fiber of type $\tilde{E}_7$
or $\tilde{E}_8$, then there exists a special 2-section. In particular, in the last case the surface is extra-special of type $\tilde{E}_8$.

Thus we infer that $S$ must admit a quasi-elliptic fibration such that the conductrix $A$ is supported on a half-fiber.

Writing $K_S$ as the difference of two half-fibers $F_1 - F_2$, we obtain that $2A + K_S \sim F_2 - B$ is not effective. If $f$ is quasi-elliptic, we get $2A = F - B + \mathcal{C}$. It follows from the above Proposition that $\mathcal{C} \cdot A = 1$ and $\mathcal{C} \cdot B = 0$. Then $2A + K_S \sim F_2 - B + \mathcal{C}$, and intersecting with $\mathcal{C}$, we obtain $(2A + K_S) \cdot \mathcal{C} = \mathcal{C}^2 + (F_2 - B) \cdot \mathcal{C} = -1$. This implies that $\mathcal{C}$ must be a component of an effective divisor in $|2A + K_S|$. Thus $F_2 - B$ must be effective, and we get again a contradiction. It follows that the support of conductrix $A$ is contained in the union of a half-fiber and the curve of cusps, if $f$ is quasi-elliptic.

Since the divisorial part $D$ of the scheme of zeros of $\omega$ does not depend on a choice of an elliptic fibration, we obtain that $D$ enters in a fiber of any other elliptic fibration on $S$. Since we know that two non-reduced non-multiple fibers must be of types $\tilde{D}_4$ and they occur only for quasi-elliptic fibrations, we see that the type of a non-reduced non-multiple fiber of an elliptic fibration on a classical Enriques surface does not depend on the fibration.

**Example 6.2.16.** Let $S$ be as above, and let $f$ be an elliptic fibration on $S$ with a non-multiple fiber $F$ of type $\tilde{E}_8$. Note that it is possible to find a classical (resp. supersingular) surface with a non-multiple fiber of type $\tilde{E}_8$.

It follows from Theorem 6.1.10 from Section 4.1 that there always exists another genus one pencil $f' : S \rightarrow \mathbb{P}^1$ on $S$ such that its general fiber intersects $F$ with multiplicity 4. Since $f$ is an elliptic fibration, the bielliptic map $S \rightarrow D$ onto a quartic del Pezzo surface must be separable. This implies that $f'$ is also an elliptic fibration. Since the conductrix does not depend on the choice of a fibration, we see that $f'$ also contains a non-multiple fiber $F'$ of type $\tilde{E}_8$. The fibers $F$ and $F'$ share all components except their reduced components $R$ and $R'$. It is easy to see that we must have the following diagram of the components

![Diagram](https://via.placeholder.com/150)

The divisor $R + R'$ (or $2R + 2R'$) defines a third genus 1 fibration $f''$ on $S$. Its other reducible fiber or a half-fiber must be of type $\tilde{E}_7$. It consists of irreducible components of $A$ and a component $R_0$ depicted on the following diagram.
Note that $C \cdot R_0 = 0$ because, otherwise, $C \cdot R_0 = 1$ and hence there exists a genus 1 fibration with a double fiber of type $\tilde{A}_7$ which is a contradiction.

Since the curve of cusps $C$ intersects $R + R'$ with multiplicity 2, the fiber $R + R'$ is not multiple. The eleven components form a crystallographic root basis in $\text{Num}(S)$, so there are no more $(-2)$-curves on $S$. The surface is $\tilde{E}_7^1$-special.

It remains to see the existence of an exceptional Enriques surface.

In the previous example, we constructed an extra-special $\tilde{E}_7^1$-surface starting from a quasi-elliptic fibration with a non-multiple fiber of type $\tilde{E}_8$. As we have shown this implies that the surface is exceptional.

**Example 6.2.17.** Take a quasi-elliptic surface with a non-multiple fiber $F$ of type $\tilde{E}_7$ and a reducible non-wild half-fiber of type $\tilde{A}_1^*$. It obviously exists as a torsor of a rational quasi-elliptic surface. This surface admits another genus one fibration $f'$ such that the conductrix $A$ is supported in a half-fiber $F'$ of type $\tilde{E}_6$. This fibration must be elliptic. The conductrix $A$ is supported in $F'$, so we get an exceptional Enriques surface. We obtain the following diagram.

![Diagram](image)

Here $C$ is the curve of cusps of the quasi-elliptic pencil. Since it has a reducible half-fiber, we obtain the following diagram

![Diagram](image)

Now we see that our diagram contains two more parabolic subdiagrams of type $\tilde{E}_7$ and each must contain another reducible half-fiber of type $\tilde{A}_1^*$. We also see that the second fibration with fiber of type $\tilde{E}_6$ has another reducible fiber, a component of the half-fiber of the first quasi-elliptic fibration. The only possible configuration is the following one.

Note that the Mordell-Weil group of the jacobian fibration of the elliptic fibration of type $\tilde{E}_6 + A_2^*$ is of order 3 and it acts on this diagram by a symmetry of order 3. The vertices of the diagram define a crystallographic root basis in $\text{Num}(S)$ of cardinality 13. We will see in Section ?? that the automorphism group of the surface $S$ is isomorphic to the symmetric group $\mathfrak{S}_3$. 

Example 6.2.18. We take an Enriques surface, denoted by $S$, with a quasi-elliptic fibration with a reducible half-fiber of type $\tilde{E}_7$ or $\tilde{E}_8$ defining the conductrix from diagram (6.2.14). In the former case the quasi-elliptic fibration must have an additional reducible fiber of type $\tilde{A}^*_1$. If this fibre were not multiple, we get the diagram (6.2.15). This defines an extra-special Enriques surface of type $\tilde{E}^*_7$. If the fiber is multiple, we get the following digram:

![Diagram](image)

(6.2.16)

This shows that $S$ is an extra $\tilde{E}^*_7$-special Enriques surface.

In the latter case, when the conductrix is supported on a half-fiber of a quasi-elliptic fibration of type $\tilde{E}_8$, we get an extra $\tilde{E}_8$-special Enriques surface.

6.3 Smooth rational curves on an Enriques surface

Let $|2F|$ be a genus one pencil on $S$. If it contains a reducible member, then its irreducible components are $(-2)$-curves. The next theorem shows that each $(-2)$-curve occurs in this way.

Recall the function $\Phi : \text{Num}(S)^+ \to \mathbb{Z}_{\geq 0}$ defined in Chapter 2 (2.4.6). We use the same formula to define $\Phi(R)$, where $R$ is a $(-2)$-curve. A $(-2)$-curve is an irreducible component of some genus one pencil if and only if $\Phi(R) = 0$.

Lemma 6.3.1. Let $h \in \text{Num}(S)$ be the class of a divisor with $h^2 > 0$. Then

$$\Phi(h) \leq \frac{1}{2} h^2.$$  

Proof. Let $\omega_0, \ldots, \omega_9$ be the fundamental weights for the Enriques lattice defined in Proposition 1.5.3. Applying an element from $W(E_{10})$, we may assume that $h = \sum m_i \omega_i$ with non-negative integers. Using the explicit formulae for $\omega_i$ in terms of isotropic vectors $f_i$ from Proposition 1.5.3, we can write

$$h = m_0 h + m_1 (h - f_1) + m_2 (2h - f_1 - f_2) + \sum_{i \geq 3} m_i (3h - f_1 - \cdots - f_i).$$
Using Remark 1.5.5 we can rewrite it in the form

\[ h = m_0(f_1 + f_2 + f_{1,2}) + m_1(f_2 + f_{1,2}) + m_2(f_1 + f_2 + 2f_{1,2}) + \sum_{i \geq 3}^9 m_i(f_{i+1} + \cdots + f_9). \]

This gives

\[ h^2 = h \cdot (m_0(f_1 + f_2 + f_{1,2}) + m_1(f_2 + f_{1,2}) + m_2(f_1 + f_2 + 2f_{1,2}) + \sum_{i \geq 3}^9 m_i(f_{i+1} + \cdots + f_{10})) \]

\[ \geq (3m_0 + 2m_1 + 4m_2 + \sum_{i \geq 3}^9 (10 - i)m_i)\Phi(h). \]

This implies that \( h^2 \geq 2\Phi(h) \) unless \( h = \omega_9 \). However, \( \omega_9^2 = 0 \). \[ \square \]

**Remark 6.3.2.** One can prove a stronger inequality \( \Phi(h) \leq \sqrt{h^2} \) (see [134, Corollary 2.7.1]).

**Theorem 6.3.3.** Suppose \( p \neq 2 \) or \( p = 2 \) and \( S \) is not extra-special of type \( \tilde{E}_8 \). For any \((-2)\)-curve \( R \), there exists a genus one pencil \(|2F|\) such that \( R \) is an irreducible component of its member.

**Proof.** We have to show that \( \Phi(R) = 0 \). Suppose \( \Phi(R) = R \cdot f_0 \geq 2 \) and let \( F_0 \) be a genus one curve with \([F_0] = f_0\). Since \( f_0 \) is nef, the divisor class \( C = F_0 + R \) is nef. Since \( C^2 = 2(F_0 \cdot R - 1) \geq 2 \), the previous lemma implies that there exists a genus one pencil \(|2F|\) such that \( C \cdot F = F \cdot F_0 + F \cdot R \leq F_0 \cdot R - 1 \). Since \( F_0 \) is nef, this gives that \( F \cdot R \leq F_0 \cdot R - 1 \) contradicting the choice of \( F_0 \).

Suppose \( \Phi(R) = R \cdot F_0 = 1 \). In this case \( C^2 = 0 \). We can find a canonical isotropic sequence \((f_1, \ldots, f_{10})\), where \( f_1 = [F_0], f_2 = [F_0 + R] = [C] \). If there exists a nef numerical class \( f_i, i > 2 \), then \( f_i \cdot f_2 = 1 \) implies that \( f_i \cdot R = 0 \), and we are done. So, assume that the sequence contains only one nef isotropic divisor class \( f_1 \). Then \( f_3 = f_1 + R + R_1, \ldots, f_{10} = f_1 + R + \cdots + R_8 \), and we have the following diagram

\[
\begin{array}{cccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
f_1 & R & R_1 & R_2 & R_3 & R_4 & R_5 & R_6 & R_7 & R_8 \\
\end{array}
\]

Let \( D \) be a member of \(|2F_0|\) that contains the curves \( R_1, \ldots, R_8 \). Since the surface is not extra-special of type \( \tilde{E}_8 \), \( D \) cannot be of type \( \tilde{E}_8 \). Hence, there are three possibilities:

\[
\begin{array}{cccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
R_9 & f_1 & R & R_1 & R_2 & R_3 & R_4 & R_5 & R_6 & R_7 & R_8 \\
\end{array}
\]

or
In the first case the curves \( R, R_1, R_2, R_3, R_4, R_9, R_8, R_7 \) support a genus one curve \( F' \) of type \( E_7 \) that contains \( R \) as an irreducible component. Hence \( R \) is a component of a member of \( |2F'| \). In the second case, \( R \) is a component of a curve \( R + R_1 + R_9 \) of type \( A_2 \). Finally, in the third case, \( R \) is a component of a curve of type \( E_7 \) formed by the curves \( R, R_1, \ldots, R_6, R_9 \). This proves the assertion.

**Definition 6.3.4.** Let \( |2F| \) be a genus one pencil on an Enriques surface. A smooth rational curve \( R \) with \( F \cdot R = 1 \) is called a special bisection.

**Theorem 6.3.5.** Assume that \( S \) contains a \((-2)\)-curve \( R \). Then there exists a genus one fibration with a special bisection.

**Proof.** If \( S \) is extra-special of type \( E_8 \), then, by its definition it contains a bisection. By Proposition 6.2.7, it is the curve \( C \) of cusps. Assume that \( S \) is not such a surface. By the previous theorem, \( R \) is contained in a fiber \( D \) of some genus one fibration \( |2F_1| \). By Theorem 6.1.10, there exists a genus one fibration \( |2F_2| \) such that \( F_1 \cdot F_2 = 1 \). Since \( D \cdot F_2 = 2 \), either \( R \cdot F_2 = 1 \), and we are done, or there exists an irreducible component \( R' \) of \( D \) such that \( R' \cdot F_2 = 0 \). Let \( U \) be the sublattice of \( \text{Num}(S) \) generated by \( \{F_1, F_2\} \). Its orthogonal complement \( U^\perp \) is isomorphic to \( E_8 \), which contains \( r' = [R'] \) and a root \( \alpha \) with \( \alpha \cdot r' = 1 \). Then \( f_3 = [F_1] + [F_2] + \alpha \) is an isotropic vector in \( \text{Num}(S) \) with \( f_3 \cdot r' = 1 \). Suppose \( f_3 \) is not nef. Let \( f_3 = [F_3] \), where \( F_3 \) is an effective divisor. Then there exists a \((-2)\)-curve \( R'' \) such that \( F_3 \cdot R'' < 0 \). This implies that \( R \) is an irreducible component of \( F_3 \) and \( (F_3 - R'')^2 = -2F_3 \cdot R'' - 2 \geq 0 \). Intersecting \( D = F_3 - R'' \) with \( F_1 \) we obtain \( D \cdot F_1 = 1 - R'' \cdot F_1 \geq 1 \). If \( F_1 \cdot R'' = 1 \), we have found a special bisection of the pencil \( |2F_1| \). So, we obtain \( F_1 \cdot R'' = 0 \) and similarly, we get \( F_2 \cdot R'' = 0 \). Thus the divisor class of any \((-2)\)-curve \( R'' \) with \( R'' \cdot F_3 < 0 \) belongs to the orthogonal complement \( U^\perp \). Applying the reflections in the classes of such curves, we may assume that \( f_3 \) is nef and \( r' \) is the sum \( R \) of classes of \((-2)\)-curves in \( U^\perp \). Denote by \( |2F_3| \) a genus one fibration defined by the class \( f_3 \). All these \((-2)\)-curves intersect \( F_1 \) with multiplicity 0, and hence they are contained in fibers of \( |2F_1| \). Since \( F_1 \cdot F_3 = 1 \), we see that \( F_3 \cdot R' = 1 \), and we are done.

**Remark 6.3.6.** It is not true that any \((-2)\)-curve is a special bisection of some genus one fibration. For example, in Figure 8.5 that gives the intersection graph of all \((-2)\)-curves on a surface \( S \) with finite automorphism group, the curves \( R_2, R_4, R_6, R_8 \) are not bisections of any genus one fibration.

**Corollary 6.3.7.** Let \( S \) be a nodal Enriques surface. Then
(i) $S$ contains a genus one fibration with a reducible fiber;

(ii) $S$ contains a genus one fibration with a special bisection;

(iii) $S$ admits a special bielliptic linear system;

(iv) if $S$ is not an extra-special of type $\tilde{E}_8$, there exists a non-special bielliptic map $S \to D$ that blows down a $(-2)$-curve to a point;

Recall that a choice of a nef and big divisor class $H$ (resp. numerical class $h$) on a smooth projective algebraic surface $X$ is called a polarization (resp. numerical polarization). The number $h^2$ is called the degree of the polarization. The pair $(X, H)$ (resp. $(X, h)$) is called a polarized (resp. numerically polarized) algebraic surface.

Let $h$ be a numerical polarization and let $R_S$ be the set of $(-2)$-curves on $S$. We set

$$\delta(h) = \min\{h \cdot R, R \in R_S\}. \quad \text{(6.3.1)}$$

Clearly $\delta(h) > 0$ if and only if $h$ is ample.

We refer for the proof of the following lemma to [135, Theorem 1.5].

**Lemma 6.3.8.** Let $\alpha$ be a root in the Enriques lattice $E_{10}$ and let $x$ be a vector in $E_{10}$ with $x^2 > 0$. Then there exists an isotropic vector $f$ such that $f \cdot \alpha = 0$ and

$$x \cdot f < x \cdot \alpha, \quad \text{or} \quad x \cdot \alpha \leq \frac{3x^2}{\Phi(x)},$$

where $\Phi(x) = \min\{|x \cdot f|, f^2 = 0\}$.

**Theorem 6.3.9.** Let $(S, h)$ be a numerical polarized nodal Enriques surface. Assume that $\Phi(h) \geq 3$. Then

$$\delta(h) \leq 3h^2/\Phi(h).$$

**Proof.** Let $R_0$ be a $(-2)$-curve on $S$. Assume

$$R_0 \cdot h > 3h^2/\Phi(h).$$

Applying the previous lemma, we find an isotropic divisor class $f$ such that $R_0 \cdot f = 0$ and $h \cdot f < R_0 \cdot h$. Obviously, we may assume that $f = [D]$, where $D > 0$. The previous inequality implies that the curve $R_0$ is not a component of $D$ but it is a component of the pencil $|2D|$. Suppose $D$ is not nef. Let $R$ be a $(-2)$-curve such that $R \cdot D < 0$. Obviously, $R$ is contained in $D$ and hence $R \cdot R_0 = 0$. We have

$$s_R(f) \cdot h = (f + (R \cdot f)R) \cdot h = f \cdot h + (R \cdot f)(R \cdot h) \leq f \cdot h < R_0 \cdot h,$$

$$s_R(f) \cdot R_0 = (f + (R \cdot f)R) \cdot R_0 = f \cdot R_0.$$

Replacing $f$ with $s_R(f)$, and continue in this way, we may assume that $f$ is nef. Thus $f = [F]$, where $|2F|$ is a genus one pencil. Since $D \cdot R_0 = 0$, $R_0$ is contained in a member $D$ of $|2F|$. Let $R'$ be another irreducible component of $D$. We have

$$h \cdot R' \leq h \cdot (D - R_0) = h \cdot D - h \cdot R_0 < 2h \cdot f - h \cdot R_0 < h \cdot f < R_0 \cdot h.$$
Continuing in this way, we obtain that \( \delta(h) = 0 \) or \( \delta(h) \leq \frac{3k^2}{\Phi(h)} \). In any case, the assertion has been proven.

Let \( H_{10} \) be a Fano polarization on \( S \) and let \( h_{10} = [H_{10}] \) be its numerical class, we can write
\[
3h_{10} = f_1 + \cdots + f_{10}
\]
for some canonical isotropic 10-sequence (see (3.5.4)).

The next result shows that the bound from the previous theorem is not the best one.

**Theorem 6.3.10.** Let \( h_{10} \) be a numerical Fano polarization of a nodal Enriques surface. Then \( \delta(h_{10}) \leq 4 \).

**Proof.** Assume that \( R \cdot h_{10} > 4 \) for any \((-2)\)-curve \( R \) on \( S \). In particular \( h_{10} \) is ample and hence each \( f_i \) is nef. Thus, \((f_1, \ldots, f_{10})\) is a non-degenerate canonical isotropic sequence.

Let us show that for every \( w \in W(\text{Num}(S)) \), \( w(h_{10}) \) is ample and \( \delta(w(h_{10})) > 4 \). The proof is by induction on the length \( \lg(w) \) of \( w \) as an element of the Coxeter group \((W(\text{Num}(S)), B)\), where \( B \) is the set of reflections corresponding to the root basis formed by \( \alpha_0 = h_{10} - f_1 - f_2 - f_3, \alpha_i = f_i - f_{i+1} \).

If \( \lg(w) = 1 \), then \( w = s_{\alpha_i} \). If \( i > 0 \), we have \( \alpha_i \cdot h_{10} = 0 \), hence \( w(h_{10}) = h_{10} \). If \( i = 0 \), then
\[
3w(h_{10}) = w(f_1) + \cdots + w(f_{10}) = f_{23} + f_{13} + f_{12} + f_4 + \cdots + f_{10},
\]
where \( f_{ij} = h_{10} - f_i - f_j \) are isotropic classes with \( h_{10} \cdot f_{ij} = 4 \). If \( f_{ij} \) is not nef, then \( f_{ij} - R' \geq 0 \) for some \((-2)\)-curve \( R' \), hence \( h_{10} \cdot R' \leq 4 \) contradicting our assumption.

Thus \((f'_1, \ldots, f'_{10}) = (w(f_1), \ldots, w(f_{10}))\) is a non-degenerate canonical isotropic sequence. If \( R \) is a \((-2)\)-curve with \( f'_i \cdot R = 0 \), then \( R \) is a component of a fiber \( F \) of a genus one fibration \([2F]\) with \([F'_i] = f'_i \), hence \( 2f'_i - R = R' \geq 0 \) for some \((-2)\)-curve \( R' \). Since \( h_{10} \cdot f'_i = w(h_{10}) \cdot f_i = (2h - f_1 - f_2 - f_3) \cdot f_i \leq 4 \), this implies that \( h_{10} \cdot R \) or \( h_{10} \cdot R' \) is less than or equal to 4, contradicting our assumption. Thus \( f'_i \cdot R \geq 1 \) for all \( i \), hence \( w(h_{10}) \cdot R \geq 10/3 \). Thus \( \delta(w(h_{10})) \geq 4 \). If \( \delta(w(h_{10})) > 4 \), we have proved our claim. Suppose the equality holds. Then \( 12 = R \cdot 3w(h_{10}) = R \cdot (f'_1 + \cdots + f'_{10}) \) implies that there exists \( f'_i \) such that \( f'_i \cdot R = 3 \) and \( f'_{j} \cdot R = 1 \) for \( j \neq i \), or \( f'_i \cdot R = f'_j \cdot R = 2 \) and \( f'_k \cdot R = 1 \) for \( k \neq i, j \).

Suppose \( f'_i \cdot R = 3 \). Then \( (w(h_{10}) - 2w(f_i)) \cdot w(f_i) = R \cdot w(f_i) = 3, \) and \( (w(h_{10}) - 2w(f_i)) \cdot w(f_j) = R \cdot w(f_j) = 1. \) Since \( w(f_1), \ldots, w(f_{10}) \) generate \( \text{Num}(S) \otimes \mathbb{Q} \), we obtain that the numerical class \([R]\) of \( R \) is equal to \( w(h_{10}) - 2w(f_i) \).

If \( i > 3 \), then \( w(f_i) = f_i \) and
\[
R \cdot f_{11} = R \cdot (h_{10} - f_1 - f_i) = (w(h_{10}) - 2f_i) \cdot (h_{10} - f_1 - f_i) = (2h_{10} - f_1 - f_2 - f_3 - 2f_i) \cdot (h_{10} - f_1 - f_i) = 0.
\]

Thus \( 2h_{10} - 2f_1 - 2f_i - R = R' \geq 0 \) and this implies that \( h_{10} \cdot R' \leq 8 - h_{10} \cdot R \leq 3 \), contradicting \( \delta(h_{10}) > 4 \). If \( i \leq 3 \), say \( i = 1 \), then \( w(f_1) = h_{10} - f_2 - f_3 \) and we can repeat the argument by taking \( f_2 \) instead of \( f_1 \).

Suppose \( f'_i \cdot R = f'_j \cdot R = 2. \) Again, by comparing the intersection numbers, we find that
\[
R = w(h_{10}) - w(f_i) - w(f_j).
\]
However, this implies that $R^2 = 0$, a contradiction. So, we have checked that $\delta(w(h_{10})) > 4$ if $\lg(w) = 1$, applying the induction, we obtain that $w(h_{10}) \cdot R > 4$ for all $w \in W(\text{Num}(S))$ and $R \in \mathcal{R}_S$. By taking $w = s_R$, we obtain $w(h_{10}) \cdot R = h_{10} \cdot w(R) = -h_{10} \cdot R < 0$ which is absurd.

**Corollary 6.3.11.** Let $h_{10}$ be a numerical Fano polarization on a nodal Enriques surface $S$. Write $3h_{10} = f_1 + \cdots + f_{10}$ for some canonical isotropic sequence. Then $S$ contains a $(-2)$-curve $R$ such that its class in $\text{Num}(S)$ is equal to the one of the following classes representing 496 cosets in $\text{Num}(S)/2 \text{Num}(S)$ of roots in $\text{Num}(S)$:

$$
R \cdot h_{10} = 0 : f_j - f_i, 1 \leq i < j \leq 10,
R \cdot h_{10} = 1 : h_{10} - f_i - f_j - f_k, 1 \leq i < j < k \leq 10,
R \cdot h_{10} = 2 : f_i + f_j + f_k + f_l - h_{10}, 1 \leq i < j < k < l \leq 10,
R \cdot h_{10} = 3 : f_i + f_j - f_k, 1 \leq i < j < k \leq 10,
R \cdot h_{10} = 4 : h_{10} - 2f_1.
$$

**Proof.** By the previous theorem, we can find a $(-2)$-curve $R$ such that $h_{10} \cdot R \leq 4$. Assume $R \cdot h_{10} = 0$. Then $R \cdot f_i \neq 0$ for some $f_i$, hence there exists some $j$ such that $R \cdot f_j < 0$. It follows from the description of a canonical isotropic sequence that there exists some $f_j$ such that $f_j = f_i + R$ with $i < j$. Thus $R \sim f_j - f_i$.

Assume $d = h_{10} \cdot R > 0$. We may assume that $S$ does not contain $(-2)$-curves $R'$ with $h_{10} \cdot R' = 0$, hence $h_{10}$ is ample. Let $R \cdot f_i = m_i$. We may assume that $m_1 \geq \cdots \geq m_{10} \geq 0$. We have $(f_1 + R)^2 = 2m_1 - 2, h_{10} \cdot (f_1 + R) = 3 + d$. Applying the Hodge Index Theorem to the rank 2 sublattice spanned by $h_{10}$ and $f_1 + R$, we obtain $20(m_1 - 1) < (3 + d)^2$. This gives $m_1 \leq 2$ if $d \leq 3$ and $m_1 = 1$ if $d = 1$.

Assume $d = 1$ so that $\sum_{i=1}^{10} m_i = 3$. Since $m_1 = 1$, we obtain $m_1 + m_2 \geq 1$. Thus, $(f_1 + f_2) \cdot R \geq 2$ and $(h_{10} - f_1 - f_2) \cdot R \leq -1$. We can write $f = h_{10} - f_1 - f_2 = g - (f \cdot R)R$, where $g$ is an isotropic vector. Since $h_{10} \cdot g \leq 3$, we obtain that $g = f_k$ for some $k \neq 1, 2$ and $f \cdot R = -1$, hence $R \sim h_{10} - f_1 - f_2 - f_k$.

Assume $d = 2$. Since $m_1 \leq 2$ and $\sum_{i=1}^{10} m_i = 6$, we find that $R \cdot (f_1 + \cdots + f_5) \geq 5$. Then $f = 2h_{10} - (f_1 + \cdots + f_5)$ is an effective and isotropic class with $f \cdot R \leq -1$ and $f \cdot h_{10} = 5$. We can write $f = g - (f \cdot R)R$, where $g$ is an effective isotropic class. Since $g \cdot h_{10} = 5 + 2f \cdot R \leq 3$, we obtain that $g = f_k$ for some $k > 5$, and $R \sim 2h_{10} - (f_1 + \cdots + f_5) - f_k$.

Assume $d = 3$. Suppose there exists $f_i + f_j$ such that $R \cdot (f_i + f_j) > 3$. Then we apply the same argument as before by taking $f = h_{10} - f_i - f_j$ to make $f \cdot R < 0$ and get a contradiction with $\Phi(h_{10}) = 3$. Thus we may assume that $v = (m_1, \ldots, m_{10}) = (2, 1, 1, 1, 1, 1, 1, 0, 0)$ or $(1, 1, 1, 1, 1, 1, 1, 1, 1, 0)$. We know that there exists a root $\alpha$ from the list (6.3.2) such that $R - \alpha \in 2 \text{Num}(S)$. Intersecting with $h_{10}$ we find that $\alpha \cdot h_{10} = 1$ or 3. In the first case $\alpha = h_{10} - f_i - f_j - f_k$ and, the vector $v' = (\alpha \cdot f_1, \ldots, \alpha \cdot f_{10})$ has three coordinates equal to 1, others are zeros. Since $v - v'$ is not divisible by two, we get the second case $\alpha \cdot h_{10} = 3$, and hence $\alpha = f_i + f_j - f_k$ and $v'$ contains two coordinates equal to 0, one coordinate equal to 2, and the remaining coordinates equal to 1. Since $R - \alpha \in 2 \text{Num}(S)$, we must get $\alpha = R$. 


Assume $R \cdot h_{10} = 4$. As before, taking $f = h_{10} - f_i - f_j$ or $f = 2h_{10} - f_i - f_j - f_k - f_l - f_m$, we obtain that $v = (m_1, \ldots, m_{10})$ is equal, after a permutation, to either $(3,1,\ldots,1)$, or $(2,2,1,\ldots,1)$, or $(2,2,2,\ldots,1,1,0)$. The only vector $\alpha$ from the list (6.3.2) for which $(\alpha \cdot f_1, \ldots, \alpha \cdot f_1)$ is congruent mod $2 \text{Num}(S)$ to one of these vectors is $h_{10} - 2f_1$, we may $v = (3,1,\ldots,1)$. This gives $R \sim h_{10} - 2f_1$.

Note that the last 10 vectors are congruent mod $2 \text{Num}(S)$ and the remaining 495 classes in the list represent different cosets.

For any numerical Fano polarization $h_{10}$ we denote by $\Pi_{h_{10}}$ the set of cosets of $\text{Num}(S)/2 \text{Num}(S)$ represented by effective classes $x$ with $x^2 = -2$, $x \cdot h_{10} \leq 4$. The previous Corollary gives a choice of possible representatives of $\Pi_{h_{10}}$.

**Proposition 6.3.12.** Let $\Pi_{h_{10}}^{\text{nod}}$ be the subset of $\Pi_{h_{10}}$ represented by the classes of smooth rational curves. Then it defines a root basis whose Dynkin graph $\Gamma$ has only simple or double edges.

**Proof.** Under the map defined by the Fano polarization $h_{10}$, each $(-2)$ curve is either contracted or is mapped to a rational curve $R_i$ of degree $\leq 4$. It follows from the description of the set $\Pi_{h_{10}}$ that $|x \cdot y| \leq 2$ for any two distinct $x,y \in \Pi_{h_{10}}$. This proves the assertion.

**Remark 6.3.13.** In fact, one observes that, if $|x \cdot y| = |x \cdot y'| = 2$ then $y = y'$. Also, if $|x \cdot y| = 2$, then $|h_{10} \cdot x| + |h_{10} \cdot y| \leq 4$.

Now we will give another (much simpler) proof of Theorem 6.3.10 and its Corollary that works if $p \neq 2$ or the canonical cover is birationally isomorphic to a K3-surface. It is based on the following lemma communicated to us by E. Looijenga.

**Lemma 6.3.14.** Assume the canonical cover $X \to S$ of an Enriques surface $S$ is birationally isomorphic to a K3 surface. Let $R$ be a $(-2)$-curve on $S$, let $H$ be an ample divisor, $\alpha \in \text{Num}(S)_{-2}$ with $R - \alpha \in 2 \text{Num}(F), \alpha \cdot H > 0$. Then $\alpha$ is the class of an effective divisor.

**Proof.** Let $\alpha = R + 2x$ for some divisor class $x$. Since $O_R(K_S) \cong O_R$, we obtain that the cover splits over $R$.

Assume first that $\pi : X \to S$ is separable, this means that $\pi^*(R) = R_1 + R_2$, where $R_1, R_2$ are disjoint $(-2)$-curves on $X$. We have

$$\pi^*(\alpha) = \pi^*(R) + 2\pi^*(x) = R_1 + R_2 + 2\pi^*(x) = (R_1 + \pi^*(x)) + (R_2 + \pi^*(x)).$$

Using that

$$-4 = \pi^*(\alpha)^2 = 2(R_1 + \pi^*(x))^2 + 2(R_1 + \pi^*(x)) \cdot (R_2 + \pi^*(x))$$

we obtain

$$(R_1 + \pi^*(x))^2 = -2, (R_1 + \pi^*(x)) \cdot (R_2 + \pi^*(x)) = 0.$$
Since 
\[(R_1 + \pi^*(x)) \cdot \pi^*(H) = \frac{1}{2} \pi^*(\alpha) \cdot \pi^*(H) = \alpha \cdot H > 0,\]
applying Riemann-Roch on $X'$ the minimal resolution of $X$, this gives $R_i + \pi^*(x) \sim R_i' > 0, i = 1, 2$. Thus, $\pi^*(x)$ is represented by the sum of two effective divisors which are permuted by the cover involution. This immediately implies that $\alpha$ is linearly equivalent to an effective divisor $D$ on $S$ equal to the image of the curve $R_i'$ on $S$.

Next we assume that $\pi$ is inseparable. Let $\sigma : X' \to X$ be a minimal resolution of singularities. Let $x' = \pi^*(x)$, we have $\pi^*(\alpha) = 2(R' + x')$ and $\sigma^*(\pi^*(\alpha)) = 2R' + 2\sigma^*(x') + \mathcal{R}$, where $\mathcal{R}$ is the exceptional divisors over the points lying on $R$. We will see later in Lemma 10.2.9 that there are two such points and $\mathcal{R}^2 = -4$ and $\tilde{R}' \cdot \mathcal{R} = 2$ (see Example 10.6.11). This gives $-4 = 4(\tilde{R}' + \sigma^*(x'))^2 + 4$, hence $(\tilde{R}' + \sigma^*(x'))^2 = -2$. As in the separable case, we check that $\tilde{R}' + \sigma^*(x')$ is an effective divisor and its image on $S$ is linearly equivalent to $\alpha$.

The proof of Theorem 6.3.10 and Corollary 6.3.11 is now easy. The coset $R + 2\text{Num}(S)$ contains a class from the list (6.3.2). By the previous lemma, it is an effective class. Since $\alpha^2 = -2$, one of the irreducible components of an effective representative of $\alpha$ is $(2)$-curve $R'$. Since $\alpha \cdot h \leq 4$, we obtain that $R' \cdot h \leq 4$.

**Lemma 6.3.15.** Let $(|2F_1|, |2F_2|)$ be a non-degenerate $U$-pair on a nodal Enriques surface $S$. There exists a $(2)$-curve $R$ with $F_1 \cdot R \leq 1, F_2 \cdot R \leq 1$.

**Proof.** We extend $f_1 = [F_1], f_2 = [F_2]$ to a canonical $\sigma$-degenerate maximal isotropic sequence $(f_1, f_2, \ldots, f_{10})$. If $c < 10$, there exists a $(2)$-curve satisfying $R \cdot f_1 = R \cdot f_2 = 0$, hence $R$ satisfies the assertion. Thus we may assume that the sequence is non-degenerate and $3h_{10} = f_1 \cdots + f_{10}$ is an numerical Fano polarization. Applying Theorem 6.3.10, we find a $(2)$-curve $R$ with $h_{10} \cdot R \leq 4$. This gives 
\[3h_{10} \cdot R = \sum_{i=1}^{10} f_i \cdot R \leq 12.\]

Assume $h_{10} \cdot R < 3$. If $R \cdot (f_1 + f_2) > 3$, then the previous equality implies that there exists $f_j$ with $j > 2$ such that $R \cdot f_j = 0$. Thus $R$ is a component of a fiber $F$ of a genus one fibration $|2F_j|$ with $[F_j] = f_j$. We have $2f_j - R - Z \geq 0$ for some nodal cycle $Z$, hence $f_j \cdot (2f_j - R - Z) = 2 - f_j \cdot R - f_j \cdot Z \leq 2$ for $i = 1, 2$. This implies that either a component of $Z$ satisfies the assertion or $R \cdot (f_1 + f_2) \leq 2$. Suppose $f_1 \cdot R = 2, f_2 \cdot R = 0$. Then $f_1 \cdot Z = 0$. Choose a component $R'$ of $Z$ such that $f_2 \cdot R' \neq 0$. Then $R' \cdot f_1 = 0, R' \cdot f_2 \leq 1$, and we are done.

Assume $h_{10} \cdot R = 4$. By Corollary 6.3.11, $R \sim h_{10} - 2f_j$ for some $j$. If $j > 2$, then $R \cdot f_1 = R \cdot f_2 = 1$, and we are done. If $j \leq 2$, say $j = 1$, then we consider the isotropic class $h_{10} = f_1 = f_2 = 3$. By above, we may assume that $h_{10} \cdot R' \geq 4$ for any $(2)$-curve $R'$. Since $h_{10} \cdot f_{1,3} = 4$, any $(2)$-curve $R'$ with $f_{1,3} \cdot R < 0$ must be a component of an effective representative of $f_{1,3}$, hence satisfies $h_{10} \cdot R' \leq 3$. Since there are no such $(2)$-curves, $f_{1,3}$ is nef. We have $(h_{10} - 2f_1) \cdot (h_{10} - f_1 - f_3) = 0$, hence $2f_{1,3} = R + Z$, where $Z$ is a nodal cycle. We have $R \cdot f_1 = (h_{10} - 2f_1) \cdot f_1 = 3$, and $f_{1,3} \cdot f_1 = 2$, hence $Z \cdot f_1 = (2f_{1,3} - R) \cdot f_1 = 1$. Also, $R \cdot f_2 = 1$ and $f_{1,3} \cdot f_2 = 1$. This gives $f_2 \cdot Z = 1$. So any irreducible component $R'$ of $Z$ satisfies $R' \cdot E_i \leq 1, i = 1, 2$. 

\[\square\]
6.3. SMOOTH RATIONAL CURVES ON AN ENRIQUES SURFACE

By the similar argument we prove the following two lemmas. We assume that $S$ is a nodal Enriques surface.

**Lemma 6.3.16.** Let $(f_1, f_2, f_3)$ be a non-degenerate canonical sequence. Then there exists a $(-2)$-curve such that $f_1 \cdot R \leq 1$ except possibly for one $f_i$ with $f_i \cdot R \leq 2$.

**Lemma 6.3.17.** Let $|2F_1|, |2F_2|$ be two genus one pencils with $F_1 \cdot F_2 = 2$. Then there exists a $(-2)$-curve $R$ such that $F_1 \cdot R \leq 2, F_2 \cdot R \leq 1$, or $F_1 \cdot R \leq 1, F_2 \cdot R \leq 2$.

**Proof.** The class $v = [F_1 + F_2] \in \text{Num}(S)$ satisfies $v^2 = 4, \Phi(v) = 2$. By Corollary 0.8.13, $v$ is equivalent modulo $W(\text{Num}(S))$ to the vector $h - f_1$ for some maximal isotropic sequence $f_1, \ldots, f_{10}$. It is easy to see that $[F_1] = f_i$ for some $i \neq 1$. Thus $[F_2] = h - f_1 - f_i$. Without loss of generality, we may assume that $i = 2$. Since $v$ is nef, for any $(-2)$-curve $R$, we have $h \cdot R - f_1 \cdot R = v \cdot R \geq 0$. This shows that $h \cdot R < 0$ implies $f_1 \cdot R < 0$. We may assume that $h$ and $f_1$ are effective classes. Thus, subtracting from $h$ and $f_1$ the same set of $(-2)$-curves, we may assume that $h$ is nef and $(f_1, \ldots, f_{10})$ is a canonical isotropic sequence.

We have

$$h = (h - f_1 - f_2) + f_1 + f_2 = v + f_1.$$ 

If $h \cdot R = 0$, then $f_1 \cdot R \geq -1$ for some $(-2)$-curve $R$, then $R \cdot v \leq 1$, hence $R$ satisfies the assertion. Thus, we may assume that $h$ is equal to the class of an ample divisor $H$. By Proposition 6.1.6, the canonical isotropic sequence $(f_1, \ldots, f_{10})$ with $h = \frac{1}{3}(f_1 + \cdots + f_{10})$ is non-degenerate. Applying Corollary 6.3.11, we find a $(-2)$-curve $R$ such that $h \cdot R \leq 4$. This gives $v \cdot R = (h - f_1) \cdot R \leq 4 - f_1 \cdot R$. If $f_1 \cdot R \geq 1$, we are done. Assume $f_1 \cdot R = 0$. Then $2f_1 = R + Z$ for some nodal cycle $Z$ and $h \cdot Z = 6 - R \cdot h$. We have $v \cdot Z = h \cdot Z = 6 - R \cdot h$ and $v \cdot R = h \cdot R$. One of these numbers is less or equal than 3. So far, we have shown that there exists a $(-2)$-curve $R$ such that $(f_1 + f_2) \cdot R \leq 3$. If $f_1 \cdot h = 3$, then $f_2 \cdot h = 0$. Replacing $R$ with a component $R'$ of a nodal cycle $Z$ such that $2f_2 = R + Z$, we find $R' \cdot f_2 = 0$ and $R' \cdot f_1 = 4 - R \cdot f_1 = 1$. This proves the assertion.

**Theorem 6.3.18.** Let $(S, h)$ be a polarized nodal Enriques surface. Then

$$\delta(h) \leq h^2.$$ 

**Proof.** If $\Phi(h) \geq 3$, this follows from Theorem 6.3.9. Assume $\Phi(h) = 1$. By Proposition 2.6.1, $h$ is one of the following form: $h = nf_1 + f_2$ for some nef isotropic classes $f_1, f_2$ with $f_1 \cdot f_2 = 1$, or $h = (n + 1)f + R$ for some nef isotropic class $f$ and a $(-2)$-curve $R$ with $f \cdot R = 1$.

Note that in both cases $h^2 = 2n$. In the first case we apply Lemma 6.3.15 to find a $(-2)$-curve $R$ such that $f_i \cdot R \leq 1, i = 1, 2$. Then $h_{10} \cdot R \leq n + 1 \leq 2n = h^2$. In the second case, $h_{10} \cdot R = n - 1 = \frac{1}{2}h^2 - 1 \leq h^2$.

Assume $\Phi(h) = 2$ and $h^2 = 4k$. We apply Proposition 2.6.6. In case (1), we have $h = kf_1 + 2f_2$, where $f_1$ are as above. We choose $R$ as above, and get $h_{10} \cdot R \leq k + 2 = \frac{1}{2}h^2 + 2 \leq h^2$. In case (2), we have $h = (k + 2)f + 2R$, where $f, R$ are as above. We get $h_{10} \cdot R = k = \frac{1}{4}h^2 \leq h^2$. In case (3), $h = kf_1 + f_2$, where $f_1, f_2$ are nef isotropic classes with $f_1 \cdot f_2 = 2$. We apply Lemma 6.3.17 and find a $(-2)$-curve $R$ such that $h_{10} \cdot R \leq 2k + 1 = \frac{1}{2}h^2 + 1 \leq h^2$. In cases (4)-(6), we take the nodal cycle $R$ in the notation of Proposition 2.6.6 to obtain $h_{10} \cdot R = k - 1 \leq \frac{1}{4}h^2 - 2 \leq h^2$. 

Assume $\Phi(h) = 2$ and $h^2 = 4k + 2$. We apply Proposition 2.6.7. In case (i), we have $h = kf_1 + f_2 + f_3$, where $f_i$ are as in Lemma 6.3.16. We take $R$ from the assertion of the lemma to obtain $h \cdot R \leq 2 + 2k = \frac{1}{2}h^2 + 1 \leq h^2$. In case (ii) there exists a curve $R$ such that $h \cdot R = 0$. In cases (iii) and (iv), $R_1 \cdot h = k - 1 = \frac{1}{4}h^2 - \frac{3}{2}$.

It follows from the proof that the bound $\delta(h) \leq h^2$ can be improved for polarizations $h$ with $\Phi(h) < 3$.

**Corollary 6.3.19.** Assume $\Phi(h) \leq 2$. Then

$$\delta(h) \leq 1 + \frac{1}{2}h^2.$$  

**Remark 6.3.20.** A long and elaborate proof of the following inequality

$$\delta(h) \leq \frac{h^2}{\Phi(h)} + \frac{\Phi(h)}{3}$$

can be found in unpublished manuscripts of F. Cossec.

We will see later that a general nodal surface has infinitely many smooth rational curves. However, in a special case the set of such curves could be finite. By Theorem 8.1.11 this happens if and only if the automorphism group of $S$ is finite. We will classify all such surfaces in Sections 8.9 and 8.10.

### 6.4 Nodal invariants

Let $\mathcal{R}_S$ be the set of $(-2)$-curves on an Enriques surface $S$. We assume that it is not empty. It spans a nonzero sublattice $N_S$ of $\text{Num}(S)$ which we will call the *nodal sublattice*. We define the *r-invariant* $\text{Nod}(S)$ of $S$ as the image of $\mathcal{R}_S$ in the vector space

$$\overline{\text{Num}}(S) := \text{Num}(S)/2\text{Num}(S) \cong E_{10}/2E_{10} \cong \mathbb{F}_2^{10}.$$  

The homomorphism of multiplication by 2

$$E_{10}(2)^\vee/E_{10}(2) \to E_{10}/2E_{10}$$

is an isomorphism from the discriminant group of the lattice $E_{10}(2)$ to the additive group of the vector space $\overline{\text{Num}}(S)$. Under this identification, the discriminant quadratic form from Chapter 0, §6 equips $\overline{\text{Num}}(S)$ with a non-degenerate symplectic bilinear form defined by

$$(x + 2\text{Num}(S)) \cdot (y + 2\text{Num}(S)) = x \cdot y \mod 2$$

and the quadratic form $q$ defined by

$$(x + 2\text{Num}(S))^2 = \frac{1}{2}x^2 \mod 2.$$  

Let

$$\langle \text{Nod}(S) \rangle \subset \overline{\text{Num}}(S)$$

be the linear span of $\text{Nod}(S)$.

It is clear that

$$N_S/2\text{Num}(S) \cap N_S \cong \langle \text{Nod}(S) \rangle.$$
Obviously,
\[ \text{Nod}(S) \subset q^{-1}(1). \]

Choose a numerical Fano polarization \( h_{10} \) and let \( \Pi_{h_{10}}^{\text{nod}} \) be the set of elements in \( \Pi_{h_{10}} \) represented by the classes of \((-2)\)-curves modulo \( 2\text{Num}(S) \). Let \( N_{S}^{h_{10}} \) be the sublattice spanned by these classes. By Proposition 6.3.12, the set \( \Pi_{h_{10}}^{\text{nod}} \) forms a root basis in \( N_{S}^{h_{10}} \) whose Dynkin diagram has only simple or double edges.

Since any element in \( N_{S} \) is congruent modulo \( 2\text{Num}(S) \) to an element from \( N_{S}^{h_{10}} \), the image of \( N_{S}^{h_{10}} \) in \( \overline{\text{Num}}(S) \) spans the same subspace \( \langle \text{Nod}(S) \rangle \). We also see that the kernel
\[ N_{0} := \text{Ker}(N_{S}/2N_{S} \to \overline{\text{Num}}(S)) = N_{S} \cap 2\text{Num}(S)/2N_{S} \]
coinsides with the kernel
\[ G_{h_{10}} := \text{Ker}(N_{S}^{h_{10}}/2N_{S}^{h_{10}} \to \overline{\text{Num}}(S)) = N_{S}^{h_{10}} \cap 2\text{Num}(S)/2N_{S}^{h_{10}}. \]

Since any \( g \in G_{h_{10}} \) is represented by an element \( 2x \), where \( x \in \text{Num}(S) \), we have \( x = \frac{1}{2}g \in (N_{S}^{h_{10}})^{\vee} \) and \( x^{2} \in 2\mathbb{Z} \), we see that \( G_{h_{10}} \) defines an isotropic subgroup of the 2-torsion group of \( D(N_{S}^{h_{10}}) \).

**Definition 6.4.1.** The pair \( (N_{h_{10}}, G_{h_{10}}) \) is called the Fano root invariant of \( S \).

We believe that the Fano root invariant \( (N_{h_{10}}, G_{h_{10}}) \) does not depend on a Fano polarization. Next we give a partial confirmation of this under the assumption that \( p \neq 2 \) or \( S \) is a \( \mu_{2} \)-surface.

Let \( \pi : X \to S \) be the K3-cover. Assume that it is an étale map, i.e. \( p \neq 2 \) or \( S \) is a \( \mu_{2} \)-surface. Let \( \sigma \) be the covering involution and let
\[ \text{Pic}(X)^{\pm} := \{ L \in \text{Pic}(X) : \sigma^{*}(L) = L^{\otimes \pm 1} \}. \]
We have \( \text{Pic}(X)^{\pm} = \pi^{*}(\text{Pic}(S)) = \pi^{*}(\text{Num}(S)) \). Let
\[ \Delta_{\pm} = \{ \delta_{\pm} \in \text{Pic}(X)^{\pm} : \delta_{\pm}^{2} = -4 \} \text{ exists } \delta_{\mp} \in \text{Pic}(X)^{\mp} \text{ such that } \delta_{\pm} + \delta_{\mp} \in 2\text{Pic}(X) \}.
\]

Since \( (\delta_{+} + \delta_{-})^{2} = -8 \), \( \frac{1}{2}(\delta_{+} + \delta_{-})^{2} = -2 \). Replacing \( \delta_{+}, \delta_{-} \) with \( -\delta_{+}, -\delta_{-} \), we may assume that \( D_{+} = \frac{1}{2}(\delta_{+} + \delta_{-}) \) and \( D_{-} = \frac{1}{2}(\delta_{+} - \delta_{-}) \) are effective divisors on \( X \) with \( D_{+}^{2} = -2 \) and \( D_{-}^{2} = \pi^{*}(C) \), where \( C \) is an effective divisor on \( S \) with \( C^{2} = -2 \). Write \( C = \sum n_{i}C_{i} + \sum m_{i}R_{i} \) as a sum of irreducible curves \( C_{i} \) of positive arithmetic genus with primitive numerical class \( [C_{i}] \) and smooth rational curves \( R_{i} \). The exact sequence
\[ 0 \to \mathcal{O}_{S}(K_{S} - C_{i}) \to \mathcal{O}_{S}(K_{S}) \to \mathcal{O}_{C_{i}}(K_{S}) \to 0 \]
together with the Vanishing Theorem, shows that the étale cover \( \pi \) does not split over \( C_{i} \). It splits over each \( R_{i} \). Thus we can write
\[ \pi^{*}(C) = \sum n_{i}\pi^{*}(C_{i}) + \sum m_{i}\pi^{*}(R_{i}) = \sum n_{i}\pi^{*}(C_{i}) + \sum m_{i}(R_{i}^{+} + R_{i}^{-}) = D_{+} + D_{-}. \]
The only way to split the sum into the sum of two positive divisors which are conjugate with respect to the covering involution \( \sigma \) is to take \( D_{+} = \frac{1}{2}\sum n_{i}\pi^{*}(C_{i}) + \sum m_{i}R_{i}^{+} \), \( D_{-} = \frac{1}{2}\sum n_{i}\pi^{*}(C_{i}) + \sum m_{i}R_{i}^{-} \). Thus, we obtain that \( \delta_{-} = D_{+} - D_{-} = \sum m_{i}(R_{i}^{+} - R_{i}^{-}) \).
Let $K'$ be the sublattice of $\text{Pic}(X)^-$ spanned by the classes $\delta_\pm = R^+ - R^-$, where $R^\pm$ is an effective divisor class on $S$ with $R^2 = -2$. Since $\delta^2 = -4$, the lattice $K = K'(\frac{1}{2})$ is a negative definite (because $\text{Pic}(X)^+$ contains an ample divisor class) and generated by the vectors of norm $-2$. Thus $K$ is a root lattice. It follows from above that $K$ is generated by the classes $\delta_R = R^+ - R^-$, where $\pi^*(R) = R^+ + R^-$ for some $(-2)$-curve $R$.

Choose a root basis in $K$ represented by vectors $\delta_R = R^+_i - R^-_i$ and let $\phi(R_i) = [R^+_i + R^-_i] = \pi^*(R_i)$. Define a homomorphism

$$\phi : K/2K \to \text{Num}(S)/2\text{Num}(S)$$

by assigning to $\delta_R$ the class of $R_i$ modulo $2\text{Num}(S)$. This definition does not depend on a choice of a root basis since under a reflection $s_i : x \mapsto x + (x, \delta_{R_i})\delta_{R_i}$, we have $\phi(s_i(x)) = \phi(x)$ (we use that $(x, \delta_{R_i})_K = \frac{1}{2}(x, \delta_{R_i})_{K'}$ so $(x, \delta_{R_i})_K \in \mathbb{Z}$ implies that $x \cdot \delta_{R_i} \in 2\mathbb{Z}$).

Let $H$ be the kernel of the homomorphism $\phi$. Let $h \in H$ be represented by an element $\sum n_i\delta_{R_i} \in K'$. Then $\frac{1}{2} \sum n_i(R^+_i + R^-_i) \in \pi^*(\text{Num}(S))$ and hence $\frac{1}{2} \sum n_i\delta_{R_i} \in \text{Pic}(X)^-$. This shows that we can identify $H$ with a subgroup of the 2-torsion subgroup of the discriminant group $D(K')$. It defines a sublattice $\tilde{K}'$ of $\text{Pic}(X)^-$ that contains $K'$. Since $\text{Pic}(X)^+$ contains an ample divisor, $\text{Pic}(X)^-$ does not contain divisor classes of smooth rational curves, i.e. the lattice $\text{Pic}(X)^-$ does not contain vectors of norm $-2$, hence $\tilde{K}'$ does not contain vectors of norm $-2$. Its elements $x$ satisfy $x^2 \in 4\mathbb{Z}$. Thus, considered as elements in $\tilde{K}'$, they satisfy $x^2 \in 2\mathbb{Z}$.

Definition 6.4.2. The pair $(K, H)$ is called the Nikulin $R$-invariant of $S$ and is denoted by $\text{Nik}(S)$.

Theorem 6.4.3. Suppose $N^{h_{10}}_S$ is negative definite. Then it is isomorphic to the lattice $K$.

Proof. If $R'$ is a $(-2)$-curve with $[R'] = [R + 2D]$ for some divisor $D$ on $S$, we have $[\pi^*(R')] = [R^+ + R^-] = [\pi^*(R) + 2\pi^*(D)] = [(R^+ + \pi^*(D))] + [(R^- + \pi^*(D))]$. It follows from the proof of Lemma 6.3.13, that $(R^+ + \pi^*(D))^2 = (R^- + \pi^*(D))^2 = -2$ and both classes $[R^+ + \pi^*(D)]$ and $[R^- + \pi^*(D)]$ are effective. Since $h^0(R^+ + R^-) = 1$, we must have $R^+ \sim R^+ + \pi^*(D)$ or $R^+ \sim R^- + \pi^*(D)$. This implies that $[R^+ + R^-] = \pm [R^+ - R^-]$. Thus, we see that the lattice $K'$ is generated by the classes of $R^+ - R^-$, where $R \in \text{Nik}^{h_{10}}_S$.

Let $R, R' \in \text{Nik}^{h_{10}}_S$ with $R \cdot R' = 0$. Then $R^+, R^-, R'^+, R'^-$ are disjoint $(-2)$-curves and hence $\delta_R \cdot \delta_{R'} = 0$. Assume $N^{h_{10}}_S$ is negative definite. Choose a root basis of $N^{h_{10}}_S$ represented by $(-2)$-curves $R_1, \ldots, R_l$. Its Dynkin diagram $\Gamma$ has no loops and we may assume that it is connected. This implies that the canonical cover splits over $R = \sum R_i$. We may assume that $\pi^*(R) = R^+ + R^-$, where $R^+ = \sum R^+_i, R^- = \sum R^-_i$ are connected with the Dynkin graph isomorphic to $\Gamma$. This implies that the matrix $(\delta_{R_i} \cdot \delta_{R_j}) = 2C$, where $C$ is the Cartan matrix of $\Gamma$. Thus $N^{h_{10}}_S \cong K'(\frac{1}{2}) = K$.

\[\square\]

Recall some terminology and facts about quadratic forms over a field $F_2$ in which we follow the terminology of [129]). A quadratic form on a linear space $q : V \to F_2$ over $F_2$ of dimension $n$ defines an associated symmetric bilinear form $b_q(x, y) = q(x + y) + q(x) + q(y)$. It is always a symplectic form, i.e. satisfies $b_q(x, x) = 0$. The kernel of $q$ is the subspace $\ker(q)$ of the radical
of \( b_q \) on which \( q \) vanishes. A quadratic form is regular if its kernel is trivial. Geometrically this means that the quadric \( V(q) \) is regular over \( \mathbb{F}_2 \). The rank of \( q \) is \( \text{rank}(q) := n - \dim \ker(q) \) and the rank of \( b_q \) is \( \text{rank}(b_q) := n - \dim \text{rad}(b_q) \). Since \( q \mid \text{rad}(b_q) \) is a semi-linear map to \( \mathbb{F}_2 \), we have \( \dim \text{rad}(b_q) - \dim \ker(q) \leq 1 \). This difference is called the defect of \( q \) and the quadratic form is called defective if it is not equal to zero. Since \( b_q \) is alternating, \( q \) is regular if and only if its radical is trivial. In this case \( b_q \) is a symplectic form. Otherwise, we have \( \dim \text{rad}(b_q) \geq 2 \). However, when \( n \) is odd, \( \text{rad}(b_q) \) is always non-trivial, and \( q \) is regular if and only if is defective and \( \dim \text{rad}(b_q) = 1 \).

A linear subspace of \( V \) is called isotropic if the restriction of \( q \) to this subspace is identically zero. A vector is isotropic if it spans an isotropic line. The Witt index of \( q \) is the maximal dimension of an isotropic subspace. A Witt index of a quadratic form is equal to \( \frac{1}{2}(r - 1) \) if \( r = \text{rank}(b_q) \) is odd and it is equal to \( \frac{1}{2}r \) or \( \frac{1}{2}(r - 2) \) if \( r \) is even. In the former case \( q \) is called of even type and in the latter case it is called of odd type.

After an appropriate choice of a basis \( (e_1, \ldots, e_n) \) in \( V \) such that the radical of \( V \) is spanned by the last \( n - 2m \) vectors, a quadratic form with defect \( \delta \) can be written in one of the following forms:

1. \( q(\sum x_ie_i) = \sum_{i=1}^{m} x_ix_{m+i} + x_{2m+1}^2 \) if \( \delta = 1 \); (6.4.2)
2. \( q(\sum x_ie_i) = \sum_{i=1}^{m} x_ix_{m+i} \) if \( \delta = 0 \);
3. \( q(\sum x_ie_i) = \sum_{i=1}^{m} x_ix_{m+i-1} + x_{2m-1}^2 + x_{2m-1}x_{2m} + x_{2m}^2 \) if \( \delta = 0 \).

In the second (resp. third) case \( q \) is even (resp. odd). We assign to it sign 1 (resp. -1) if it is even (resp. odd). Viewing the sign as an element \( \epsilon \) of the group \( \{+1, -1\} \cong \mathbb{Z}/2\mathbb{Z} \), the sign of the direct sum of quadratic forms (defined in a natural way) is equal to the product of the signs of the summands. We refer to a linear space equipped with a quadratic form as a quadratic space.

We denote by \( O(V, q) \) or just \( O(V) \) the orthogonal group of a quadratic space \( (V, q) \). If \( q \) is regular and \( n \) is odd, \( O(V, q) \cong \text{Sp}(V/\text{rad}(V), b_q) \). If \( q \) is a regular quadratic form on \( \mathbb{F}_2^n \) given by formula (6.4.2), we denote the orthogonal group by \( \text{Sp}(n-1, \mathbb{F}_2) \) if \( n \) is odd and by \( O(n, \mathbb{F}_2)^+ \) (resp. \( O(n, \mathbb{F}_2)^- \)) if \( n \) is even and \( q \) is even (resp. odd). Note that here we differ from the terminology in [129], where this notation is reserved to the subgroup of index 2, the kernel of the spin norm. Our groups are denoted by \( GO_n^\pm(q) \).

The type of a quadratic form of rank \( r = 2m \) and defect \( \delta \) on a quadratic space of dimension \( n \) can be recognized by the number \( \#q^{-1}(1) \) of non-zero non-isotropic vectors (see [19, Chapter III, §6]).

\[
\#q^{-1}(1) = \begin{cases} 
0 & \text{if } q = 0, \\
n - 1 & \text{if } \delta = 1, \\
n - m - 1(2m - 1) & \text{if } q \text{ is even}, \delta = 0, \\
n - m - 1(2m + 1) & \text{if } q \text{ is odd}, \delta = 0.
\end{cases}
\] (6.4.3)

We will also use the following Witt’s Theorem (see [256, Theorem 12.10]).
Theorem 6.4.4. Let $L_1$ and $L_2$ be two isomorphic linear subspaces of a non-defective quadratic space $V$. Then there exists an orthogonal transformation of $V$ that maps $L_1$ onto $L_2$ and an orthogonal transformation that maps $L_1^\perp$ onto $L_2^\perp$.

Let $M$ be an even lattice of rank $l$ and let $M = M/2M$, considered as a vector space over $\mathbb{F}_2$. We equip $\tilde{M}$ with the quadratic form defined by

$$q(x + 2M) = \frac{1}{2} x^2 \mod 2\mathbb{Z}.$$  

The polar bilinear form $b_q$ associated with $q$ is defined by

$$b_q(x + 2M, y + 2M) = x \cdot y \mod 2\mathbb{Z}.$$  

Lemma 6.4.5. Let $M$ be a root lattice and let $(r, \delta, \epsilon)$ be the rank of $b_q$, defect and the sign of the quadratic form $q$ on $M$. Then

$$(r, \delta, \epsilon) = \begin{cases} 
(2m, \frac{1}{2}(1 + (-1)^m), (-1)^{\frac{1}{2}m(m+1)}) & \text{if } M = A_{2m+1}, \\
(2m, 0, (-1)^{\frac{1}{2}m(m+1)}) & \text{if } M = A_{2m}, \\
(4m - 2, 0, (-1)^m) & \text{if } M = D_{4m}, \\
(4m, 1, (-1)^{(m+1)} m) & \text{if } M = D_{4m+2}, \\
(2m, 0, (-1)^{\frac{1}{2}m(m+1)}) & \text{if } M = D_{2m+1}, \\
(6, 0, -1) & \text{if } M = E_6, \\
(6, 1, 1) & \text{if } M = E_7, \\
(8, 0, 1) & \text{if } M = E_8.
\end{cases}$$

Proof. The rank of $b_q$ is equal to $l - s$, where $s$ is the order of the 2-torsion group of the discriminant group of $M$. The known discriminant groups given, for example, in Bourbaki’s Tables [89], check the assertion for the rank.

Let $M = A_{2m+1}$ with the standard root basis $\alpha_1 = e_1 - e_2, \alpha_1 = e_2 - e_3, \ldots, \alpha_{2m+1} = e_{2m+1} - e_{2m+2}$, where $(e_1, \ldots, e_{2m+2})$ is the standard basis of $\mathbb{Z}^{2m+2}$. The radical $\tilde{M}^\perp$ of $M$ is spanned by the vector $r = \alpha_1 + \alpha_3 + \cdots + \alpha_{2m+1} \mod 2M$. We have $q(r) = m + 1 \mod 2$. Thus the defect $\delta$ is equal to $1 + (-1)^m$. We check that $\tilde{M}/\tilde{M}^\perp$ decomposes into the direct sum of $m$ subspaces $V_1, \ldots, V_m$, where $V_i$ is spanned by the cosets of $\alpha_1 + \alpha_3 + \cdots + \alpha_{2i-1} + \alpha_{2i}$. The quadratic form on $V_i$ is odd if $i$ is odd and even otherwise. Thus the sign of the quadratic form on $\tilde{M}/\tilde{M}^\perp$ is $-1$ if $m \equiv 1, 2 \mod 4$ and it is equal to 1 otherwise.

Suppose $M = A_{2m}$. The bilinear form is non-degenerate and, as above, we find the decomposition $\tilde{M} = V_1 \oplus \cdots \oplus V_m$ into the sum of quadratic forms with the signs indicated in the above paragraph.

If $M = D_l$, we use our notation for the simple roots $\alpha_0, \alpha_1, \ldots, \alpha_{l-1}$, where $\alpha_2$ is the trivalent vertex in the Dynkin diagram and $\alpha_0, \alpha_1$ correspond to the short arms. If $l = 2m$ (resp. $l = 2m + 1$), the radical $\tilde{M}^\perp$ of $M$ is generated by the cosets of $\alpha_0 + \alpha_1$ if $l = 2m + 1$ and cosets of $\alpha_0 + \alpha_1, \alpha_1 + \alpha_3 + \cdots + \alpha_{l-1}$ if $l$ is even. If $l$ is odd, $\tilde{M}$ is not defective. If $l = 2m$, then $\tilde{M}$ is defective if and only if $m$ is even. The quotient by the radical $\tilde{M}/\tilde{M}^\perp$ is spanned by the cosets of $\alpha_2, \ldots, \alpha_{l-1}$ if $l$ is even and $\alpha_1, \ldots, \alpha_{l-1}$ is $l$ is odd. If $l$ is even (resp. odd) the quadratic space is isomorphic to $\tilde{A}_{l-2}$ (resp. $\tilde{A}_{l-2}$). Now we can apply the previous case and find the sign of the reduced quadratic form.
6.4. NODAL INVARIANTS

Suppose \( M = E_6 \). The quadratic form on \( M \) is regular of dimension 6. It contains 36 non-isotropic vectors, the cosets of positive roots. Comparing the known number of such vectors, we obtain that the form is odd.

If \( M = E_7 \), the radical is spanned by the coset of the vector \( r = \alpha_0 + \alpha_4 + \alpha_6 \). Modulo the radical, \( M \cong E_6 \). Since \( q(r) = 1 \), we obtain \( (r, \delta, \epsilon) = (6, 1, 1) \). It is also confirmed by computing the number of positive roots which is equal to 63. Their cosets are non-isotropic vectors.

Finally, if \( M = E_8 \), then the form is non-degenerate and contains 120 non-isotropic vectors, the cosets of positive roots. This shows that \( q \) is even. \( \square \)

Using that \( E_{10} \cong E_8 \oplus U \), and \( U/2U \) is an even quadratic space of rank 2, we obtain

**Corollary 6.4.6.** The quadratic form on \( \bar{E}_{10} = E_{10}/2E_{10} \) is a regular even form of rank 10.

Let \( (N_{S}^{h_{10}}, G_{h_{10}}) \) be the Fano root invariant of an Enriques surface \( S \). To determine the image of \( N_{S}^{h_{10}} = N_{S}^{h_{10}}/2N_{S}^{h_{10}} \) in the quadratic space \( \text{Num}(S) \), we write \( N_{S}^{h_{10}} \) as the direct sum of irreducible root lattices and use the previous lemma. If \( N_{S}^{h_{10}} \) is a regular quadratic space, the group \( G_{h_{10}} \) is trivial. Otherwise, it is a subgroup of the radical of \( E_8 \). Its structure depends on the embedding of \( N_{S}^{h_{10}} \) in \( \text{Num}(S) \).

The following well-known fact from the theory of Coxeter groups (see [89, Chapter V, §4, Exercise 2(d)]) will be useful for us here and later.

**Lemma 6.4.7.** Let \( H \) be a finite subgroup of a Coxeter group \( (W, S) \). Then there exists a finite subset \( J \) of \( S \) such that the subgroup \( W_J \) generated by \( s \in J \) is finite and \( H \) is conjugate to a subgroup of \( W_J \).

A subgroup \( W_J \) of a Coxeter group is called a **parabolic subgroup**. It is a Coxeter group \( (W_J, J) \) and its Coxeter graph is called the type of \( W_J \). In our case \((W, S)\) is the Weyl group \( W(E_{10}) \) with the set of generators corresponding to the canonical root basis with the Coxeter-Dynkin diagram of type \( T_{2,3,7} \). Finite parabolic subgroups of \( W(E_{10}) \) with connected Coxeter-Dynkin diagrams are of types \( A_k, D_k, E_6, E_7, E_8 \).

Note that different subdiagrams of the same type in the Coxeter graph \( T_{2,3,7} \) of \( W_{2,3,7} \) may define non-conjugate subgroups. In general, there is an algorithm, due to V. Deodhar and R. Howlett (see [592]) that decides whether two subsets \( J, J' \) of the set \( S \) of Coxeter generators of a Coxeter group \((W, S)\) are \( W \)-equivalent. In our case, it works as follows. For any \( J \subseteq S \) generating a finite Coxeter group and \( s \in S \), let us denote by \( L = L(J, s) \) the connected component of \( J \cup \{s\} \) which contains \( s \). Let \( A(J) \) be the set of elements \( s \in S \) such that the Coxeter group \( W_L \) generated by \( L(J, s) \) is of type \( A_k, k > 1, D_{2k+1}, \) or \( E_6 \). Let \( s \in A(J) \) and \( s' = w_0(s) \), where \( w_0 \) is the element of the maximal length (it acts as the symmetry of the diagram) in \( W_L \). We set \( K(s, J) = (J \cup \{s'\}) \setminus \{s\} \). We say that \( J \) and \( K \) are related by an elementary equivalence. Then two subsets \( J, J' \) define \( W \)-equivalent subgroups \( W_J, W_{J'} \) if and only if there is a chain of elementary equivalences relating \( J \) and \( J' \).

Any parabolic subgroup \( W_J \) of \( W(E_{10}) \) is contained in a parabolic subgroup \( W_{J'} \), where \( \#J' = 9 \).
They are of the following types
\[ A_9, D_9, E_8 + A_1, A_1 + A_8, A_6 + A_2 + A_1, A_4 + A_5, E_6 + A_3, E_7 + A_2, D_5 + A_4. \] (6.4.4)

A subgroup of a Coxeter group generated by reflections is always a Coxeter group, however it is not necessary conjugate to a parabolic subgroup. For example, the lattice \( E_8 \) contains a sublattice isomorphic to \( A_1^{\oplus 8} \) but its Weyl group is not conjugate to a parabolic subgroup of \( W(E_8) \) because \( E_8 \) does not contain 8 orthogonal simple roots.

However, if \( W \) is the Weyl group of some root lattice, then any reflection subgroup of \( W \) is conjugate to the Weyl group of some root system of rank \( r \) (see, for example, [193]). The latter can be found using the following the Borel-de Siebenthal-Dynkin algorithm that can be derived from [85] or [197].

Let \( M \) be an irreducible root lattice with Dynkin diagram \( \Gamma \). Extend it to the affine Dynkin diagram \( \tilde{\Gamma} \) by adding the maximal root \( \alpha_{\text{max}} \). Then throw away one vertex \( v \). The remaining diagram \( \Gamma_v = \tilde{\Gamma} \setminus \{v\} \) is the Dynkin diagram of a root sublattice of \( M \) of the same rank. We repeat the process by choosing a connected component of \( \Gamma_v \) and delete a vertex from the extended Dynkin diagram \( \tilde{\Gamma}_v \). We continue the process until nothing is left. All root sublattices of the same rank are obtained in this way. Note that this algorithm allows us to classify all semi-simple Lie subalgebras conjugate parabolic subgroups. However, the subdiagrams \( \{85\} \) or \( \{197\} \).

If \( M \) is reducible, we write it as a direct sum of irreducible root lattices \( M_i \). Since the set of roots of a sublattice \( N \) of \( M \) is equal to the disjoint sum of the subsets of roots of each direct summand, we obtain that \( N \) is the direct sum of root sublattices of each \( M_i \).

**Example 6.4.8.** Let \( \alpha_0, \ldots, \alpha_9 \) is the canonical root basis of \( E_{10} \) from Figure 6.1.1. The subdiagrams of type \( A_8 \) in the Dynkin diagram \( T_{2,3,7} \) of \( E_{10} \) defined by the sets of simple roots \( J = \{\alpha_0, \alpha_3, \ldots, \alpha_9\} \) and \( \{\alpha_2, \ldots, \alpha_9\} \) are related by one elementary equivalence and define conjugate parabolic subgroups. However, the subdiagrams \( \alpha_1, \ldots, \alpha_8 \) and \( \alpha_0, \alpha_3, \ldots, \alpha_8 \) are not \( W \)-equivalent. On the other hand, all subsets of type \( A_7 \) are \( W \)-equivalent.

For another example, let \( M \) be the root lattice of type \( A_1^{\oplus 4} \) spanned by \( \alpha_0, \alpha_2, \alpha_4, \alpha_6 \) embedded in \( E_7 \) which we consider as a sublattice of \( E_{10} \) spanned by \( \alpha_0, \ldots, \alpha_6 \). Using the Borel-de Siebenthal-Dynkin algorithm, one easily shows that all parabolic subgroup corresponding to a set of disjoint vertices of the Dynkin diagram are conjugate. However, if we take roots \( (\beta_1, \beta_2, \beta_3, \beta_4) = (\alpha_{\text{max}}, \alpha_2, \alpha_4, \alpha_6) \), where \( \alpha_{\text{max}}^E = 2\alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 \) is the maximal root of the lattice \( E_7 \), we find a non-conjugate root sublattice isomorphic to \( A_1^{\oplus 4} \). To see that we have obtained different embeddings, one check that the sum \( \alpha_0 + \alpha_1 + \alpha_4 + r \notin 2E_7 \) in the parabolic embedding and \( \beta_1 + \beta_2 + \beta_3 + \beta_4 \in 2E_7 \) in the non-parabolic embedding.

One can consider another embedding of \( N = A_1^{\oplus 4} \) in \( E_7 \) defined by \( (\beta_1, \beta_2, \beta_3, \beta_4) = (\alpha_{\text{max}}^D, \alpha_0, \alpha_2, \alpha_4) \), where \( \alpha_{\text{max}}^D = \alpha_0 + \alpha_2 + \alpha_4 + 2\alpha_3 \) is the maximal root of the lattice \( D_4 \). We leave to the reader to check that it is conjugate to the previous one. Note that \( \frac{1}{2}(\beta_1 + \beta_2 + \beta_3 + \beta_4) \) defines an isotropic vector in \( D(N) \). In the non-parabolic case it corresponds to an embedding \( A_1^{\oplus 4} \) in \( D_4 \).

To decide whether the group \( G_{910} \) in the Fano root invariant \( (N_S^{h10}, G_{910}) \) is non-trivial, one checks whether the discriminant group of \( N_S^{h10} \) has a non-trivial 2-torsion subgroup. We refer to Table 1 for the information about the discriminant forms of root lattices. We also use that the discriminant
form of $U(2)$ is equal to $u_2$.

We will also use the following lemma.

**Lemma 6.4.9.** Let $\pi : X \to S$ be the canonical cover which we assume to be étale. Let $R_1, \ldots, R_k$ be $(-2)$-curves on $S$. Let $M$ be the sublattice of $\text{Pic}(X)$ spanned by the divisor classes $\delta_{R_i}$.

(i) Suppose that $R_1, \ldots, R_k$ span a negative definite lattice $N$. Then $M$ is isomorphic to $N(2)$.

(ii) Suppose $\sum_{i=1}^{k} R_i$ forms a half-fiber of type $\tilde{A}_{k-1}$ on $S$. Then $M$ is isomorphic to $A_1 \oplus A_1$ if $k = 2$, $A_3$ if $k = 3$ and $D_k$ if $k \geq 4$.

(iii) Suppose that $R_1, \ldots, R_k$ span a half-fiber of type $\tilde{A}_{k-1}$. Let $R_0$ be a special bisection. Then $k \leq 8$ and $\delta_{R_0}, \ldots, \delta_{R_k}$ span the following lattice

$$M = \begin{cases} A_1 \oplus A_2 & \text{if } k = 2, \\ A_4 & \text{if } k = 3, \\ D_5 & \text{if } k = 4, \\ E_6 & \text{if } k = 5, \\ E_7 & \text{if } k = 6, \\ E_8 & \text{if } k = 7, 8. \end{cases}$$

**Proof.**

(i) Since $\mathcal{R} = R_1 + \cdots + R_k$ is simply-connected, the cover splits over $\mathcal{R}$ into a disjoint sum of two nodal cycles whose components span a lattice isomorphic to $N$. Choosing the notation $R_i^\pm$ for irreducible components of $\pi^*(R_i)$, we can write

$$\pi^*(\mathcal{R}) = \sum_{i=1}^{k} R_i^+ + \sum_{i=1}^{k} R_i^-,$$

where $R_i^+ \cdot R_j^- = 0$. We have

$$(R_i^+ - R_i^-) \cdot (R_j^+ - R_j^-) = R_i^+ \cdot R_j^+ + R_i^- \cdot R_j^- = (R_i^+ + R_i^-) \cdot (R_j^+ + R_j^-) = 2R_i \cdot R_j.$$  

Thus $\delta_{R_1}, \ldots, \delta_{R_k}$ is a root basis of $M(\frac{1}{2})$ isomorphic to the root basis of $N$.

(ii) As above we can choose the notation to write $\pi^*(R)$ as in (6.4.5), where $R_i^+$ and $R_i^-$ are opposite vertices in the dual graph of the components $R_i^\pm$ isomorphic to a $2k$-gon. If $k = 2$, we see that $M(\frac{1}{2}) \cong A_1 \oplus A_1$. Assume $k \geq 3$. We see that $(R_i^+ - R_i^-) \cdot (R_j^+ - R_j^-) = -2$ and $(R_i^+ - R_i^-) \cdot (R_j^+ - R_j^-) = 2$ if $i = 1, \ldots, k - 1, j = i + 1$, and $(R_i^+ - R_i^-) \cdot (R_j^+ - R_j^-) = 0$ if $i, j = 1, \ldots, k - 1, j \neq i + 1$. Replacing the basis $(e_1, \ldots, e_k) = (\delta_{R_1}, \ldots, \delta_{R_k})$ of $\mathcal{M}$ with $(e_{k-1}, e_{k-2}, \ldots, e_2, -(e_2 + \cdots + e_k), e_1)$ we find a root basis in $M(\frac{1}{2})$ of type $A_3$ if $k = 3$ and $D_k$ if $k \geq 4$. This proves (ii).

(iii) Throwing away $R_0$, we find a root basis $(e_{k-1}, e_{k-2}, \ldots, e_2, -(e_2 + \cdots + e_k), e_1)$ as in (ii). We may assume that $\pi^*(R_0) = R_0^+ + R_0^-$, where $R_0^+$ intersects $R_1^+$ and $R_0^-$ intersects $R_1^-$. Adding to the previous basis $\delta_{R_0}$ as the last vector, we obtain a root basis $\delta_{R_0}$ of type $A_1 \oplus A_2$ if $k = 2$, $A_4$ if $k = 3$, $D_5$ if $k = 4$ and $E_{k+1}$ if $k = 5, 6, 7$. If $k = 8$, the Gram matrix of the set we obtain the Gram matrix of type $\tilde{E}_8$. Since $M$ is negative definite, this means that the vectors
are linearly dependent. By deleting some $R_i$, and using (i) and (ii) we see that the lattice $M(\frac{1}{2})$ contains sublattices isomorphic to $A_8, D_8, A_1 \oplus A_7, E_8$. Since the rank of $M$ is equal to 8, we see that the only possibility is that $M(\frac{1}{2}) \cong E_8$. The sublattices are embedded by using the Borel-de Siebenthal-Dynkin algorithm.

All of this makes feasible to classify all possible Fano root invariants and the corresponding quadratic subspaces $\langle \text{Nod}(S) \rangle$ of $\text{Num}(S)$. We restrict ourselves only with the cases where the Fano root invariants are of rank $l \leq 4$.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$l$ & $\mathbb{N}^{\oplus 10}_S$ & $K$ & $\tilde{K}$ & $\dim(\text{Nod}(S))$ & $(r, \delta)$ & $\epsilon$
\hline
1 & $\mathbb{A}_1$ & $\mathbb{A}_1$ & $\mathbb{A}_1$ & 1 & $(0, 1)$ & \\
2 & $\mathbb{A}_4^{\oplus 2}$ & $A_1^2$ & $A_1^2$ & 2 & $(0, 1)$ & \\
2 & $\mathbb{A}_2$ & $\mathbb{A}_2$ & $\mathbb{A}_2$ & 2 & $(2, 0)$ & -1
\hline
2 & $\mathbb{A}_1$ & $\mathbb{A}_1^2$ & $A_1^2$ & 2 & $(0, 1)$ & \\
3 & $\mathbb{A}_4^{\oplus 3}$ & $A_1^{\oplus 3}$ & $A_1^3$ & 3 & $(0, 1)$ & \\
3 & $\mathbb{A}_1 \oplus \mathbb{A}_1$ & $A_1^{\oplus 3}$ & $A_1^3$ & 3 & $(0, 1)$ & \\
3 & $\mathbb{A}_1 \oplus \mathbb{A}_2$ & $A_1^{\oplus 2} \oplus A_2$ & $A_1^2 \oplus A_2$ & 3 & $(2, 1)$ & \\
3 & $\mathbb{A}_3$ & $A_3$ & $A_3$ & 3 & $(2, 0)$ & -1
\hline
3 & $\mathbb{A}_2$ & $A_2$ & $A_2$ & 2 & $(2, 0)$ & -1
\hline
4 & $\mathbb{A}_4^{\oplus 3}$ & $A_1^{\oplus 3}$ & $D_4$ & 3 & $(0, 1)$ & \\
4 & $\mathbb{A}_4^{\oplus 3}$ & $A_1^{\oplus 3}$ & $D_4^2$ & 4 & $(0, 1)$ & \\
4 & $\mathbb{A}_1 \oplus \mathbb{A}_1$ & $A_1^{\oplus 4}$ & $A_1^4$ & 4 & $(0, 1)$ & \\
4 & $\mathbb{A}_1 \oplus \mathbb{A}_2$ & $A_1^{\oplus 2} \oplus A_2$ & $A_1^2 \oplus A_2$ & 4 & $(2, 1)$ & \\
4 & $\mathbb{A}_2 \oplus \mathbb{A}_2$ & $A_1^{\oplus 2} \oplus A_2$ & $A_1^2 \oplus A_2$ & 4 & $(2, 1)$ & \\
4 & $\mathbb{A}_4$ & $A_4$ & $A_4$ & 4 & $(4, 0)$ & -1
\hline
4 & $\mathbb{A}_1 \oplus \mathbb{A}_3$ & $A_1 \oplus A_3$ & $A_1 \oplus A_3$ & 4 & $(2, 1)$ & \\
4 & $\mathbb{A}_4$ & $A_4$ & $A_4$ & 4 & $(4, 0)$ & -1
\hline
4 & $\mathbb{D}_4$ & $D_4$ & $D_4$ & 4 & $(2, 0)$ & -1
\hline
\end{tabular}
\caption{Fano root invariants of rank $l \leq 4$}
\end{table}

Remark 6.4.10. We see from the previous Table that two different Fano root invariants may define the same quadratic space $\langle \text{Nod}(S) \rangle$. Also note that for larger rank the Fano root invariant is not necessary defined by a subdiagram of the Dynkin diagram of $E_{10}$. An example is the lattice $A_1^{\oplus 8}$ that can be realized by a quasi-elliptic surface.

The next invariant of a nodal Enriques surface is the Reye lattice defined by

$$\text{Rey}(S) := \{ x \in \text{Num}(S) : x \cdot R \in 2\mathbb{Z} \text{ for all } R \in \mathcal{R}_S \} = \{ x \in \text{Num}(S) : \frac{1}{2} x \in \mathbb{N}^{\oplus}_S \}. \quad (6.4.6)$$

It is clear that

$$\text{Rey}(S) = p^{-1}(\langle \text{Nod}(S) \rangle)^{\perp},$$
where \( p : \text{Num}(S) \to \text{Num}(S)/2\text{Num}(S) \) is the projection to the factor-space. Applying Witt’s Theorem 6.4.4 and the fact that the homomorphism \( \text{O}(E_{10}) \to \text{O}(\bar{E}_{10}) \) is surjective, we obtain

**Proposition 6.4.11.** The isomorphism class of \( \text{Rey}(S) \) depends only on the isomorphism class of the quadratic space \( \langle \text{Nod}(S) \rangle \).

It follows that

\[
\begin{align*}
\text{Num}(S)/\text{Rey}(S) &\cong \text{Num}(S)/\text{Nod}(S)\perp \cong \mathbb{F}_2^a, \\
\text{Rey}(S)/2\text{Num}(S) &\cong \mathbb{F}_2^{10-a}.
\end{align*}
\]

We have

\[
\text{Rey}(S) \subset \text{Num}(S) \subset \text{Rey}(S)^\vee.
\]

Thus the discriminant group \( D(\text{Rey}(S)) \) fits into an exact sequence

\[
0 \to (\mathbb{Z}/2\mathbb{Z})^a \to D(\text{Rey}(S)) \to (\mathbb{Z}/2\mathbb{Z})^a \to 0.
\]

Since \( \text{Num}(S) \) is a unimodular lattice, we infer that the maximal isotropic subgroup of \( D(\text{Rey}(S)) \) is of order \( 2^a = \frac{1}{2} \# D(\text{Rey}(S)) \) and

\[
D(\text{Rey}(S)) = (\mathbb{Z}/2\mathbb{Z})^{\oplus a} \oplus (\mathbb{Z}/4\mathbb{Z})^{\oplus b}.
\]

It follows from the classification of finite discriminant forms on such groups that the discriminant quadratic form on the group \( (\mathbb{Z}/2\mathbb{Z})^{\oplus a} \) is the orthogonal sum of \( w_{2,1}^\bot, u_1 \) and \( v_1 \). Also, the discriminant quadratic form on the group \( (\mathbb{Z}/4\mathbb{Z})^{\oplus b} \) is the orthogonal sum of \( w_{2,2}^\perp, w_{2,2}^5, u_2, v_2 \). Since the maximal isotropic subgroup of \( w_{2,2}^\perp \) is trivial, we obtain that \( \beta \) must be even and the quadratic form on \( (\mathbb{Z}/4\mathbb{Z})^{\oplus b} \) must be direct sum of quadratic forms \( u_2, v_2, w_{2,2}^1 \oplus w_{2,2}^{-1} \) or \( w_{2,2}^5 \oplus w_{2,2}^{-5} \).

It follows from [532, Proposition 1.11.2], that the lattice with discriminant form isomorphic to \( v_2 \) must have signature congruent to 0 modulo 8. This happens in our case since it is either a hyperbolic lattice of rank 10 (we will see later that it is isomorphic to \( E_{4,4} \)) or a negative definite lattice of rank 8 (we will see that it is a direct summand of \( E_{4,4} \)).

Note that the decomposition of the discriminant quadratic form into the orthogonal sum is not unique. For example,

\[
v_k \oplus v_k \cong u_k \oplus u_k.
\]

It follows from the definition that the image \( \overline{\text{Rey}}(S) \) in \( \overline{\text{Num}}(S) \) coincides with the orthogonal complement of the quadratic space \( \langle \text{Nod}(S) \rangle \)

\[
\overline{\text{Rey}}(S) = \langle \text{Nod}(S)\perp \rangle.
\]

Since \( \dim \overline{\text{Rey}}(S) = \dim \text{Rey}(S)/2\text{Num}(S) \), we obtain that

\[
a = \dim \langle \text{Nod}(S) \rangle.
\]

**Lemma 6.4.12.** Let \( L \) be a sublattice of \( E_{10} \) with \( E_{10}/L \cong (\mathbb{Z}/2\mathbb{Z})^a \). Let \( \bar{L} \) be its reduction modulo \( 2E_{10} \) in the quadratic space \( \bar{E}_{10} = E_{10}/2E_{10} \). Then its orthogonal complement \( \bar{L}^\perp \) is isomorphic to the quotient group \( D(L)/D(L)_0 \) of \( D(L) \) by the maximal isotropic subspace \( D(L)_0 \) that corresponds to the primitive embedding \( L \hookrightarrow E_{10} \). The quadratic form on \( \bar{L}^\perp \) is defined by \( q(x) = 2x^2 \mod 2\mathbb{Z}, \) where \( x^2 \) is the quadratic form on \( D(L) \).
follows from the definition. The plane is spanned by isotropic vectors of type \( \tilde{\alpha} \). We find that the roots \( \alpha \) in Table 6.4. Theorem 6.4.15. This defines the odd quadratic form \( x \cdot x \) corresponding to a basis of the quadratic space of vectors \( \tilde{\alpha} \). One can embed this lattice in \( E \) that can be identified with the maximal isotropic subgroup \( D(L) \) of \( D(L) \) that corresponds to the embedding \( L \rightarrow E \). The image of \( \phi \) is equal to \( L^\perp \). The assertion about the quadratic form follows from the definition.

\[ \Box \]

Example 6.4.13. Let \( (\alpha_0, \ldots, \alpha_{10}) \) be a generator of \( L = D_4 \oplus D_4 \oplus U \) given in (6.2.5). Here \( \alpha_0, \alpha_1, \alpha_2, \alpha_3 \) generate one copy of \( D_4 \), \( \alpha_4, \alpha_5, \alpha_9, \alpha_{10} \) generate another copy of \( D_4 \) and the summand \( U \) is generated by \( f = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_0, g = f + \alpha_5 \). The lattice \( L \) can be embedded in \( E_{10} \) of index \( 2^2 \). The discriminant group of \( L \) is generated by \( v_1 = \frac{1}{2}(\alpha_1 + \alpha_2), v_2 = \frac{1}{2}(\alpha_1 + \alpha_3), v_3 = \frac{1}{2}(\alpha_5 + \alpha_8), v_4 = \frac{1}{2}(\alpha_7 + \alpha_9) \). The discriminant quadratic form is isomorphic to \( v_1 \oplus v_1 \cong u_1 \oplus u_1 \). The maximal isotropic subspace corresponding to this embedding is generated by \( v_1 + v_3, v_2 + v_4 \). The quotient is generated by the cosets of \( v_1, v_2 \). Since \( v_1^2 = v_2^2 = 1, v_1 \cdot v_2 = \frac{1}{2} \) in \( D(L) \), we see that the subspace \( \tilde{L} \) is isomorphic to a regular quadratic space with quadratic form \( q = x_1x_2 \).

Example 6.4.14. Consider the lattice \( E_{4,4,4} \) spanned by a root basis defined by the following diagram

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]

We already noticed that its discriminant form is isomorphic to \( \nu_2 \) [95], [493].

One can embed this lattice in \( E_{10} \) by using the following diagram

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]

As we will see later it is realized as as the dual graph of \((-2)\)-curves on an exceptional classical Enriques surface of type \( \tilde{E}_6 \) (see Table 8.11). We use the notation of the vertices of any \( T_{pqr} \)-diagram from Part I, Figure 2. We find that \( v_1 = \frac{1}{4}(\alpha_0 + 2\alpha_1 + 3\alpha_2 + \alpha_5 + 2\alpha_4 + 3\alpha_3), v_2 = \frac{1}{4}(\alpha_0 + 2\alpha_8 + 3\alpha_7 + \alpha_5 + 2\alpha_4 + 3\alpha_3) \) generate \( D(L) \) and \( 2v_1, 2v_2 \) generate \( D(L) \). We have \( v_1^2 = v_2^2 = 1, v_1 \cdot v_2 = \frac{1}{2} \) as expected. We see that under an isomorphism \( D(L)/D(L)_0 \rightarrow L^\perp \), this corresponds to a basis of the quadratic space of vectors \( e_1, e_2 \) with \( e_1^2 = e_2^2 = 1 \) and \( e_1 \cdot e_2 = 1 \). This defines the odd quadratic form \( \tilde{x_1^2} + x_1 x_2 + x_2^2 \).

For the future use, let us observe that the lattice \( E_{4,4,4} \cong U \oplus E'_{4,4,4}, \) where \( E'_{4,4,4} \) is generated by the roots \( \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_2 \) spanning a copy of \( E_6 \) and vectors \( \alpha_3 - \alpha_0, \alpha_9 - \alpha_0 \). The hyperbolic plane is spanned by isotropic vectors of type \( \tilde{E}_6 \) and the half of an isotropic vector of type \( \tilde{E}_7 \).

Theorem 6.4.15. The Reye lattice \( \text{Rey}(S) \) with \( a \leq 5 \) is isomorphic to one of the lattices from the following Table 6.4.
6.4. NODAL INVARIANTS

<table>
<thead>
<tr>
<th>$a$</th>
<th>$\text{Rey}(S)$</th>
<th>rank</th>
<th>defect</th>
<th>sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$A_1 \oplus E_7 \oplus U \cong E_{2,4,6}$</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>$E_{4,4,4}$</td>
<td>2</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>$A_1 \oplus E_7 \oplus U(2)$</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$A_1 \oplus E_7 \oplus U(4)$</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$E_{4,4,4} \oplus U(2)$</td>
<td>2</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>3</td>
<td>$D_4 \oplus A_1^{\oplus 4} \oplus U$</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$D_4 \oplus D_4 \oplus U(4)$</td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>$E_{4,4,4} \oplus U(4)$</td>
<td>4</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>4</td>
<td>$D_6 \oplus A_1^{\oplus 2} \oplus U(4)$</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$L_1 \oplus U$</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>$A_1^{\oplus 8} \oplus U$</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$A_1^{\oplus 4} \oplus D_4 \oplus U(4)$</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$A_1^{\oplus 8} \oplus U(2)$</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$L_2 \oplus U$</td>
<td>4</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$L_3 \oplus U$</td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 6.2: Reye lattices with $a \leq 5$

Here the lattices $L_1, L_2, L_3$ are given by the matrices $M_1, M_2, M_3$ below. The triple $(r, \delta, \text{sign})$ denote the rank, the defect and the sign of the quadratic space $L^\perp \subset E_{10}$.

**Proof.** The embedding $\text{Rey}(S) \subset \text{Num}(S)$ corresponds to a 2-elementary isotropic subgroup of $D(\text{Rey}(S))$ of order $a$. This gives a restriction for a possible discriminant quadratic form. This number is always less than or equal to 8 unless $(\alpha, \beta) = (10, 0), (8, 2), (6, 4), (4, 6), (2, 8), (0, 10)$. In the first case, $\text{Rey}(S)$ is a maximal isotropic subspace of $E_{10}$, hence its orthogonal complement does not contain a vector $v$ with $q(v) = 1$. Hence it cannot be realized as $\langle \text{Nod}(S) \rangle$. In the last case we obtain that $\text{Rey}(S) = 2 \text{Num}(S)$, hence $\text{Rey}(S) \cong E_{10}(4)$. Except this, we have $\alpha + \beta \leq 8$, so we may apply Theorem 0.8.5 to obtain that the lattice is uniquely determined by its discriminant form.

In the following we denote $\text{Rey}(S)$ by $L$. Assume $a = 1$. There is only one isomorphism class of a non-trivial one-dimensional quadratic space. Applying Proposition 6.4.11 we obtain that there is only one possible Reye lattice. Since $D(L) = (\mathbb{Z}/2\mathbb{Z})^\oplus 2$, the possible discriminant forms with non-trivial isotropic subgroup are $u_1$ and $w_1^{1} \oplus w_2^{1}$. In the first case $D(L)/D(L)_0$ is generated by an isotropic vector, so this case is discarded since $\langle \text{Nod}(S) \rangle = L^\perp$ is generated by a non-isotropic vector. The remaining case is realized by a lattice $L \cong A_1 \oplus E_7 \oplus U$.

Assume $a = 2$. In this case there are two possible 2-dimensional quadratic spaces generated by non-isotropic vectors. They are defined by non-defective odd quadratic form or a defective quadratic form of rank 0. To realize the first case we can take the lattice $L = A_1 \oplus E_7 \oplus U(2)$ with discriminant form $D(L) \cong (w_1^{1} \oplus w_2^{1}) \oplus u_1$. The second possible quadratic space corresponds to the case $(\alpha, \beta) = (0, 2)$. The only possible discriminant form satisfying our conditions is $v_2$. It is realized by the lattice $E_{4,4,4}$. 


Assume $a = 3$. There are three possible 3-dimensional quadratic spaces generated by non-isotropic vectors: a non-defective odd quadratic form of rank 2, a defective quadratic form of rank 2 and a quadratic form of rank 0. Using the previous case, we realize the first quadratic form by the lattice $L = E_4^{r,4,4} \oplus U(2)$ with discriminant form $v_2 \oplus u_1$. Here $E_4^{r,4,4}$ is the negative definite lattice of rank 8 with finite discriminant quadratic form $v_2$ introduced in Example 6.4.14. We have $(\alpha, \beta) = (6, 0), (2, 2)$. So, the lattice $L$ corresponds to the case $(\alpha, \beta) = (2, 2)$. Another possibility in this case is the discriminant quadratic form $(w_{2,1}^1 \oplus w_{2,1}^{-1}) \oplus u_2$. It is realized by the lattice $L = A_1 \oplus E_7 \oplus U(4)$. The case $(\alpha, \beta) = (6, 0)$ corresponds to a lattice of rank 10 with a 2-elementary discriminant group. According to Nikulin [533] there are two isomorphism classes of such lattices of rank 10. They are represented by the lattices $A_1^{\oplus 2} \oplus U$ and $E_8(2) \oplus U$. The second lattice does not embed in $E_{10}$ (a maximal isotropic subgroup corresponds to an embedding into an odd unimodular lattice $l^{10}$). Since $D(L)$ in the first case is generated by orthogonal vectors with value of the quadratic form equal to $\frac{1}{2}$, we see that $L$ has rank 0. So, $L$ realizes the quadratic space of rank 0.

Assume $a = 4$. We have $(\alpha, \beta) = (8, 0), (4, 2)$, or $(0, 4)$. In the first case, the discriminant group is 2-elementary. According to Nikulin, there are two isomorphism classes of such lattices of rank 10. They are represented by the lattices $A_1^{\oplus 2} \oplus U$ and $E_8(2) \oplus U$. The second lattice does not embed in $E_{10}$ (a maximal isotropic subgroup corresponds to an embedding into an odd unimodular lattice $l^{10}$). Since $D(L)$ in the first case is generated by orthogonal vectors with value of the quadratic form equal to $\frac{1}{2}$, we see that $L$ has rank 0. So, $L$ realizes the quadratic space of rank 0.

Assume now that $(\alpha, \beta) = (4, 2)$. Possible discriminant quadratic forms are $(w_{2,1}^1)^{\oplus 2} \oplus (w_{2,1}^{-1})^{\oplus 2} \oplus u_2$ and $u_1^{\oplus 2} \oplus u_2$. The first one is realized by the lattice $L = D_6 \oplus A_1^{\oplus 2} \oplus U(4)$. The quadratic space $L^\perp$ has $r = 2$ and $\delta = 1$. The second quadratic form is realized by the lattice $D_4 \oplus D_4 \oplus U(4)$. The quadratic space $L^\perp$ is non-defective, even of rank 4.

In the case $(\alpha, \beta) = (0, 4)$, we must have $D(L) = u_2 \oplus v_2$ or $u_2^{\oplus 2}$. The first case is realized by the lattice $L = E_4^{r,4,4} \oplus U(4)$. The quadratic space is the direct sum of an odd rank 2 space and an even rank 2 space. So, $L$ has $(r, \delta, \epsilon) = (4, 0, -1)$. Note that we do not see an $R$-invariant $M$ of rank $\leq 5$ with $M$ isomorphic to such quadratic space. However, this space is realized by a Fano root invariant $M \cong D_6$ of rank $l = 6$. By Proposition 1.11.2 from [532], the signature of $L$ realizing $u_2^{\oplus 2}$ must be equal to 0 mod 8. We have not realized the possibility that the quadratic space $L^\perp$ is even of rank 2. So, this must be the case when $L$ has the discriminant quadratic form $u_2^{\oplus 2}$. We use that $L$ is orthogonal mod 2 to the Fano root invariant $M = D_4$. By embedding $D_4$, in an obvious way, into $E_{10}$ we find that $L$ is generated by $\alpha_0 + \alpha_2 + \alpha_4, 2\alpha_i, i \neq 4, 6, 7, 8, 9$ and $\alpha_i, i = 6, 7, 8, 9$. Thus $L \cong L_1 \oplus U$, where $L_1$ is given by the following matrix:

$$M_1 = \begin{pmatrix} -6 & 4 & 2 & -4 & 0 & 2 & 0 & 0 \\ -4 & -8 & 4 & 0 & 0 & 0 & 0 & 0 \\ 2 & 4 & -8 & 4 & 0 & 0 & 0 & 0 \\ -4 & 0 & 4 & -8 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & -8 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & -8 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \end{pmatrix}$$

The integral Smith normal form of $M_1$ gives that $(\alpha, \beta) = (2, 4)$.

Assume $a = 5$. We have $(\alpha, \beta) = (10, 0), (6, 2)$ or $(2, 4)$. According to [533], there are only two even hyperbolic lattices of rank 10 with 2-elementary discriminant group of rank 10. They are
E_{10}(2) and \( E^{1,9}(2) \approx A_1^8 \oplus U(2) \). Only the second one embeds in \( E_{10} \) as \( A_1^8 \oplus U(2) \hookrightarrow E_8 \oplus U \). Using the previous case, we find that \( \bar{L}^\perp \) is an even quadratic form with \((r, \delta) = (0, 1)\). The lattice \( L = D_8(2) \oplus U \) realizes the case \((6, 2)\). Its discriminant quadratic form is \( u_1^{\oplus 3} \oplus u_2 \). We see that \( \bar{L}^\perp \) is a quadratic space with \((r, \delta) = (2, 0, 1)\).

We can also realize the case \((6, 2)\) by the lattice \( A_1^{10} \oplus D_4 \oplus U(4) \) with discriminant quadratic form \( (w_{2,1}^{-1})^{\oplus 4} \oplus v_1 \oplus u_2 \). We can choose \( D(A_1^{10} \oplus D_4)_0 \) generated by \( v_1 + v_2 + w_1, v_1 + v_3 + w_2, v_1 + v_2 + v_3 + v_4 \), where \( (v_1, v_2, v_3, v_4) \) is an orthogonal basis of \( A_1^{10} \) and \( w_1, w_2 \) is the standard basis of \( v_1 \). The quadratic space \( \bar{L}^\perp \) is defective of rank 2.

Unfortunately, we do not see how to realize other three possible cases by using the orthogonal sum of root lattices. So we have to find them by straightforward computations.

Assume that the quadratic space is defective of rank 4. We may take \( M = A_1 \oplus A_4 \) generated by simple roots \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6 \).

The Reye lattice is spanned by \( 2\alpha_0, 2\alpha_1, 2\alpha_2, 2\alpha_3, \alpha_0 + \alpha_4, \alpha_1 + \alpha_3 + \alpha_5 + \alpha_7, \alpha_6, 2\alpha_7, \alpha_8, \alpha_9 \). The lattice is the orthogonal sum of \( U \) and the lattice \( L_2 \) given by the matrix

\[
M_2 = \begin{pmatrix}
-8 & 0 & 0 & 4 & -4 & 2 & 0 & 0 \\
0 & -8 & 4 & 0 & 0 & -4 & 0 & 0 \\
0 & 4 & -8 & 4 & 0 & 4 & 0 & 0 \\
4 & 0 & 4 & -8 & 4 & -4 & 0 & 0 \\
-4 & 0 & 0 & 4 & -4 & 3 & 0 & 0 \\
2 & -4 & 4 & -4 & 3 & -8 & 2 & -4 \\
0 & 0 & 0 & 0 & 0 & 2 & -2 & 2 \\
0 & 0 & 0 & 0 & 0 & -4 & 2 & -8
\end{pmatrix}.
\]

The integral Smith normal form of the matrix \( M_2 \) shows that the discriminant group has \((\alpha, \beta) = (2, 4)\).

Note that there is another, non-isomorphic, embeddings of \( A_1 \oplus A_4 \) in \( E_{10} \). It is represented by the diagrams

In both cases \( M \) defines the Fano root invariant \( (N_S^{h_{10}}, G_{h_{10}}) \) with \( G_{h_{10}} = \{0\} \) and \( \bar{M} \) does not depend on the isomorphism class of the embedding.

Finally, to realize the quadratic space with \((r, \delta, \epsilon) = (4, 0, 1)\) we take \( N_S^{h_{10}} = A_2 \oplus A_3 \) generated by \( \alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6 \)

The Reye lattice is spanned by \( \alpha_0, 2\alpha_1, 2\alpha_2, 2\alpha_3, \alpha_4 + \alpha_6, 2\alpha_4, 2\alpha_5, 2\alpha_7, \alpha_8, \alpha_9 \). It is isomorphic
to $U \oplus L_3$, where $L_4$ is given by the matrix

$$M_3 = \begin{pmatrix}
-2 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \\
0 & -8 & 4 & 0 & 0 & 0 & 0 & -4 \\
0 & 4 & -8 & 4 & 0 & 0 & 0 & 4 \\
2 & 0 & 4 & -8 & 2 & 4 & 0 & -4 \\
0 & 0 & 0 & 2 & -4 & -4 & 4 & 4 \\
0 & 0 & 0 & 4 & -4 & -8 & 4 & 4 \\
0 & 0 & 0 & 4 & 4 & -8 & -4 & -4 \\
1 & -4 & 4 & -4 & 4 & 4 & -4 & -8
\end{pmatrix}$$

The integral Smith normal form of $M_3$ shows that the discriminant group has $(\alpha, \beta) = (2, 4)$.

**Example 6.4.16.** In this example we will compute the Fano root invariants and the Reye lattices of extra-special Enriques surfaces.

Suppose that $S$ is extra-special of type $\tilde{E}_8$ with crystallographic nodal basis equal to the canonical root basis of $\text{Num}(S) \cong E_{10}$. Obviously, $N_S \cong E_{10}$ and $\text{Rey}(S) = 2 \text{Num}(S) \cong E_{10}(4)$. There is only one numerical Fano polarization $h_{10}$ defined by the canonical isotropic sequence

$$(f, f + R_9, f + R_9 + R_8, \ldots, f + R_9 + \cdots + R_2).$$

We have $h_{10} \cdot R_i = 0$, $i \neq 0$ and $h_{10} \cdot R_0 = 1$. Thus the Fano model has one line and a rational double point of type $A_8$ lying on this line. Obviously, $N_S^{h_{10}} = N_S$.

Let $S$ be an extra-special surface of type $\tilde{E}_7$. Then the $(-2)$-curves on $S$ are given by vertices of the diagram $\tilde{E}_7^2$ from Theorem 6.2.3. It is obvious that $N_S \cong A_1 \oplus \tilde{E}_7 \oplus U$.

There is only one numerical Fano polarization $h_{10}$ defined by the canonical isotropic sequence

$$(f, f + R_9, f + R_9 + R_8, \ldots, f + R_9 + \cdots + R_3, g, g + R_{10}),$$

where $f$ is the class of half-fiber of type $\tilde{E}_7$ and $g$ is the class of a half-fiber of the genus one fibration with simple fiber of type $\tilde{E}_8$. We find that $h_{10} \cdot R_i = 0$ for $i \neq 1, 10, 11$ and $h_{10} \cdot R_1 = 1$, $h_{10} \cdot R_{10} = h_{10} \cdot R_{11} = 3$. Thus $N_S^{h_{10}} = N_S$. We see that $\frac{1}{2}N_S/2N_S$ is generated by $\frac{1}{3}R_1 + R_6 + R_8 + R_{10}$, hence $G_{h_{10}} \cong \mathbb{Z}/2\mathbb{Z}$. Thus the Fano root invariant of $S$ is equal to $(E_{2,4,5}, \mathbb{Z}/2\mathbb{Z})$.

It follows now that the quadratic space $N_S^{\text{mod}}$ is 9-dimensional, defective of rank 8. Its orthogonal complement is a defective quadratic space of rank 0 and dimension 1. The Reye lattice coincides with its pre-image under the map $p : \text{Num}(S) \rightarrow \overline{\text{Num}}(S)$, hence it must be isomorphic to $2E_9 \oplus A_1$.

Finally, let $S$ be an extra-special surface of type $\tilde{D}_8$. In notation of the diagram (6.2.4), we see that $D(N_S)$ is generated by $\frac{1}{2}R_1 + R_5 + R_7 + R_9$ and $\frac{1}{2}R_9 + R_{10}$. Thus, again $N_S$ has 2-elementary discriminant form. Computing the discriminant quadratic form, we find that $N_S \cong E_8 \oplus U(2)$.

We can confirm the previous computations by computing $N_S^{h_{10}}$. Note that $R_1 + R_5 + R_7 + R_9 \in 2 \text{Num}(S)$ if we assume that it is supported in a simple fiber of type $\tilde{E}_8$. We take a Fano polarization corresponding to a canonical isotropic sequence $(f, f + R_9, f + R_9 + R_8, \ldots, f + R_9 + \cdots + R_4, f + R_9 + \cdots + R_4 + R_1, g, g + R_2)$, where $f$ is the numerical class of the half-fiber of the genus one fibration of type $\tilde{D}_8$ and $g$ is the numerical class of a half-fiber of the genus one fibration with a simple fiber of type $\tilde{E}_8$. We assume that the latter contains the component $R_{10}$. We check that
Remark 6.4.17. In [46, p.393], the author define the nodal type of a complex Enriques surface. Let \( \pi : X \to S \) be the canonical cover of \( S \). Consider the following sublattices of \( L = H^2(X, \mathbb{Z}) \cong L_{K3} \). Let \( L^+ \oplus L^- \subseteq L \) be the orthogonal direct sum of the invariant and anti-invariant sublattices with respect to the deck transformation of \( X/S \). First let \( L_1 = \pi^*(N_S)' \subseteq L^+ \), where \( (N_S)' \) is the primitive closure of \( N_S \). Let \( L_2 \) be the smallest sublattice of \( L^- \) containing the classes \( R^+ - R^- \), where \( \pi^*(R) = R^+ + R^- \) for some \((-2)\)-curve \( R \) on \( S \). Let \( L_3 = L_2^+ \cap \text{Pic}(X) \cap L^- \). The nodal type of \( S \) is defined to be the smallest primitive sublattice \( N \) of \( L \) containing \( T_X = \text{Pic}(X)^\perp \) and \( L_3 \). It is clear that \( N \) determines \( L_2 \) which is the lattice \( K \) in the Nikulin \( R \)-invariant \( \text{Nik}(S) \).

Assume that \( k = \mathbb{C} \). Following S. Mukai, one can give the following refinement of the Nikulin \( R \)-invariant. We identify the fundamental group \( \pi_1(S) \) with \( (\sigma) \), where \( \sigma \) is the Enriques involution of \( X \). Let \( \mathbb{Z}_S^\omega \) be the local coefficient system on \( S \) defined by the unique non-trivial homomorphism \( \mathbb{Z}/2\mathbb{Z} = \pi_1(S) \to \text{GL}(\mathbb{Z}) = \{ \pm 1 \} \). We have \( \pi^*(\mathbb{Z}_S^\omega) = \mathbb{Z}_X \) and, hence, a canonical homomorphism \( \mathbb{Z}_S^\omega \to \pi_*(\mathbb{Z}_X) \). It identifies \( \mathbb{Z}_S^\omega \) with the subsheaf of \( \pi_*\mathbb{Z}_X \). The quotient sheaf is the sheaf \( \mathbb{Z}_S \), so that we get an exact sequence of local coefficient systems on \( S \)

\[
0 \to \mathbb{Z}_S^\omega \to \pi_*\mathbb{Z}_X \to \mathbb{Z}_S \to 0
\]

The map \( \pi_*(\mathbb{Z}_X) \to \mathbb{Z}_S \) is the local trace map. The cup-product defines a perfect pairing

\[
\mathbb{Z}_S^\omega \times \mathbb{Z}_S^\omega \to \mathbb{Z}_S.
\]

It allows one to extend the usual Poincaré type dualities and the universal coefficient theorem from Chapter (0.9) to the cohomology with coefficients in \( \mathbb{Z}_S^\omega \). Since \( \mathbb{Z}_S^\omega \) is not trivial, we have \( H^0(S, \mathbb{Z}_S^\omega) = \{ 0 \} \). The map \( H^0(S, \pi_*\mathbb{Z}_X) = H^0(X, \mathbb{Z}) \to H^0(S, \mathbb{Z}_S) \) is the multiplication by 2 map \( \mathbb{Z} \to \mathbb{Z} \). This gives \( H^1(S, \mathbb{Z}_S^\omega) \cong \mathbb{Z}/2\mathbb{Z} \) and \( H^3(S, \mathbb{Z}_S^\omega) = 0 \). Since \( H^1(S, \mathbb{Z}_S) = 0 \), we obtain an exact sequence

\[
0 \to H^2(S, \mathbb{Z}_S^\omega) \to H^2(X, \mathbb{Z}) \to H^2(S, \mathbb{Z}) \to 0.
\]

This gives

\[
H^2(S, \mathbb{Z}_S^\omega) \cong \mathbb{Z}^{12}.
\]

The Poincaré Duality defines a structure of a unimodular quadratic lattice on \( H^2(S, \mathbb{Z}_S^\omega) \) of signature \((2, 10)\). The lattice \( H^2(S, \mathbb{Z}_S^\omega)(2) \) is a sublattice of \( H^2(X, \mathbb{Z}) \). Since \( H^2(S, \mathbb{Z}) \) has torsion subgroup of order 2, it is not a primitive sublattice.

For any \( \gamma \in H^2(X, \mathbb{Z}) \), we can write \( 2\gamma = (\gamma + \sigma^*(\gamma)) + (\gamma - \sigma^*(\gamma)) \). Thus, for any \( \alpha \in H^2(S, \mathbb{Z}) \), we have

\[
(2\gamma, \pi^*(\alpha))_X = (\gamma + \sigma^*(\gamma), \pi^*(\alpha))_X = 2(\beta, \alpha)_S,
\]
where \( \pi^*(\beta) = \gamma + \sigma^*(\gamma) \). This shows that the homomorphism \( H^2(X, \mathbb{Z}_X) \to H^2(S, \mathbb{Z}) \) in (6.4.13) coincides with the Gysin map \( \pi_* : H^2(X, \mathbb{Z}) \to H^2(S, \mathbb{Z}) \). We saw before that it is a surjective map (see another proof in [53]). Thus,

\[
H^2(S, \mathbb{Z}_S^2)(2) = \text{Ker}(\pi_*).
\]

The saturation of the sublattice \( H^2(S, \mathbb{Z}_S^2) \) in \( H^2(X, \mathbb{Z}) \) coincides with the sublattice \( \text{Ker}(1 + \sigma^*) \).

We see in Chapter 5, that one can find a decomposition \( H^2(X, \mathbb{Z}) = E_{10} \oplus U \) such that, for any \( x, y \in E_{10}, a \in U \), we have \( \sigma^*(x, y, a) = (y, x, -a) \). This shows that

\[
\text{Ker}(1 + \sigma^*) = \{(x, -x, a), x \in E_{10}, a \in U \} \cong E_{10}(2) \oplus U.
\]

It is equal to the orthogonal complement of \( \pi^*(H^2(S, \mathbb{Z})) \cong E_{10}(2) \) in \( H^2(X, \mathbb{Z}) \). It is easy to see that the only unimodular lattice \( M \) of rank 12 of signature \( (2, 10) \) such that \( M(2) \) embeds in \( E_{10}(2) \oplus U \) as a sublattice of index 2 is the odd lattice

\[
I^{2,10} \cong E_{10} \oplus (1) \oplus (-1) \cong (1) \oplus (2) \oplus (-1)^{10}
\]

where \( I^{1,1}(2) = \langle 2 \rangle \oplus \langle -2 \rangle \) embeds in \( U \) as a sublattice generated by \( f + g \) and \( f - g \), where \( f, g \) is a canonical basis of \( U \). This realization of the lattice \( I^{2,10} \) was first introduced in [6] who used it to simplify the theory of periods for Enriques surfaces. Thus, we obtain an isomorphism of quadratic lattices

\[
H^2(S, \mathbb{Z}_S^2) \cong I^{2,10}.
\]

It follows from the previous discussion that

\[
\text{Ker}(1 + \sigma^*) = \langle H^2(S, \mathbb{Z}_S^2)(2), \alpha \rangle,
\]

where \( \pi_*(\alpha) = K_S \).

Suppose now that \( k \) is not necessary the field of complex number but the canonical cover \( \pi : X \to S \) is étale. Following Mukai, we set

\[
\text{Pic}^\omega(S) := \text{Ker}(\text{Nm}),
\]

where \( \text{Nm} : \text{Pic}(X) \to \text{Pic}(S) \) is the norm map defined in Section 1.3. It follows from [233, Theorem 1.4] that it coincides with the map \( \pi_* : \text{Pic}(X) \to \text{Pic}(S) \) defined on the divisor classes. If \( k = \mathbb{C} \), we identify \( \text{Pic}^\omega(S)(\frac{1}{2}) \) with a sublattice of \( H^2(S, \mathbb{Z}_S^2) \). It is clear now that

\[
\text{Im}(1 - \sigma^*) \subset \text{Pic}^\omega(S) \subset \text{Ker}(1 + \sigma^*),
\]

and the first factor is non-trivial if and only if the homomorphism \( \text{Br}(S) \to \text{Br}(X) \) is the zero map, and the second factor is non-zero if and only if \( K_S \in \text{Im}(\text{Nm}) \). Since \( 2x = (x + \pi^*(x)) + (x - \pi^*(x)) \), we obtain that \( \text{Ker}(1 + \sigma^*)/\text{Im}(1 - \sigma^*) \) is killed by 2.

It is easy to describe elements of the group \( \text{Im}(1 - \sigma^*) \). Note that, for any irreducible curve \( C \) on \( X \), we have \( C \cdot \sigma^*(C) \) is even (otherwise \( \tau \) has a fixed point on \( C \)). This implies that \( (C - \sigma^*(C))^2 \equiv 0 \mod 4 \), and hence \( \text{Im}(1 - \sigma^*) \) is an even sublattice of \( \text{Pic}^\omega(S)(\frac{1}{2}) \). Thus if \( \text{Pic}^\omega(S)(\frac{1}{2}) \) is an odd sublattice of \( H^2(S, \mathbb{Z}_S^2) \), we have \( \text{Pic}^\omega(S) \neq \text{Im}(1 - \sigma^*) \), hence the map \( \pi^* : \text{Br}(S) \to \text{Br}(X) \) is the zero map. Conversely, if this map is the zero map, then, by Corollary 5.7 in [53], there exists a divisor class \( D \in \text{Ker}(\text{Nm}) \) such that \( D^2 \equiv 2 \mod 4 \). This shows that \( \text{Pic}^\omega(S)(\frac{1}{2}) \) is an odd lattice in this case.

Note that the definition of \( \text{Pic}^\omega(S) \) makes sense in any characteristic \( p \neq 2 \) and in characteristic 2 if \( S \) is a \( \mu_2 \)-surface. It is a quadratic lattice such that \( \text{Pic}^\omega(S)(2) \) coincides with \( \text{Ker}(\text{Nm}) \). In
particular, \( \text{Pic}^\omega(S) \) lies in the orthogonal complement of \( \pi^*(\text{Pic}(S)) \) and hence must be a negative definite quadratic lattice. Also, by Riemann-Roch, any divisor of norm \(-2\) must be effective or anti-effective, and, since \( \pi^*(\text{Pic}(S)) \) contains an ample divisor class, we see that \( \text{Pic}^\omega(S) \) does not contain elements of norm \(-1\).

For any smooth compact oriented 4-manifold \( M \), the exact sequence

\[
0 \to \mathbb{Z} \overset{[2]}{\to} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0
\]

defines an exact sequence

\[
0 \to H^2(M, \mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z} \to H^2(M, \mathbb{Z}/2\mathbb{Z}) \to H^3(M, \mathbb{Z})[2] \to 0.
\]

The Poincaré Duality defines a non-degenerate symmetric form on \( H^2(M, \mathbb{Z}/2\mathbb{Z}) \) with values in \( \mathbb{Z}/2\mathbb{Z} \). The cohomology \( H^2(M, \mathbb{Z}/2\mathbb{Z}) \) contains a special class, the Stiefel-Whitney class \( w_2(M) \) such that, for any \( x \in H^2(M, \mathbb{Z}/2\mathbb{Z}) \), we have \( (w_2, x) = x^2 \). If \( M \) is a complex surface, then \( w_2(M) \) is the image of \( c_1(M) = -K_M \) in \( H^2(M, \mathbb{Z}/2\mathbb{Z}) \).

Applying this to \( M = X \), we get

\[
H^2(X, \mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z} \cong H^2(X, \mathbb{Z}/2\mathbb{Z}).
\]

Applying this to \( M = S \), we get an exact sequence

\[
0 \to H^2(S, \mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z} \overset{p}{\to} H^2(S, \mathbb{Z}/2\mathbb{Z}) \overset{\phi}{\to} H^3(S, \mathbb{Z})[2] \to 0.
\]

We have \( H^2(S, \mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z} \cong \overline{\text{Num}}(S) \mathbb{Z}/2\mathbb{Z} \mathbb{Z}/2\mathbb{Z} \), and \( H^3(S, \mathbb{Z})[2] \cong \text{Br}(S) \). For any \( x \) in the image of \( p \), we have \( w_2(S) \cdot x = 0 \), hence \( x^2 = 0 \).

One has the analogue of the exact sequence (6.4.14) for cohomology with coefficient in local systems. For our need we use the exact sequence

\[
0 \to \mathbb{Z}^\omega_S \overset{[2]}{\to} \mathbb{Z}^\omega_S \to \mathbb{Z}/2\mathbb{Z}^\omega_S \to 0.
\]

Since \( \text{GL}(\mathbb{Z}/2\mathbb{Z}) = \{1\} \), the local coefficient system \( (\mathbb{Z}/2\mathbb{Z})^\omega_S \) is trivial, hence isomorphic to \( (\mathbb{Z}/2\mathbb{Z})^\omega_S \). The exact sequence of cohomology gives an exact sequence

\[
0 \to H^1(S, \mathbb{Z}/2\mathbb{Z}) \to H^2(S, \mathbb{Z}^\omega_S) \otimes \mathbb{Z}/2\mathbb{Z} \to H^2(S, \mathbb{Z}/2\mathbb{Z}) \to 0.
\]

Let

\[
c : \text{Pic}^\omega(S) \to H^2(S, \mathbb{Z}^\omega_S) \to H^2(S, \mathbb{Z}/2\mathbb{Z})
\]

be the composition of the inclusion of \( \text{Pic}^\omega(S) \) in \( H^2(S, \mathbb{Z}^\omega_S) \) with the reduction mod 2 map \( H^2(S, \mathbb{Z}^\omega_S) \to H^2(S, \mathbb{Z}^\omega_S) \otimes \mathbb{Z}/2\mathbb{Z} \) followed by the projection map \( H^2(S, \mathbb{Z}^\omega_S) \otimes \mathbb{Z}/2\mathbb{Z} \to H^2(S, \mathbb{Z}/2\mathbb{Z}) \).

For any \( x \in H^2(X, \mathbb{Z}) \), we can write \( 2x = (x + \sigma^*(x)) + (x - \sigma^*(x)) \). This shows that the restriction of \( c \) to the subgroup \( \text{Im}(1 - \sigma^*) \) is defined as follows. Let \( x = \sigma^*(y) - y \in \text{Pic}^\omega(S) \). Then \( \sigma^*(y) + y = \pi^*(z) \) for some \( z \in \text{Pic}(S) \). It is immediate to see that \( z \mod 2 \text{Pic}(S) \) is independent of a choice of \( y \) and hence defines an element in \( \overline{\text{Num}}(S) \subset H^2(S, \mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z} \). Its image in \( H^2(S, \mathbb{Z}/2\mathbb{Z}) \) is equal to \( c(x) \).

Let \( w_2(S) \) be the image of \( K_S \) in \( H^2(S, \mathbb{Z}/2\mathbb{Z}) \). Suppose it belongs to \( c(\text{Pic}^\omega(S)) \). Then, by Wu’s formula (0.10.12), for any \( x \) in the orthogonal complement of \( \text{Pic}^\omega(S)(\frac{1}{2}) \) in \( H^2(S, \mathbb{Z}^\omega_S) \), we have \( x^2 \equiv 0 \mod 2 \). This shows that \( \text{Pic}^\omega(S)(\frac{1}{2}) \) is an even lattice. The converse is also true, if \( \text{Pic}^\omega(S)(\frac{1}{2}) \) is an even sublattice of \( H^2(S, \mathbb{Z}^\omega_S) \), then \( w_2(S) \in c(\text{Pic}^\omega(S)) \). Also note that the
latter happens if and only if there exists a divisor class $D$ in $\Pic(X)$ such that $\Nm(D) = K_S$.

Let $L$ be a negative definite lattice that does not contain elements of norm $-1$. We say that an Enriques surface is of Mukai's nodal type $L$ if there is a primitive embedding of $L$ in $\Pic^0(S)(1/2)$. We say that the nodal type is odd (resp. even) if the orthogonal complement of $L$ in $\Pic^0(S)(1/2)$ is odd (resp. even). When we want to distinguish these cases, we will write $(L, \text{even})$ (resp. $(L, \text{odd})$).

Let $(K, H)$ be the Nikulin $R$-invariant of $S$. It is clear that $K$ is a sublattice of $\Pic^0(S)(1/2)$. It is the maximal even sublattice of $\Pic^0(S)(1/2)$. Also, one can show that the maximal even sublattice of $\Pic^0(S)(1/2)$ is generated by $K$ and elements $\frac{1}{2}h \in K^\vee$, such that $h \mod 2K \in H$. Note that, in general, the maximal even sublattice $L$ of $\Pic^0(S)(1/2)$ could be larger than $K$. For example, if $S$ admits an elliptic fibration with a double fiber $F = 2F_0$ of type $\tilde{A}_1$, we have $\pi^*(F_0) = R_1 + R_2$, where $R_1 + R_2$ is a fiber of type $A_1$ on $X$, and $R_1 - R_2 \in L(1/2)$ but $R_1 - R_2 \notin K(1/2)$.

**Example 6.4.18.** Let $C = V(F)$ be a nonsingular cubic surface in $\mathbb{P}^3$ over $\mathbb{k}$ of characteristic $\neq 2, 3$. It is called Sylvester non-degenerate if the homogeneous cubic form $F$ can be written as $l_1^3 + \cdots + l_5^3 = 0$, where $l_i$ are linear forms spanning the 4-dimensional linear space of such forms no two of which are proportional. It is known that a general cubic surface is Sylvester non-degenerate. The Sylvester Theorem asserts that the linear forms $(l_1, \ldots, l_5)$ are defined uniquely by $C$ up to common scaling ([180, Theorem 9.4.1]). One can rewrite its equation in the form

$$t_0 + \cdots + t_4 = a_1t_0^3 + \cdots + a_5t_4^3 = 0, \quad a_i \neq 0.$$  

The Hessian surface of $C$ is a quartic surface $H(C)$ defined by the determinant of the matrix of second order partial derivatives of the polynomial $F$. In variables $t_1, \ldots, t_5$, the equation of the Hessian surface can be written in the form

$$t_1 + \cdots + t_5 = \frac{1}{a_1t_1} + \cdots + \frac{1}{a_5t_5} = 0. \quad (6.4.15)$$

The last sum is shorthand for the quartic polynomial got by clearing denominators. It contains 10 ordinary double points $P_{ab}$ satisfying $t_i = 0, i \neq a, b$ and 10 lines $\ell_{ab}$ satisfying $t_i = 0, i \in \{a, b\}$. For any 2-subsets $\alpha, \beta$ of $\{1, 2, 3, 4, 5\}$, we have $P_{\alpha} \in \ell_{\beta}$ if and only if $\alpha \cap \beta = \emptyset$. The union $\Pi$ of the hyperplanes $t_i = 0$ in $\mathbb{P}^4$ identified with the hyperplane $t_1 + \cdots + t_5 = 0$ is called the Sylvester pentahedron. The points $P_{\alpha}$ are its vertices and the lines $\ell_{\alpha}$ are its edges.

The Hessian surface may acquire other singular points. It is easy to see that they do not lie on the coordinate hyperplanes $t_i = 0$ and satisfy the equations $a_it_i^2 - a_jt_j^2 = 0$. They coincide with singular points of the cubic surface $C$.

The birational involution of $\mathbb{P}^4$ defined by

$$T : (t_1, \ldots, t_5) \mapsto \left(\frac{1}{a_1t_1}, \ldots, \frac{1}{a_5t_5}\right)$$

leaves the surface $H(C)$ invariant. It extends to a biregular involution $\tau$ of a surface $X'$ obtained from $H(C)$ by resolving the ten nodes $P_{\alpha}$. Let $E_{\alpha}$ be the exceptional curve over the point $P_{\alpha}$ and $L_{\alpha}$ be the proper transform of the line $\ell_{\alpha}$. The involution switches $E_{\alpha}$ with $L_{\alpha}$. The fixed points of $T$ coincide with singular points of $H(C)$ different from the nodes $P_{\alpha}$.
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In particular, if $C$ is a smooth cubic surface, we see that $X'$ is a K3 surface $X$ and the involution $\tau$ has no fixed points. Thus $\pi: X \to S = X/(\tau)$ is the canonical cover of an Enriques surface. We call it an Enriques surface of Hessian type. The curve $E_\alpha + L_\alpha$ descends to a $(-2)$-curve $U_\alpha$ on $S$. The intersection graph of the curves $U_\alpha$ is the famous Petersen graph given in Figure 6.1.

![Petersen graph](image)

Figure 6.1: Petersen graph

One computes the determinant of the intersection matrix of the curves $U_\alpha$ and finds that it is equal to $-256$. Thus, the classes $[U_\alpha]$ span $\text{Num}(S) \otimes \mathbb{Q}$. In fact, any $U_\alpha$ is a component of a singular fiber of an elliptic pencil $|2F_\alpha|$ with a simple fiber $\sum_{\beta, \# \beta = 1} U_\beta$ of type $A_5$. The pre-image of this elliptic pencil on $H(C)$ is the pencil of residual cubics in hyperplane sections passing through the line $\ell_\alpha$. We have

$$h = \sum_\alpha U_\alpha$$

(6.4.16)

is a Fano polarization on $S$ and

$$3h_{10} = \sum_\alpha F_\alpha.$$  

(6.4.17)

The pre-image of $h$ on the canonical cover is equal to $h + \tau^*(h)$, where $h$ is the class of a hyperplane section (see [189]).

The sublattice $L$ of $\text{Pic}(X)$ generated by the curves $E_\alpha$ and $L_\alpha$ is of rank 16 and discriminant 48. Its discriminant quadratic form is $u_1 \oplus q_{A_2}(-2)$. There exists a unique (up to isometry) hyperbolic lattice with same rank and discriminant form. This lattice primitively embeds into the K3-lattice and orthogonal complement isomorphic to $U \oplus U(2) \oplus A_2(2)$. If $C$ is a general cubic surface, the sublattice $L$ coincides with the Picard lattice $\text{Pic}(X)$.

We assume now that $C$ is a general cubic surface. It is clear that the divisor classes $r_\alpha = L_\alpha - E_\alpha$ generate the subgroup $\text{Im}(1 - \sigma^*)$ of $\text{Pic}^c(S)$. They also generate the whole group $\text{Ker}(1 + \sigma^*)$. In fact, suppose $D = \sum n_\alpha L_\alpha + \sum m_\alpha E_\alpha$ belongs to this group. Then $\sigma^*(D) = \sum n_\alpha E_\alpha + \sum m_\alpha L_\alpha$.

$$D + \sigma^*(D) = \sum (n_\alpha + m_\alpha)(L_\alpha + E_\alpha) = \sum (n_\alpha + m_\alpha)\pi^*U_\alpha = 0.$$  

One can compute the determinant of the intersection matrix $(U_\alpha \cdot U_\beta)$ to obtain that it is equal to $-256$. In particular, the divisor classes $U_\alpha$ on $S$ are linearly independent. This implies that $n_\alpha + m_\alpha = 0$, hence $D \in \text{Im}(1 - \sigma^*)$. 
We have \( r_\alpha^2 = -4 \) and \( r_\alpha \cdot r_{\alpha'} = -2U_\alpha \cdot U_{\alpha'} \). Consider a subgraph of the Petersen graph obtained by deleting the vertices \( U_{23}, U_{24}, U_{25}, U_{14} \). It is the Coxeter-Dynkin diagram of the root system of type \( E_6 \). The vectors \( r_\alpha \) corresponding to the vertices of this diagram span the lattice \( M \) isomorphic to \( E_6(2) \). It is shown in [175] that, for any \( i = 1, 2, 3, 4, 5 \),

\[
r_{ia} + r_{ib} + r_{ic} + r_{id} = 2(h' - \sigma^+(h')) ,
\]

(6.4.18)

where \( \{a, b, c, d, i\} = \{1, 2, 3, 4, 5\} \) and \( h' \) is the class of a plane section of \( H(C) \). Using relations (6.4.18), one can easily show that the remaining divisor classes \( r_{14}, r_{23}, r_{24}, r_{25} \) are linear combination of the previous 6 divisor classes and \( 2(h - \tau^+(h)) \). This shows that the sublattice \( \text{Im}(1 - \sigma^+) \) of \( \text{Pic}^\omega(S) \) is isomorphic to \( E_6 \).

Let us now compute the lattice \( N^h_{10} \). It follows from (6.4.16) that each \( U_\alpha \) becomes a line in the Fano model defined by \( |h| = |h_{10}| \). Suppose \( R \) is a \((-2)\)-curve with \( h_{10} \cdot R \leq 4 \), then \( R \cdot \sum_\alpha U_\alpha \leq 4 \) implies that \( R \) coincides with one of the curves \( U_\alpha \). It follows that \( N^h_{10} \) is generated by the ten classes \( [U_\alpha] \) and the Dynkin graph of its root basis formed by \( [U_\alpha] \) is equal to the Petersen graph.

The elliptic fibration \( |2F_\alpha| \) contains a simple fiber of type \( \tilde{A}_5 \) with components \( U_\beta \), where \( \alpha \cap \beta \neq \emptyset \). This shows that \( \sum_{\beta < \alpha \neq \emptyset} [U_\beta] \subseteq 2\text{Num}(S) \). Computing the matrix \( A = (U_\alpha \cdot U_\beta) \), where we order \( \alpha \)'s as \( (12, 13, 14, 15, 23, 24, 25, 34, 35, 45) \), we find that the null-space of \( A \) over \( \mathbb{F}_2 \) is generated by 4 linearly independent vectors \( v_1 = (1, 1, 1, 0, 0, 0, 0, 0, 0) \), \( v_2 = (1, 0, 0, 1, 1, 1, 1, 0, 0, 0) \), \( v_3 = (0, 1, 0, 1, 0, 1, 0, 1, 0, 1) \), \( v_4 = (0, 0, 1, 0, 0, 1, 0, 1, 0, 1) \). This shows that the discriminant quadratic form has isotropic subgroup isomorphic to \((\mathbb{Z}/2\mathbb{Z})^4 \). Thus the Fano root invariant with respect to \( h_{10} \) is equal to \( (N^h_{10}, \mathbb{G}_{h_{10}}) \), where \( \mathbb{G}_{h_{10}} = (\mathbb{Z}/2\mathbb{Z})^4 \). In particular, the quadratic space \( \overline{N}^h_{10} \) is of dimension 6. By taking the representatives of \( \Pi_{h_{10}}^{\text{nod}} / \mathbb{G}_{h_{10}} \), we find that it is a regular odd quadratic space that coincides with the \( r \)-invariant of \( S \) defined by the Nikulin \( R \)-invariant \( (E_6, 0) \).

### 6.5 General nodal surfaces

A nodal Enriques surface \( S \) is said to be general if \( \text{Nod}(S) \) consists of one element (in other words, all \((-2)\)-curves are congruent modulo \( 2\text{Num}(S) \)).

**Lemma 6.5.1.** Assume \( S \) is general nodal. Then for any two \((-2)\)-curves \( R \) and \( R' \)

\[
R \cdot R' \equiv 2 \mod 4 .
\]

**Proof.** We have \( R' - R = 2x \) for some \( x \in \text{Num}(S) \). This yields \(-4 - 2(R \cdot R') = 4x^2 \equiv 0 \mod 8\), hence \( R \cdot R' \equiv 2 \mod 4 \). \( \square \)

**Corollary 6.5.2.** Let \( S \) be a general nodal Enriques surface. Then \( S \) has neither chains of \((-2)\)-curves nor disjoint \((-2)\)-curves.

The following corollary follows from Theorems 6.3.5 and 6.3.3.

**Corollary 6.5.3.** Every \((-2)\)-curve on a general nodal Enriques surface is realized as an irreducible component of some genus one fibration or as a special bisection.
Two \((-2)\)-curves \(R\) and \(R'\) are called \textit{\(f\)-equivalent} if there exists a sequence of genus one fibrations \(|2F_1|, \ldots, |2F_{k-1}|\) and a sequence of \((-2)\)-curves \(R_1 = R, \ldots, R_k = R'\) such that

\[
R_1 + R_2 \in |2F_1|, R_2 + R_3 \in |2F_2|, \ldots, R_{k-1} + R_k \in |2F_{k-1}|.
\]

Obviously, the \(f\)-equivalence is an equivalence relation on the set of \((-2)\)-curves.

The following result gives some characterizations of general nodal Enriques surfaces.

**Theorem 6.5.4.** The following properties are equivalent.

(i) \(S\) is a general nodal Enriques surface;

(ii) Any genus one fibration on \(S\) contains at most one reducible fiber that consists of two irreducible components. A half-fiber is irreducible.

(iii) The degeneracy invariant \(c\) of any canonical isotropic 10-sequence is larger than or equal to 9.

(iv) Any two \((-2)\)-curves are \(f\)-equivalent.

(v) For any Fano polarization, the set \(\Pi^{\text{nod}}_{h_{10}}\) consists of one element.

(vi) For any \(d\leq 4\), \(S\) admits a Fano polarization such that \(\Pi^{\text{nod}}_{h_{10}}\) consists of one vector represented by a \((-2)\)-curve \(R\) with \(R \cdot h_{10} = d\).

(vii) A genus one pencil that admits a special bisection does not contain reducible fibres.

**Proof.** (i) \(\Rightarrow\) (ii) The first assertion follows from Corollary 6.5.2. It remains to show that a half-fiber \(F\) is irreducible. If not, then, by the same corollary, it must consist of two components \(R_1, R_2\) intersecting with multiplicity 2. Then \(R_1 + R_2 \equiv 2R_1 \mod 2\text{Num}(S)\), hence \([F]\) is divisible by 2, a contradiction.

(ii) \(\Rightarrow\) (ii)'. Obvious.

(ii)' \(\Rightarrow\) (iii) Suppose we find a canonical isotropic sequence \((f_1, \ldots, f_{10})\) with degeneracy invariant \(\leq 8\). It follows from (ii)' that no nodal cycle of type \(A_k, k > 1\), has zero intersection with a nef isotropic class \(f\), without loss of generality, we may assume that \(f_1, f_2\) are nef and \(f_3 = f_1 + R_1, f_4 = f_2 + R_2\) with notations from Proposition 6.1.5. We have \(f_1 \cdot R_1 = 1, f_1 \cdot R_2 = 0\). Again since there are no chains of \((-2)\)-curves of length larger than 1, we can find a nef class \(f_i\) for some \(i > 4\), then \(f_i \cdot R_1 = f_i \cdot R_2 = 0\), and we obtain that a genus one fibration that does not satisfy property (ii)'.

(iii) \(\Rightarrow\) (ii) Suppose (ii) is not true. Then there exist two \((-2)\)-curves \(R_1\) and \(R_2\) with \([R_1] - [R_2] \not\in 2\text{Num}(S)\) and a nef isotropic vector \(f_1\) such that \(f_1 \cdot [R_1] = f_1 \cdot [R_2] = 0\). Let us extend \(f_1\) to a canonical isotropic sequence \((f_1, \ldots, f_{10})\). By assumption its degeneracy invariant \(\geq 9\). Consider the vectors \(v_i = (f_1 \cdot R_i - 10 \cdot R_i) \mod 2\). Since the images of \(f_i\) form a basis in \(\overline{\text{Num}}(S)\)
these vectors are nonzero and $v_1 \neq v_2$. Moreover, since $R_1 \cdot R_2$ is equal to 0 or 1 (if they lie in the same fiber), the image of $R_1 + R_2$ in $\overline{\text{Num}}(S)$ is of square norm 0. This implies that the vectors $v_1, v_2$ have at least two different coordinates. Without loss of generality, we may assume $f_2 \cdot R_1 = f_3 \cdot R_2 = 1$ and $f_3 \cdot R_1 = f_2 \cdot R_2 = 0$. By assumption, only one of the classes $f_2, f_3$ is not nef. Suppose that $f_2$ is nef. If $R_1, R_2$ are components of the same fiber with $R_1 \cdot R_2 = 1$, then $(f_1, f_2, f_2 + R_1, f_2 + R_1 + R_2)$ can be extended to a canonical isotropic sequence with degeneracy invariant $\leq 8$. Thus we have proved that no fiber contains more than two irreducible components, so we may assume that $R_1$ and $R_2$ are components of different fibers.

Suppose that $f_2, f_3$ are both nef. Then $(f_1, f_2 + R_1, f_3 + R_2)$ extends to a canonical isotropic sequence with degeneracy invariant $\leq 8$. If $f_2$ is not nef and $f_3$ is nef, then one of $f_1, i > 3$ is equal to $f_2 + R$. Since $f_1 \cdot f_1 = f_1 \cdot f_2 = 1$, we obtain $R \cdot R_1 = f_1 \cdot R = 0$. Thus $(f_1, f_2, f_2 + R, f_3, f_3 + R_1)$ extends to a canonical isotropic sequence with degeneracy invariant $\leq 8$.

(iii) $\Rightarrow$ (iv) Let $R$ and $R'$ be two $(-2)$-curves with $R \cdot R' = n$. We use induction on $n$. If $n = 2$, by (iii), $R + R'$ must be a simple fiber of a genus one fibration, hence $R$ and $R'$ are $f$-equivalent. Assume $n > 2$. Since $S$ has no chains of $(-2)$-curves of length larger than one, by Theorem 6.3.3, there exists an nef isotropic vector $f_1$ with $f_1 \cdot R = 0$. We use the argument and the notation from the previous argument to find an isotropic sequence $(f_1, \ldots, f_{10})$ with $f_1$ nef and $f_2 \cdot R = 1$. Now we extend $(f_1, f_2, f_3 + R)$ to a canonical isotropic sequence $(g_1, g_2, g_3, g_4, \ldots, g_{10})$. By assumption (iii), all $g_i, i \neq 3$, are nef. Since $g_j \cdot R = 0$ for $j \geq 3$, the curve $R$ is contained in a simple fiber of each pencil $\{2G_j\}, j \neq 3$, with $[G_j] = g_j$. Let $R_j$ be another component of a fiber of $\{2G_j\}$. If $R_j \cdot R' < n$, then we are done by induction, so we may assume that $R_j \cdot R' \geq n$ for $j \neq 3$. Let $h = \frac{1}{2}(g_1 + \ldots + g_{10})$. We know that $R' \cdot g_j \geq R' \cdot R_j \geq n$ for $j \neq 3$ and also $R' \cdot g_1 \geq R' \cdot R_j \geq n$. Since we may assume that $R'$ is not a fiber component of $\{2G_1\}$, we have $R' \cdot g_3 = R' \cdot (R + R') \geq n$. Intersecting $h_{10}$ with $R'$, we get

$$h_{10} \cdot R' \geq \frac{1}{3}(10n) \geq 10 = \frac{3h^2}{\Phi(h)}.$$ 

Applying Theorem 6.3.9, we find a genus one pencil $\{2F'\}$ such that $F' \cdot R' = 0$, and $0 < h_{10} \cdot F' < h_{10} \cdot R'$. Let $R'' + R''$ be a fiber of $\{2F'\}$. We have $h_{10} \cdot R'' = 2h_{10} \cdot F' - h_{10} \cdot R' < h_{10} \cdot R'$. The curve $R''$ is $f$-equivalent to $R'$, if $R'' \cdot R, R'' \cdot R > n$, we repeat the argument, replacing $R'$ with $R''$ to decrease further $h_{10} \cdot R''$. Continuing in this way we get a $(-2)$-curve equivalent to $R'$ which intersects $R$ or $R'$ with multiplicity $< n$. Note that if $h_{10} \cdot R'' = 0$, the curve $R''$ must be in one of the fibers of $\{2G_j\}$.

(iv) $\Rightarrow$ (i) We use that $f$-equivalence implies that two $(-2)$-curves are congruent modulo $2 \text{Num}(S)$.

(i) $\Leftrightarrow$ (v) Obvious.

(v) $\Leftrightarrow$ (vi) We know that $S$ contains a curve $R$ with $d = h_{10} \cdot R \leq 4$. Suppose $d \leq 4$. We use Corollary 6.3.11. If $d = 0$, we represent $R$ by some $f_i - f_j$. Then $\alpha = h_{10} - f_i - f_k - f_k, j \neq k, l$, satisfies $\alpha^2 = -2, \alpha \cdot h = 1$. If it were effective, it must be linearly equivalent to a $(-2)$-curve $R''$. However, $\alpha$ is not congruent to $R \mod 2 \text{Num}(S)$. Hence, it is not effective, and we can apply the reflection $x \mapsto x + (x \cdot \alpha)\alpha$ to transform $h$ to a new polarization $h'$ such that the class of $R$ becomes equal to $h_{10} - f_j - f_k - f_j$. We have now $h' \cdot R = 1$. Now take $\alpha = h_{10} - f_m - f_n - f_r$, where $\{j, k, l\} \cap \{m, n, r\} = \emptyset$. As above, applying the reflection in $\alpha$, we obtain the class of a $(-2)$-curve $R$ with $h_{10} \cdot R = 2$ representing $2h - f_j - f_k - f_j - f_m - f_n - f_r = -h + f_a + f_b + f_c + f_d$. 

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Next we use \( \alpha = h_{10} - f_c - f_d - f_j \) to transform \( h_{10} \) to \( h'_{10} \) such that the \((-2)\)-curve with the class \( f_a + f_b - f_j \) satisfies \( R \cdot h'_{10} = 3 \). Finally, we use \( \alpha = h_{10} - f_a - f_b - f_j \) to find a \((-2)\)-curve with \( R \cdot h'_{10} = 4 \).

(i) \( \Rightarrow \) (vii) Suppose \( |2F| \) is a genus one fibration with a special bisection \( R \) and a reducible fiber \( F \). Then \( R \) intersect one of the components \( R' \) of a \( F \) with multiplicity 1. This contradicts Lemma 6.5.1.

(vii) \( \Rightarrow \) (iii) Suppose there is a canonical isotropic sequence \((f_1, \ldots, f_{10})\) with the degeneracy invariant \( c \leq 8 \). Without loss of generality we may assume that \((f_1, f_2, f_3, f_4) = (f_1, f_1 + R, f_3, f_3 + R)\) or \((f_1, f_2, f_3, f_4) = (f_1, f_1 + R, f_1 + R + R', f_4)\). In the first case \( R \cdot f_1 = 1, R' \cdot f_1 = 0 \) contradicting condition (vii). In the second case, \( R' \cdot f_1 = 0, R \cdot f_1 = 1 \), again contradicting (vii).

Remark 6.5.5. Some of the properties of elliptic fibrations on a general nodal Enriques surface follow also from the analysis of isomorphism classes of elliptic fibrations on such surfaces based on the known structure of the automorphism group (see 8.4). For example, this analysis shows that a general nodal surface does not admit two elliptic fibrations \(|2F_1|\) and \(|2F_2|\) with \( F_1 \cdot F_2 = 2 \) that share a common irreducible fiber components.

Remark 6.5.6. We will prove later in Corollary 7.9.9 that, under the assumption of the previous Corollary, the Picard lattice of the canonical cover of \( S \) is isomorphic to \( U \oplus E_8(2) \oplus A_1(2) \).

The following corollary follows from the discussions from the previous section.

**Corollary 6.5.7.** Let \( S \) be a general nodal Enriques surface. The lattice \( N_S^{h_{10}} \) is isomorphic to \( A_1 \) and the \( R \)-invariant is \( (A_1, \{0\}) \). It coincides with the Nikulin \( R \)-invariant when the canonical cover is étale. The Reye lattice is isomorphic to \( E_{2,4,6} \), and the \( r \)-invariant is a one-dimensional defective quadratic space. Conversely, any of these properties characterize a general nodal Enriques surface.

**Corollary 6.5.8.** Let \( R \) be a \((-2)\)-curve on a general nodal Enriques surface \( S \). Then there exists a canonical isotropic 10-sequence \((f_1, \ldots, f_{10})\) of degeneracy invariant 9 such that \( f_1, \ldots, f_9 \) are nef, and \( f_{10} = f_9 + R \).

**Proof.** By Theorem 6.5.4 (vi), there exists a numerical Fano polarization \( h_{10} \) and a \((-2)\)-curve \( R \) with \( h_{10} \cdot R = 0 \). Thus, if we write \( 3h_{10} = f_1 + \cdots + f_{10} \), as usual, the non-degeneracy invariant \( c \) of the isotropic sequence \((f_1, \ldots, f_{10})\) will be \( \leq 9 \). Thus we may assume that \( f_1, \ldots, f_9 \) are nef and \( f_{10} = f_9 + r \), where \( r \) is the class of a \((-2)\)-curve. Intersecting with \([R]\), we get \( 0 = f_1 \cdot [R] + \cdots + 2f_9 \cdot [R] + r \cdot [R] \). This obviously implies \( r = [R] \).

Since \( R \cdot f_9 = 1 \), we obtain

**Corollary 6.5.9.** Any \((-2)\)-curve on a general nodal Enriques surface is a special bisection of some elliptic fibration.

**Corollary 6.5.10.** Let \( S \) be a general nodal surface. Then its canonical cover is a normal surface. Moreover, \( H^0(S, \Theta_S) = \{0\} \) unless \( S \) is an \( \alpha_2 \)-surface.
Proof. We know from Proposition 10.2.7 that the singular points of the canonical cover lie over singular points of fibers in any genus one fibration on \( S \). In our case, there is at most one reducible fiber and its type is \( \tilde{A}_1 \). Thus the canonical cover has only isolated singular points, hence must be a normal surface birationally isomorphic to a K3 surface or a rational surface with one elliptic singularity. If \( K_S \neq 0 \), then the second assertion follows from Theorem 1.4.10. We also know that a \( \mu_2 \)-surface has no nonzero regular global fields and an \( \alpha_2 \)-surface has them always.

Remark 6.5.11. It is not true that \( X \) is always birationally isomorphic to a K3-surface. There is an example due to S. Schroer [612] and Y. Matsumoto [474] of an unnodal Enriques surface with all elliptic fibrations have one singular irreducible fiber such that its canonical cover is a normal rational surface. However, Matsumoto proves that the surface must be an \( \alpha_2 \)-surface in this case [474, Proposition 3.2]

Bibliographical notes

The notion of an isotropic 10-sequences of the numerical classes of elliptic curves on an Enriques surface goes back to G. Fano [221]. He used it for his study of a degree 10 birational model of an Enriques surfaces, the Fano models which we discussed in volume 1. The first use of them in modern literature can be found in [46]. The first systematic study of such sequences, and in particular, introducing the notion of a canonical isotropic sequence and the proof of Proposition 6.1.5 was given by F. Cossec [131]. The notion of the non-degeneracy invariant \( nd(S) \) of an Enriques surface was introduced in [134]. The proof of Theorem 6.1.10 that asserts that \( nd(S) \geq 2 \) for any Enriques surface in characteristic different from 2 was first given by Cossec [131, Proposition 3.4]. It was extended to Enriques surfaces in characteristic 2 different from an extra-\( \tilde{E}_8 \)-special surface can be found in [134]. The proof of Theorem 6.1.13 in the case \( p \neq 2 \) was indicated by Cossec [131, Theorem 3.5] who left the full proof to the reader. The same proof that works in all characteristics and was based on considering many cases arising from possible extensions of a non-degenerate isotropic pair to a canonical maximal isotropic sequence was undertaken in [134]. It required almost 32 pages and almost surely the analysis missed some cases. The proof in the case \( p \neq 2 \) that is given in this Chapter is new and based on completely different ideas.

The notion of an extra-special Enriques surfaces of types \( \tilde{E}_8 \), \( \tilde{D}_8 \) and \( \tilde{A}_1 + \tilde{E}_7 \) was introduced in [134]. It was proven that the non-degeneracy invariant of these surfaces is less than 3 (it was mistakenly asserted that both possibilities for extra-special surfaces of type \( \tilde{A}_1 + \tilde{E}_7 \) have \( nd \leq 2 \)). In section 6.2 of this Chapter we defined an extra-special surface as a surface with \( nd \leq 2 \) and prove in Theorem 6.2.6 that the definitions are equivalent. An extra-special surface of type \( \tilde{E}_8 \) was considered earlier by W. Lang [421, Appendix A], where it was called a surface of hyperbolic type. The first construction of extra-special surfaces was given in an unpublished paper of Salomonsson [607].

The facts that any rational smooth curve is contained in a fiber of some genus one fibration was proven by Cossec [131, Theorem 4.1] in the case \( p \neq 2 \). He also proved that nodal Enriques surface contains an elliptic fibration with a special bisection. The extension of this result to characteristic 2 is due to W. Lang [421, Theorem A3].

Theorems 6.3.10 and 6.3.18 from section 6.3 were first proven in [135].

The notions of nodal and Nikulin \( R \)-invariant from section 6.4 were first introduced in [535]. The Fano root invariant that can be defined in all characteristics seems to be new. The notion of the Reye lattice coincides with the notion of the Reye lattice introduced in [134] only in the case when the nodal invariant consists of one vector. We also discuss in this section the slightly different definition of \( R \)-invariant given by S. Mukai
and follow A. Beauville [53] to relate it with the Brauer group of the K3 cover of an Enriques surface.

The systematic study and different characterization of general nodal Enriques surfaces is new. Over the field of complex numbers the fact that a general, in the sense of moduli, nodal surface can be characterized by the condition that the classes of all smooth rational curves are congruent modulo 2 follows from the work of Nikulin [535].
Chapter 7

Reye congruences

7.1 Congruences of lines

Recall that a congruence of lines in \( \mathbb{P}^3 \) is an irreducible surface \( S \) in the Grassmann variety \( \mathbb{G} = G_1(\mathbb{P}^3) \) of lines in \( \mathbb{P}^3 \). The line \( \ell_s \) in \( \mathbb{P}^3 \) corresponding to a point \( s \in S \) is a ray of the congruence.

We will use some general facts about the geometry of \( G_1(\mathbb{P}^3) \) that can be found in many sources, e.g. in [233] or [180], Chapter 9. For any point \( x \), a line \( \ell \) and a plane \( \Pi \) in \( \mathbb{P}^3 \), let

\[
\sigma_x = \{ \ell \in \mathbb{G} : x \in \ell \},
\]

(7.1.1)

\[
\sigma_\ell = \{ \ell' \in \mathbb{G} : \ell' \cap \ell \neq \emptyset \},
\]

(7.1.2)

\[
\sigma_\Pi = \{ \ell \in \mathbb{G} : \ell \subset \Pi \},
\]

(7.1.3)

\[
\sigma_{x,\Pi} = \{ \ell \in \mathbb{G} : x \in \ell \subset \Pi \}.
\]

(7.1.4)

be the Schubert varieties. The Chow ring \( A^*(\mathbb{G}) \) is generated by the classes of these varieties, the class of a point and the class of \( \mathbb{G} \):

\[
A^*(\mathbb{G}) = \bigoplus_{i=0}^{4} A^i = \mathbb{Z}[\mathbb{G}] \oplus \mathbb{Z}[\sigma_\ell] \oplus (\mathbb{Z}[\sigma_x] \oplus \mathbb{Z}[\sigma_\Pi]) \oplus \mathbb{Z}[\sigma_{x,\Pi}] \oplus \mathbb{Z}[\text{point}].
\]

The multiplication in \( A^*(\mathbb{G}) \) is determined by the formulas:

\[
[\sigma_\Pi]^2 = [\sigma_x]^2 = [\text{point}], \quad [\sigma_x] \cdot [\sigma_\Pi] = 0, \quad [\sigma_\ell] \cdot [\sigma_\ell] = [\sigma_x,\Pi], \quad [\sigma_\Pi] \cdot [\sigma_\ell] = [\sigma_{x,\Pi}], \quad [\sigma_\ell]^2 = [\sigma_{x,\Pi}] + [\sigma_x].
\]

(7.1.5)

The class \([S]\) of a congruence is determined by the two numbers \((m,n)\) called the order and the class of \( S \):

\[
[S] = m[\sigma_x] + n[\sigma_\Pi].
\]

It follows from the previous formulas that

\[
m = [S] \cdot [\sigma_x], \quad n = [S] \cdot [\sigma_\Pi], \quad m + n = [S] \cdot [\sigma_\ell]^2.
\]

The Grassmann variety \( \mathbb{G} \) is isomorphic to a nonsingular quadric in \( \mathbb{P}^5 \) embedded via the Plücker embedding. The class \([\sigma_\ell]\) is the class of a hyperplane section of \( \mathbb{G} \). Thus the number \( m + n \) coincides with the degree of \( S \) in the Plücker embedding. The pair \((m,n)\) will be called the bidegree of \( S \).
CHAPTER 7. REYE CONGRUENCES

In coordinate-free way, we consider \( \mathbb{P}^3 \) as the variety of lines in a linear space \( E \) of dimension 4 over \( k \). In Grothendieck notation, \( |E| = \mathbb{P}(E^\vee) \), where \( E^\vee \) is the dual linear space. The Plücker space becomes \( |\wedge^2 E| = \mathbb{P}(\wedge^2 E^\vee) \). A line in \( |E| \) is \( |U| \), where \( U \) is a 2-dimensional subspace of \( E \), and the Plücker embedding is \( |U| \mapsto |\wedge^2 U| \subset |\wedge^2 E| \). The Grassmann variety comes equipped with a natural exact sequence of locally free sheaves

\[
0 \to S_G \to E \otimes O_G \to Q_G \to 0
\]

and the dual exact sequence

\[
0 \to Q_G^\vee \to E^\vee \otimes O_G \to S_G^\vee \to 0.
\]

The fiber of the geometric vector bundle \( V(S_G) \) over a point \( \ell = [U] \in G \) is the subspace \( U \) of the fiber of \( V(E \otimes O_G) = E \), where we consider \( E \) as the associated affine space over \( k \). The locally free sheaf \( S_G \) identified with the corresponding vector bundle \( V(S_G) \) is called the universal vector subbundle. The locally free sheaf \( Q_G \) is called universal quotient bundle. The surjection \( E^\vee \otimes O_G \to S_G^\vee \) defines a canonical closed embedding

\[ Z_G := \mathbb{P}(S_G^\vee) \to \mathbb{P}(E \otimes O_G) = |E| \times G. \]

The Plücker embedding is given by the surjection \( \wedge^2 E^\vee \otimes O_G \to \wedge^2 S_G^\vee \). In particular,

\[ \wedge^2 S_G^\vee \cong O_G(1). \]

The usual properties of the Chern classes and equalities \((7.1.5)\) give

\[
\begin{align*}
c_1(Q_G) &= [\sigma_\ell] = c_1(O_G(1)), \\
c_2(S_G) &= [\sigma_\Pi], \\
c_2(Q_G) &= [\sigma_x].
\end{align*}
\]

Let

\[
\begin{array}{c}
Z_G \\
\downarrow p_1 \\
|E| \\
\downarrow p_2 \\
G
\end{array}
\]

be the projections. The variety \( Z_G \) coincides with the flag variety of points-lines in \( |E| \):

\[ Z_G = \{(x, \ell) \in |E| \times G : x \in \ell \}. \]

The projection \( p_2 : Z_G \to G \) is the projective bundle. It is the projective subbundle of the trivial bundle \( E_G := \mathbb{P}(E^\vee \otimes O_G) \). We have \( O_{E_G}(1) \cong p_2^*O_{|E|}(1) \). The restriction \( O_Z(1) \) of \( O_{E_G}(1) \) to \( Z_G \) is the tautological invertible sheaf corresponding to the choice of an isomorphism \( Z_G \cong \mathbb{P}(S_G^\vee) \).

The projection \( p_1 : Z_G \to |E| \) is a projective bundle over \( |E| \). The Euler exact sequence

\[
0 \to \Omega^1_{|E|} \to E^\vee \otimes O_{|E|}(-1) \to O_{|E|} \to 0
\]

defines a canonical isomorphism of the projective bundles

\[ Z_G \cong \mathbb{P}(\Omega^1_{|E|}(1)). \]
Twisting (7.1.8) by \( O_{|E|}(2) \) we obtain a canonical isomorphism

\[
H^0(|E|, \Omega^1_{|E|}(2)) \cong \text{Ker}(E^\vee \otimes E^\vee \to S^2 E^\vee) = \bigwedge^2 E^\vee. \tag{7.1.10}
\]

Let \( Z_S = p_2^{-1}(S) \subset Z_G \) and

\[
\begin{array}{ccc}
S & \xrightarrow{\pi_1} & Z_S \\
& \searrow & \downarrow \\
& & \pi_2 \to |E|
\end{array}
\]

be the restriction of the projections \( p_1 \) and \( p_2 \) to \( Z_S \). For any point \( x \in |E| \), the projection \( \pi_2 \) defines an isomorphism of the fiber \( \pi_1^{-1}(x) \cong S \cap \sigma_x \). In particular, if \( m > 0 \), \( \pi_1 : Z_S \to |E| \) is a morphism of degree equal to \( m \). It is known, and it is easy to prove, that a congruence \( S \) with \( m = 0 \) is equal to a Schubert variety \( \sigma_\Pi \) for some plane \( \Pi \).

There is a natural duality isomorphism \( G_1(|E|) \cong G_1(\mathbb{P}(E)) \) that assigns to a line \( \ell \) the pencil of planes containing \( \ell \). Under this isomorphism a congruence of bidegree \( (m, n) \) is mapped to a congruence of bidegree \( (n, m) \). In particular, a congruence with \( n = 0 \) has order \( m = 1 \) and coincides with a Schubert variety \( \sigma_x \). The varieties \( \sigma_x, x \in |E| \) (resp. \( \sigma_\Pi, \Pi \in \mathbb{P}(E) \)) are planes, classically called \( \alpha\)-planes (resp. \( \beta\)-planes). They form two rulings of the quadric \( G \) by planes.

From now on we assume that \( S \) is smooth and \( m, n > 0 \), if not stated otherwise.

Let \( R(S) \subset Z_S \) be the set of points \( z = (x, \ell) \in Z_S \) such that the morphism \( \pi_1 \) is not smooth at \( z \). Since \( S \) is smooth, \( Z_S \) is also smooth and consists of points at which the relative differential sheaf \( \Omega^1_{Z_S/P^3} \) is not zero. Thus the set \( R(S) \) is the support of the closed subscheme of \( Z_S \) given by the Fitting ideal defined by the map of locally free sheaves \( \pi_1^* \Omega_{P^3} \to \Omega^1_{Z_S} \). We equip \( R(S) \) with the structure of this subscheme. Recall that the Fitting ideal sheaf is the image of the canonical map

\[
\bigwedge^3 \pi_1^* \Omega_{P^3} \otimes \left( \bigwedge^3 \Omega^1_{Z_S} \right)^{-1} \to O_{Z_S}.
\]

Locally, it is given by the determinant of a \( 3 \times 3 \)-matrix, in particular \( R(S) \) is an effective divisor on \( Z_S \) such that

\[
K_{Z_S} \sim R(S) + \pi_1^*(K_{P^3}). \tag{7.1.11}
\]

The following lemma computes the canonical sheaf of \( Z_G \) and \( Z_S \) (see [180, 10.1.1]).

**Lemma 7.1.1.**

\[
\omega_{Z_G} \cong \pi_1^* O_{P^3}(-2) \otimes \pi_2^* O_G(-3),
\]

\[
\omega_{Z_S} \cong \pi_1^* O_{P^3}(-2) \otimes \pi_2^* \omega_S(1).
\]

Applying (7.1.11), we obtain

\[
O_{Z_S}(R(S)) \cong \pi_1^* \omega_S(1) \otimes \pi_1^* O_{P^3}(2) \cong \pi_2^* \omega_S(1) \otimes O_{Z_S}(2). \tag{7.1.12}
\]

The adjunction formula gives

\[
\omega_{R(S)} \cong \pi_2^* \omega_S(1)^{\otimes 2} \otimes O_{R(S)}. \tag{7.1.13}
\]
Thus the ramification divisor of \( \pi_2 : R(S) \to S \) belongs to the linear system \( |\pi_2^*O_S(2) \otimes O_{R(S)}| \).

Assume \( p \neq 2 \). We have \( \omega_{R(S)} = \pi_2^*(\omega_S \otimes \mathcal{L}) \), where a section of \( \mathcal{L}^{\otimes 2} \) defines the branch divisor \( B \) of \( \pi_2 \). It follows from (7.1.13) that

\[
O_S(2B) \cong \omega_S^{\otimes 2} \otimes O_S(4). \tag{7.1.14}
\]

Let \( \pi_1 : Z_S \to Z'_S \to \mathbb{P}^3 \) be the Stein decomposition. The map \( \alpha : Z_S \to Z'_S \) is a birational morphism onto a normal variety and the map \( \pi_1^1 : Z'_S \to \mathbb{P}^3 \) is a finite morphism of degree \( m \). If \( m > 1 \), and \( \pi_2 \) is separable, the branch divisor \( \text{Foc}(S) \) of this map is a surface in \( \mathbb{P}^3 \) called the focal surface of \( S \). The image \( \text{Fund}(S) \) of the exceptional divisor of \( \alpha \) under the map \( \pi_1^1 \) is called the fundamental locus of \( S \). In the classical terminology, its points are singular points of \( S \) but this terminology is somewhat confusing because it is usually assumed that \( S \) is smooth. Fundamental points often exist even when \( S \) is a smooth surface. The one-dimensional part of \( \text{Fund}(S) \) is called the fundamental curve. One can show that it is always an irreducible curve. This fact together with the description of such curves can be found [16]. For every singular point \( x \in \text{Fund}(S) \), the fiber \( \pi_1^{-1}(x) \) is one-dimensional and isomorphic to the intersection of the plane \( \sigma_x \) with \( S \).

The fibers over points of \( \text{Fund}(S) \) are points in \( R(S) \) such that the rank of the determinant of \( f : \pi_1^1|_{\text{Fund}(S)} \to \Omega_{Z_S}^1 \) drops by 2. It is contained in the closure of points where the corank is equal to 1. This shows that \( \text{Fund}(S) \) is a closed subset of \( \text{Foc}(S) \).

If \( p \neq 2 \), the corank of \( f \) at a point \( z = (x, \ell) \in R(S) \) is equal to the dimension of the fiber \( \pi_1^{-1}(x) \) [246]. In particular, ramification points of \( R(S) \to \text{Foc}(S) \) over points in the complement of \( \text{Fund}(S) \) are all simple ramification points.

Applying the projection formula for the morphism \( \pi_1 \) to (7.1.12), we obtain that a general fiber of the projection \( \pi_2 : Z_S \to S \) intersects \( R(S) \) with multiplicity 2. This shows that \( R(S) \) contains at most two irreducible components that are mapped onto \( S \), all other components are blown down to irreducible curves on \( S \).

We will assume that \( \text{Fund}(S) \) consists of isolated points. In this case, the ramification divisor \( R(S) \) is reduced, and the projection \( \pi_2 : R(S) \to S \) is a double cover. Every ray \( \ell_x \) of the congruence is touching \( R(S) \) at two points, the image of the intersection points \( \pi_2^{-1}(s) \) with \( R(S) \). Thus

\[
S \subset \text{Bit}(\text{Foc}(S)), \tag{7.1.15}
\]

where, for any reduced surface \( X \) in \( \mathbb{P}^3 \), the bitangent congruence of \( X \) is defined to be the closure in \( G \) of the set of lines in \( \mathbb{P}^3 \) that are tangent to \( X \) at two points (maybe equal). Note that \( \text{Bit}(\text{Foc}(S)) \) could be a reducible surface, so \( S \) is one of its irreducible components.

For any singular point \( x \in \text{Fund}(S) \) of the congruence, the intersection \( \sigma_x \cap S \) is a plane curve \( F(x) \). The image of \( \pi_2^{-1}(F(x)) \) under the projection \( \pi_1 \) is a cone over the curve \( \tilde{F}(x) \) equal to the projection of \( \pi_2^{-1}(F(x)) \cap R(S) \to \text{Foc}(S) \). Note that a line contained in \( \text{Fund}(S) \) or \( \text{Foc}(S) \) may be a ray of \( S \). It is called a multiple ray.

Let \( g \) be the genus of a general hyperplane section of \( S \). It is called the sectional genus of \( S \).

**Proposition 7.1.2.** Let \( S \) be a smooth congruence of bidegree \((m, n)\). Assume \( \pi_1 \) is a separable
map, $R(S)$ is reduced and $\text{Fund}(S)$ consists of points. Then 
\[
\deg \text{Foc}(S) = 2g - 2 + 2m.
\]

Proof. Let $\ell$ be a general line in $\mathbb{P}^3$. Its pre-image under $\pi_1$ consists of pairs $(x, s) \in \ell \times S$ such that $x \in \ell_s$. Since $\ell$ is not a ray and two lines intersect at one point, the projection $\pi_2$ is an isomorphism from $\pi_1^{-1}(\ell)$ to a curve $C$ in $S$ that is cut out in $S$ by a hyperplane section $\sigma_\ell$ of $G$. Thus the genus of $C$ is equal to the sectional genus of $S$. The cover $\pi_1 : C \rightarrow \ell$ is a degree $m$ finite cover. By Hurwitz’s formula the degree of the ramification divisor is equal to $2g - 2 + 2m$. We have already observed that all ramification points of $R(S) \rightarrow \text{Foc}(S)$ that do not belong to $\text{Fund}(S)$ are simple ramification points. Thus the number of branch points is equal to $2g - 2 + 2m$. This is the intersection number of $\ell \cap \text{Foc}(S)$. Note that if the map $\pi_1 : R(S) \rightarrow \text{Foc}(S)$ is of degree $d > 1$, we have to consider $\text{Foc}(S)$ as non-reduced scheme with $[\text{Foc}(S)] = d[\text{Foc}(S)_{\text{red}}]$. So, the formula is still true.

The following formula can be found in [293]. For completeness sake, we supply the proof.

**Proposition 7.1.3.** Let $S$ be a smooth congruence of bidegree $(m, n)$ and let $g$ be its sectional genus. Then
\[
m^2 + n^2 = 3(m + n) + 8(g - 1) + 2K_S^2 - 12\chi(O_S).
\]  
(7.1.16)

Proof. Using the intersection theory on $G$, we obtain $[S]^2 = m^2 + n^2$. On the other hand, this number is equal to the second Chern class of the normal sheaf $N_{S/G} = (I_S/I_S^2)^\vee$ of $S$ in $G$. The standard exact sequence
\[
0 \rightarrow I_S/I_S^2 \rightarrow \Omega^1_G \otimes O_S \rightarrow \Omega^1_S \rightarrow 0,
\]
after passing to the dual sequence and taking the Chern classes, gives
\[
c_1(N_{S/G}) = -K_G \cdot S + K_S = -4 \deg c_1(O_S(1)) + K_S = 4c_1(O_S(1)) + K_S,
\]
\[
c_2(N_{S/G}) = c_2(G) \cdot S - c_2(S) + K_S \cdot c_1(N_{S/G}).
\]
The second class of the quadric $G$ in $\mathbb{P}^5$ is computed using the exact sequence
\[
0 \rightarrow \Theta_G \rightarrow \Theta_{\mathbb{P}^5} \otimes O_G \rightarrow O_G(2) \rightarrow 0
\]
that easily gives $c_2(G) = 7c_1(O_G(1))^2$. Now we apply Noether’s formula $c_2(S) + K_S^2 = 12\chi(O_S)$ and obtain
\[
c_2(N_{S/G}) = 7(m + n) - (12\chi(O_S) - K_S^2) + 4c_1(O_S(1)) \cdot K_S + K_S^2
\]
\[
= 7(m + n) - 12\chi(O_S) + 2K_S^2 + 4c_1(O_S(1)) \cdot (K_S + c_1(O_S(1))) - 4c_1(O_S(1))^2
\]
\[
= 7(m + n) - 12\chi(O_S) + 2K_S^2 + 4(2g - 2) - 4(m + n) = 3(m + n) - 12\chi(O_S) + 2K_S^2 + 8(g - 1).
\]

We finish this section with an example.
Example 7.1.4. Let \( C \) be a smooth connected curve of degree \( d \) and genus \( g \) in \( \mathbb{P}^3 \). Consider the congruence \( S = \text{Bis}(C) \) of bisecants of \( C \). Projecting from a general point \( x \in \mathbb{P}^3 \), we obtain a plane curve of degree \( d \) and genus \( g = \frac{1}{2}(d-1)(d-2) - m \), where \( m \) is the order of \( S \). Thus
\[
m = \frac{1}{2}(d-1)(d-2) - g.
\]
A general plane intersects \( C \) at \( d \) points. The class is equal to the number of joints of two points in this set, i.e.
\[
n = \frac{1}{2}d(d-1).
\]
In similar way, we obtain the bidegree of the congruence \( \text{Join}(C_1, C_2) \) of joints of two disjoint smooth projective curves of degrees \( d_1, d_2 \) and genus \( g_1, g_2 \). A joint is a line \( x_1, x_2 \), where \( x_i \in C_i \). As before, the projection of \( C_i \) is a plane curve of genus \( g_i = \frac{1}{2}(d_i - 1)(d_i - 2) - m_i \), where \( m_i \) is the order of \( \text{Bis}(C_i) \). The number of common bisecants is expected to be equal to \( \# \text{Bis}(C_1) \cap \text{Bis}(C_2) = m_1 m_2 + n_1 n_2 \). We take the center of the projection not lying on any common bisecant. Taking \( x \) even more general, we may assume that the projections intersect transversally at \( d_1 d_2 \) points. Thus \( m = d_1 d_2 \). The class is equal to the number of joints of \( d_1 \) on \( C_1 \) and \( d_2 \) points on \( C_2 \), i.e.
\[
n = m = d_1 d_2.
\]
It is easy to see that \( \text{Join}(C_1, C_2) \) is a complete intersection of two hypersurfaces of degrees \( d_1 \) and \( d_2 \) in \( G_1(\mathbb{P}^3) \). The divisors are the Chow forms of the curves.

We will need two special cases. In the first one, we take \( d = 4 \) and \( g = 1 \). The curve \( C \) is the intersection of two quadrics. We get a congruence \( S \) of bidegree \((2, 6)\). It is isomorphic to the symmetric product \( C^{(2)} \) of \( C \). The natural map \( C^{(2)} \to \text{Pic}^2(C) \cong C \) is a \( \mathbb{P}^1 \)-bundle. Its fiber over \( c \in \text{Pic}^2(C) \) is equal to the linear system \(|c|\) of degree 2. The surface is a minimal elliptic ruled surface. We have \( K_S^2 = 0, g = 1, p_g = 0 \). By formula (7.1.16), its sectional genus is equal to 3. Note that \( \text{Bis}(C) \) contains four special lines. They parameterize rays through a point \( x_i \) such that the projection from \( x_i \) is a double cover of a plane conic. The points \( x_i \) are the singular points of four quadric cones containing \( C \). They also are the fundamental points of the congruence. The focal surface is the union of the four singular quadrics containing \( C \). Its degree is 8, that agrees with formula from Proposition 7.1.2. The ramification divisor \( R(S) \subset Z_S \) is the union of the preimages of the four special fibers of the ruled surface \( S \).

Another special case is when \( S = \text{Join}(C_1, C_2) \), where \( d_1 = d_2 = 1 \). The congruence is a nonsingular quadric in \( G_1(\mathbb{P}^3) \) cut out by a linear space of codimension 2. The lines \( C_1, C_2 \) are the fundamental curves of \( S \). The map \( \pi_1 : Z_S \to \mathbb{P}^3 \) is the blow-up of \( C_1 \) and \( C_2 \). The focal surface is not defined.

7.2 Hyperwebs of quadrics

Let \( W = |L| \) be an \( n \)-dimensional linear system of quadrics in \( \mathbb{P}^n = |E| \), a hyperweb of quadrics. Assume first that \( p = \text{char}(k) \neq 2 \). We refer to [180, Chapter 1], for the following varieties associated with \( W \).
The discriminant variety $D(W) \subset W$ parameterizing singular quadrics in $W$. It is either the whole $\mathbb{P}^n$ or a hypersurface of degree $n + 1$ given by the intersection of $W$ with the discriminant variety of singular quadrics on $\mathbb{P}^n$. If $(\lambda_0, \ldots, \lambda_n)$ are projective coordinates in $W$ corresponding to a basis formed by quadrics associated to symmetric matrices $A_0, \ldots, A_n$, then

$$D(W) : \det(\lambda_0 A_0 + \cdots + \lambda_n A_n) = 0.$$ (7.2.1)

The Steinerian variety $\text{St}(W) \subset \mathbb{P}^n$, the union of $\text{Sing}(Q), Q \in W$. It is either the whole $\mathbb{P}^n$ or a hypersurface in $\mathbb{P}^n$ of degree $n + 1$. If $(t_0, \ldots, t_n)$ are projective coordinates in $\mathbb{P}^n$, then in the notation from above,

$$\text{St}(W) : \det([A_0 \begin{pmatrix} t_0 \\ \vdots \\ t_n \end{pmatrix}, \ldots, A_n \begin{pmatrix} t_0 \\ \vdots \\ t_n \end{pmatrix}]) = 0$$

(here the matrix is written as the collection of its columns).

The base locus $\text{Bs}(W)$ of $W$, the intersection of quadrics in $W$.

The polar base locus $\text{PB}(W) \subset \mathbb{P}^n \times \mathbb{P}^n$ equal to the base locus of the linear system $|\tilde{W}| = |\tilde{L}|$ of divisors of bidegree $(1, 1)$ on $\mathbb{P}(E) \times \mathbb{P}(E)$ obtained by taking the associated bilinear forms of quadrics in $E$ defined by quadrics $Q \in W$. In the notation from above, it is given by $n + 1$ bilinear equations

$$\text{PB}(W) : (t_0, \ldots, t_n) \cdot A_i \cdot \begin{pmatrix} t'_0 \\ \vdots \\ t'_n \end{pmatrix} = 0, \ i = 0, \ldots, n.$$ (7.2.2)

The Reye variety $\text{Rey}(W) \subset G_1(|E|)$ parameterizing lines in $|E|$ contained in the base locus of linear subspace of $W$ of codimension $\leq 2$. They are called Reye lines.

Suppose that $D(W)$ is a reduced hypersurface. Then it contains an open subset $D(W)^0$ of quadrics of corank 1. The map that assigns to such a quadric its unique singular point is the Steinerian map

$$\text{st} : D(W)^0 \to \text{St}(W).$$

If the Steinerian is a hypersurface, then this map is a birational morphism. Note that the open subset $D(W)^{\text{sm}}$ of nonsingular point in $D(W)$ is contained in $D(W)^0$. The image $\text{st}(D(W)^{\text{sm}})$ is the open subset $\text{St}(W)^0$ of points $x$ such that there exists a unique quadric $Q \in W$ with $x \in \text{Sing}(Q)$.

Let

$$\hat{D}(W) = \{(x, Q) \in \mathbb{P}^n \times D(W) : x \in \text{Sing}(Q)\}.$$

In coordinates, $(\lambda_0, \ldots, \lambda_n)$ in $W$ and $(t_0, \ldots, t_n)$ in $|E|$, it is given by the equations

$$\sum_{k=0}^{n} \lambda_i A^{(k)} \cdot \begin{pmatrix} t_0 \\ \vdots \\ t_n \end{pmatrix} = 0.$$
The variety $\tilde{D}(W)$ comes with the projections

\[
\begin{array}{ccc}
\tilde{D}(W) & \xrightarrow{pr_W} & D(W) \\
& & \downarrow \quad pr_{|E|} \\
& & St(W)
\end{array}
\]

For any $Q \in D(W)$, the fiber $pr_{|E|}^{-1}(Q)$ is isomorphic to $\text{Sing}(Q)$. For any $x \in St(W)$, the fiber $pr_W^{-1}(x)$ is isomorphic to the linear subspace of $D(W)$ of quadrics $Q$ with $x \in \text{Sing}(Q)$.

Recall that a quadratic form $q \in S^2E^\vee$ defines a bilinear form $b_q \in (S^2E)^\vee$. We identify it with a linear map $b_q : E \rightarrow E^\vee$. In the language of projective geometry, the map is given by $x \mapsto P_x(Q)$, where $P_x(Q)$ is the first polar hypersurface of the quadric hypersurface $Q = V(q)$. A quadric $Q = V(q)$ is singular at $x$ if $x \in \text{Ker}(b_q)$, or, equivalently, $P_x(Q) = |E|$. For any point $x = [v] \in |E|$, the intersection $\bigcap_{q \in L} b_q(v) \subset E^\vee$ is equal to $\{0\}$ unless there exists $q \in L$ such that $b_q(v) = 0$. In the language of projective geometry this means that $\bigcap_{q \in W} P_q(Q) \neq \emptyset$ if and only if there exists $y \in \mathbb{P}^n$ such that $y \in P_q(Q)$ for all $Q \in W$. This is also equivalent to that there exists $Q \in D(W)$ such that $x \in \text{Sing}(Q)$. This implies the following.

**Proposition 7.2.1.** Let $p_1, p_2 : \text{PB}(W) \rightarrow \mathbb{P}^n$ be the projection map induced by the projections $\mathbb{P}^n \times \mathbb{P}^n \rightarrow \mathbb{P}^n$. Then the image of $p_1$ is equal to $\text{St}(W)$ and the projection $p_1$ defines an isomorphism $p_2^{-1}(x) \cong \bigcap_{Q \in W} P_x(Q)$.

The varieties $\tilde{D}(W)$ and $\text{PB}(W)$ both project to $\text{St}(W)$ and have isomorphic fibers. However, they are, in general, not isomorphic. In fact they are transpose to each other in the following sense (see [202, Exercise A3.22]). Let us consider the natural polarization map $L \rightarrow E^\vee \otimes E^\vee, q \mapsto b_q$. We view this map as a homomorphism of locally free sheaves on $|E|$:

\[
p : L(-1) \rightarrow E^\vee,
\]

where, for any linear space $V$, we denote by $V$ the sheaf of sections of the trivial vector bundle associated to $V$. We assume that a general quadric in $|L|$ is nonsingular, i.e. the discriminant variety $D(W)$ is a hypersurface. Then the map $p$ is injective at a generic point of $W$. Since a locally free sheaf does not contain torsion sheaves, the homomorphism $p$ is injective. Let $C$ be the cokernel of $p$. Its support is equal to $D(W)$. Passing to fibers, we find that $C([v]) = \text{Coker}(L \rightarrow E^\vee, q \mapsto b_q(v, \cdot))$ and $C([v])^\vee = \{w \in E : b_q(v, w) = 0, \text{ for all } q \in L\}$. This shows that $\mathbb{P}(C([v]) = \{y \in |E|, y \in P_x(Q), \text{ for all } Q \in W\}$ and $\text{Proj} (\text{Sym}(C^\vee))$ is isomorphic to $\text{PB}(W)$ embedded in $\mathbb{P}(E^\vee)$ via the surjection $E^\vee \rightarrow C$. Since the dimensions of the cokernel of a linear map of the linear spaces of the same dimension and its transpose are equal, we obtain that the fibers $\tilde{D}(W)_x$ and $\text{PB}(W)_x$ of the projections $\tilde{D}(W) \rightarrow \text{St}(W)$ and $\text{PB}(W) \rightarrow \text{St}(W)$ have the same dimension. However, there is no canonical isomorphism of these fibers except when the fibers are of dimension 1.

Also observe, it follows from the above discussion, that the variety $\text{St}(W)$ is the degeneracy scheme of the homomorphism of vector bundles $p$ in the sense of [233]. Let

\[
\text{St}(W)_r = \{x \in \text{St}(W) : \dim \tilde{D}(W)_x \geq r\}.
\]

It is known that the expected codimension (i.e. for general $W$) of $\text{St}(W)_r$ in $\text{St}(W)$ is equal to $r^2$.
and
\[ \deg \text{St}(W)_r = \prod_{i=0}^{r} \frac{(n+1+i)!}{(n-r+i)!(r+1+i)!} . \] (7.2.4)

The proof of the following proposition can be found in [180, Proposition 1.1.28 and Proposition 1.1.30].

**Proposition 7.2.2.** A point \((x, y) \in \text{PB}(W)\) is singular if and only if \(x = y\), or there exists a quadric \(Q \in W\) such that \(x, y \in \text{Sing}(Q)\). In this case the point \((Q, x) \in \text{Sing}(\tilde{D}(W))\). Conversely, \((Q, x)\) is a singular point of \(\tilde{D}(W)\) if and only if \(x \in \text{Bs}(W)\) or there exists a point \(y \neq x\) such that \(x, y \subset \text{Sing}(Q)\).

Recall that the bilinear form \(b_q \in E^\vee \otimes E^\vee\) associated to a quadratic form \(q \in S^2 E^\vee\) satisfies \(b_q(v, w) = q(v + w) - q(v) - q(w)\), for any \(v, w \in E\). It defines a hypersurface of bidegree \((1, 1)\) in \(|E| \times |E|\). It consists of points \((x, y) \in |E| \times |E|\) such that \(y \in P_x(V(q))\), where \(P_x(V(q))\) is the polar hyperplane of the quadric \(V(q)\).

Assume that \(\text{Bs}(W) = \emptyset\). Then \(\text{PB}(W)\) does not intersect the diagonal \(\Delta\) of \(\mathbb{P}^n \times \mathbb{P}^n\). Consider a map \(f : \text{PB}(W) \to G_1(|E|)\) that assigns to \((x, y) \in \text{PB}(W)\) the line \(\ell = \overline{x, y}\) spanned by \(x\) and \(y\).

**Proposition 7.2.3.** The map \(f\) is of degree 2 onto \(\text{Rey}(W)\).

**Proof.** A quadric \(Q = V(q)\) contains \(\ell\) if and only if \(q\) vanishes at three distinct points on \(\ell\). Let \(Q\) contain \(x = [v]\) and \(y = [w]\), then \(q(v + w) = b_q(v, w) + q(v) + q(w) = 0\). This shows that \(Q\) contains \(\ell\). Thus \(\ell\) imposes only two conditions on quadrics to contain it. This implies that \(\ell \in \text{Rey}(W)\). It is clear that \(f\) factors through the involution \((x, y) \mapsto (y, x)\) of \(\mathbb{P}^n \times \mathbb{P}^n\). For any \(\ell \in \text{Rey}(W)\), the restriction of \(W\) to \(\ell\) is of dimension \(\leq 1\). Since \(W\) is base-point-free, the dimension is equal to 1. Let \(|L - \ell|\) be the linear system of quadrics in \(W\) that contain \(\ell\). A base-point free pencil of divisors of degree 2 on a line contains two ramification points. If \(\ell = \overline{x, y}\) for some \((x, y) \in \text{PB}(W)\), then there exists a quadric \(Q \in W\) such that \(x \in \text{Sing}(Q)\). This easily shows that all quadrics in \(W\) intersect \(\ell\) at \(x\) with multiplicity 2. Thus, \(x\) and \(y\) are the two ramification points. Conversely, if \(x, y\) are the ramification points of the pencil, then, for any quadric \(Q \in W\), the line \(\ell\) is contained in the tangent plane \(P_x(Q)\) and \(P_y(Q)\). Thus \(y \in P_x(Q)\) and \(y \in P_y(Q)\), i.e. \((x, y) \in \text{PB}(W)\).

We see that any Reye line comes equipped with two points \(x, y\) such that \((x, y) \in \text{PB}(W)\).

**Proposition 7.2.4.** Assume \(\text{Bs}(W) = \emptyset\). The two points \(x, y\) lie on the Steinerian hypersurface \(\text{St}(W)\). Let \(x \in \text{Sing}(Q)\) for some quadric \(Q \in W\). Then \(Q\) contains \(\ell\) if and only if there exists a pencil of quadrics with singular point at \(x\).

**Proof.** The restriction of \(W\) to \(\ell\) is a pencil \(\mathcal{P}\) without base points. It defines a degree 2 map \(f : \ell \to \mathbb{P}^*\) with two ramification points \(x', y'\). Let \(Q \in W\) be a quadric that intersects \(\ell\) at \(x'\) with multiplicity 2 but does not contain \(\ell\). Together with the \((n - 2)\)-dimensional linear system \(|L - \ell|\) of quadrics containing \(\ell\) they span a hyperplane in \(W\) of quadrics touching \(\ell\) at the point \(x'\). By dimension count, one of the quadrics in the linear system must have the tangent
space equal to the whole \( \mathbb{P}^n \), i.e. it is singular at \( x' \). Thus \( x' \in \text{St}(W) \). Let \( Q = V(q_0) \) with \( x' = [v] \in \text{Sing}(Q) \). Then \( b_{q_0}(v, w) = 0 \) for all \( w \in E \), hence the map \( L \to E^\vee, q \mapsto b_q(v, \cdot) \), is contained in a hyperplane of \( E^\vee \). This shows that we can find \( w \in E \) such that \( b_q(v, w) = 0 \) for all \( q \in L \). Therefore, \( (x', [w]) \in \text{PB}(B) \) and \( x' \) is equal to \( x \) or \( y \).

Choose projective coordinates such that \( \ell = \{t_2 = \ldots = t_n = 0\} \) and \( x = [1, 0, \ldots, 0], y = [0, 1, 0, \ldots, 0] \). Then we can find a basis in \( L \) such that \( W \) consists of quadrics of the form

\[
q(\lambda) = \lambda_0 t_0^2 + \lambda_1 t_1^2 + \sum_{i=2}^{n} \lambda_i L_i(t_0, \ldots, t_n) = 0,
\]

where the coefficients in \( L_i \) are linear forms in \( t_2, \ldots, t_n \). Computing the partials at the point \( x \), we obtain that the conditions for \( x \) to belong to \( Q(\lambda) \) are

\[
L_j(1, 0, \ldots, 0) = 0, \ j = 2, \ldots, n, \ \lambda_0 = 0.
\]

Let \( A^{(j)}_0 \) be the coefficients in \( L_j \) at \( x_0 \). The linear forms \( A^{(j)}_0 \) are linearly dependent if and only if there exists a quadric containing \( \ell \) with singular point at \( x \). In this case we can find \( a_2, \ldots, a_n \) not all zeros such that the quadrics \( Q(a) = V(\sum a_i q_i) \) and \( Q_1 = V(q_1) \) are linearly dependent and have \( x \) as their singular point.

**Definition 7.2.5.** A hyperweb of quadrics \( W \) is called regular if \( \text{PB}(W) \) (or, equivalently, \( \hat{D}(W) \)) is smooth.

Note that, by the adjunction formula, \( \text{PB}(W) \) has trivial canonical class, so, when it is nonsingular, it is a Calabi-Yau variety.

**Theorem 7.2.6.** Let \( W \) be a regular \( n \)-dimensional linear system of quadrics in \( \mathbb{P}^n \). The following properties hold:

(i) \( \text{Bs}(W) = \emptyset \);

(ii) The map \( f : \text{PB}(W) \to \text{Rey}(W) \) is an étale finite map of degree 2;

(iii) \( \text{Rey}(W) \) is smooth;

(iv) \( \text{Sing}(\hat{D}(W)) \) consists of quadrics \( Q \in W \) of corank \( > 1 \);

(v) \( \hat{D}(W) \to D(W) \) is a resolution of singularities;

(vi) The projections \( \text{PB}(W) \to \text{St}(W) \) and \( \hat{D}(W) \to \text{St}(W) \) are weak resolutions of singularities.\(^1\)

**Proof.** (i) follows from Proposition 7.2.2.

Let us prove (ii). By Proposition 7.2.3, a fiber of the map \( f \) consists of two distinct points. We have to show that the differential at each point is bijective. Let \( (v_0, w_0) \in E \times E \) represent such a point. The fiber of the map \( F : |E| \times |E| \to G_1(|E|), (x, y) \mapsto x, y \) over a line \( \ell \) is equal to

\(^1\)A weak resolution of singularities requires only to be a birational morphism, not necessary an isomorphism over the set of nonsingular points.
7.2. HYPERWEBS OF QUADRICS

\( \ell \times \ell \subset |E| \times |E| \). If \( \ell = f(x, y) \), \( F^{-1}(\ell) \) consists of pairs \( ([\lambda v_0 + \mu w_0], [\lambda' v_0 + \mu' w_0]) \), so the fiber has the natural coordinates \( [\lambda, \lambda', \lambda', \lambda'] \) in the Segre embedding of \( \ell \times \ell \). The intersection of \( F^{-1}(\ell) \cap F^{-1}(\ell) \) is given by the equations

\[
\mathcal{g}(\lambda v_0 + \mu w_0, \lambda' v_0 + \mu' w_0) = (\lambda' + \lambda') \mathcal{g}(v_0, w_0) + \lambda \lambda' q(v_0) + \mu \mu' q(w_0) = 0,
\]

where \( V(q) \notin W \). These are bilinear equations on \( \ell \times \ell \). By (i), not all coefficients are zeros. Passing to the symmetric product \( \ell^{(2)} \) we see that these are linear equations in the plane. Since, we know that there is only one solution of these equations, we obtain that the fibers of the differential consist of one point, hence the map is étale.

(iii) follows from (ii).

(iv) Let \( D_n \) be the discriminant variety of singular quadrics in \( \mathbb{P}^n \) and \( D_n(k) \) be the closed subvariety of quadrics of corank \( \geq k \). It is known that \( D_n(k + 1) = \text{Sing}(D_n(k)) \). In particular, the singular locus of \( D_n \) consists of quadrics of corank \( > 1 \). The discriminant variety \( D(W) \) is equal to the intersection \( W \cap D_n(1) \). The tangent space of \( D_n \) at a nonsingular point \( Q \) can be canonically identified with the space of quadrics passing through the unique singular point of \( Q \) (see [180, p. 32]). Thus, a quadric of corank 1 in \( W \) is a singular point in \( D(W) \) if and only if the singular point of \( Q \) is a base point of \( W \). Thus a quadric of corank 1 in a regular web \( W \) is always nonsingular point of \( D(W) \). On the other hand, a quadric \( Q \) of corank \( \geq 2 \) is a singular point of the discriminant hypersurface \( D_n(1) \), hence it is a singular point of \( D(W) \) if we assume that \( D(W) \neq \mathbb{P}^n \).

(v) By Proposition 7.2.2, \( \bar{D}(W) \) is smooth. It is known that a nonsingular point \( Q \) of \( D(W) \) is of corank 1. Thus the fiber \( \text{pr}_W^{-1}(Q) \) consists of one point, hence \( \text{pr}_W \) is an isomorphism over the subset of nonsingular points. Note that a quadric \( Q \in D(W) \) of corank 1 could be a singular point of \( D(W) \) but this implies that \( D(W) \) is singular over \( Q \).

(vi) We know that \( \text{pr}_{|E|} : \bar{D}(W) \to \text{St}(W) \) and \( p_2 : \text{PB}(W) \to \text{St}(W) \) are isomorphic as schemes over \( \text{St}(W) \). It remains only to prove that \( p_2 \) is a birational morphism. By (iv), the set of quadrics in \( W \) of corank 1 (i.e. with isolated singular point) is the open subset of smooth points on \( D(W) \). The Steinerian map \( st : D(W)^{\text{sm}} \to \text{St}(W) \) has linear spaces as fibers. Since both the source and the target are hypersurfaces in an \( n \)-dimensional projective space, the map is a birational morphism. Thus \( \text{St}(W) \) contains an open subset \( \text{St}(W)^0 \) that consists of points \( x \) such that there exists a unique quadric \( Q_x \) in \( W \) with \( x \in \text{Sing}(Q) \). Moreover, this quadric is of corank 1. This implies that \( \dim \cap_{Q \in W} P_x(Q) = 0 \), hence, by Proposition 7.2.1, the projection \( \text{PB}(W) \to \text{St}(W) \) is an isomorphism over \( x \).

\[\square\]

**Definition 7.2.7.** A hyperweb of quadrics in \( \mathbb{P}^n \) is called excellent if it is regular and the projection map \( \text{PB}(W) \to \text{St}(W) \) is an isomorphism.

**Proposition 7.2.8.** A regular hyperweb \( W \) is excellent if and only if \( D(W) \) does not contain lines.

**Proof.** Suppose \( W \) is excellent. Then, for any point \( x \in \text{St}(W) \), the intersection of polar hyperplanes \( P_x(Q), Q \in W \), consists of a single point. Thus there exists a unique quadric \( Q \) with \( x \in \text{Sing}(Q) \). Suppose \( D(W) \) contains a pencil. By Bertini’s Theorem, there exists a point \( x \in \mathbb{P}^n \) such that all quadrics in this pencil have \( x \) as its singular point. This gives a contradiction. This argument also proves the converse. \[\square\]
Example 7.2.9. Assume \( n = 2 \), so we are dealing with a net \( W \) of conics in \( \mathbb{P}^2 \). The classification of nets of conics up to projective equivalence over \( \mathbb{C} \) is due to C. Jordan \[344\] (see a modern survey in \[1\]). It consists of 15 isomorphism classes. For our application we restrict ourselves with the case when the net is base-point-free. There are the following four non-projectively equivalent nets:

(i)  
\[
\lambda(2x_0x_1) + \mu(2x_0x_2 + x_1^2) + \gamma((-a - \frac{3}{4}\alpha^2)x_0^2 - \alpha x_1^2 + x_2^2 + \alpha x_0x_2) = 0,
\]
where \( \alpha \) is one of the three distinct roots of the equation \( \alpha^3 + a\alpha + b = 0 \). The discriminant curve is a nonsingular plane cubic
\[
D(W) : \lambda^2\gamma + \mu^3 + a\mu\gamma^2 + b\gamma^3 = 0, \quad 4\alpha^3 + 27b^2 \neq 0.
\]
The Steinerian curve is a nonsingular plane cubic curve:
\[
St(W) : x_0^2x_2 + \frac{3}{2}\alpha x_0x_1^2 - x_0x_2^2 - (a - \frac{3}{4}\alpha^2)x_0^3 = 0.
\]
The curve \( PB(W) \) and \( \tilde{D}(W) \) are isomorphic to \( D(W) \). The Reye curve lies in the dual plane and parameterizes line components of singular conics in \( W \). It coincides with the Cayleyan curve of \( St(W) \) \[180, 3.2\]. The map \( PB(W) \to Rey(W) \) is an étale double cover.

(ii)  
\[
\lambda x_0^2 + 2\mu x_1 x_2 + \gamma(x_1^2 + x_2^2 + 2x_0x_1) = 0.
\]
The discriminant curve is a nodal irreducible cubic
\[
D(W) : \gamma^3 + \lambda\mu^2 - \lambda\gamma^2 = 0.
\]
The Steinerian curve is the union of a line and a conic intersecting transversally:
\[
St(W) : x_0(x_2 - x_1^2 - x_0x_1) = 0.
\]
The curve \( PB(W) \) and \( \tilde{D}(W) \) are isomorphic to \( St(W) \). The Reye curve is a nodal cubic. The map \( PB(W) \to Rey(W) \) is an étale double cover.

(iii)  
\[
\lambda x_0^2 + \mu x_1^2 + \gamma(2x_0x_1 + x_2^2) = 0.
\]
The discriminant curve is the union of a line and a conic intersecting at two points:
\[
D(W) : \gamma(\lambda\mu - \gamma^2) = 0.
\]
The Steinerian curve is the union of three non-concurrent lines:
\[
St(W) : x_0x_1x_2 = 0.
\]
The curves \( PB(W) \) consists of four irreducible components. In the Segre embedding of \( \mathbb{P}^2 \times \mathbb{P}^2 \), two components are lines and other two are conics. The curve \( D(W) \) is isomorphic to \( PB(W) \). The Reye curve is isomorphic to \( D(W) \), the map \( PB(W) \to Rey(W) \) is a double cover with the deck transformation switching two conics and two lines.

(iv)  
\[
\lambda x_0^2 + \mu x_1^2 + \gamma x_2^2 = 0.
\]
The discriminant curve is the union of three non-concurrent lines
\[ \lambda \mu \gamma = 0. \]

The Steinerian curve is the union of three non-concurrent lines:
\[ \text{St}(W) : x_0 x_1 x_2 = 0. \]

The curve \( \text{PB}(W) \) consists of six irreducible components. In the Segre embedding of \( \mathbb{P}^2 \times \mathbb{P}^2 \), they are lines. The curve \( D(W) \) is isomorphic to \( \text{PB}(W) \). The Reye curve is isomorphic to \( D(W) \), the map \( \text{PB}(W) \to \text{Rey}(W) \) is a double cover with the deck transformation switching two skew lines.

A net of conics is regular if and only if the discriminant curve \( D(W) \) is nonsingular. A regular net of conics is an excellent net.

The proof of the following theorem can be found in \([281]\) or \([348]\)).

**Theorem 7.2.10.** Let \( D_n(k) \) be the variety of quadrics of corank \( k \) in \( \mathbb{P}^n \).

- \( D_n(k) \) is an irreducible Cohen-Macaulay subvariety of codimension \( \frac{1}{2} k(k + 1) \);
- \( \text{Sing}(D_n(k)) = D_n(k + 1) \);
- \( \deg D_n(k) = \prod_{0 \leq i \leq k - 1} \left( \frac{n + 1 + i}{k - i} \right) \).

A general \( W \) intersects \( D_n(k) \) transversally and \( D(W) \) inherits all properties from Theorem 7.2.10. We do not know if a regular hyperweb of quadrics satisfies these properties.

Finally, we introduce one more variety associated to a hyperweb of quadrics \( |L| \). We know that \( E^\vee \otimes E^\vee = \bigwedge^2 E^\vee \oplus S^2 E^\vee \). The map given by the linear system \( |\bigwedge^2 E^\vee| \) maps \( |E| \to |E| \) to \( |\bigwedge^2 E| \). The map given by the linear system \( |S^2 E^\vee| \) maps \( |E| \times |E| \) to \( |S^2 E| \). Consider the restriction of these maps to \( \text{PB}(W) \). The image of \( \text{PB}(W) \) under the first map is the Reye variety \( \text{Rey}(W) \). The image of \( \text{PB}(W) \) under the second map is a subvariety of quadrics of rank 2 in \( |L^\perp| \subset |S^2 E| \). It is called the Cayley variety of \( W \) and it will be denoted by \( \text{Cay}(W) \). Note that, by Theorem 7.2.10, one expects that \( \text{Cay}(W) \) is of codimension \( n(n - 1)/2 \) in \( |L^\perp| \cong \mathbb{P}^{2n(n+1)-1} \), i.e. it is of dimension \( n - 1 = \dim \text{PB}(W) \). Its degree is equal to \( \frac{1}{2} \binom{2n}{n} \). The complete linear system \( |E^\vee \otimes E^\vee| \) defines the Segre map \( |E| \times |E| \to |E \otimes E| \). The Reye variety (resp. the Cayley variety) is the projection of the Segre variety from the subspace \( |L^\perp| \) (resp. \( |S^2 E| \)).

**Proposition 7.2.11.** The Cayley variety of a regular hyperweb of quadrics is a smooth subvariety of \( \mathbb{P}^{\frac{1}{2} n(n+1)-1} \) of degree \( \frac{1}{2} \binom{2n}{n} \). It is isomorphic to \( \text{Rey}(W) \) and to the quotient of \( \text{PB}(W) \) by a fixed-point involution \( (x, y) \mapsto (y, x) \).

**Proof.** The Cayley variety is equal to the image of the restriction of the map \( |E| \times |E| \to |S^2 E| = \mathbb{P}(S^2 E^\vee) \) given by \( (x, y) \mapsto H_x \cup H_y \), where \( H_x \) and \( H_y \) are hyperplanes in \( |S^2 E^\vee| \) corresponding to the points \( x, y \in |E| \). Since \( W \) is regular, the base locus of \( W \) is empty. This implies that \( \text{PB}(W) \) does not intersect the diagonal. Hence \( \text{Cay}(W) \cong \text{PB}(W)/\tau \), where \( \tau \) is the involution \( (x, y) \mapsto (y, x) \).

\[ \square \]
Next we assume that $p = 2$. The difference here is that a quadric is not determined by its polar bilinear form. In coordinates, a quadratic form $q \in S^2 E^\vee$ can be written in the form

$$q = \sum_{0 \leq i \leq j \leq n} a_{ij} t_i t_j,$$

(7.3.1)

the polar bilinear form $b_q$ is written in coordinates $t_i t'_j$ on $E \otimes E$ as

$$b_q = 2 \sum_{i=0}^n a_{ii} t_i^2 + \sum_{0 \leq i < j \leq n} a_{ij} (t_i t'_j + t_j t'_i)$$

and it could be zero without $q$ being zero. We have

$$\text{Sing}(V(q)) \subset V(\text{Ker}(b_q)),$$

where $\text{Ker}(b_q) = \{ v \in E : b_q(v, w) = 0 \text{ for all } w \in E \}$. The projective space $| \text{Ker}(b_q) |$ is called the nullspace of $Q$. A quadric is called defective if the equality does not hold. In this case, the set of singular points of $Q$ is the subspace of codimension 1 in the nullspace. In geometric terms, a point $x$ with $P_x(Q) = \mathbb{P}^n$ is not necessary a singular point of $Q$ as it is in the case $p \neq 2$. Recall that, any quadratic form $q(t_0, \ldots, t_n)$ over an algebraically closed field $k$ of characteristic 2 can be reduced to the form $t_0 t_1 + \cdots + t_{2k-2} t_{2k-1}$ or $t_0 t_1 + \cdots + t_{2k-2} t_{2k-1} + t_{2k}^2$, where $2k$ is the rank of the bilinear form $b_q$. Such a quadric $Q = V(q)$ is nonsingular if and only if $k = \left[ \frac{n}{2} \right]$. In the first case the discriminant of the polar bilinear form is zero, and, in the second case, it is not zero.

If $n = 2k + 1$ is odd, the quadric $V(q)$ is nonsingular if and only if the polar bilinear form is non-degenerate. In coordinates, this means that the determinant of its matrix is nonzero. Since the associated bilinear form is alternate and the determinant of a general alternate form is the square of the pfaffian determinant, we see that the discriminant hypersurface $D_{2k}$ is of degree $k + 1$ and is given by the pfaffian determinant of the matrix $(a_{ij})$.

If $n = 2k$ is even, the analogue of the discriminant of a quadric is the half-discriminant [382]. In order to define it, one considers a polynomial $q = \sum \sum_{0 \leq i \leq j \leq n} A_{ij} T_i T_j \in \mathbb{Z}[A_{ij}, T_i]$. Let $\text{discr}(q)$ be the determinant of the symmetric matrix defining the associated symmetric bilinear form $\sum_{i=0}^n 2 A_{ii} + \sum_{0 \leq i < j \leq n} A_{ij} (T_i T'_j + T_j T'_i)$. Reducing modulo 2, we get a polynomial in $\mathbb{F}_2 [A_{ij}]$. If $n$ is even, evaluating it on any quadratic form, we get zero. This implies that all coefficients of $\text{discr}(q)$ are even integers. We define the universal discriminant $\text{discr}'(q)$ by setting $\text{discr}'(q) = \frac{1}{2} \text{discr}(q)$.

We define the half-discriminant $\text{discr}'(q)$ of any quadratic form $q(t_0, \ldots, t_n)$ to be the value of $\text{discr}'(q)$ on $q$ if $n$ is even. If $n$ is odd, we do not correct the definition of the discriminant. As is for the usual discriminant, the half-discriminant does not depend on a choice of basis up to a multiplicative factor of nonzero squares in the field $k$.

One can give an explicit formula for the half-discriminant as follow. Let $q \in S^2 E^\vee$ and $b_q \in \bigwedge^2 E^\vee$ be the associated polar bilinear form. The symbolic power $b_q^{(k)} \in \bigwedge^{2k} E^\vee$ can be identified, via a choice of a basis in $\bigwedge^{2k+1} E^\vee$ with a vector $\text{pf}(q)$ from $E$. In coordinates in $E$, the coordinates of $\text{pf}(q)$ are the pfaffians of the principal minors of the alternate matrix of $b_q$ (see [202, Exercise
A2.11). We have
\[ \text{discr}'(q) = q(\text{pf}(q)). \]

This allows one to write an explicit formula for the half-discriminant hypersurface \( D(n) \) in the space of quadrics \( S^2E^\vee \). If we use the coefficients \( a_{ij} \) in (7.3.1), we get
\[
D(2) : a_{00}a_{12}^2 + a_{11}a_{12}^2 + a_{22}a_{01}^2 + a_{01}a_{12}a_{02} = 0. \tag{7.3.2}
\]

If \( n = 4 \), we get
\[
D(4) : (a_{00}a_{12}a_{34}^2 + \cdots) + (a_{01}a_{23}a_{34}a_{24} + \cdots) + (a_{01}a_{12}a_{23}a_{34}a_{04} + \cdots) = 0. \tag{7.3.3}
\]

We leave to the reader to guess the general formula.

We denote the discriminant variety by \( D(W) = W \cap D(n) \) by using the pfaffian equation if \( n \) is odd and the half-discriminant if \( n \) is even.

Recall from the previous section that a linear subspace \( L \) of \( S^2E^\vee \) defines a linear subspace \( \tilde{L} \) of the linear space \( (S^2E)^\vee \) of symmetric bilinear forms. It is equal to the image of \( L \) under the polarization map
\[
p : S^2E^\vee \to (S^2E)^\vee.
\]

Since \( p = 2 \), the image is contained in the subspace \( \bigwedge^2 E^\vee \) of alternating forms. The kernel of \( p \) consists of quadratic forms \( l^2 \), where \( l \in E^\vee \). We assume that \( L \) does not contain such quadratic form, and hence can be identified with a linear subspace of \( \bigwedge^2 E^\vee \).

We have the canonical exact sequence
\[
0 \to \bigwedge^2 E^\vee \to E^\vee \otimes E^\vee \to S^2E^\vee \to 0 \tag{7.3.4}
\]
that comes from the definition of the symmetric square of a linear space. In characteristic \( \neq 2 \), the polarization map splits this exact sequence.

The base locus of the linear system \( \tilde{W} = |\tilde{L}| \) on \( |E| \times |E| \) now contains the diagonal \( \Delta \). We denote by \( \text{Bs}(\tilde{W})^0 \) the residual component of dimension 2 of the base locus.

Let \( (S^2E)^\vee \) be the subspace of \( E^\vee \otimes E^\vee \) of symmetric bilinear forms. As in the previous section, we view \( |(S^2E)^\vee| \) as the space of symmetric divisors of type \((1,1)\) on \( |E| \times |E| \). Restricting to \( \text{Bs}(\tilde{W}) \subset |E| \times |E| \), we obtain a linear system \( |(S^2E)^\vee|/\tilde{L}| \). It defines a rational map
\[
\text{Bs}(\tilde{W}) \dashrightarrow |\tilde{L}| \subset |S^2E| = |(S^2(E)^\vee)|.
\]

This is the analogue of the Cayley map in characteristic \( \neq 2 \). It assigns to \( (x,y) \in \text{Bs}(\tilde{W}) \) the reducible quadric \( H_x + H_y \) in \( |E^\vee| \), where \( H_x, H_y \) are hyperplanes associated to the points \( x, y \in |E| \).

Note that the polarization map \( L \to S^2(E^\vee) \to \text{Sym}^2(E)^\vee \) defines a linear map
\[
p : L \to \bigwedge^2 E^\vee.
\]

We will assume that this map is injective. This happens if and only if \( L \) does not contain quadrics of rank 1. Let
\[
D(W)' = \{ Q = V(q) \in W : \text{Ker}(b_q) \neq \{0\} \}.
\]
If $n$ is even, it is equal to the whole $|L|$, if $n$ is odd, it is a hypersurface of degree $\frac{1}{2}(n+1)$ given by the pfaffian of the matrix $b_q$, where $q = \sum_{i=0}^{n} \lambda_i q_i$ for some basis $(q_i)$ of $L$.

**Proposition 7.3.1.** Let $\mathbb{A}^{(n+1)}$ be the variety of alternating matrices of size $n+1$. Let Pf$_n(k)$ be the (reduced) closed subvariety of matrices of rank $k$.

(i) Pf$_n(2c) = \text{Pf}(2c + 1)$, $2c \leq n$, is defined by pfaffians of principal $(2c \times 2c)$-submatrices of size $2c$;

(ii) Pf$_n(2c)$ is a Cohen-Macaulay variety of codimension $\frac{1}{2}(m - 2c + 2)(m - 2c + 1)$;

(iii) Pf$_n(2c - 2) = \text{Sing}(\text{Pf}_n(2c))$ if $2c < n + 1$;

(iv) $\deg \text{Pf}_n(2c) = 2^{-n+2c}\prod_{i=0}^{n-2c-1} \frac{(n+1+i)}{(2i+1)}$.

The proof of properties (i)-(iii) can be found in [380]. The formula for the degrees can be found in [348] or [281].

Let $D(n)_k$ be the set of quadrics $V(q)$ in $|E|$ with dim $\text{Sing}(V(q)) \geq k$. If $n = 2k + 1$, then $D(n)_k = \text{Pf}_n(k)$, hence $D(n)_{2s} = D(n)_{2s+1}, s \geq 0$, and $D(n)_0 = D(n)$. We do not know the analogue of Proposition 7.2.10 in the case $n$ is even. However, it is not known that $\text{Sing}(D(n)) = D(n)_2$ (see [186]).

We set

\[
\tilde{D}(W)' = \{([v], Q) \in |E| \times |L| : v \in \text{Ker}(b_q)\},
\]

\[
\tilde{D}(W) = \{(x, Q) \in |E| \times W : x \in \text{Sing}(Q)\}.
\]

**Proposition 7.3.2.** The projection $\tilde{D}(W)' \to |E|$ is surjective. It is a birational morphism if a general quadric in $W$ is of corank $\leq 2$.

*Proof.* In coordinates, $\tilde{D}(W)'$ is given by (7.2.1), where the matrices $A_i$ are the matrices of the polar bilinear form. Since the matrices $A_i$ are alternating, for any point $x = [x_0, \ldots, x_n]$, we have $\langle x, A_i \rangle x = 0$. This shows that $x$ belongs to the left kernel of the matrix $\tilde{B}(x) = [A_0 \cdot x, \ldots, A_n \cdot x]$. Thus $\det \tilde{B}(x) = 0$, and there exists a quadric $Q = V(q)$ such that $x \in |\text{Ker}(q)|$. Note that the right kernel of $\tilde{B}(x)$ can be identified with the linear space of $q \in L$ such that $x \in |\text{Ker}(b_q)|$. The assumption on $W$ implies that $\dim \tilde{D}(W) = n$. This implies the assertion. \(\square\)

Let St($W$) be the image of the projection of $\tilde{D}(W)$ to $|E|$. We assume that $\dim \tilde{D}(W) = n - 1$, i.e. a general member of $W$ is a nonsingular quadric, and a general point of $D(W)$ represents a quadric with isolated singular point. Then the image St($W$) of the projection $\tilde{D}(W) \to D(W)$ is a proper subvariety of $\mathbb{P}^n$. We continue to call it the Steinerian hypersurface. It follows from the following proposition that it is, in fact, a hypersurface.

**Proposition 7.3.3.** Assume that $\dim D(W) = n - 1$. Then St($W$) is a hypersurface of degree $n + 1$. 
7.3. HYPERWEBS OF QUADRICS IN CHARACTERISTIC 2

Proof. A point $x \in \mathbb{P}^n$ belongs to $\text{St}(W)$ if and only if there exists a quadric $Q = V(q)$ such that $x \in \text{Ker}(b_q)$ and $x \in Q$. Choose a basis $q_0, \ldots, q_n$ in $L$ and coordinates $(t_0, \ldots, t_n)$ in $|E|$. Then $Q(\lambda) = V(\sum \lambda_i q_i)$ contains $x = [x_0, \ldots, x_n]$ in its singular locus if and only if $B(x) \cdot \lambda = 0$ and $x \in Q(\lambda)$, where $B(x)$ is the matrix from the proof of the previous proposition. Assume that $\text{St}(W)$ is not contained in the subset $\{x : \text{corank } B(x) > 1\}$. Then, for a general point $x$, there exists a unique quadric $Q(\lambda)$ such that $x \in \text{Sing}(Q(\lambda))$. We can take $[\lambda_0, \ldots, \lambda_n]$ to be a column of the adjugate matrix $\text{adj}(B(x))$. Thus each $\lambda_i$ is a polynomial of degree $n$ in coordinates of $x$. Since $x$ belongs to the right kernel of the matrix $B(x)$, we see that each entry $C_{ij}$ in a row $(C_{i0}, \ldots, C_{in})$ of $\text{adj}(B(x))$ is divisible by $x_j$. This shows that each column $(C_{0j}, \ldots, C_{nj})$ of $\text{adj}(B(x))$ is divisible by $x_j$. Thus we can take $\lambda_i$ to be polynomials of degree $n - 1$ in $x$. This gives the equation of $\text{St}(W)$ of degree $n + 1$. 

To extend the construction of the double cover $\text{PB}(W) \to \text{Rey}(W)$ to characteristic 2, we have to use a different definition of $\text{PB}(W)$. In fact, in characteristic 2, the old definition of $\text{PB}(W)$ shows that it contains the diagonal of $|E| \times |E|$. We will define $\text{PB}(W)$ now as a certain surface in the projective completion $\mathbb{T}(|E|)$ of the tangent bundle $T(|E|)$ that is projected to the Steinerian surface of $W$.

Recall that for any quasi-coherent sheaf $\mathcal{F}$ on a scheme $Y$, one defines the following schemes:

\begin{align*}
C &= \mathbb{V}(\mathcal{F}) = \text{Spec } (\mathcal{S}(\mathcal{F})), \\
X &= \mathbb{P}(\mathcal{F}) = \text{Proj } (\mathcal{S}(\mathcal{F})), \\
\hat{C} &= \mathbb{V}(\mathcal{F}) := \mathbb{P}(\mathcal{F} \oplus \mathcal{O}_Y) \cong \text{Proj } (\mathcal{S}(\mathcal{F})[z]), \\
C_0 &= \text{the closed subscheme of } \hat{C} \text{ defined by the surjection } \mathcal{S}(\mathcal{F}) \to S^0(\mathcal{F}) := \mathcal{O}_Y, \\
C^* &= C \setminus C_0, \\
s_0 : Y = \mathbb{P}(\mathcal{O}_Y) \to \hat{C} &= \text{the closed embedding corresponding to the surjection } \mathcal{F} \oplus \mathcal{O}_Y \to \mathcal{O}_Y, \\
i : C \hookrightarrow \hat{C} &= \text{the open embedding with the complement } s_0(Y), \\
C^* &= \hat{C} \setminus s_0(Y), \\
s_{\infty} : X \to \hat{C} &= \text{closed embedding corresponding to the surjection } \mathcal{F} \oplus \mathcal{O}_Y \to \mathcal{F}; \\
p : \hat{C}^* \to X &= \text{the morphism corresponding to the natural inclusion } \mathcal{S}(\mathcal{F}) \subset \mathcal{S}(\mathcal{F})[z]; \\
\pi : C^* \to X &= \text{the composition } p \circ i.
\end{align*}

We specialize by taking $\mathcal{F}$ to be a locally free sheaf of rank $r + 1$. Then $C = \mathbb{V}(\mathcal{F})$ is the vector bundle associated to $\mathcal{F}$. The sheaf $\mathcal{F}^\vee$ is the sheaf of local sections of $\mathbb{V}(\mathcal{F})$. The scheme $X = \mathbb{P}(\mathcal{F})$ is the projectivization of the vector bundle $C$. It is a projective $r$-bundle. Its fibers $X_y$ are the projective spaces $\mathbb{P}(\mathcal{F}(y)) = |\mathcal{F}(y)^\vee|$ of dimension $r$. The scheme $\hat{C} = \mathbb{V}(\mathcal{F})$ is the projective completion of the vector bundle $\mathbb{V}(\mathcal{F})$.

To specialize it further, we take $S = \mathbb{P}^n$ and $\mathcal{F} = \Omega^1_{\mathbb{P}^n}$. Then $\mathbb{V}(\Omega^1_{\mathbb{P}^n})$ is the tangent bundle $T(\mathbb{P}^n)$ of $\mathbb{P}^n$, $\mathbb{P}(\Omega^1_{\mathbb{P}^n}) = \mathbb{P} T(\mathbb{P}^n)$ is its projectivization, $\mathbb{V}(\Omega^1_{\mathbb{P}^n}) = \mathbb{T}(\mathbb{P}^n)$ is the completion of the tangent bundle. We denote by $\mathbb{T}(\mathbb{P}^n)^\infty$ (resp. $\mathbb{T}(\mathbb{P}^n)^0$) the image of the projectivized tangent bundle $\mathbb{P} T(\mathbb{P}^n)$ (resp. of $\mathbb{P}^n$) under the morphism $s_{\infty}$ (resp. $s_0$). Note that $\mathbb{T}(\mathbb{P}^n)^0$ is the analogue of the diagonal of $\mathbb{P}^n \times \mathbb{P}^n$. 

Recall that there is a canonical isomorphism (7.1.10)
\[ \bigwedge^2 E^\vee \rightarrow H^0(|E|, \Omega^1_{|E|}(2)), \]
In coordinates, it is given by \( t_i \wedge t_j \mapsto t_j^2 \frac{dt_i}{t_j} \). Thus, if we write
\[ T(|E|) = \mathbb{P}(\Omega^1_{|E|}) = \mathbb{P}(\Omega^1_{|E|}(2) \oplus \mathcal{O}_{|E|}(2)), \]
then, we can identify the vector space \( \bigwedge^2 E^\vee \oplus S^2 E^\vee \) with the space of sections of \( \mathcal{O}_{T(|E|)}(2) \). If \( p \neq 2 \), we have \( \bigwedge^2 E^\vee \oplus S^2 E^\vee \cong E^\vee \oplus E^\vee \), and sections of the latter are identified with sections of \( \mathcal{O}_{|E|}(1,1) \) to which polar symmetric bilinear forms belong.

Let \( \mathcal{O}_{T(|E|)}(1) \) be the invertible sheaf on \( T(|E|) \) whose direct image in the base is equal to \( \Omega^1_{|E|}(1) \oplus \mathcal{O}_{|E|}(2) \).

Recall that a tangent vector \( t_x \in T(|E|)_x \) at a point \( x \) is canonically identified with a linear map \( t_x : x \rightarrow E/x \), where \( x \) is considered as the line in \( E \) corresponding to the point \( x \in |E| \). We identify the space \( T(|E|)_x \) with \( |\text{Hom}(x, E/x) \oplus k| \). We denote vectors from \( \text{Hom}(x, E/x) \oplus k \) by \( t_x \) and write them \( (t_x, a) \) and denote the corresponding point in \( T(|E|)_x \) by \( \tilde{x} \) or by \( [t_x] \). The tangent space \( T(|E|)_x = \text{Hom}(x, E/x) \) embeds in \( T(|E|)_x \) as the set of points \([t_x, 1]\). The projectivized tangent space \( |T(|E|)_x| \) embeds as the hyperplane of points \([t_x, 0]\). The point \([0, 1]\) belongs to \( T(|E|)^0 \).

By definition, the corrected polar bilinear form \( b'_q \) of a quadratic form \( q \in S^2 E^\vee \) is defined to be a section \( (b_q, q) \) of \( \bigwedge^2 E^\vee \oplus S^2 E^\vee \). Considered as a section of the tautological invertible sheaf \( \mathcal{O}_{T(|E|)}(2) \) it defines a divisor \( Z(b'_q) \) on \( T(|E|) \). We have \( t_x = [t_x, a] \in Z(b'_q) \) if and only if \( b_q(v, t_x(v)) + aq(v) = 0 \), where \( x = [v] \).

In characteristic 2, we define the polar base variety \( \text{PB}(W) \) of a hyperweb \( W \) as the intersection scheme \( \cap_{q \in W} Z(b'_q) \). Note that
\[ \text{PB}(W)^0 := \text{PB}(W) \cap T(|E|)^0 = s_0(\text{Bs}(W)), \quad (7.3.5) \]
\[ \text{PB}(W)^\infty := \text{PB}(W) \cap T(|E|) = \bigcap_{q \in L V(b'_q) \subset \mathbb{P} T(|E|). \quad (7.3.6) \]
Here \( b_q \in \bigwedge^2 E^\vee \) is considered as a section of the invertible sheaf \( \mathcal{O}_{T(|E|)}(1) \otimes p^* \mathcal{O}_{|E|}(2) \) on \( \mathbb{P} T(|E|) = \mathbb{P}(\Omega^1_{|E|}) \). Recall from [180, 10.2.3] that a point in \( |\bigwedge^2 E^\vee| \), viewed as a hyperplane in the Plücker space \( |\bigwedge^2 E| \), is identified with a linear complex, i.e. a hyperplane section of the Grassmann variety \( G(2, E) \). We assume that the polarization map \( L \rightarrow \bigwedge^2 E^\vee \) is injective, i.e. \( W \) does not contains quadrics of rank 1. Then \( W \) defines a linear system of linear complexes in \( G \). In this terminology, the variety \( \text{PB}(W)^\infty \) becomes isomorphic to the preimage in \( Z_G \) of the base locus of the corresponding linear system of linear line complexes in \( G \). Its dimension is equal to \( n - 2 \). Its degree is equal to the degree of the Grassmannian \( G \), known to be equal to \( \frac{(2n-2)!}{(n-1)!(n)!} \).

We keep the definition of the Reye variety \( \text{Rey}(W) \) and a Reye line.

**Lemma 7.3.4.** For any \( \tilde{x} \in \text{PB}(W) \setminus \text{PB}(W)_0 \), the line \( \ell_{\tilde{x}} \) is a Reye line. All quadrics in \( W \) are tangent to \( \ell_{\tilde{x}} \) at the point \( x \). For any Reye line \( \ell \) not passing through a base point of \( W \), there exists
Proposition 7.3.6. It is equal to the restriction to 

\[ PB(\mathbb{P}^1) \]

Proof. Let \( Q = V(q) \) be a quadric from \( W \) containing \( \ell = \ell_\tilde{x} \). Let \( x = [v] \) and \( y = [w] \in \ell_\tilde{x}, y \neq x \). Since \( q(v) = q(w) = q(v + w) = 0 \), we get \( b_q(v, w) = 0 \). Conversely, if \( q(v) + b_q(v, w) = 0 \) and \( q(w) = 0 \) for some \( y = [w] \in \ell, y \neq x \), we obtain that \( \ell \subset Q \). This shows that \( \ell \) imposes two (instead of expected three) linear conditions on a quadric \( Q \in W \) to contain \( \ell \). This implies that \( \ell \) is a Reye line. Also observe that \( q(v) = 0 \) implies \( b_q(v, w) = 0 \), hence the line \( \ell \) is tangent at the point \( x \) to all quadrics from \( W \).

Conversely, let \( \ell \) be a Reye line. Then the restriction of \( W \) to \( \ell \) is of dimension \( \leq 1 \). If the dimension is equal to zero, all quadrics in \( W \) have a base point \( x \) on \( \ell \). Thus we may assume that the dimension is equal to 1 and the pencil does not have a base point. Let \( \phi : \ell \to \mathbb{P}^1 \) be the corresponding map of degree 2.

Assume \( \phi \) is a separable map. Then \( \phi \) has a unique ramification point \( x = [v] \in \ell \). Thus there exists a quadric in the pencil that it is tangent to \( \ell \) at \( x \). This implies that there exists a hyperplane \( H \) of quadrics in \( W \) such that all quadrics from \( H \) are tangent to \( \ell \) at \( x \). For any \( t_x \in T(\ell)_x \), we have \( b_q(v, t_x(v)) = q(v) = 0 \) for all \( V(q) \in H \). We can find a scalar \( a \) such that \( b_q(v, t_x(v)) + aq(v) = 0 \) for all \( q \in L \). This implies that the point \( \tilde{x} = [t_x, a] \in PB(W) \) and \( \ell = \ell_{\tilde{x}} \).

Assume \( \phi \) is an inseparable map. Then, for each point \( x \in \ell \), there exists a hyperplane \( H(x) \subset W \) of quadrics tangent to \( \ell \) at \( x \). Take two different points \( x_1 = [v_1], x_2 = [v_2] \) on \( \ell \) and let \( Q_i = V(q_i) \) be quadrics in \( W \) that touch \( \ell \) at the points \( x_i \). Take a point \( \tilde{x}_i \in \mathbb{T}(|E|)_x^{\infty} \) defined by \( [t_{x_1}] = [v_2 - v_1] \in \mathbb{P}(T(|E|))_x \). Since \( H(x_1) \) and \( H(x_2) \) generate \( W \), we find that \( b'_q(\tilde{x}_1) = b_q(v_1, t_{x_1}(v_1)) = 0 \) for all \( q \in L \). Thus \( \ell = \ell_{\tilde{x}_1} \).

\[ \square \]

Definition 7.3.5. We say that a Reye line that does not pass through a base point of \( W \) is separable (resp. inseparable) if the restriction of \( W \) to \( \ell \) defines a separable (resp. inseparable) map \( \ell \to \mathbb{P}^1 \) of degree 2.

It follows from the proof of the previous proposition that a separable (resp. inseparable) Reye line is equal to a line \( \ell_{\tilde{x}} \) for some point \( \tilde{x} \in PB(W) \setminus (PB(W)_0 \cup PB(W)^\infty) \) (resp. \( \tilde{x} \in PB(W)^\infty \)). We know from the description of the subvariety \( PB(W)^\infty \) that the subvariety \( \text{Rey}(W)_{\text{ins}} \subset \text{Rey}(W) \) of inseparable Reye lines is of dimension \( n - 3 \) and its degree in the Plücker embedding is equal to \( \text{deg} \, G = \frac{[2n-2]!}{n!(n-1)!} \). The formula for the canonical class of \( G = G_1(\mathbb{P}^m) \) shows that

\[ \omega_{\text{Rey}(W)_{\text{ins}}} \cong \mathcal{O}_{\text{Rey}(W)_{\text{ins}}} \].

(7.3.7)

Assume that \( B_8(W) = \emptyset \). Consider the Reye map

\[ \tau : PB(W) \to \text{Rey}(W), \quad \tilde{x} \mapsto \ell_{\tilde{x}}. \]

It is equal to the restriction to \( PB(W) \) of the composition of the projections \( \mathbb{T}(|E|)^* \to \mathbb{P} T(|E|) = Z_G \to G \).

Proposition 7.3.6. Let \( \tau : PB(W) \to PB(W)' \to \text{Rey}(W) \) be the Stein factorization of the Reye map. Then map \( \tau' \) is an inseparable finite map of degree 2 and \( \sigma \) is the blowing down of \( PB(W)^\infty \) onto \( \tau'^{-1}(\text{Rey}(W)_{\text{ins}}) \).
Proof. Consider the map $\text{pr} : \mathbb{T}(|E|)^* \to Z_G$. The fiber over a Reye line $\ell$ is equal to the fiber of $\text{pr}$ over the preimage of $\ell$, considered as a point of $G$ in the $Z_G$. Consider the restriction of $\mathbb{T}(\mathbb{P}^n) \to \mathbb{P}^n$ over $\ell$. Its intersection with $\text{PB}(W)$ is isomorphic to the intersection of $\text{PB}(W)$ with the surface $\mathbb{T}(\ell) \cong F_2$ embedded in $\mathbb{T}(\mathbb{P}^n)|_\ell$ via the surjection $\Omega_{\mathbb{P}^n}^2 \oplus \mathcal{O}_T \to \Omega^1_{\mathbb{P}^n} \oplus \mathcal{O}_T$. Since $\ell$ is a Reye line, the restriction of the linear system $[b'_q]$ of corrected polar forms to $\mathbb{T}(\ell)$ is the scheme-theoretical intersection of two sections $s_1$ and $s_2$ of the ruled surface $F_2$. If $\ell$ is a separable Reye line $\ell_{x}$, then $s_1$ and $s_2$ do not intersect the exceptional section $e = \mathbb{T}(\ell) \cap \text{PB}(W)_0$. They intersect at one point $\tilde{x}$ with multiplicity 2. If $\ell$ is an inseparable line, then $s_1 = s_2 = e$. So, the fiber is the section taken with multiplicity 2. This proves the assertion.

\[ \square \]

Remark 7.3.7. Note that the projection $\mathbb{T}(\mathbb{P}^n)^* \to \mathbb{P}\mathbb{T}(\mathbb{P}^n)$ is a line bundle, a torsor over the group $\mathbb{G}_m$. The subscheme $\text{PB}(W)$ is invariant with respect to the subgroup $\mu_2$ and the quotient $\text{PB}(W)/\mu_2$ is isomorphic to the blow-up of the locus of inseparable Reye lines in $\text{Rey}(W)'$.

Note that, for any locally free sheaf $\mathcal{V}$ over a scheme $S$, the relative Euler exact sequence gives an isomorphism $\omega_{\mathcal{V}/S} \cong \text{det}(p^*\mathcal{E}(-1))$. This gives

$$\omega_{\mathbb{T}(|E|)/S} \cong p^*\omega_{|E|} \otimes \omega_{\mathbb{T}(|E|)/|E|} \cong p^*\omega_{|E|} \otimes p^*\omega_{|E|}(-n - 1) = \mathcal{O}_{\mathbb{T}(|E|)}(-n - 1).$$

Applying the adjunction formula, we obtain

$$\omega_{\text{PB}(W)} \cong \mathcal{O}_{\text{PB}(W)}.$$

However, as we will see later, the variety $\text{PB}(W)$ is always singular.

Consider the canonical projection $p : \mathbb{T}(|E|) \to |E|$. The restriction of the projection to $\text{PB}(W)$ defines a morphism $q : \text{PB}(W) \to |E|$. The scheme-theoretical image is the Steinerian variety $\text{St}(W)$. In fact, $x = [v] \in p(\mathbb{T}(|E|))$ if and only if $b_q(v, t_x(v)) + a_q(v) = 0$ for some $[t_x, a] \in \mathbb{T}(|E|)_x$ and all $q \in L$. Let $q_0$ be an element of the kernel of the linear map $L \to |E'/x^\vee|, q \mapsto (t_x \mapsto b_q(v, t_x(v))).$ Then $b_{q_0}(v, w) = 0$ for all $w \in E$ and $q_0(v) = 0$, i.e. $[v] \in \text{Sing}(V(q_0))$.

Proposition 7.3.8. A point $(x, Q) \in \mathcal{D}(W)$ (resp. $\mathcal{D}(W)'$) is singular if and only if $x \in \text{Bs}(W)$, or there exists a point $\tilde{x} \in \text{PB}(W)$ (resp. $\tilde{x} \in \text{PB}(W)_{\text{tr}}$) such that the line $\ell_{\tilde{x}} \subset |\text{Ker}(b_q)|$.

Proof. Suppose $(x, Q)$ is a singular point of $\mathcal{D}(W)$. Choose coordinates $(\lambda_0, \ldots, \lambda_n)$ in $L$ and coordinates $(t_0, \ldots, t_n)$ in $E$. Let $q_k = \sum a_{ij}^{(k)} t_it_j, k = 0, \ldots, n$ be the corresponding basis in $L$ and let $B^{(k)} = (b_{ij}^{(k)})$ be the alternating matrices defining the polar bilinear form $b_{q_k}$. We have $b_{ij}^{(k)} = 0, b_{ij}^{(k)} = b_{ji}^{(k)} = a_{ij}^{(k)}$. We write $A(\lambda) = \sum \lambda_k A^{(k)}, B(\lambda) = \sum \lambda_k B^{(k)}$. The equations of $\mathcal{D}(W)$ are

$$F_i = \sum_{0 \leq j, k \leq n} \lambda_k b_{ij}^{(k)} t_j = 0, \quad i = 0, \ldots, n,$$

$$G = \sum_{0 \leq j, k \leq n} \lambda_k a_{ij}^{(k)} t_j t_j = 0.$$
The Jacobian matrix at the point $\lambda_i = \alpha_i, t_i = x_i$ is equal to
\[
\begin{pmatrix}
\sum_{j=0}^{n} b_{0j}^{(0)} x_j & \cdots & \sum_{j=0}^{n} b_{nj}^{(0)} x_j & \sum_{k=0}^{n} \alpha_k b_{0k}^{(k)} & \cdots & \sum_{k=0}^{n} \alpha_k b_{kn}^{(k)} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\sum_{j=0}^{n} b_{nj}^{(0)} x_j & \cdots & \sum_{j=0}^{n} b_{nj}^{(0)} x_j & \sum_{k=0}^{n} \alpha_k b_{nj}^{(k)} & \cdots & \sum_{k=0}^{n} \alpha_k b_{kn}^{(k)} \\
\sum_{i,j=0}^{n} a_{ij}^{(0)} x_i x_j & \cdots & \sum_{i,j=0}^{n} a_{ij}^{(0)} x_i x_j & \sum_{k=0}^{n} \alpha_k b_{0j}^{(k)} x_j & \cdots & \sum_{k=0}^{n} \alpha_k b_{kn}^{(k)} x_j
\end{pmatrix}
\]
Note that the last $n+1$ entries in the last row are equal to zero because $B(\alpha) \cdot x = 0$. Since the point $(\alpha, x)$ is singular, there exist a nonzero vector $(\beta_0, \ldots, \beta_{n+1})$ such that the linear combination of the rows with coefficients $\beta_i$ is equal to zero. This gives
\[
\sum_{0 \leq i,j \leq n} a_{ij}^{(0)} \beta_i x_j + \beta_{n+1} a_{ij}^{(0)} x_i x_j = 0 \quad \Rightarrow \quad \sum_{0 \leq i,j \leq n} b_{ij}^{(n)} \beta_i x_j + \beta_{n+1} b_{ij}^{(n)} x_i x_j = 0,
\]
and let $a : x \rightarrow \alpha$ be defined by $\beta_0(x) = 0, \ldots, \beta_n(x) = 0$. The last $n+1$ equations imply that $A(\alpha) \cdot b = 0$. Let $t_x : x \rightarrow E = \mathbb{K}^{n+1}$ be defined by $(\beta_0, \ldots, \beta_n)$ and let $a : x \rightarrow x$ be defined by $\beta_{n+1}$. This shows that the point $\tilde{x} = (t_x, a)$ belongs to $PB(W)$ and the line $\ell_{\tilde{x}}$ belongs to the nullspace of the matrix $B(\alpha)$. Conversely, if this happens the point $(x, Q)$ is a singular point.

If we follow the previous proof by taking $G = 0$, we obtain the assertion about singularities of $\mathcal{D}(W) \gamma$.

Assume $n$ is odd. One can restate the previous proposition in terms of the projections $\pi : \mathcal{D}(W) \rightarrow D(W)$ and $\pi : \mathcal{D}(W) \gamma \rightarrow D(W)$. A point $(x, Q) \in \mathcal{D}(W)$ (resp. $(x, Q) \in \mathcal{D}(W) \gamma$) is singular if and only if $x \in BS(W)$, or the fiber $\pi'^{-1}(Q)$ does not contain a Reye line $\ell_{\tilde{x}}$ (resp. an inseparable Reye line $\ell_{\tilde{x}}$).

**Proposition 7.3.9.** A point $\tilde{x} = [t_x, a] \in PB(|E|)$ is a singular point if and only if $\tilde{x} \in PB(W)_0$ (equivalently, $x \in BS(W)$), or there exists a quadric $Q = V(q) \in W$ such that $x \in Sing(Q)$ and the line $\ell_{\tilde{x}} \subset |Ker(b_q)|$.

**Proof.** We keep the notation from the proof of the previous proposition. Let $\tilde{x} \in Sing(PB(W))$. We may assume that $x = [e_0]$, where $(e_0, \ldots, e_n)$ is a basis in $E$ corresponding to coordinates $(t_0, \ldots, t_n)$. We identify $E/x$ with the span of the basis vectors $e_1, \ldots, e_n$. A point $\tilde{x}$ is defined by linear maps $t_x : x \rightarrow E/k$ and $a \in k$. We identify these maps with a vector $\alpha = (\alpha_0, \ldots, \alpha_n)$ and a scalar $a = \alpha_{n+1}$, respectively. The point $\tilde{x}$ belongs to $PB(W)_0$ if and only if $t_x = 0$, or, equivalently, $x \in BS(W)$.

The equations of $PB(W)$ are
\[
F_k = \sum_{i,j=0}^{n} b_{ij}^{(k)} t_j y_i + y_{n+1} a_{ij}^{(k)} t_i t_j = 0, \quad k = 0, \ldots, n. \tag{7.3.8}
\]
We have
\[
\frac{\partial F_k}{\partial t_j}(x, \alpha) = \sum_{i=0}^{n} b_{ij}^{(k)} \alpha_i + a_{ij}^{(k)} x_i, j = 0, \ldots, n, \quad (7.3.9)
\]
\[
\frac{\partial F_k}{\partial y_i}(x, \alpha) = \sum_{j=0}^{n} b_{ij}^{(k)} x_i, i = 0, \ldots, n, \quad (7.3.10)
\]
\[
\frac{\partial F_k}{\partial y_{n+1}}(x, \alpha) = \sum_{i,j=0}^{n} a_{ij}^{(k)} x_i x_j. \quad (7.3.11)
\]

The point \( \bar{x} \) is singular if and only if there exists \([\beta_0, \ldots, \beta_n] \in \mathbb{P}^n\) such that

\[
\sum_{k=0}^{n} \beta_k \frac{\partial F_k}{\partial t_j}(x, \alpha) = \sum_{k=0}^{n} \beta_k \frac{\partial F_k}{\partial y_j}(x, \alpha) = 0
\]

for all \( i = 0, \ldots, n \) and \( j = 0, \ldots, n + 1 \). The last two equations imply that \( x \cdot A(\beta) = 0 \) and \( B(\beta) \cdot x = 0 \), i.e. \( (x, Q(\beta)) \in \tilde{D}(W) \). The first equation implies that the line \( \bar{x}, [\alpha] \) belongs to \( \text{Ker}(b_q(\beta)) \). Conversely, if this happens, the point \( \bar{x} \) is a singular point of \( \text{PB}(W) \). \( \square \)

Assume \( n \) is odd. Let \( D(W)_2 \) be the subset of quadrics \( Q = V(q) \) such that \( \text{dim Ker}(q) = 2 \). We can view \( D(W)_2 \) and let \( \tilde{D}(W)_2 \) be its preimage in \( D(W)' \). Then the projection \( \pi : \tilde{D}(W)_2' \to D(W)_2 \) is a \( \mathbb{P}^1 \)-bundle. Consider the map \( a : D(W)_2 \to G_1(\mathbb{P}^n) \) that sends \( Q = V(q) \) to \( \text{Ker}(b_q) \). The preimage \( a^{-1}(\ell) \) of a line \( \ell \in G_1(\mathbb{P}^n) \) is the linear system of quadrics containing \( \ell \) in its nullspace. If the fiber is of positive dimension, then all quadrics from \( a^{-1}(\ell) \) are tangent to \( \ell \) at some points. Let \( P \) be a pencil of quadrics contained in \( a^{-1}(\ell) \). We may assume that \( \ell \) is given by equations \( t_2 = \ldots = t_n = 0 \). Thus the coefficients at \( t_0 t_i, i = 1, \ldots, n \) in equation of any quadric in \( P \) are equal to zero. We can find a quadric \( Q = V(q) \) in \( P \) such that the rank of \( b_q \), considered as a bilinear form in \( t_2, \ldots, t_n \), is less than \( n - 1 \). Thus \( b_q \) has corank \( > 2 \). The set of such quadrics is of expected codimension 15. Assume that \( D(W) \) does not consist of quadrics with nullspace of dimension \( > 1 \). In this case the map \( a \) is birational. So, the null-lines of quadrics in \( W \) are parameterized by a subvariety of dimension \( n - 1 \) in \( G_1(\mathbb{P}^n) \). Since the Reye variety is of dimension \( n - 1 \), we expect that they intersect. By Proposition 7.3.8, the point \( (Q, x) \in \tilde{D}(W) \) such that \( x \) lies on a Reye line contained in a nullspace of \( Q \) is a singular point. Thus we expect that \( \tilde{D}(W) \) is always singular. In the same way, we see that \( \text{PB}(W) \) is expected to be singular.

Applying Proposition 7.2.2, we also see that \( D(W) \) is singular.

**Proposition 7.3.10.** The image of the projection \( p : \text{PB}(W) \to \mathbb{P}^n \) is equal to \( \text{St}(W) \).

**Proof.** Let \( \bar{x} \in \text{PB}(W) \) be represented by \( \bar{t}_x : x \to E/x \) and \( a \in k \). Taking coordinates such that \( x = [1,0,\ldots,0] \), we may assume that \( E/x \) has coordinates \( y_1, \ldots, y_n \) so \( t_x = (t_x, a) \). Choose a basis \( (q_0, \ldots, q_n) \) in \( L \) and let \( q_k = \sum a_{ij}^{(k)} t_i t_j \) and let \( b_{q_k} \) be represented by an alternating matrix \( A^{(k)} \). The equations of \( \text{PB}(W) \) imply that \( (y_0, y_1, \ldots, y_n) \) satisfy the equations

\[
\sum_{j=0}^{n} \sum_{i=1}^{n} y_i x_j \lambda_k b_{ij}^{(k)} + y_0 \sum_{0 \leq i,j \leq n} a_{ij}^{(k)} t_i t_j = 0, \quad k = 0, \ldots, n.
\]
A point \( x \) belongs to the image of the projection if and only if
\[
\det \left( \begin{array}{cccc}
\sum_{j=0}^{n} \tilde{b}_{ij}^{(0)} x_j & \cdots & \sum_{j=0}^{n} \tilde{b}_{ij}^{(n)} x_j & \sum_{0 \leq i,j \leq n} a_{ij}^{(0)} t_i t_j \\
\vdots & \vdots & \vdots & \vdots \\
\sum_{j=0}^{n} \tilde{b}_{ij}^{(n)} x_j & \cdots & \sum_{j=0}^{n} \tilde{b}_{ij}^{(n)} x_j & \sum_{0 \leq i,j \leq n} a_{ij}^{(n)} t_i t_j \\
\end{array} \right) = 0.
\]
Expanding the determinant along the last column of the matrix, we obtain that the equation of \( p(D(W)) \) coincides with the equation of \( \text{St}(W) \).

**Example 7.3.11.** One can classify the projective equivalence classes of nets of conics in characteristic 2. There are 19 projective equivalence classes \([44]\). We restrict ourselves with base-point-free nets. They are represented by the following families:

(i) \[
\lambda x_0 x_1 + \mu (x_0 x_2 + x_1^2) + \gamma \left( \frac{a_6}{a_1} x_0^2 + x_2^2 + a_1 x_1 x_2 + \sqrt{a_4} x_0 x_2 \right) = 0,
\]
where \( a_1 \neq 0 \). The half-discriminant curve is an ordinary elliptic curve:
\[
D(W) : \lambda^2 \gamma + a_1 \lambda \gamma + a_1 \sqrt{a_4} \lambda \gamma^2 + \mu^3 + a_4 \gamma^2 + a_6 \gamma^3 = 0.
\]
The curves \( D(W), PB(W), \tilde{D}(W) \) are isomorphic, there is a separable isogeny of degree \( D(W) \to \text{Rey}(W) \) of elliptic curves corresponding to the unique non-trivial 2-torsion point on \( D(W) \). The curve \( \text{Rey}(W) \) lies in the dual plane and coincides with the Cayleyan curve of \( D(W) \).

(ii) \[
\lambda x_0 x_1 + \mu (x_0 x_2 + x_1^2) + \gamma (a x_0^2 + x_2^2) = 0, \ a \neq 0
\]
The discriminant curve is a cuspidal cubic:
\[
D(W) : \mu^3 + \lambda^2 \gamma = 0.
\]
The curve \( \tilde{D}(W) \) consists of two smooth rational curves \( C_1 + C_2 \), the projection \( \tilde{D}(W) \to D(W) \), restricted to \( C_1 \), is the normalization map, it maps \( C_2 \) to the cusp \([0, 0, 1] \). The Steinerian curve \( \text{St}(W) \) is a non-reduced reducible cubic
\[
x_0 (a x_0^2 + x_2^2) = 0.
\]
The curve \( PB(W) = C_1' + C_2' \) is isomorphic to \( \tilde{D}(W) \). It is mapped bijectively onto \( \text{St}(W) \). The Reye curve is isomorphic to \( D(W) \) and consists of lines \( \alpha x_0 + \beta x_1 + \gamma x_2 \) satisfying the equations
\[
\text{Rey}(W) : \alpha \beta^2 + \gamma (\alpha^2 + a \gamma^2) = 0.
\]
The line \( x_0 = 0 \) corresponding to the singular point \([1, 0, 0] \) of \( \text{Rey}(W) \) is the unique inseparable Reye line of \( W \). Under the map \( PB(W) \to \text{Rey}(W) \), one component is mapped to this point, another component is a degree two inseparable cover of \( \text{Rey}(W) \).

(iii) \[
\lambda x_0^2 + \mu (x_0 x_1 + a^{-1} x_1^2) + \gamma (x_2^2 + a x_0 x_2 + x_1^2 + x_1 (\epsilon a^{-2} x_1 + x_2)) = 0, \ a \neq 0, \epsilon = 0, 1.
\]
The discriminant curve is the union of a conic and its tangent line:
\[
D(W) : \gamma (a^2 \lambda \gamma + (\epsilon + a^2) \gamma^2 + a^2 \mu^2) = 0.
\]
The curve $\tilde{D}(W)$ consists of 3 smooth rational curves $C_1 + C_2 + C_3$, the projection $\tilde{D}(W) \rightarrow D(W)$, restricted to $C_1 + C_2$ an isomorphism, it maps $C_3$ to the tangency point $[1, 0, 0]$. The Steinerian curve $St(W)$ is a non-reduced reducible cubic, the union of two lines $ax_0 + x_1 = 0$ and $x_0 = 0$, the latter taken with multiplicity 2. The intersection point of the two lines is the base point of the pencil $\gamma = 0$. The Reye curve is isomorphic to $D(W)$ and has the equation

$$\text{Rey}(W) : \delta(a\delta + a\beta^2 + (a + \epsilon a^{-1})\delta^2) = 0.$$ 

Its singular point $(1, 0, 0)$ corresponds to the line $x_0 = 0$. There are inseparable Reye lines and the map $PB(W) \rightarrow \text{Rey}(W)$ is an inseparable map of degree 2.

(iv)

$$\lambda x_0^2 + \mu x_1(x_0 + x_1) + \gamma(x_2^2 + ax_0x_1 + x_0x_2) = 0, \ a \neq 0.$$ 

The discriminant curve, the Steinerian curve and the Reye curve are the unions of three concurrent lines

$$D(W) : \gamma(\mu \gamma + \mu^2 + a^2\gamma^2) = 0,$$

$$St(W) : x_0x_1x_2 = 0,$$

$$\text{Rey}(W) : \beta(\beta^2 + \beta \delta + a\delta^2) = 0.$$ 

(v)

$$\lambda x_0^2 + \mu x_1^2 + \gamma x_2(x_0 + x_2) = 0.$$ 

The discriminant curve is given by the equation:

$$D(W) : \gamma^2 \mu = 0.$$ 

The Steinerian curve is the line $x_2 = 0$ taken with multiplicity 3. The Reye curve has the equation

$$\text{Rey}(W) : \delta^2 \beta = 0.$$ 

(vi)

$$\lambda x_0^2 + \mu x_1^2 + \gamma x_2^2 = 0.$$ 

All curves in the net are singular.

### 7.4 Reye congruences: $p \neq 2$

Now let us specialize. We assume that $n = 3$ and $W = |L|$ is a regular web of quadrics in $\mathbb{P}^3 = |E|$. The surface $D(W)$ is a quartic in $W$, the surface $St(W)$ is a quartic in $\mathbb{P}^3$. The morphisms $\pi_1 : \tilde{D}(W) \rightarrow D(W)$ and $\pi_2 : PB(W) \rightarrow \text{D}(W)$ are resolutions of singularities. Both of these surfaces are K3 surfaces, so the resolutions are minimal, hence there is an isomorphism

$$\sigma : \text{PB}(W) \cong \tilde{D}(W).$$

Since the exceptional curves of the resolutions are isomorphic to $\mathbb{P}^1$, we obtain that all singular points of $D(W)$ are ordinary double points. Their number is equal to the degree of the locus of quadrics of corank 2, which is equal to 10. Thus $D(W)$ is a quartic surface with 10 nodes. It is called a Cayley quartic symmetroid.
If \( W \) is an excellent web, the quartic surface \( \text{St}(W) \) is nonsingular. The projection \( \pi : \hat{\mathcal{D}}(W) \to \text{St}(W) \) is a minimal resolution of singularities. The image of an irreducible exceptional curve of the resolution under the projection to \( \mathbb{P}^3 \) is a line on \( \mathcal{D}(W) \). Conversely, a line on \( \mathcal{D}(W) \) defines a pencil of singular quadrics. By Bertini’s theorem they have a common singular point. Thus this pencil is the projection of an exceptional curve of \( \pi \). A pencil of singular quadrics with a common singular point \( x \) is mapped to a pencil of conics under the projection from \( x \). A pencil of conics has three singular conics unless it contains a double line. Since no quadrics in \( W \) is a double plane, we obtain that the line of singular quadrics passes through three nodes of \( \mathcal{D}(W) \).

The surface \( \text{PB}(W) \) is a K3 surface in \( \mathbb{P}^3 \times \mathbb{P}^3 \). The Reye surface is isomorphic to the quotient of \( \text{PB}(W) \) by a fixed-point-free involution. It is an Enriques surface embedded in \( G_1([E]) \), i.e. it is a smooth congruence in \( G_1([E]) \). It is called the Reye congruence associated to \( W \). The surface \( \text{PB}(W) \) is isomorphic to its K3-cover.

**Proposition 7.4.1.** Assume \( W \) is an excellent web. Then no Reye line is contained in the Steinerian surface.

**Proof.** Suppose a Reye line is contained in \( \text{St}(W) \). This means that, for any point \( x \in \ell \), there exists a quadric in \( W \) with singular point at \( x \). Suppose \( \ell \) is not the singular line of a quadric from \( |L - \ell| \). Then, there are at most two singular points of the base curve of the pencil that lie on \( \ell \). Therefore, there are at most two quadrics from \( |L - \ell| \) that have singular points on \( \ell \). This shows that, for a general point \( x \) on \( \ell \), there exists a quadric \( Q \) not containing \( \ell \) and containing \( x \) as its singular point. This implies that the restriction of \( W \) to \( \ell \) is a linear series of non-reduced divisors of degree 2. Since \( p \neq 2 \), this is impossible.

Assume now that \( \ell \) is the singular locus of a quadric \( Q \in |L - \ell| \). Since \( \ell \) is a Reye line, by Proposition 7.2.4, it contains two points \( (x, y) \in \text{PB}(W) \) such that there exists a quadric not containing \( \ell \) but containing \( x \) as its singular point. Then there is a pencil of quadrics with singular point at \( x \), hence the fiber of \( \hat{\mathcal{D}}(W) \to \mathcal{D}(W) \) is one-dimensional. This contradicts the assumption that \( W \) is an excellent web, \( \Box \).

According to classical terminology, the complete intersection of \( G_1(\mathbb{P}^n) \) and a hypersurface of degree \( d \) in the Plücker space is called a complex of lines of degree \( d \).

**Lemma 7.4.2.** Let \( N \) be a net of quadrics in \( \mathbb{P}^n \). Assume that \( N \) does not consist of singular quadrics, does not contain quadrics of corank \( \geq 2 \), and its base locus is of expected codimension 3. Then the set of lines in \( \mathbb{P}^n \) contained in some quadric from \( N \) is a cubic complex \( M(N) \) in \( G_1(\mathbb{P}^n) \).

**Proof.** Take a general line in \( G = G_1(\mathbb{P}^n) \) represented by a pencil \( \sigma_{x, \Pi} \) of lines in a general plane \( \Pi \). The restriction of \( N \) to \( \Pi \) is a net \( N(\Pi) \) of conics. If all conics are singular, then all quadrics in \( N \) are touching \( \Pi \). By Bertini’s Theorem, all conics in \( N(\Pi) \) have a common singular point \( x_0 \), hence there exists a pencil in \( N \) of quadrics with singular point at \( x_0 \). Since \( N \) does not contain quadrics of corank \( \geq 2 \) and has the curve of singular quadrics, the set of singular points of quadrics from \( N \) is a curve in \( \mathbb{P}^n \). By taking a general \( \Pi \), we get a contradiction.

Thus, we may assume that not all conics in \( N(\Pi) \) are singular, hence the set of singular conics is a plane cubic \( F \) in \( N(\Pi) \). Taking \( \Pi \) away from \( \text{Bs}(N) \), we can further assume that \( N(\Pi) \) has no.
base points. This easily implies that $F$ is reduced. Take a general point $x_0$ in II and consider the pencil of conics in $N(II)$ that contain $x_0$. Then it is a general pencil of conics, hence it intersects $F$ at three different points corresponding to three singular conics in the pencil. This implies that there are three singular conics passing through $x$. By taking $x_0$ general enough, we may assume that none of these conics have a singular point at $x_0$. Hence $x_0$ belongs to three line components of singular conics from the net. This implies that a general pencil of lines in II contains three line components of reducible conics in $N(II)$. Therefore, $\sigma_{x,II}$ intersects $M(N)$ at three points. This shows that the class $[M]$ of $M$ in $A^2(G)$ is equal to $3[\sigma]$; i.e. $M(N)$ is a complete intersection of $G$ and a cubic hypersurface.

The cubic complex of lines $M(N)$ constructed from the lemma is called the Montesano complex of lines.

Theorem 7.4.3. The Reye congruence $\text{Rey}(W)$ is a smooth congruence of bidegree $(7, 3)$ and its sectional genus is equal to 6. Any smooth congruence of bidegree $(7, 3)$ and sectional genus equal to 6 is isomorphic to an Enriques surface.

Proof. Let $N$ be a general net in $W$. It easy to see that it satisfies the assumption of the previous lemma. Let $M_1$ and $M_2$ be the Montesano complexes of two general nets. They intersect at a congruence of bidegree $9[\sigma]^2 = (9, 9)$. A line $\ell$ from $M_1 \cap M_2$ is either contained in a quadric from the pencil $N_1 \cap N_2$, or it is contained in a quadric $Q_1$ from $N_1$ and a quadric $Q_2$ from $N_2$ not belonging to $N_1 \cap N_2$. In the latter case, the line is a Reye line of $W$. In the former case, $\ell$ intersects the base curve of $N_1 \cap N_2$ at two points. Conversely, every such line is contained in a quadric from $N_1 \cap N_2$. This shows that $M_1 \cap M_2 = \text{Rey}(W) \cup S$, where $S$ is the congruence $\text{Bis}(B)$ of bisecants of the base curve $B$ of $N_1 \cap N_2$ from Example 7.1.4. We know that the order of $S$ is equal to 2 and the class is equal to 6. This implies that $\text{Rey}(W)$ is a congruence of bidegree $(9, 9) - (2, 6) = (7, 3)$.

The Reye variety $\text{Rey}(W)$ admits an étale double cover $\text{PB}(W) \to \text{Rey}(W)$. Since $\text{PB}(W)$ is smooth and is a complete intersection of 4 divisors in $\mathbb{P}^3 \times \mathbb{P}^3$, it is a K3 surface. Therefore, $\text{Rey}(W)$ is a smooth Enriques surface embedded as a Fano model in $\mathbb{P}^5$. By the adjunction formula, the sectional genus of $\text{Rey}(W)$ is equal to 6.

Let us prove the converse. By formula (7.1.16), we have $K_S^2 = 6(\chi(O_S) - 1)$. Let $C$ be a general plane section of $S$. Since $g(C) = 6$, by adjunction formula, $C \cdot K_S = -C^2 + 2g - 2 = -10 + 10 = 0$. This implies that $K_S \equiv 0$ or no nonzero integer multiple of $K_S$ is effective. In the first case, $K_S^2 = 0$, hence $\chi(O_S) = 1$. By Proposition 1.1.5, $S$ is an Enriques surface. In the second case, by Riemann-Roch, $K_S^2 = 0$, thus, again $\chi(O_S) = 1$. Let $\phi : S \to S'$ be the morphism from $S$ to its minimal model. We have $K_{S'}^2 \geq K_S^2 = 0$ and $K_S - \phi^*(K_{S'}) \geq 0$. This implies that the Kodaira dimension of $S$ is equal to $-\infty$ and $S'$ is a minimal ruled surface. If it is not rational, $\chi(O_S) \leq 0$, so it must be a rational surface. It is easy to see that the anticanonical linear system $|-K_S|$ of a rational surface is non-empty. This contradiction proves the assertion.

We will prove later that, if $p \neq 2$, any smooth congruence of bidegree $(7, 3)$ and sectional genus 6 is equal to a Reye congruence $\text{Rey}(W)$ for some regular web of quadrics in $\mathbb{P}^3$. 

Let \( \ell \) be a Reye line of \( W \). It is spanned by the ramification points \( x_1, x_2 \) of the restriction of \( W \) to \( \ell \). The points \( x, y \) are the singular points of quadrics \( Q_1, Q_2 \) such that \( x_1 \in \Sing(Q_1) \). This shows that \( \{x_1, x_2\} = \ell \cap \St(W) \), in fact, \( \ell \) is tangent to \( \St(W) \) at these points. It is known that the variety of bitangent lines to a normal quartic surface is a congruence of lines. Its class \( n \) is equal to the number of bitangents of a plane quartic curve, known to be equal 28. Its order is equal to the number of bitangents passing through a general point in \( \mathbb{P}^3 \). It is equal to 12 [606, p. 283]. Thus \( \Rey(W) \) is one of irreducible components of the bitangent surface \( \Bit(\St(W)) \). However, the focal surface of \( \Rey(W) \) is not \( \St(W) \).

Recall that the focal surface \( \Foc(S) \) of a congruence \( S \) is the branch surface of the projection map \( Z_S \to \mathbb{P}^3 \). In our case it consists of points \( x \in \mathbb{P}^3 \) such that the net of quadrics \( N(x) \) in \( W \) containing \( x \) has less than eight base points. This is equivalent to that \( N(x) \) does not intersect \( D(W) \) transversally.

The linear system \( W \) of quadrics defines a map
\[
    f : |E| \to \mathbb{P}(L) = |L^\vee|, \quad x \mapsto N(x). \tag{7.4.1}
\]
For any point \( H \in \mathbb{P}(L) \), considered as a plane in \( W \), the fiber \( f^{-1}(H) \) consists of base points of the net \( H \). Thus the map is of degree 8. Its branch divisor is the set of nets in \( W \) that have less than 8 base points. This happens if and only if one of the quadrics in the net has a singular point at one of the base points. This base point must be on the Steinerian surface \( \St(W) \). Thus we see that the branch locus \( B(f) \) of \( f \) is equal to \( f(\St(W)) \). Since a transversal plane section of \( D(W) \) has 8 base points, we obtain that \( B(f) \) is contained in the dual surface \( D(W)^* \) of \( D(W) \). Since \( \St(W) \) and \( D(W)^* \) are irreducible,
\[
    B(f) = D(W)^*.
\]
It is known that the degree of the dual surface of a quartic surface with 10 ordinary nodes is equal to 16 (see [180, 1.2.3]). The canonical class formula \( K_{|E|} = \Ram(f) + f^*(K_{|L^\vee|}) \) shows that the degree of the ramification divisor is equal to 4. Thus
\[
    \Ram(f) = \St(W).
\]
The restriction of \( W \) to \( \St(W) \) is given by a linear subsystem of \( |O_{\St(W)}(2)| \). The image of the map is a surface of degree 16, this implies that the degree of the map \( f : \St(W) \to D(W)^* \) is equal to 1 and therefore
\[
    f^{-1}(D(W)^*) = \St(W) \cup F,
\]
where \( \deg F = 32 - 8 = 24 \).

**Remark 7.4.4.** One can compute the sectional genus of \( \Rey(W) \) without appealing to the fact that \( \Rey(W) \) is an Enriques surface. In the notation of the proof, a hyperplane section \( H \) of \( \Rey(S) \cap \Bis(B) \) is a complete intersection of two hypersurfaces of degree 3 and a quadric in \( \mathbb{P}^4 \). Its arithmetic genus is equal to 28. On other hand, it is a reducible curve \( H_1 \cup H_2 \), where \( H_1 \) is a hyperplane section of \( \Rey(W) \) and \( H_2 \) is a hyperplane section of \( \Bis(W) \). The curves intersect at \( \delta \) points, where \( \delta \) is the degree of the intersection curve \( \Rey(W) \cap \Bis(B) \). Let us compute the degree. Consider the restriction of \( W \) to the base curve \( B \) of a general pencil \( P_1 \) of quadrics in \( W \). A general quadric in \( W \) intersects \( B \) at 8 points, a member of the linear system \( |O_B(2)| \). Any join of two intersection points is a Reye line. Take a general line in \( \mathbb{P}^3 \) and consider a pencil \( P_2 \) contained in \( |O_B(1)| \). A line from \( \Rey(W) \cap \Bis(W) \) intersects \( B \) at two points that are common to a divisor from \( P_1 \) and
Proposition 7.4.5. Assume that \( W \) is an excellent web of quadrics in \( \mathbb{P}^3 \). Then the Reye congruence \( \text{Rey}(W) \) has no fundamental points and has no multiple rays. The residual part \( F \) in \( f^{-1}(\mathcal{D}(W)^*) \) is the focal surface of the Reye congruence. It is irreducible and reduced.

Proof. Let \( S = \text{Rey}(W) \) and let \( \pi : Z_S \to \mathbb{P}^3 \) be the natural projection. Assume that \( x \in \mathbb{P}^3 \) is a fundamental point of \( S \). This means that \( \dim \pi^{-1}(x) > 0 \). By definition, \( \pi^{-1}(x) \) is the set of Reye lines passing through \( x \). Let \( N(x) \) be the net of quadrics from \( W \) with one of the base points equal to \( x \). The restriction of \( W \) to a Reye line \( \ell \) containing \( x \) is the same as the restriction of \( N(x) \) to \( \ell \). Thus \( \ell \) is contained in a pencil of quadrics from \( N(x) \), or, equivalently, the restriction of \( N(x) \) to \( \ell \) is a set of two base points of \( N(x) \) which may coincide. Thus, either all quadrics in \( N(x) \) have a multiple base point at \( x \), or the base locus of \( N(x) \) is a curve. In the former case, there exists a pencil of quadrics singular at \( x \), contradicting the assumption that \( W \) is excellent. In the latter case, a quadric from \( W \setminus N(x) \) intersects the base curve of \( N(x) \) at a base point of \( W \). This contradicts the assumptions on \( W \). This proves that \( \text{Fund}(S) = \emptyset \).

Assume \( \ell \) is a multiple ray of \( S \). Recall that this means that \( \ell \) is a Reye line contained in the focal surface. By above, for any point \( x \in \ell \), \( \pi_1^{-1}(x) \) consists of Reye lines joining \( x \) with another base point of \( N(x) \). The line \( \ell \) is multiple if, for each \( x \in \ell \), the net \( N(x) \) has a multiple base point on \( \ell \). A multiple base point of a net is a singular point of one of its members. Thus, for each \( x \in \ell \), there exists a quadric in \( N(x) \) singular at some point \( y_x \in \ell \). Since two different \( N(x) \) and \( N(y) \) span \( W \), and \( W \) has no base points, we obtain that the set of points \( y_x \) is equal to the whole \( \ell \). This implies that \( \ell \) is contained in the Steinerian surface \( \text{St}(W) \). This contradicts Proposition 7.4.1.

We know that \( \text{Foc}(S) \) is the image of the ramification divisor \( \text{Ram}(\pi_1) \subset Z_S \) of the map \( \pi_1 : Z_S \to |E| \) and the projection \( \pi_2 : \text{Ram}(\pi_1) \to S \) is of degree 2. Suppose \( \text{Ram}(\pi_1) \) is not reduced or it is reducible. Then it contains an irreducible components mapping birationally onto \( S \). On the other hand, it maps dominantly onto \( \text{Foc}(S) \) and hence onto \( \mathcal{D}(W)^* \). The latter surface is birationally isomorphic to a K3 surface. Since an Enriques surfaces cannot be mapped birationally onto a K3 surface, we obtain a contradiction.

Finally, let us see that \( F = \text{Foc}(S) \). Let us consider the map (7.4.1). For any \( x \in \text{Foc}(S) \), the net \( N(x) \) has less than 8 base points, hence \( f(N(x)) \in \mathcal{B}(f) \). This shows that \( x \in f^{-1}(\mathcal{B}(f)) = \text{St}(W) \cup F \) and \( \text{Foc}(S) \subset \text{St}(W) \cup F \). Let \( x' \) be a multiple base point of \( N(x) \) and let \( \ell = x, x' \) be the Reye line joining the two points. If \( x \in \text{St}(W) \), then there exists a quadric \( Q \in N(x) \) with \( x \in \text{Sing}(Q) \). Since \( x' \in Q \), the line \( \ell \) is contained in \( Q \). Applying Proposition 7.2.4, we obtain that the fiber of \( \text{PB}(W) \to \mathbb{P}^3 \) over \( y \) is one-dimensional. The number of such points is finite and it is equal to zero when the web is excellent. By above, \( \text{Foc}(S) \) is an irreducible component of \( F \). Comparing the degree of \( F \) with the degree of the focal surface found in Proposition 7.1.2, we find that \( F = \text{Foc}(S) \).

Let \( \text{Rey}(W)^* \) be the dual congruence of lines in \( \mathbb{P}(E) = |E^\vee| \). It consists of pencils of planes containing a Reye line. Its bidegree is equal to \((3, 7)\). Its fundamental locus consists of planes \( \Pi \)}
that contain infinitely many Reye lines. If \( P \) is not contained in any quadric from \( W \) of corank 2, the restriction of \( W \) to \( \Pi \) is a web of conics. One can classify web of conics (as apolar linear systems to pencils of conics) to check that a web of conics without base points contains only finitely many pencils with reducible base locus. This shows that \( \Pi \) has only finitely many (in fact, \( \leq 3 \)) Reye lines. On the other hand, if we take \( \Pi \) to be an irreducible component of a corank 2 quadric, the restriction of \( W \) to \( \Pi \) is a net of conics without base points. Its Reye variety is isomorphic to a cubic curve. We have 20 such planes, and a Reye line from this plane will be called an exceptional Reye line.

Let \( \ell \) be a Reye line and let \( |L - \ell| \) be the pencil of quadrics containing \( \ell \), a line in \( W \). Its base locus contains \( \ell \). If \( \ell \) is a general Reye line, the pencil contains only two singular quadrics corresponding to the tangency points of \( \ell \) and the focal surface \( \text{Foc}(\text{Rey}(W)) \). This shows that \( |L - \ell| \) intersects \( D(W) \) at two points instead of expected four. It is easy to check that these points are the tangency points. Thus the line \( |L - \ell| \) is a bitangent of the Cayley quartic symmetroid \( D(W) \). Thus we obtain a map

\[
\nu : \text{Rey}(W) \to \text{Bit}(D(W)), \tag{7.4.2}
\]

where \( \text{Bit}(D(W)) \) is the closure in \( G_1(W) \) of lines bitangent to \( D(W) \) at two nonsingular points.

**Theorem 7.4.6.** The bidegree of the congruence \( \text{Bit}(D(W)) \) is equal to \((12, 28)\), in particular, it is a surface of degree 40 in the Plücker space \( \mathbb{P}^5 \). It has 45 singular points corresponding to lines \( \overline{y_i, y_j} \) through two singular points of \( D(W) \). Its singular locus consists of 10 pairs of plane cubic curves \( B_i, B'_i \) representing the generators of the enveloping cone of \( D(W) \) at one of the 10 nodes. The union \( B_i \cup B'_i \) is equal to the intersection of \( \text{Bit}(D(W)) \) with the plane of lines through the node \( y_i \). Two cubics \( B_i \) and \( B'_i \) intersect at 9 points corresponding to the lines \( \ell_{ij} = \overline{y_i, y_j}, j \neq i \). Two cubics from different pairs \((B_i, B'_i)\) and \((B_j, B'_j)\) intersect at one point \( \ell_{ij} \). Under the normalization map \( \nu : \text{Bit}(D(W))^3 \to \text{Bit}(D(W)) \) the map \( \nu : \nu^*(B_i) \to B_i \) is an unramified map of degree 2 (same for \( B'_i \)). The pre-image of \( \ell_{ij} \) consists of 4 points. The map \( \nu \) is the normalization map.

**Proof.** A general plane section of \( D(W) \) is a plane quartic curve. It has 28 bitangents. Thus the class of \( D(W) \), i.e. the intersection of the surface with the set of lines in a general plane, is equal to 28. Let \( x \) be a general point in \( \mathbb{P}^3 \). We already observed that the number of bitangents to a normal quartic surface passing through \( x \) is equal to 12. Thus gives the order of \( \text{Bit}(D(W)) \). We can also argue without referring to [606]. A bitangent to \( D(W) \) defines a pencil of quadrics in \( W \) containing only two singular quadrics. The base locus of such a pencil is either an irreducible rational cubic and its bisecant, or a conic plus two lines. The latter correspond to the case when the pencil contains a reducible quadric, i.e. the bitangent line passes through a singular point of \( D(W) \). In both cases the line components are Reye lines of \( W \). Conversely, a Reye line is contained in a pencil of quadrics with a singular base locus. Such a pencil in an excellent web \( W \) has only two singular quadrics, and the base locus is either a line plus an irreducible cubic, or an irreducible conic plus two lines, or 4 lines. The latter corresponds to the case when the pencil passes through two nodes of \( D(W) \). In this way we see that the map

\[
\nu : \text{Rey}(W) \to \text{Bit}(D(W)).
\]

is a normalization map satisfying the properties from the assertions of the theorem.

Let us see that the order of \( \text{Bit}(D(W)) \) is indeed equal to 12. Consider the regular map \( f : \mathbb{P}^3 \to W^* \) given by the linear system \( W \). The pre-image of a plane is a quadric from \( W \). It is singular if
and only if the plane is tangent to the branch locus of $f$. Thus the dual hypersurface of the branch locus is $D(W)$, so by the duality, the branch locus is $D(W)^*$. In other words, the congruence of bitangent lines to a surface is dual, in the sense of Grassmannian of lines, to the congruence of bitangents of the dual surface. Under this duality, the order and the class interchange. So to compute the order $\text{Bit}(D(W))$ we need to compute the class of $\text{Bit}(D(W)^*)$. A plane in $W^*$ is a point in $W$, i.e. a quadric $Q$ from $W$. A bitangent to $D(W)^*$ contained in this plane corresponds to a Reye line contained in $Q$. Let $\mathcal{N}$ be a general net of quadrics which together with $Q$ generates $W$. The number of Reye lines contained in $Q$ is equal to the number of lines in $\mathcal{N}$ contained in $\mathcal{N}$ and in $Q$. By Lemma 7.4.2, the set of lines contained in at least one quadric from a general net of quadrics in $\mathbb{P}^3$ is a cubic line complex. Now the set of lines in a nonsingular quadric is parametrized by the union of two disjoint conics in $G_1(\mathbb{P}^3)$. Hence the cubic complex intersects the union at $4 \times 3 = 12$ points.

**Remark 7.4.7.** In the proof of the previous Theorem we showed that the map (7.4.2) is the normalization map. Let us add that this map is given by a linear subsystem in $|O_{\text{Rey}(W)}(2)|$. To see this, we may assume that $W = \mathbb{P}(E)$ is spanned by 4 quadrics $Q_i$ which we represent by symmetric matrices $A_i$. Take a Reye line $\ell$ represented by 2 points $([v], [w]) \in \mathbb{P}B(W) \subset \mathbb{P}^3 \times \mathbb{P}^3$. Consider the evaluation map

$$E \cong k^4 \to k^3, \quad A \mapsto (t^i v^A v, t^i w^A w, t^i v^A w).$$

Its kernel consists of quadrics containing $\ell$. Since $\ell$ is a Reye line, we have $t^i v^A w = 0$, hence the evaluation map is $k^4 \to k^2$, and its kernel is the pencil of quadrics containing $\ell$. The Plücker coordinates of the pencil are equal to the maximal minors of the matrix

$$\begin{pmatrix}
  t^{i} v^{A_1} v & t^{i} v^{A_2} v & t^{i} v^{A_3} v & t^{i} v^{A_4} v \\
  t^{i} w^{A_1} w & t^{i} w^{A_2} w & t^{i} w^{A_3} w & t^{i} w^{A_4} w
\end{pmatrix}$$

It is easy to see that they are expressed by quadratic polynomials in Plücker coordinates of $\ell$.

**Remark 7.4.8.** As we saw in the proof, for any normal quartic surface $X$ the bidegree of the congruence of bitangents is equal to $(12, 28)$. However, $\text{Bit}(X)$ could be highly reducible. For example, when $X$ is a Kummer quartic surface with 16 nodes, the surface $\text{Bit}(X)$ consists of 6 components of bidegree $(2, 2)$ and 16 $\beta$-planes corresponding to tropes of $X$. We do not know how many irreducible components the bitangent surface $\text{Bit}(\text{St}(W))$ has.

Let $X$ be an irreducible quartic surface in $\mathbb{P}^3$ and let $x_0$ be its ordinary double point. Choosing the coordinates such that $x_0 = [1, 0, 0, 0]$, we can define $X$ by an equation

$$t_0^2 A_2(t_1, t_2, t_3) + 2t_0 A_3(t_1, t_2, t_3) + A_4(t_1, t_2, t_3) = 0,$$

where $A_i$ are homogeneous forms of degree indicated in the subscript. Projecting from $x_0$, we see that $X$ is birationally isomorphic to the double cover of $\mathbb{P}^2$ with branch curve $B$ of degree 6 given by equation $A_3^2 - A_2 A_4 = 0$. The conic $A_2 = 0$ is equal to the image of the exceptional curve over the node after we extend the projection map to a minimal resolution of the quartic surface. Its splits under the cover $z^2 = A_3^2 + A_2 A_4$ into two rational curves with equations $z \pm A_3 = 0$. Conversely, a double cover $\pi: X \to \mathbb{P}^2$ branched along a plane sextic curve $B$ such that there exists a conic that splits under the cover admits a birational model isomorphic to a quartic surface with an ordinary node. The linear system that map the cover to $\mathbb{P}^3$ is equal to $|\pi^*(\ell) + C|$, where $\ell$ is a line in $\mathbb{P}^2$ and $C$ is one of the two parts of the split conic.
7.4. REYE CONGRUENCES: \( P \neq 2 \)

For example, taking \( B \) to be an irreducible plane sextic with 9 ordinary nodes and an everywhere tangent conic, we obtain a quartic surface with 10 nodes. However, it is not a Cayley quartic symmetroid. In fact, it follows from Theorem 7.4.6 that the Cayley quartic symmetroid has a special property that the enveloping cone breaks into the sum of two cubic cones. In other words, the branch curve \( B_i \) for the projection from a node \( q_i \) is the union of two cubic curves intersecting at 9 points.

The next theorem, due to A. Cayley, shows that the converse is true.

**Theorem 7.4.9.** A quartic surface \( X \) with 10 ordinary nodes is a Cayley quartic symmetroid if and only if one of the following conditions is satisfied:

(i) Let \( \pi : X' \to X \) be the minimal resolution of \( X \) and let \( \theta_1, \ldots, \theta_{10} \) be the classes of exceptional curves. Then \( 3\pi^*(c_1(O_X(1)) - \theta_1 - \cdots - \theta_{10}) \) is divisible by 2 in \( \text{Pic}(X') \).

(ii) At every node, the enveloping cone is the union of two cubic cones.

**Proof.** (i) follows from the theory of symmetric determinantal representations of hypersurfaces, see [180, 4.2.6]. To prove (ii) we use that the Steinerian map \( \text{st} : D(W) \to \text{St}(W) \) is given by the linear system of polar cubics that generate a linear subsystem of \( 3c_1(O_X(1)) - \theta_1 - \cdots - \theta_{10} \). By (i), it is equal to \( |\text{O}_{\text{St}(W)}(2)| \). Let \( Q \in |\text{O}_{\text{St}(W)}(2)| \) be the corank 2 quadric corresponding to the singular point \( p_i \) of \( D(W) \). It corresponds to the polar of \( D(W) \) with pole at \( p_i \). Since \( Q_i \) is the union of two planes \( \pi_i, \pi'_i \), the pre-image of \( Q_i \) under the Steinerian map intersects \( D(W) \) at the union of two cubics, each isomorphic to the residual cubic in \( \pi_i \cap \text{St}(W) \) and \( \pi'_i \cap \text{St}(W) \). \( \square \)

Consider the map \( f : \mathbb{P}^3 \to W^* = |L^V| \) from (7.4.1). Its restriction to a Reye line \( \ell \) is given by the pencil obtained by restriction of the web \( W \) to \( \ell \). The map is a degree 2 cover onto a line \( \ell^* \) in \( W^* \) with ramification points \( x, y \), where \( (x, y) \in \text{PB}(W) \). The images of the two points are the points on the branch divisor equal to \( D^* \). The line \( \ell^* \) is tangent to \( D^* \) at these points. The set of lines \( \ell^* \) is the dual congruence of lines \( \nu(\ell) = |L - \ell| \). This shows that the lines \( \ell^* \) are parameterized by the dual congruence \( \text{Bit}(D)^* \) of bidegree \( (28, 12) \). The set of 28 lines \( \ell^* \) passing through a general point \( H \in W^* \) is the set of lines joining two base points of the net of quadrics in \( W \) defined by \( H \).

Recall that we have defined the Cayley variety of \( W \). This is a subvariety of the projective space \( |L^\perp| \cong \mathbb{P}^5 \). Since \( \text{PB}(W) \) is smooth, by Theorem 7.2.11, Cay(\( W \)) is a smooth surface of degree 10 in \( \mathbb{P}^5 \). We will see later that it is not contained in a quadric, although \( \text{Rey}(S) \) is.

Consider the universal family \( U = \{ (\ell, Q) \in G(2, E) \times W : \ell \subset Q \} \) of lines contained in some quadric from \( W \) and its two natural projections

\[
\begin{array}{c}
\text{U} \\
\downarrow \quad \downarrow
\end{array}
\begin{array}{c}
G(2, E) \\
p_1 \quad p_2
\end{array}
\begin{array}{c}
W
\end{array}
\]

By definition of a Reye line, the first projection \( p_1 : U \to G(2, E) \) is isomorphic to the blow-up of \( \text{Rey}(W) \). The fiber of the second projection \( p_2 : U \to W \) over a quadric \( Q \) is isomorphic under the projection \( p_1 \) to the subvariety of \( G(2, E) \) parameterizing lines in \( Q \). If \( Q \) is nonsingular, then
the fiber is the union of two conics. If $Q$ is a singular quadric of corank 1 with vertex $x_0$, then the fiber is a conic in the plane $\sigma_x$ taken with multiplicity 2, and if $Q$ is the union of two planes $\Pi \cup \Pi'$, the fiber is the union of two planes $\sigma_{\Pi} \cup \sigma_{\Pi'}$ intersecting at one point $\Pi \cap \Pi'$. Using the Stein factorization, we factor $p_2$ as the composition of a morphism $g: U \to X$ and the double cover $\pi: X \to W$ branched along the quartic symmetroid $D(W)$. The variety $X$ is singular over the 10 nodes of $D(W)$. The morphism $g: U \to X$ is birationally isomorphic to a conic bundle over $X$. Its fibers over $\pi^{-1}(W \setminus D(W))$ are conics, its fibers over nonsingular points of $\pi^{-1}(D(W))$ are double conics, and its fibers over the preimages of singular points of $D(W)$ are the unions of two planes. The variety $X$ is birationally isomorphic to the Artin-Mumford threefold [36]. We refer to [412] for the description of small resolutions of $X$ (in the category of algebraic spaces) by choosing a plane $\sigma_{\Pi}$, or $\sigma_{\Pi'}$, in each fiber consisting of two planes. Note that under the first projection the image of each plane intersects $\text{Rey}(W)$ along a cubic curve $F_i$ or $F'_i$.

Let $f: G(2, E) \to W$ be the composition of the rational maps $p_2 \circ p_1^{-1}$. The preimage of a plane in $W = |L|$, i.e. a net $N$ of quadrics in $W$, is the Montesano complex of lines of $N$. Thus the map $f$ is given by a linear system in $|\mathcal{O}_{G(2, E)}(3)|$ isomorphic to the dual space $W^* = |L^*|$ of $W$. We refer to a later Proposition 7.10.8, where this observation is used to find a resolution of the ideal sheaf of $\text{Rey}(W)$ in $G(2, E)$, or, equivalently, the base locus of the map $f$.

### 7.5 Catalecticant quartic symmetroid

In this section we assume that $p = 0$. Let us first briefly remind some basic facts from the theory of *apolarity* (see [180, Chapter 1].

Let $E$ be a linear space over $k$ of dimension $n + 1$. An element $f$ of a symmetric power $S^k E$ of $E$ is called *apolar* to an element $g$ from $S^m E^\vee$ if $f(g) = 0$, where $S^k E$ is identified with a linear subspace of $(S^m E^\vee)^\vee$ by the natural extension to symmetric products of the isomorphism $E \to (E^\vee)^\vee$. Two forms are called apolar if $f(g) = g(f) = 0$. In a basis $(e_0, \ldots, e_n)$ in $E$ and the dual basis $(t_0, \ldots, t_n)$ in $E^\vee$, we identify $S^k E$ with the space of degree $k$ homogeneous polynomials $k[u_0, \ldots, u_n]$ and the space $S^m E^\vee$ with the space of degree $m$ homogeneous polynomials $k[t_0, \ldots, t_n]_m$, then we view each $e_i$ as the differential operator $\frac{\partial}{\partial t_i}$ and apply $f(\frac{\partial}{\partial t_0}, \ldots, \frac{\partial}{\partial t_n})$ to $g(t_0, \ldots, t_n)$. For example, if $m = k = 2$, a quadratic form $f = \sum_{i=0}^n a_{ii} t_i^2 + 2 \sum_{0 \leq i < j \leq n} a_{ij} t_i t_j$ is apolar to a quadratic form $g = \sum_{i=0}^n b_i t_i^2 + \sum_{0 \leq i < j \leq n} b_{ij} t_i t_j$ if and only if

$$\sum_{0 \leq i < j \leq n} a_{ij} b_{ij} = 0.$$ 

For any form $g \in S^m E^\vee$ and $k \leq m$, the apolarity defines a homomorphism

$$\text{ap}_g: S^k E \to S^{m-k} E, \quad f \mapsto f(g)$$

(for $k > m$ such a homomorphism is obviously zero). For example, if $e \in E$ and $g \in S^m E^\vee$, the value of $g$ at $e$ is defined to be $e^m(g)$. If we use a basis as above and consider $e$ as a vector in $E$ with coordinates $(\alpha_0, \ldots, \alpha_d)$ and $g$ as a polynomial in $k[t_0, \ldots, t_d]_m$, then $e^d(g)$ is obtained by plugging in $t_i = \alpha_i$ and coincides with the value of the polynomial $g$ at the vector $e$ multiplied by $m!$. 

In the special case $k = m$, the apolarity defines a perfect pairing $S^mE \otimes S^mE^\vee$ equal to the $m$th symmetric power of the canonical pairing $E \otimes E^\vee \rightarrow \mathbb{k}$. In particular, it defines a canonical polarization isomorphisms
\[
S^mE^\vee \rightarrow (S^mE)^\vee, \quad S^mE \rightarrow (S^mE^\vee)^\vee
\]  
(7.5.1)
which we will constantly use in order to identify these spaces.

Now let $U$ denote a linear space of dimension 2 over $\mathbb{k}$. Elements from $S^dU^\vee$ are called binary forms on $U$ of degree $d$. Let $\nu_d : \mathbb{P}^1 = |U| \rightarrow |S^dU| \cong \mathbb{P}^d$ be the $d$th Veronese map defined by the map $U \rightarrow S^dU, u \mapsto u^d$. Its image is the Veronese curve $R_d$ of degree $d$. It follow from above that the Veronese map coincides with the map $|U| \rightarrow |S^dU^\vee|^\ast = |(S^dU^\vee)^\vee|$ given by the complete linear system $|\mathcal{O}_{|U|}(d)| = |S^dU^\vee|$. Thus a hyperplane $H$ in the projective space $|S^dU|$ of binary forms of degree $d$ on $U^\vee$ can be considered as an element $[b_H]$ of $|S^dU^\vee|$ defined by a binary form $b_H \in S^dU^\vee$. Geometrically, $H$ cuts $R_d$ along the positive divisor $D_H = \nu_d(V(b_H))$. A hyperplane $H$ is called an osculating hyperplane of $R_d$ if the support of $D_H$ is equal to one point. In this case $b_H = l^d$, where $l \in U^\vee$, therefore the set of osculating hyperplanes can be identified with the Veronese curve in the dual projective space $|S^dU^\vee|$, the image of the Veronese map $\nu_d^\ast : |U^\vee| \rightarrow |S^dU^\vee|$, $l \mapsto l^d$. We call it the dual Veronese curve and denote it by $R_d^\ast$. If we use a correlation isomorphism $c : U \rightarrow U^\vee$ defined by an isomorphism $\wedge^2U \cong \mathbb{k}$, then the composition $\nu_d^\ast \circ c : |U| \rightarrow |S^dU^\vee|$ will assign to $[u] \in |U|$ the osculating hyperplane at the point $\nu_d^\ast([u])$.

The projective space $|S^2(S^dU^\vee)|$ is the complete linear system $|\mathcal{O}_{|S^dU^\vee|}(2)|$ of quadrics in $|S^dU|$. The restriction of a quadric to $R_d$ comes from a natural (meaning $SL(U)$-equivariant) homomorphism $S^2(S^dU^\vee) \rightarrow S^{2d}U^\vee$ whose kernel is the linear space $I(R_d^\ast)$ of elements of degree 2 in the homogeneous ideal $I(R_d)$ of $R_d$ in $|S^dU|$. The restriction homomorphism splits and defines an isomorphism of linear representation of $SL(2)$

\[
S^2(S^dU^\vee) \cong S^{2d}U^\vee \oplus I(R_d^\ast). \tag{7.5.2}
\]
It is a special case of the plethysm isomorphism (see [235, §11]). Replacing $U$ with the dual space $U^\vee$, we obtain a decomposition of linear representations of $SL(U)$

\[
S^2(S^dU) \cong S^{2d}U \oplus I(R_d^\ast), \tag{7.5.3}
\]
where $R_d^\ast$ is the dual Veronese curve. We have

\[
S^{2d}U = (S^{2d}U^\vee)^\vee = (S^2(S^dU^\vee)/I(R_d^\ast))^\vee = I(R_d^\ast)^\perp \subset S^2(S^dU) = S^2(S^dU^\vee)^\vee.
\]

Thus the space $|S^{2d}U|$ can be naturally identified with the space of quadrics in $|S^dU^\vee| = |S^dU|^\ast$ apolar to quadrics vanishing on the Veronese curve $R_d$. They are also known as harmonic quadrics with respect to the Veronese curve.

Choose a basis $(u_0, u_1)$ in $U$ and the dual basis $(t_0, t_1)$ in $U^\vee$. It defines a monomial basis $(u_0^d, u_0^{d-1}u_1, \ldots, u_1^d)$ in $S^dU$. In coordinates, the Veronese map $\nu_d$ is given by sending

\[
[\alpha u_0 + \alpha u_1] \mapsto \sum_{k=0}^d \binom{d}{k} \alpha_0^{d-k} \alpha_1^k u_0^{d-k} u_1^k.
\]

If we modify the monomial basis by inserting the binomial coefficients $\binom{d}{k}$ in from of $u_0^{d-k} u_1^k$, then
the map is given by familiar formula

\[ [t_0, t_1] \mapsto [t_0^d, t_0^{d-1} t_1, \ldots, t_0 t_1^{d-1}, t_1^d]. \]

The dual of the modified basis of \( S^d U \) is \((t_0^d, t_0^{d-1} t_1, \ldots, t_0 t_1^{d-1}, t_1^d)\). The dual Veronese map \( \nu_d^* \) is given by

\[ [u_0, u_1] \mapsto [u_0^d, du_0^{d-1} u_1, \ldots, (d_i u_0^{d-k} u_1^k, \ldots, du_0 u_1^{d-1}, u_1^d]. \]

The composition \( \nu_d^* \circ c : |U| \to |S^d U^\vee| \) is given now by

\[ [t_0, t_1] \mapsto [-u_1, u_0] \mapsto [u_0^d, -du_0^{d-1} u_0, \ldots, (-1)^k u_0^{d-k} u_0^k, \ldots, (-1)^d u_0^d]. \]

If \((x_0, \ldots, x_d)\) are coordinates in \( |S^d U| \) with respect to the modified monomial basis, then the equations of the Veronese curve \( R_d \) are given the 2 \times 2-minors of the matrix

\[ A = \begin{pmatrix} x_0 & x_1 & \cdots & x_{d-1} \\ x_1 & x_2 & \cdots & x_d \end{pmatrix}. \]

The equations of the dual Veronese curve \( R_d^\ast \) are given by 2 \times 2-minors of the matrix obtained from the matrix \( A \) by replacing \( x_k \) with \((-1)^k(d_k-1) \xi_{d-k} \), where \((\xi_0, \ldots, \xi_d)\) are the dual coordinates.

For example, if we take \( d = 2 \), the equation of the Veronese conic is \( t_0 t_2 - t_1^2 = 0 \) and the equation of the dual Veronese conic is \( 4 \xi_0 \xi_2 - \xi_1^2 = 0 \). If \( d = 3 \), the equations of the Veronese cubic \( R_3 \) are

\[ x_0 x_2 - x_1^2 = x_0 x_3 - x_1 x_2 = x_1 x_3 - x_2^2 = 0 \]

and the equations of the dual Veronese cubic \( R_3^\ast \) are

\[ 9 \xi_0 \xi_3 - \xi_1 \xi_2 = 3 \xi_0 \xi_2 - \xi_1^2 = 3 \xi_1 \xi_3 - \xi_2^2 = 0. \] (7.5.4)

Thus a quadratic form \( q = \sum_{i=0}^{d} a_{ii} x_i^2 + 2 \sum_{0 \leq i < j \leq d} a_{ij} x_i x_j \) is harmonic with respect to \( R_d \) if and only if each \( h_{ij} \), considered as a differential operator in \( \frac{\partial}{\partial a_i}, \ldots, \frac{\partial}{\partial a_d} \), vanishes at \( q \). This happens if and only if

\[ q = \sum_{k=0}^{2d} A_k (\sum_{i+j=k} x_i x_j). \] (7.5.5)

The projection of this harmonic quadratic form in \( S^{2d} U^\vee \) to \( S^{2d} U^\vee \) is equal to \( \sum_{k=0}^{2d} A_k t_0^{2d-k} t_1^k \).

A subspace \( \Lambda_{k-1} \) of dimension \( k-1 \) is called a \( k \)-secant of \( R_d \) if the linear system of hyperplanes with base locus \( \Lambda_{k-1} \) cuts out in \( R_d \) a base-point free linear system of divisors of degree \( d-k \). A 1-secant is a point on \( R_d \), a 2-secant is a line called a bisecant of \( R_d \).

**Proposition 7.5.1.** Let \( g \in S^{2d} U \). Under the inclusion \( \iota : S^{2d} U^\vee \hookrightarrow S^2 (S^d U^\vee) \) defined by (7.5.2), the following conditions are equivalent

(i) \( \text{Ker}(ap_g) \neq \{0\} \);

(ii) the quadric \( V(\iota(g)) \) is singular;

(iii) the point \([g] \in |S^{2d} U^\vee|\) belongs to a \( d \)-secant of the Veronese curve \( R_{2d} \).
7.5. CATALECTICANT QUARTIC SYMMETROID

Proof. It is known that the first condition is equivalent to that \( g \) belongs to the closure of the locus of binary form that can be written as a sum \( t_1^{2d} + \cdots + t_a^{2d} \) of powers of linear forms \( l_i \in U^\vee \). This easily implies that conditions (i) and (iii) are equivalent. Since \( \dim S^dU = d + 1 \), a quadratic form \( Q \in S^2(S^dU^\vee) \) is degenerate if and only if it can be written as a sum of \( d \) powers of squares of linear forms.

Suppose (i) holds to show that (ii) holds it is suffices to show that, for any \( l \in U^\vee \), we have \( \nu(l^{2d}) = L^2 \) for some \( L \in S^dU^\vee \). Choosing coordinates \((t_0, t_1)\) in \( U \), we may assume that \( l^{2d} = t_0^{2d} \).

It follows from (7.5.5) that the harmonic quadric \( q \) which is projected to \( t_0^{2d} \) must coincide with \( q = x_0^2 \). \( \Box \)

Let \( \text{Cat}_{2d} \) denote the subvariety of \( |S^{2d}U^\vee| \) parametrizing binary forms \( g \) which admit an apolar form \( f \in S^dU \). Since \( \text{ap}_g \) is a linear map of spaces of the same dimension \( d + 1 \), \( \text{Cat}_{2d} \) is a hypersurface of degree \( d + 1 \) in \( |S^{2d}U^\vee| \). It is called the catalecticant hypersurface of binary forms of degree \( 2d \). If we choose a modified monomial basis in \( S^{2d}U^\vee \) to write \( f = \sum_{i=0}^{2d} (2d) a_i l_0^{2d-i} t_1^i \), the equation of \( \text{Cat}_{2d} \) is given by the determinant of the catalecticant matrix (or Hankel matrix):

\[
\det \begin{pmatrix}
  a_0 & a_1 & \cdots & a_{d-1} & a_d \\
  a_1 & a_2 & \cdots & a_d & a_{d+1} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  a_d & a_{d+1} & \cdots & a_{2d-1} & a_{2d}
\end{pmatrix} = 0. \tag{7.5.6}
\]

Let \( \text{Sec}_k(R_d) \) be the union of \((k-1)\)-secants of \( R_d \). Using Proposition 7.5.1, we can identify \( \text{Cat}_{2d} \) with the discriminant hypersurface \( D \) in the space of harmonic quadrics \( |S^2(S^dU^\vee)|_{\text{harm}} \). The argument from the proof of the Proposition 7.5.1 can be used to prove that

\[
\text{Sec}_k(R_d) = |S^2(S^dU^\vee)|_{\text{harm}} = D_{d-k},
\]

where \( |S^2(S^dU^\vee)|_{\text{harm}} \) is the closure of the locus of harmonic quadrics of corank \( \geq d - k \).

The following proposition [314, Theorem 1.56] (that goes back to J. Sylvester and S. Gundelfinger) shows that the subspace \( |S^{2d}U^\vee| \) of harmonic quadrics intersects the discriminant hypersurface \( D_{2d} \) of quadrics in \( |S^dU| \) transversally, i.e. the catalecticant hypersurface \( \text{Cat}_{2d} \) inherits the nice properties of \( D_{2d} \) from Theorem 7.2.10.

**Proposition 7.5.2.** For any \( 1 \leq k \leq d \), the kth secant variety \( \text{Sec}_k(R_{2d}) \) is projectively normal and Cohen-Macaulay. Its singular locus is \( \text{Sec}_{k-1}(R_{2d}) \) and its degree is \( (2d-k+1) \). Its homogeneous ideal is generated by the size \( k + 1 \) minors of the catalecticant matrix.

We specialize by letting \( d = 3 \). Then \( \text{Cat}_6 \) is a quartic hypersurface in \( \mathbb{P}^6 = |S^6(U^\vee)| \) equal to the intersection of the discriminant hypersurface \( D(3) \) of quadrics in \( \mathbb{P}^3 = |S^3U| \) with the 6-dimensional hyperweb of harmonic quadrics in \( |S^3U| \) with respect to the Veronese curve in the dual space \( |S^3U| \).

Its singular points are the intersection points of \( W \) with \( \text{Sec}_2(R_6) \). Since the degree of the singular locus of the discriminant hypersurface of quadrics in \( \mathbb{P}^3 \) is equal to 10, we see from Theorem 7.2.10 that, for a general \( W \), the surface \( D(W) \) has 10 nodes, as expected.

Recall that a rational normal curve \( C \) of degree \( d \) in \( |V| = \mathbb{P}^d \) is the image of the Veronese curve \( R_d \) under an isomorphism \( |S^dU| \to |V| \) defined by a linear isomorphism \( S^dU \to V \). Assume
that \( d = 3 \), the space of quadrics in \( \mathbb{P}^3 = |V^\perp| \) apolar to quadrics vanishing on \( C \) is of dimension 6. A choice of a web of quadrics in this space depends on \( \dim G_3(\mathbb{P}^6) = 12 \) parameters. On the other hand we know that the variety of rational normal cubics also depends on the same number of parameters. It is a natural guess that a general web \( W \) of quadrics in \( \mathbb{P}^3 \) is equal to the space of quadrics apolar to \( C \) and some other rational normal curve \( C' \). This is the assertion of a classical theorem of T. Reye (see a modern proof in [203, Lemma 4.3]).

**Theorem 7.5.3** (T. Reye). *For any general web of quadrics \( W \) in \( \mathbb{P}^3 \) there exists exactly one pair of rational normal curves \( C_1, C_2 \) such that \( W \) is equal to the space of quadrics apolar to \( C_1 \) and \( C_2 \).

Given a reducible quadric \( Q = V(q) \) in \( W \), we can write \( q \) in the form \( l_1^2 + l_2^2 \), where \( l_1, l_2 \) are some linear forms. The hyperplane \( Q^\perp \) of quadrics in the dual space apolar to \( Q \) contains the subspace of quadrics vanishing at the points \([l_1],[l_2]\]. Hence it contains a hyperplane \( H \) of quadrics containing the line \( \ell = ([l_1],[l_2]) \). Let \([I(C_1)] \) denote the net of quadrics containing \( C_1 \).

The hyperplane \( H \) intersects the planes \([I(C_2)] \) in \( Q^\perp \) along a line of quadrics vanishing on \( \ell \) and \( C_1 \). Thus \( \ell \) is a common bisecant line of \( C_1 \) and \( C_2 \). We also see that the singular line of \( Q \) is the dual line of this bisecant.

The linear systems \([I(C_1)] \) (resp. \([I(C_2)] \)) maps \( C_2 \) (resp. \( C_1 \)) to a rational curve \( \Sigma_2 \subset |I(C_2)|^* \) (resp. \( \Sigma_1 \subset |I(C_1)|^* \)) of degree 6. The images of ten common bisecants are the 10 nodes of the sextics. Thus we see that a quartic symmetroid defines a pair of rational plane sextics whose nodes correspond to the nodes of the symmetroid. We will explore this connection in the last chapter of the book.

**Remark 7.5.4.** The remarkable fact that two general rational normal cubics have ten common bisecants is originally due to Luigi Cremona. Nowadays, it easily follows from the intersection theory on the Grassmannian \( G(2,4) \). The variety of bisecants of \( R_3 \) is a congruence of lines in the cohomology class \([\sigma_2] + 3[\sigma_1]\), where \( \sigma_2 \) (resp. \( \sigma_1 \)) is the Schubert variety of lines through a point \( x \) (resp. lines in a plane \( \Pi \)). The intersection of these two surfaces in \( G(2,4) \) is equal to 10.

## 7.6 Reye congruences: \( p = 2 \)

We specialize the discussion from Section 7.3 to the case \( n = 3 \). It starts with a 4-dimensional linear space \( B \) of symmetric bilinear forms on the 4-dimensional linear space \( E \). We still consider the subvariety \( \text{Bs}(|B|) \) of base points of the divisors of bi-degree \((1,1)\) in \(|E| \times |E|\) defined by binary forms from \( B \). It is a complete intersection of 4 divisors of bi-degree \((1,1)\). For general \( B \), its intersection with the diagonal is empty, and we can define a fixed-point-free involution \( \tau \) with the quotient Enriques surface embedded in the Grassmann quadric \( G_4(|E|) \) in \( \mathbb{P}^5 \) as the congruence \( S \) of lines \( \ell = \langle x, y \rangle \), where \( (x, y) \in \text{Bs}(|B|) \). If we consider \( b \in B \) as a linear map \( E \to E^\vee \), then the condition that \( b(x, y) = 0 \) for all \( b \in B \) implies that the line \( \ell \) is contained in the null-space of \( b \). Thus the congruence \( S \) coincides with the congruence \( \text{Nul}(|B|) \) of null-planes of binary forms from \( B \).

We also have the same attributes as in the case \( p \neq 2 \). In particular, \( \text{PB}(|B|) \) is birationally isomorphic to a quartic symmetroid \( D(|B|) \) and the Steinerian quartic surface \( \text{St}(|B|) \). Let us compute the class of the congruence \( \text{Nul}(B) \). Take a general plane \( \Pi \) in \( \mathbb{P}^3 \). The restriction of \( |B| \) to
II is a 5-dimensional linear system of conics. The variety of singular conics is the half-discriminant cubic hypersurface. It is isomorphic to the quotient of $\mathbb{P}^2 \times \mathbb{P}^2$ by the switch involution. The image of our surface $X = \text{Bs}(|B|)$ is isomorphic to the image of a complete intersection of 4 divisors of type $(1, 1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$. It consists of 3 points. Since the degree of $S$ is equal to 10, the order must be equal to 7. Thus the bidegree of the congruence is equal to $(7, 3)$. So, we get an analogue of a Reye congruence in this case. The Enriques surface $\text{Nul}(|B|)$ is a $\mu_2$-surface with a smooth canonical cover $\text{Bs}(|B|)$.

We have also a construction of a classical Enriques surface as a Reye congruence of a web $W$ of quadrics in $\mathbb{P}^3 = |E|$ from section 7.3. In this case the linear system $|B|$ of polar symmetric bilinear forms contains the diagonal in its set $\text{Bs}(|B|)$ and $D(W)$ is a quadric surface (taken with multiplicity 2). The set of quadrics of rank 2 in $\mathbb{P}^3$ is a 6-dimensional variety isomorphic to the quotient of $\mathbb{P}^3 \times \mathbb{P}^3$ by the involution $(x, y) \mapsto (y, x)$. It is the quotient of the Segre variety of degree $(\binom{6}{3}) = 20$ by the involution. Its degree in the space $|S^2 E^\vee|$ is equal to 10. Its intersection with the 3-dimensional linear subspace of $|S^2 E^\vee|$ is expected to consist of 10 points. We denote this intersection by $D_2(W)$.

By analogy with the case $p \neq 2$, we say that $W$ is excellent web of quadrics if the following properties hold:

(i) $\text{Bs}(W) = \emptyset$;

(ii) the polarization map $L \to \wedge^2 E^\vee$ is injective;

(ii) $D(W)$ is a nonsingular quadric;

(iii) $D(W)_2$ consists of 10 points;

(iv) the projection $\tilde{D}(W) \to \text{St}(W)$ is an isomorphism.

We leave to the reader to adjust the proof of Proposition 7.4.1, to show that condition (iv) implies that no Reye line belongs to $D(W)_2$ and no separable Reye line is contained in $\text{St}(W)$.

One can show that a general web $W$ is excellent. From now on, we assume that $W$ is excellent.

The projection $\tilde{D}(W)' \to D(W)$ is a $\mathbb{P}^1$-bundle over $D(W)$. It is a preimage of the variety $Z_S$, where $S$ is the congruence $\text{Nul}(W)$ of nullspaces $|\text{Ker}(q)|, q \in L$. Since $L$ does not contain quadrics with the polar bilinear form of rank 1, the polarization map $p : L \to \wedge^2 E^\vee$ is an inclusion. This implies that the congruence $\text{Nul}(W)$ is equal to the intersection of $G_1(|E|)$ with the 4-dimensional linear subspace $|p(L)|$ of $|\wedge^2 E^\vee| = |(\wedge^2 E)^\vee|$. It is isomorphic to a $|p(L)|$. Since a nonsingular quadric cannot be mapped onto a singular quadric, we obtain that it is a nonsingular quadric and the map $D(W) \to \text{Nul}(W), V(q) \mapsto |\text{Ker}(q)|$, is an isomorphism. Since deg $\text{Nul}(W) = 2$, and $\text{Nul}(W)$ is not a plane in the Plücker space, we obtain that its bidegree is equal to $(1, 1)$. It is well-known that a congruence of bidegree $(1, 1)$ has two fundamental skew lines $\ell_1$ and $\ell_2$. In fact, the projection $Z_{\text{Nul}(W)} \to \mathbb{P}^3$ is a birational morphism of nonsingular 3-folds, and computing the Euler characteristics, we find that it is the blow-up of two skew lines. This shows that $\text{Nul}(W)$ is a 2-dimensional family of bisecants of $\ell_1 \cup \ell_2$. 

7.6. REYE CONGRUENCES: $P = 2$
 Proposition 7.6.1. The fundamental lines of the congruence $\text{Nul}(W)$ are the two inseparable Reye lines of $W$.

Proof. Let $\ell$ be an inseparable Reye line. For any $x = [v], y = [w] \in \ell$ and any $V(q) \in |L - \ell|$, we have $b_q(v, w) = 0$. Also, because $\ell$ is inseparable, we can find a quadric $V(q_1) \not\in |L - \ell|$ such that $b_{q_1}(v, w') = 0$ for all $y' = [w'] \in \ell$ and a quadric $V(q_2)$ such that $b_{q_2}(v', w) = 0$, for all $x' = [v'] \in \ell$. This shows that $b_q(v, w) = 0$ for all $q \in L$. Consider the matrix $B(x)$ from the proof of Proposition 7.3.2. Its rank is equal to 2, hence there is a pencil of quadrics $Q$ such that $x$ is contained in the nullspace of $Q$. This shows that $\ell$ is a fundamental line of the congruence $\text{Nul}(W)$. 

Note that the ten points in $D(W)_2$ define two rays in $\text{Nul}(W)$. Hence the singular lines of reducible quadrics from $W$ intersect the two inseparable Reye lines.

Let us look at the Reye variety of $W$. It follows from formula (7.3.2) that the discriminant variety $D(2)$ of singular plane conics is isomorphic to the cubic hypersurface in $\mathbb{P}^5$, with coordinates $(a_{11}, \ldots, a_{33})$ taken as the coefficients of a quadratic form,

$$a_{11}a_{23}^2 + a_{22}a_{12}^2 + a_{33}a_{12}^2 + a_{12}a_{13}a_{23} = 0.$$ 

Its singular locus is the plane $a_{12} = a_{13} = a_{23}$ of quadrics of rank 1. In fact, scheme-theoretically, it should be considered as the plane taken with multiplicity 4. It is equal to the image of the inseparable Veronese map $\mathbb{P}^2 \to \mathbb{P}^5, [l] \mapsto [l^2]$.

Theorem 7.6.2. The Reye variety $\text{Rey}(W)$ is a smooth congruence of bidegree $(7, 3)$ and sectional genus 6. It does not have fundamental points nor does not contain multiple rays. The surface $\text{Rey}(W)$ is an Enriques surface. Conversely, any smooth congruence of bidegree $(7, 3)$ and sectional genus equal to 6 is isomorphic to an Enriques surface.

Proof. To compute the bidegree, we use the same argument based on the Montesano complexes associated to general nets of quadrics in $W$. In order to do this we have to modify the proof of Lemma 7.4.2 where we used Bertini’s Theorem to exclude the possibility that any quadric from $W$ restricts to a singular conic in a general plane $\Pi \subset |E|$. We used Bertini’s Theorem to deduce from this that the net of conics has a base point. We know from Example 7.3.11 that it is not in characteristic 2 and such a net must be given by an equation $\lambda x_0^2 + \mu x_1^2 + \gamma x_2^2 = 0$. So all conics are double lines.

Let $W_\Pi$ be the net of conics obtained by restrictions of quadrics from $W$ to the plene $\Pi$. Suppose one of the conics in $W_\Pi$ is a double line $2\ell$. This happens when there exists a quadric $Q = V(q)$ touching $\Pi$ along $\ell$. We can choose projective coordinates to assume that $\Pi = V(t_0)$ and $q = t_0 A(t_0, \ldots, t_3) + B(t_1, t_2, t_3)^2$, where $A$ and $B$ are linear forms. Since derivative $\frac{\partial}{\partial x_0} = A$, we see that the line $\ell = V(t_0) \cap V(A)$ is the null-line of $Q$. Since the class of the congruence $\text{Nul}(W)$ is equal to 1, there is only one ray $\lambda$ from $\text{Nul}(W)$ contained in $\Pi$, hence there is only one double line in $W_\Pi$, and we get a contradiction. The rest of computation for the bidegree of $\text{Rey}(W)$ goes without change.

Assume $\text{Rey}(W)$ has a fundamental point $x$. This means that $\sigma_x$ intersects $\text{Rey}(W)$ along a curve, or, equivalently, the net of quadrics $N(x)$ in $W$ containing $x$ has ne-dimensional component in its
Let us now show that \( \text{Rey}(W) \) does not have multiple rays. As we have noted in the beginning of the section, a multiple ray of \( \text{Rey}(W) \) must be an inseparable Reye line \( \ell \). We know that \( \ell \) is a fundamental line of the congruence \( \text{Nul}(W) \). Thus each of the 10 singular lines of reducible quadrics from \( W \) intersect \( \ell \). Since \( W \) is excellent, no two intersect at one point. Thus, \( \ell \) contains ten points which are singular points of quadrics from \( W \). Not all of these quadrics contain \( \ell \). Choose one such quadric \( Q \) that does not contain \( \ell \) and intersects \( \ell \) at a point \( x \). Since \( \ell \) is a multiple ray, the net \( N(x) \) has a multiple base point on \( \ell \). It must lie in one of the irreducible components of \( Q \). Hence \( \ell \) lies in this component. This contradiction proves the assertion.

Assume \( \ell \) is a singular point of \( S = \text{Rey}(W) \). For any point \( x \in \ell \), the \( \alpha \)-plane \( \sigma_x \) contains \( \ell \) and intersects \( S \) at \( \ell \) with some multiplicity. This implies that, for each \( y \in \ell \), the net \( N(y) \) of quadrics from \( W \) with base point at \( y \) has a multiple base point. Since \( \ell \) is contained in the base locus of a pencil \( L - \ell \subset N(y) \), the multiple base point must be in \( \ell \). In particular, we see that \( \ell \) is a multiple ray of the congruence \( \text{Rey}(W) \). As we know for each multiple base point there exists a quadric in the net which is singular at this point. This shows that \( \ell \) is a multiple ray \( \text{Rey}(W) \), a contradiction.

To compute the sectional genus of \( \text{Rey}(W) \), we use the same argument as in Remark 7.4.4 to find that it is equal to 6. The rest of the assertions is proved in the same way as in the case when \( p \neq 2 \).

We will see later that, in the case \( p = 2 \), not every Enriques congruence of bidegree \((7, 3)\) and sectional genus 6 is equal to a Reye congruence.

Recall that the surface \( \text{PB}(W) \) is not smooth. Its singular points are the points \( \tilde{x} \) such that the Reye line \( \ell_{\tilde{x}} \) is a null-line of some quadric. The number of such points is equal to the number of intersection points of the Reye congruence \( \text{Rey}(W) \) and the congruence \( \text{Nul}(W) \) of the null-lines. Since the bidegree of \( \text{Nul}(W) \) is equal to \((1, 1)\), the number of intersection points is expected to be equal to \([\text{Rey}(W)] \cdot [\text{Nul}(W)] = (7[\sigma_x] + 3[\sigma_{11}]) \cdot ([\sigma_x] + [\sigma_{11}]) = 10 \). The Stein factorization \( \text{PB}(B) \to \text{PB}(W) \to \text{Rey}(W) \) gives two additional singular points on \( \text{PB}(W) \) corresponding to inseparable Reye lines. The cover \( \text{PB}(W) \to \text{Rey}(W) \) is a principal \( \mu_2 \)-cover of \( \text{Rey}(W) \) that coincides with the canonical cover of the Enriques surface \( \text{Rey}(W) \). In particular, \( \text{Rey}(W) \) is a classical Enriques surface. It is expected that the canonical cover \( \text{PB}(W) \) is a normal surface with 12 ordinary double points. However, it could degenerate to a non-normal rational surface. This happens when \( \text{Rey}(W) \cdot \text{Nul}(W) \) contains one-dimensional components.

Remark 7.6.3. In the last chapter we will study in detail canonical covers of Enriques surfaces \( S \) in characteristic 2 with the canonical cover birationally isomorphic to a K3 surface. We will show that it has rational double points such that the exceptional curves of its minimal resolution of singularities generate a lattice of rank 12. The case when we have 12 points of type \( A_1 \) is a general case. Their images on \( S \) are called canonical points. They are singular points of simple fibers of elliptic fibrations on \( S \). As we saw the realization of \( S \) as a smooth Reye congruence gives a choice of two of these points corresponding to two inseparable Reye rays. Recall from Corollary 7.9.13 that a Reye congruence contains a \((-2)\)-curve \( R \) of degree 4 with respect to to the Reye polarization. It follows from Lemma 10.2.9 that each \((-2)\)-curve on \( S \) passes through two canonical points. So the plausible explanation is that the points corresponding to \( R \) are the inseparable rays in \( \text{Rey}(W) \).
7.7 The Picard group of a Reye congruence: \( p \neq 2 \)

Let \( W = |L| \) be a regular web of quadrics in \( \mathbb{P}^3 = |E| \). We know that \( X = \tilde{D}(W) \cong \text{PB}(W) \) are minimal resolutions of the discriminant surface \( D \). We will identify both surfaces with a K3 surface \( X \), the canonical cover of the Enriques surface \( \text{Rey}(W) \). Let \( \pi_1 : X \to D(W), \pi_2 : X \to St(W) \) be the projections. Let \( \text{Sing}(D(W)) = \{ q_1, \ldots, q_{10} \} \), and let \( \Theta_i \) be the exceptional curves of the minimal resolution \( \pi_1 \). We have the following divisor classes in \( \text{Pic}(X) \):

\[
\eta_H = \pi_1^*(c_1(\mathcal{O}_{L_1}(1))), \quad \eta_S = \pi_2^*(c_1(\mathcal{O}_{E_1}(1))), \quad \theta_i = [\Theta_i], \quad i = 1, \ldots, 10. \tag{7.7.1}
\]

The map \( p : X = \text{PB}(B) \to \text{Rey}(W) \subset |\bigwedge^2 E| \) is given by the restriction of the map given by a linear system of divisors of type \( (1, 1) \) on \( \mathbb{P}^3 \times \mathbb{P}^3 \). We set

\[
\eta = p^*(c_1(\mathcal{O}_{\text{Rey}(W)}(1))).
\]

We have

\[
\eta_S^2 = \eta_H^2 = 4, \quad \eta^2 = 20, \quad \eta_S \cdot \theta_i = 1, \quad \eta_H \cdot \theta_i = 0. \tag{7.7.2}
\]

**Proposition 7.7.1.** The following relation holds in \( \text{Pic}(X) \):

\[
2\eta_S = 3\eta_H - \theta_1 - \cdots - \theta_{10}.
\]

The proof can be found in [180, 4.2.25]. We only comment that this relation identifies the web of quadrics \( W \) with the linear system of first polars of the Cayley quartic symmetroid \( D(W) \).

We know that the image of the second projection \( X \to |E| \) is the Steinerian surface \( \text{St}(W) \). If \( W \) is an excellent web, then \( \text{St}(W) \) is a smooth quartic surface. Otherwise the fibers of \( \pi_2 \) over a singular point \( x \) of \( \text{St}(W) \) are isomorphic to \( \mathbb{P}^1 \). The image of such a fiber under the first projection is a line \( \lambda \) on \( D(W) \) equal to the pencil of quadrics with singular point at \( x \). This pencil consists of cones over plane conics. A singular conic is the projection of a reducible quadric in the pencil. Since there are no quadrics of rank 1 in \( W \), we see that the line \( \lambda \) passes through three nodes of \( D(W) \). Conversely, any such line gives rise to a singular point of \( \text{St}(W) \). Also, it implies that a line on \( D(W) \) through two nodes necessary passes through a third node.

The image \( \Theta'_i \) of \( \Theta_i \) under the second projection is a line on \( \text{St}(W) \). It is one of the 10 singular lines of reducible quadrics in \( W \). The preimage in \( X \) of a plane containing the line \( \Theta'_i \) contains the union of \( \Theta_i \) and proper transform of the lines \( P_i \) joining \( q_i \) with other two nodes. We denote the union of such curves by \( Z_i \). The union \( \Theta_i + Z_i \) form a nodal cycle of \((-2)\)-curves on \( X \). It is of type \( A_1 \) if \( Z_i = \emptyset \), of type \( A_2 \) (resp. \( A_3 \), resp. \( D_4 \)) if \( Z_i \) consists of 1 (resp. 2, resp. 3) curves. In the latter case, \( D(W) \) has three lines joining \( q_i \) with 3 disjoints pairs of other nodes. Let

\[
E_i = \eta_S - \theta_i - [Z_i].
\]

The linear system \( |E_i| \) is equal to the preimage under \( \pi_2 \) of the linear system of plane sections of \( \text{St}(W) \) that contain the line \( \Theta'_i \). We have \( E_i^2 = \eta_S^2 - 2\theta_i \cdot \eta_S + (\theta_i + [Z_i])^2 = 4 - 2 - 2 = 0 \). This confirms that the linear system \( |E_i| \) is a genus one pencil on \( X \).
Let $\tau : X \rightarrow X$ be the canonical involution of $X = \text{PB}(W)$. The corresponding birational automorphism of the Steinerian quartic $\text{St}(W)$ permutes the tangency points of $\text{St}(W)$ with a Reye line. If $E$ is an exceptional curve of $\pi_2$, then $\pi_1(\tau(E))$ is a line in $D(W)$.

**Proposition 7.7.2.** Let $Z = \sum Z_i$.

(i) $\eta = \tau^*(\eta_S) + \eta_S$;

(ii) $\theta_i \cdot \tau^*(\theta_i) = 0$, $\theta_i \cdot \tau^*(\theta_j) = 2$, $i \neq j$, $i = 1, \ldots, 10$;

(iii) $\eta_S \cdot \tau^*(\theta_i + [Z_i]) = 3$, $i = 1, \ldots, 10$;

(iv) $\tau(E_i) \sim E_i$;

(v) $3\eta = E_1 + \cdots + E_{10}$;

(vi) $\eta_H = 4\eta_S - \eta - Z$;

(vii) $2\eta_S - \eta_H = \tau^*(\theta_i + [Z_i]) - \theta_i - [Z_i] + [Z]$, $i = 1, \ldots, 10$;

(viii) $\tau^*(\eta_H) = 2\eta - \eta_H - \tau^*([Z]) = 2\tau^*(\eta_S) + \eta_S - \eta_H$.

**Proof.** (i) The involution $\tau$ is induced by the standard involution of $\mathbb{P}^3 \times \mathbb{P}^3$ which permutes the factors. Let $\text{pr}_1 : \text{PB}(W) \rightarrow \mathbb{P}^3$ be the two projections. We can identify the second one with the projection $\text{PB}(W) \rightarrow \text{St}(W)$, so that $O_X(\eta_S) \cong \text{pr}_2^* O_{\mathbb{P}^3}(1)$. Since

$$O_X(\eta) \cong \text{pr}_1^* O_{\mathbb{P}^3}(1) \otimes \text{pr}_2^* O_{\mathbb{P}^3}(1),$$

we obtain (i).

(ii) We know that no Reye line is one of the lines $\Theta'_i$ on $\text{St}(W)$. This implies that, for any $x \in \Theta'_i$, the point $y = \tau(x) \not\in \Theta'_i$ (otherwise $(x, y) \in \text{PB}(W)$ and $\Theta'_i$ is a Reye line). Thus $\Theta'_i \cap \tau(\Theta'_i) = \emptyset$, hence $\Theta_i \cdot \tau(\Theta_i) = 0$. On the other hand, if $i \neq j$, the intersection number $\Theta_i \cdot \tau(\Theta_j)$ is equal to the number of pairs $(x, y) = (x, \tau(x)) \in \text{PB}(W)$ such that $x \in \Theta_i$, $y \in \Theta_j$. This is the same as the intersection number of the surfaces $\Theta_i \times \Theta_j$ and $\text{PB}(W)$ in $|E| \times |E|$. It is clear that $\Theta_i \times \Theta_j$ is contained in $Z(b_i) \cap Z(b_j)$, where $b_i, b_j$ are the polar bilinear forms of the quadrics $q_i, q_j \in D(W)$. Thus the intersection $\Theta_i \times \Theta_j \cap \text{PB}(W)$ is equal to the intersection of two divisors of type $(1, 1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. It is equal to 2.

(iii) The image of $x \in \Theta_i$ under the map $\tau$ is equal to the point $y$ such that $(x, y) \in \text{PB}(W)$. If $x$ is the singular point of a unique quadric $q_i$, then the point $y$ is the unique point equal to $\cap_{Q \in W} P_x(Q)$. In coordinates, it generates the null space of a system of 3 equations with 4 unknowns with coefficients linear forms in coordinates of $x$. It is clear that its coordinates are cubic polynomials in coordinates of $x$. If $x$ is a point such that $Z = \dim \pi_2^{-1}(x) = 1$, then the curve $\pi_1(\tau(Z))$ is a line. This gives us

$$\tau(\theta_i + [Z_i]) \cdot \eta_S = 3.$$

Note that, if $W$ is an excellent web, $\pi_1(\tau(\Theta_i))$ is a rational normal cubic contained in $D(W)$.
(iv) First observe that
\[ \eta_S \cdot \tau^*(\eta_S) = \frac{1}{2}((\eta_S + \tau^*(\eta_S))^2 - 2\eta_S^2) = \frac{1}{2}(\eta^2 - 2\eta^2_S) = 6. \]
This yields
\[ E_i \cdot \tau^*(E_i) = (\eta_S - \theta_i - [Z_i]) \cdot (\tau^*(\eta_S) - \tau^*(\theta_i) - \tau^*(\eta_S)) = 0. \]
Therefore \( E_i \) and \( \tau(E_i) \) define the same genus one pencil on \( \text{St}(W) \).

(v) We have
\[ E_i \cdot E_j = (\eta_S - \theta_i - [Z_i]) \cdot (\eta_S - \theta_j - [Z_j]) = 2, \]
\[ E_i \cdot \eta = 2E_i \cdot \eta_S = 2(\eta_S^2 - \theta_i \cdot \eta_S - [Z_i] \cdot \eta_S) = 6. \]
This shows that \( (E_1, \ldots, E_{10}) \) is the inverse transform under \( p \) of a maximal isotropic sequence \( (f_1, \ldots, f_{10}) \) on \( \text{Rey}(W) \) with,
\[ 3h = f_1 + \cdots + f_{10}, \]
where \( h = [c_1(\mathcal{O}_{\text{Rey}(W)}(1))] \in \text{Num}(\text{Rey}(W)), \) and
\[ 3\eta = 3\pi^*(h) = E_1 + \cdots + E_{10}. \]

(vi) By Proposition 7.7.1 and (v),
\[ 2\eta_S + \theta_1 + \cdots + \theta_{10} = 2\eta_S + \Sigma(\eta_S - E_i - [Z_i]) = 12\eta_S - 3\eta - [Z'] = 3\eta_H. \]
This implies that \([Z']\) is divisible by 3 and
\[ \eta_H = 4\eta_S - \eta - \frac{1}{3}[Z']. \]
The fact that \([Z']\) is divisible by 3 is not surprising. We already know that each exceptional curve of the map \( \pi_1 : \mathcal{D}(W) \to \text{St}(W) \) comes from a line on \( \mathcal{D}(W) \) that passes through three singular points. This shows that each irreducible component of \( Z \) enters in exactly three divisors \( \Theta_i + Z_i, \) and hence enters into \( Z' \) with coefficient 3. In particular, we see that \( 3Z = Z' \).

(vii) By (vi)
\[ \eta_H = 4\eta_S - \eta - Z = 3\eta_S - \tau^*(\eta_S) - [Z], \]
\[ 4\eta_S - \eta_H = \eta + [Z]. \]
Thus
\[ 2\eta_S - \eta_H = \eta - 2\eta_S + [Z] = \tau^*(\eta_S) - \eta_S + [Z]. \]
On the other hand,
\[ \tau^*(\eta_S) = E_1 + \tau^*(\theta_i) + \tau^*(\eta_S), \]
\[ \eta_S = E_i + \theta_i + [Z_i], \]
gives, after subtracting,
\[ 2\eta_S - \eta_H = -\eta_S + \tau^*(\eta_S) + [Z] = \tau^*(\theta_i + [Z_i]) - \theta_i - [Z_i] + [Z]. \]
This proves (vii).

(viii) Applying \( \tau^* \) to equality (vii):
\[ 2\eta_S - \eta_H = \tau^*(\theta_i + [Z_i]) - \theta_i - [Z_i] + [Z], \]
we obtain
\[ 2\tau^*(\eta_S) - \tau^*(\eta_H) = \theta_i + [Z_i] - \tau^*(\theta_i + [Z_i]) + \tau^*[Z]. \]
Adding up gives (viii).

Define the following divisors on the Reye surface \( \text{Rey}(W) \):

\begin{align*}
R_i &= \pi(\Theta_i + Z_i), \quad i = 1, \ldots, 10, \quad (7.7.3) \\
F_i &= \pi(E_i), \quad (7.7.4) \\
F_{ij} &= \pi(\Theta_i - \Theta_j), \quad i \neq j. \quad (7.7.5)
\end{align*}

Applying Proposition 7.7.2, we obtain

**Theorem 7.7.3.** Let \( W \) be a regular web of quadrics in \( \mathbb{P}^3 \). Then the Enriques surface \( S = \text{Rey}(W) \) contains ten curves \( R_i \) of degree 4 each irreducible component of which is a nonsingular rational curve, 10 genus one pencils \( |2F_i| \), and 45 genus one pencils \( |2F_{ij}| \), where \( 2F_{ij} \sim R_i + R_j, i \neq j \).

Let \( H = c_1(O_{\text{Rey}(W)}(1)) \). We have

(i) \( F_i \cdot F_j = 1 \) for \( i \neq j \);
(ii) \( R_i \cdot F_i = 3, \ R_i \cdot F_j = 1, i \neq j \);
(iii) \( R_i \cdot R_j = 2, i \neq j \);
(iv) \( R_i \cdot F_{ij} = 0, \ R_k \cdot F_{ij} = 4, k \neq i, j \);
(v) \( 3H \sim F_1 + \cdots + F_{10} \);
(vi) \( 4H \sim R_1 + \cdots + R_{10} \);
(vii) \( H \cdot F_i = 3, \ h \cdot F_{ij} = 4 \);
(viii) \( H \sim 2F_i + R_i + K_S \);
(ix) \( H \equiv F_i + F_j + F_{ij} \).

**Proof.** Properties (i) -(vii) immediately follow from the definitions and Proposition 7.7.2. Let us prove (viii). By (i) from loc. cit., we have \( [p^*(H)] = \tau^*(\eta_S) + \eta_H \) on \( \text{PB}(W) \). Applying (ii) and (iv) from loc. cit., we obtain, by definition of the curves \( F_i \) and \( R_i \), that \( H \equiv 2F_i + R_i \). Let \( F_i' \in |F_i + K_S| \). Since each curve \( F_i \) (resp. \( F_i' \)) is a plane cubic, it is contained in a plane \( \Pi_i \) (resp. \( \Pi_i' \)). Obviously this plane must lie in the Grassmann quadric \( G \) containing \( S \). Now, since \( F_i \cdot F_j = F_i' \cdot F_j = 1, i \neq j \), we find that \( \Pi_i \cap \Pi_j \neq \emptyset, \Pi_i' \cap \Pi_j \neq \emptyset \). This easily implies that \( \Pi_i \) and \( \Pi_i' \) belong to the same family of planes in \( G \) and hence intersect at one point. Then the unique hyperplane containing \( \Pi_i \) and \( \Pi_i' \) cuts out in \( S \) an isolated curve from the linear system \( |h - F_i - F_i'| \). Since \( (H - F_i - F_i')^2 = -2 \), this must be a nodal cycle \( R \). If \( R_i \in |H - 2F_j| \neq \emptyset \), we obtain that \( 2R_i \sim 2R, R_i \neq R, \) that is absurd. Therefore we have only one possibility that \( R_i \sim H - F_i - F_j \).
In particular, $R_i + R_j \sim 2(H - F_i - F_j)$ is divisible by 2 in $\text{Pic}(S)$. So we can define $F_{ij}$. It follows from (iii) that $F_{ij}^2 = 0$, thus $|2F_{ij}| = |R_i + R_j|$ is a genus one pencil.

(ix) By (iv) we have $R_i \cdot (H - F_i - F_j) = R_j \cdot (H - F_i - F_j) = 0$. Hence $F_{ij} \cdot (H - F_i - F_j) = 0$. Since $(H - F_i - F_j)^2 = 0$, we obtain (ix). \hfill \Box

Recall from Theorem 7.4.6 that under the map $\nu : \text{Rey}(W) \to \text{Bit}(D(W))$ the set of Reye lines $\ell$ such that the pencil $|L - \ell|$ contains one of the 10 nodes of $D(W)$ is mapped to the union of two cubic curves, the union of the cones over these curves form the enveloping cone of the node. One can easily check that the set of such lines is equal to the union of two genus one curves $F_i, F'_i$ defining a genus one pencil $|2F_i|$. On the canonical cover $\tilde{D}(W)$ the preimages of these two curves are the curves $E_i, E'_i$. Under the double cover $D(W) \to \mathbb{P}^2$ defined by the projection from the node, the curve $F_i + F'_i$ is equal to the preimage of the branch curve.

We say that a smooth Reye congruence $\text{Rey}(W)$ is general if $X = \tilde{D}(W)$ and rank $\text{Pic}(X) = 11$.

**Proposition 7.7.4.** Assume $\text{Rey}(W)$ is general. Then $\eta_H, \eta_S, \theta_i, 1 \leq i \leq 9$ is an integral basis of $\text{Pic}(X)$. There is an isomorphism of lattices

$$\text{Pic}(X) \cong \mathbb{U} \oplus \mathbb{E}_8(2) \oplus \mathbb{A}_1(2).$$

**Proof.** Since $(\theta_1, \ldots, \theta_{10}, \eta_H)$ is a basis of $\text{Pic}(X)_{\mathbb{Q}}$, for any $D \in \text{Pic}(X)$, we can write

$$D = \frac{D \cdot \eta_H}{4} \eta_H - \sum_{i=1}^{10} \frac{D \cdot \theta_i}{2} \theta_i.$$ 

Since $\theta_{10} = 3\eta_H - 2\eta_S - \sum_{i=1}^{9} \theta_i$, we obtain

$$D = \left(\frac{D \cdot \eta_H}{4} - \frac{3D \cdot \theta_{10}}{2}\right)\eta_H + (D \cdot \theta_{10})\eta_S - \sum_{i=1}^{9} \frac{D \cdot (\theta_i - \theta_{10})}{2} \theta_i. \quad (7.7.6)$$

We have to show that all coefficients here are integers. The images of the $(−2)$-curves $\Theta_i$ under the map given by the linear system $|\eta_S|$ are disjoint lines $\ell_i$ on a smooth quartic surface $S(W)$. For any point $x$ outside $\ell_i \cup \ell_j$, the unique line $\ell(x)$ in $\mathbb{P}^3$ that passes through $x$ and intersects $\ell_i$ and $\ell_j$ intersects $S(W)$ at another point $x'$. The rational map $x \mapsto x'$ extends to a biregular involution $\Phi$ of $S(W)$. This involution is the restriction of birational transformation $\tilde{\Phi}$ (an Arguesian involution) of $\mathbb{P}^3$ that assigns to a general point $x \in \mathbb{P}^3$ the point $x'$ on the line $\ell(x)$ such that the pairs of points $\{x, x'\}$ and $\ell(x) \cap S(W)$ are harmonically conjugate on $\ell(x)$ (see [303]). It is known that $\tilde{\Phi}$ is given by the linear system of quadric surfaces passing through the lines $\ell_i$ and $\ell_j$. Restricting it to $S(W)$, we see that the involution $\Phi$ is given by the linear system $|2\eta_S - \theta_i - \theta_j|$ of dimension 3 and degree 4. It acts on $\text{Pic}(S(W))$ by the formula

$$\Phi^*(D) = -D + \frac{(2\eta_S - \theta_i - \theta_j) \cdot D}{2}(2\eta_S - \theta_i - \theta_j).$$

In particular, we see that $D \cdot \theta_i \equiv D \cdot \theta_j \mod 2$, so the coefficients in (7.7.6) at $\theta_i$ are integers. Also, we can write $D \cdot \theta_i = k + 2n_i$, hence, applying Proposition 7.7.1, we get $3D\eta_H = 2D \cdot \eta_S + 10a + 2\sum n_i$. This shows that $\eta_H \cdot D$ is even. Thus, there exist some integers $a, b, c_1, \ldots, c_9$ such
that \( D = \frac{9}{2} \eta_H + b \eta_S + \sum_{i=1}^{9} c_i \theta_i \). This gives

\[
D^2 = a^2 + 2(3ab + 2b^2) + b \sum_{i=1}^{9} c_i - \sum_{i=1}^{9} c_i^2.
\]

Since \( D^2 \) is even, we get that \( a \) is also even, hence all the coefficients in (7.7.6) are integers.

The last assertion follows from computing the integral Smith form of the intersection matrix of the integral basis \((\eta_S, \eta_H, \theta_1, \ldots, \theta_9)\) that shows that the discriminant group of \( \text{Pic}(X) \) is isomorphic to \((\mathbb{Z}/2\mathbb{Z})^8 \oplus \mathbb{Z}/4\mathbb{Z}\). Since \( \text{Pic}(X) \) contains a sublattice \( \pi^*(\text{Rey}(S)) \oplus \mathbb{Z}(\theta_i - \tau^*(\theta_i)) \) of index 2 isomorphic to \( U(2) \oplus E_8(2) \oplus (-4) \), the assertion easily follows. We may also argue as follows.

In the notation of Proposition 7.7.2, the sublattice generated by \( \eta - F_1 - F_2 - F_3, F_1, \ldots, F_8 \) is isomorphic to \( E_8(2) \). The vector \( 2\eta_2 - \eta_1 \) spans a lattice \( (-4) \) orthogonal to the previous sublattice. Finally, the sublattice spanned by \( F_{10} = \eta_2 - R_{10} \) and \( F_{10} - R_{10} \) is isomorphic to \( U \) and is orthogonal to the previous two sublattices. Computing the determinant of the Gram matrix of the basis \((\eta_S, \eta_H, r_1, \ldots, r_9)\), we find that it is equal to \( 2^{10} \). Thus the orthogonal sum of the three sublattices coincides with the whole lattice. \( \square \)

### 7.8 Smooth congruences of bidegree \((7, 3)\)

By Theorem 7.4.3, the Reye surface \( \text{Rey}(W) \) of a regular web of quadrics \( W \) is an Enriques surface. In this section we will prove that, if \( p \neq 2 \), every smooth congruence \( S \) of bidegree \((7, 3)\) and sectional genus 6 is equal to the Reye congruence of a regular web of quadrics in \( \mathbb{P}^3 \).

We will start with the following.

**Lemma 7.8.1.** Let \( S \) be a smooth congruence of lines in \( \mathbb{G} = \mathbb{G}_1(\mathbb{P}^3) \) of bidegree \((7, 3)\) and sectional genus 6. Let \( H = c_1(\mathcal{O}_S(1)) \). Then \( S \) contains 20 plane cubic curves \( F_i, i = \pm 1, \ldots, \pm 10 \), such that

(i) \( F_i \cdot F_j = 1, \) if \( i + j \neq 0 \);

(ii) \( F_i \cdot F_i = 0, \) if \( i = 1, \ldots, 10 \);

(iii) \( 3H \sim F_1 + \cdots + F_{10} \);

(iv) \( F \cdot H \geq 3 \) for every genus one curve, and equality holds if and only if \( F = F_i \) for some \( i \);

(v) \( |H - F_i - F_j|, i \neq -j, \) consists of an isolated divisor which is either a genus one curve \( F_{ij} \) of degree 4, or the sum of some \( F_k \) (\( k \neq i, j \)), and a line;

(vi) for each curve \( F_i \) there exists a unique \( \beta \)-plane \( \Lambda_i \) such that \( \Lambda_i \cap S = F_i \);

(vii) \( H - F_i - F_{-i} \sim R_i, \) where \( R_i \) is a nodal cycle such that \( R_i \cdot H = 4, R_i \cdot F_i = 3 \);

(viii) \( F_i \cap F_j \cap F_k = \emptyset \) if \( |i| \neq |j| \neq |k| \).
Proof. We have $H^2 = 10$. Since $H$ is an ample divisor class, $\Phi(h) \geq 3$, hence, $H = H_{10}$ is a Fano polarization, and there exists a non-degenerate isotropic sequence $(f_1, \ldots, f_{10})$ such that $3h_{10} = f_1 + \cdots + f_{10}$. Each isotropic vector $f_i \in \mathrm{Num}(S)$ is nef and is equal to the numerical class of half-fibers $F_i$ and $F_{-i}$ of a genus one fibration $|2F_i| = |2F_{-i}|$. Since $H_{10} \cdot F_i = 3$, we obtain that all 20 curves $F_i$ are plane cubics in $S$. They satisfy properties (i)-(iv).

(v) We have $(H_{10} - F_i - F_j)^2 = 0$ and $H_{10} \cdot (H_{10} - F_i - F_j) = 4$. Thus $|H_{10} - F_i - F_j| = \{D\}$ for some effective divisor. Its linear part is a genus one curve $E$ so that $D = F$ or $D = F + R$, where $R \cdot H_{10} = 1$, i.e. $R$ is a line on $S$. In the first case, $H_{10} \cdot F = 4$, we denote this curve by $F_{ij}$. In the second case $F \cdot H_{10} = 3$, so $F = F_k$ for some $k \neq i, j$.

(vi) Since each curve $F_i$ is of degree 3, it must lie in a unique plane $\Lambda_i \subset \mathbb{G}$. The linear system $H_{10} - F_i$ is cut out by hyperplanes containing $\Lambda_i$. Therefore our assertion follows from the fact that $|H_{10} - F_i|$ has no fixed components and base points. Obviously, each fixed components or a base point must lie in the plane $\Lambda_i$. Assume $C$ is a fixed component. Then $C \cdot F \geq 3$, hence $(H_{10} - F_i - C) \cdot F_i = 3 - C \cdot F_i$ shows that $C \cdot F_i = 3$, i.e. $C$ is a line. So $C^2 = -2$, $H_{10} \cdot C = 1$ and $(H_{10} - F_i - C)^2 = 6$. By Riemann-Roch, $\dim |H_{10} - C - F_i| \geq 3$ which is absurd. To show that $|H_{10} - F_i|$ has no isolated base points, we apply Corollary 2.6.8. We have to verify that for every nef divisor $D$ with $D^2 = 0$, one has $(H_{10} - F_i) \cdot D \geq 2$.

Take $F_j$ such that $F_j \cdot D > 0$. By (v), $H_{10} - F_i - F_j \sim F_{ij}$ or $H_{10} - F_i - F_j \sim F_k + R$. Assume $(H_{10} - F_i) \cdot D = 1$. Take $j \neq \pm i, \pm k$, then,

$$(H_{10} - F_i) \cdot D = (H_{10} - F_i - F_j) \cdot D + F_j \cdot D \geq 1$$

with the equality only if $(H_{10} - F_i - F_j) \cdot D = 0$ and $F_j \cdot D = 1$. This implies that $D = F_{ij}$, hence $(H_{10} - F_i) \cdot D = 4 - 2 = 2$.

(vii) If $i + j = 0$, then $H_{10} - F_i - F_{-i} \equiv H_{10} - 2F_i$ has self-intersection equal to $-2$. Let us see that it is an effective divisor. Since each curve $F_i$ is of degree 3, it must lie in a unique plane $\Lambda_i$. Recall that $\mathbb{G}$ has two families of planes, $\alpha$-planes and $\beta$-planes. Two different planes from the same family intersect at one point, and two planes from different families are either disjoint or intersect along a line. If $\Lambda_i$ and $\Lambda_j$ intersect along a line, the linear system $|H_{10} - F_i - F_j|$ contains a pencil of curves cut out by hyperplanes through this line. If $i + j \neq 0$ this contradicts (v). If $i = \pm j$, this contradicts the equality $(H_{10} - F_i - F_j)^2 = -2$. Since $\Lambda_i \cap \Lambda_j = F_i \cap F_{-j} \neq \emptyset$ for $i \neq \pm j$, we obtain that all 20 planes $\Lambda_i$ belong to the same family. We claim that this is the family of $\beta$-lanes. In fact, suppose one of these planes is an $\alpha$-plane $\sigma_x$. Take a general plane $\Pi$ in $\mathbb{P}^3$. Its preimage under the map $Z_S \to S$ is a cover of degree 7. On the other hand, by property (v), the plane $\Lambda_i$ intersects the cone of rays in $\sigma_x$ at 3 points. This contradiction shows that the congruence $S$ has 20 fundamental planes $\Pi$ (as it is expected since it is a Reye congruence).

In particular, we have $\Lambda_i \cap \Lambda_{-i} \neq \emptyset$. Let $H$ be a hyperplane section of $S$ cut out by the hyperplane $\langle \Lambda_i, \Lambda_{-i} \rangle$ in the Plücker space spanned by the planes $\Lambda_i$ and $\Lambda_{-i}$. Then $|H - F_i - F_{-i}| \neq \emptyset$ and consists of a divisor $R$ with $R^2 = -2$ and $H_{10} \cdot R = 4$. Suppose $R$ has a part $R_1$ with $R_1^2 \geq 0$. If $R_1^2 > 0$, then the Hodge Index Theorem gives $10R_1^2 < (R_1 \cdot H_{10})^2 \leq 9$, a contradiction. Thus $R_1^2 = 0$. By putting some components of $R_1$ to $R - R_1$, we may assume that $R_1$ is nef. Then $10 = H_{10}^2 = 2H_{10} \cdot F_1 + H_{10} \cdot R_1 + H_{10} \cdot (R - R_1)$. Assume that $R_1 \cdot R_1 = 3$ and hence $R_1$ coincides with some $F_j$ and $H_{10} \cdot (R - R_1) = 1$. Thus $R - R_1$ consists of one irreducible component
with self-intersection $-2$ taken with multiplicity 1. Suppose $j \neq \pm i$. Then $-2 = (R - R_i)^2 = (H_{10} - 2F_i - F_j)^2 = -4$, a contradiction. If $j = \pm i$, then $(R - R_1)^2 = (H_{10} - 3F_i)^2 = -8$, a contradiction again.

Thus $R$ consists of $(-2)$-curves of total degree 4 and the sublattice of $\text{Num}(S)$ generated by its components is negative definite. Since $R^2 = -2$, the class of $R$ in $\text{Num}(S)$ is a root in some root system of finite type of rank $\leq 4$. It is easy to list all possibilities and get that $R$ is a nodal cycle. In general, it is an irreducible rational normal quartic in $(\mathbb{P}, \Lambda_{-1}) \cong \mathbb{P}^4$.

(viii) Assume $F_i \cap F_j \cap F_k \neq \emptyset$ for three distinct indices with no two add up to 0. Then the curves $F_j$ and $F_k$ cut out the same point on $F_i$, and we have an exact sequence:

$$0 \to \mathcal{O}_S(F_j - F_k - F_i) \to \mathcal{O}_S(F_j - F_k) \to \mathcal{O}_{F_i} \to 0.$$ 

Since $(F_j - F_k - F_i)^2 = -2$, and neither $F_j - F_k - F_i$ nor $F_i - F_k$ is effective, we have $h^1(F_j - F_k - F_i) = h^0(F_j - F_k) = 0$. Considering the exact cohomology sequence, this immediately leads to a contradiction. This proves the last assertion of the lemma. \hfill \Box

Let $\iota : S \hookrightarrow \mathbb{G}$ be the closed embedding, and let $\mathcal{V} = \iota^*(\mathcal{S}_G^\vee)$ be the restriction of the dual of the universal subsheaf of the constant sheaf $\mathbb{P}$ on $\mathbb{G}$. We know from Section 7.1 that

$$c_1(\mathcal{V}) = \iota^*(c_1(\mathcal{S}_G^\vee)) = \iota^*(c_1(\mathcal{Q}_G)) = h,$$

$$c_2(\mathcal{V}) = \iota^*(c_2(\mathcal{S}_G^\vee)) = \iota^*(\{\sigma_{\mathbb{P}}\}).$$

Since $\mathcal{V}$ defines the Plücker embedding $S \hookrightarrow \mathbb{G}$, it is generated by global sections. The restriction of the tautological exact sequence (7.1.6) to $S$

$$0 \to \iota^*(\mathcal{Q}_G)^\vee \to E^\vee \otimes \mathcal{O}_S \to \mathcal{V} \to 0$$

gives $h^0(\mathcal{V}) = 4$, $h^1(\mathcal{V}) = 0$, $i > 0$.

**Lemma 7.8.2.** The locally free sheaf $\mathcal{V}$ is indecomposable, i.e. it does not split into the direct sum of two invertible sheaves.

**Proof.** Assume that $\mathcal{V} \cong \mathcal{O}_S(D_1) \oplus \mathcal{O}_S(D_2)$. Suppose $H^0(S, \mathcal{O}_S(D_1)) = 0$. Then every section of $\mathcal{V}$ vanishes on a curve from the linear system $|D_2|$. On the other hand, we know that scheme of zeroes of a non-zero section of $\mathcal{V}$ is equal to the intersection of $S$ with some plane $\sigma_{\mathbb{P}}$ in $\mathbb{G}$. Thus a general section has only three isolated zeros. Since $H^1(S, \mathcal{V}) = 0$, we obtain that $H^i(S, \mathcal{O}_S(D_1)) = H^i(S, \mathcal{O}_S(D_1)) = 0$, $i > 0$. By Riemann-Roch, $D_1^2 = 2(h^0(D_1) - 1)$. We may assume that $h^0(D_1) \leq 2$. If the equality holds, then $D_1^2 = 2$. Since $c_2(\mathcal{V}) = D_1 \cdot D_2 = 3$, we get a contradiction to the Hodge Index Theorem. Thus we may assume that

$$h^0(D_1) = 1, \ h^0(D_2) = 3, \ D_1^2 = 0, \ D_2^2 = 4.$$ 

Since $h^1(D_1) = 0$, we have $D_1 = F + R$, where $F$ is a half-fiber of some genus one fibration and $R$ is a nodal cycle. By the Hodge Index Theorem, $40 = H^2 D_2^2 < (H \cdot D_2)^2$. This implies $H \cdot D_2 \geq 7$.

Since $H \cdot D_1 \geq 3$, we have $H \cdot D_2 = H^2 - H \cdot D_1 \leq 7$, we get $H \cdot D_2 = 7, H \cdot D_1 = 3$, and $D_1 = F$. We know that $H \cdot F = 3$ implies that $F = \overline{F_i}$ for some $i = \pm 1, \ldots, \pm 10$. Thus

$$\mathcal{V} = \mathcal{O}_S(\overline{F_i}) \oplus \mathcal{O}_S(H - F_i).$$
A nonzero section of $\mathcal{O}_S(F_i)$ defines a section of $\mathcal{V}$ that vanishes on the curve $F_i \subset \Lambda_i$. The direct sum decomposition implies that every section of $\mathcal{V}$ vanishes on $F_i$. This is obviously absurd.

It follows from Lemma 7.8.1 (v) that $h^0(H - F_i - F_j) = h^0(H - 2F_i + K_S) \neq 0$. Since $(H - 2F_i)^2 = -2$, this implies that

$$\dim H^1(S, \mathcal{O}_S(H - 2F_i + K_S)) = \dim H^1(S, \mathcal{O}_S(-H + 2F_i)) = 1.$$

A nonzero element in $H^1(S, \mathcal{O}_S(H - F_i)) \cong \text{Ext}^1(\mathcal{O}_S(H - F_i), \mathcal{O}_S(F_i))$ defines a non-split extension

$$0 \to \mathcal{O}_S(F_i) \to \mathcal{V} \to \mathcal{O}_S(H - F_i) \to 0.$$

The proof of the next result can be found in [171, Theorem 2].

**Theorem 7.8.3.** In the notation from the previous lemma, for any $i = \pm 1, \ldots, \pm 10$, there is an exact non-split sequence

$$0 \to \mathcal{O}_S(F_i) \to \mathcal{V} \to \mathcal{O}_S(H - F_i) \to 0.$$

Two such extensions are isomorphic.

**Corollary 7.8.4.**

$$\mathcal{V} \cong \mathcal{V} \otimes \omega_S.$$

**Theorem 7.8.5.** Assume $p \neq 2$. A smooth congruence of lines in $\mathbb{P}^3$ of bidegree $(7, 3)$ and sectional genus 6 is isomorphic to the Reye congruence of a regular web of quadrics in $\mathbb{P}^3$.

**Proof.** The tautological exact sequence (7.1.6) gives an isomorphism

$$E^\vee \cong H^0(S, \mathcal{V}).$$

Let $\pi : X \to S$ be the canonical cover of $S$ and let $\tau$ be the corresponding fixed-point-free involution of $X$. Let $\tilde{V} = \pi^*(\mathcal{V})$. Applying the corollary, we find an isomorphism

$$\sigma : \tilde{V} \to \tilde{V}.$$

Considered as an automorphism of the associated projective bundles, it is an involution. This easily implies that $\mathbb{P}(\tilde{V})$ has two disjoint sections corresponding to the locus of fixed points. This shows that $\tilde{V}$ splits into the direct sum of invertible sheaves

$$\tilde{V} \cong \mathcal{L}_+ \oplus \mathcal{L}_-,$$

where

$$\mathcal{L}_\pm \cong \tau^*(\mathcal{L}_\mp). \quad (7.8.1)$$

We have $c_1(\tilde{V}) = \pi^*(c_1(\mathcal{V})) = \pi^*(h)$ and $c_2(\tilde{V}) = \pi^*(c_2(\mathcal{V})) = 6$. Let $\eta_S$ denote $c_1(\mathcal{L}_+)$. We have

$$\eta := \eta_S + \tau^*(\eta_S) = \pi^*(h), \quad \eta_S \cdot \tau^*(\eta_S) = 6.$$

This immediately implies that $\eta_S^2 = 4$. Let

$$\tau_\pm^* : H^0(X, \mathcal{L}_\pm) \to H^0(X, \mathcal{L}_\mp)$$

be the isomorphism corresponding to the isomorphism (7.8.1). By correcting with a scalar automorphism of an invertible sheaf, we may assume that $\tau_+^* \circ \tau_-^*$ is the identity.
We have
\[ H^0(X, \pi^*(\mathcal{V})) \cong H^0(S, \mathcal{V}) \oplus H^0(S, \mathcal{V} \otimes \omega_S) \cong H^0(X, \mathcal{L}_+) \oplus H^0(X, \mathcal{L}_-). \]

Using Corollary 7.8.4, we choose an isomorphism
\[ E^\vee \cong H^0(X, \mathcal{L}_+). \]

Thus we will be able to identify \( H^0(X, \mathcal{L}_+) \) with \( E^\vee \), so that \(|\eta_S|\) defines a rational map \( X \to |E| \) whose image is a quartic surface. Of course, this will be the Steiner surface \( \text{St}(|W|) \).

We have \( \eta^2 = 2h^2 = 20 \). By Riemann-Roch, \( h^0(\eta) = 12 \) and
\[ H^0(X, \mathcal{O}_X(\eta)) = H^0(S, \mathcal{O}_S(h) \oplus H^0(S, \mathcal{O}_S(h + K_S))). \]

The first summand can be identified with \( \wedge^2 E^\vee \).

Since \( \eta = \eta_S + \tau^*(\eta_S) \), we have a canonical map
\[ \phi : E^\vee \otimes E^\vee = H^0(X, \mathcal{L}_+) \otimes H^0(X, \mathcal{L}_-) \to H^0(X, \mathcal{O}_X(\eta)). \]

Assume that \(|\eta_S|\) has no fixed components (we will prove this property later). By the Base-Point-Free Pencil Trick from [13, p. 126], for any 2-dimensional subspace \( V \) of \( H^0(X, \mathcal{L}_+) \), the kernel of the restriction \( \phi_V \) of \( \phi \) to \( V \otimes H^0(X, \mathcal{L}_-) \) is isomorphic to \( H^0(X, \mathcal{L}_- \otimes \mathcal{L}_+^{-1}) \). Since \( \eta \) is ample and \( \eta \cdot (\eta_S - \tau^*(\eta_S)) = 0 \), the linear system \(|\eta_S - \tau^*(\eta_S)|\) is empty. This shows that the kernel is trivial. Thus the image of \( \phi \) contains a 6-dimensional linear subspace. Since the map is \( \tau \)-equivariant, the image is \( \tau \)-invariant subspace. However, the image of \( \phi \) is obviously not \( \tau \)-invariant, and smallest \( \tau \)-invariant subspace it contains must coincide with the whole space. Thus \( \phi \) is surjective.

The kernel of the composition of this map with the canonical projection \( H^0(X, \mathcal{O}_X(\eta)) \to \wedge^2 E^\vee \) can be identified with \( S^2 E^\vee \). Thus the kernel of \( \phi \) is a 4-dimensional linear subspace \( L \) of \( S^2 E^\vee \) and the summand \( H^0(S, \mathcal{O}_S(H + K_S)) \) can be identified with \( S^2 E^\vee / L \). This shows that \(|H + K_S|\) maps \( S \) to \( |\mathbb{P}(S^2 E^\vee / L)| \cong \mathbb{P}^5 \). The image is of course should be the Cayley model of the Reye congruence.

Now everything is ready to finish the proof. Consider the map
\[ f_{|\eta|} = f_{|\eta_S|} \times f_{|\tau^*(\eta_S)|} : X \to |E| \times |E| \subset |E \otimes E|. \]

By the above its image is contained in the subspace of zeros of \( \text{Ker}(\phi) \subset E^\vee \otimes E^\vee \). This implies that the image of \( X \) is contained in a complete intersection of 4 divisors of type \((1, 1)\) in \(|E| \times |E|\). The degree of this surface in the Segre embedding is equal to 20, and we know that the map is given by the linear system of degree 20. So, the image is equal to the complete intersection. It is a K3 surface birationally isomorphic to \( X \). Since \( \eta \) is ample, the surface is nonsingular. It can be identified with the surface PB\((W)\), where \( W = |\text{Ker}(\phi)| \) is a regular web of quadrics. We leave to the reader to see that \( S \) is the Reye congruence of \( W \).

It remains to pay the debt and check that \(|\eta_S|\) has no fixed components. Consider the exact sequence from Theorem 7.8.3 and take its pull-back to the canonical cover \( X \). After twisting by \( \mathcal{O}_X(-F_i) \), we get an exact sequence
\[ 0 \to \mathcal{O}_X \to \mathcal{O}_X(\eta_S - E_i) \oplus \mathcal{O}_X(\tau^*(\eta_S) - E_i) \to \mathcal{O}_X(\eta - 2E_i) \to 0. \]

The image of \( 1 \in H^0(X, \mathcal{O}_X) \) gives an effective divisor \( D = D_1 + D_2 \sim \eta - 2E_i \), where
$D_1 \sim \eta_S - E_i, D_2 \sim \tau^*(\eta_S) - E_i$. Since $\tau^*(E_i) = E_i$, we obtain $\tau^*(D_1) = D_2$. Without loss of generality, we may assume that

$$D_1 \in |\eta_S - E_i|, \ i = 1, \ldots, 10.$$  

We know that $D_1 + D_2 \sim \eta - 2E_i = \tau^*(\eta - E_i - E_{-i})$. By Lemma 7.8.1, $|H - F_i - F_{-i}|$ is a nodal cycle $R_i$. It must split under the cover into the disjoint sum of nodal cycles $\tilde{R}_i, \tau^*(\tilde{R}_i)$. Thus, we may assume that $D_1 = \tilde{R}_i$. We have $F_i \cdot \tilde{R}_i = \frac{1}{2}(F_i \cdot R_i) = 3$. Hence $\eta_S \cdot F_i = 3, \eta_S \cdot \tilde{R}_i = 1$.

Suppose $A$ is the fixed part of $|\eta_S|$. Then each irreducible component of $A$ is a part of $E_i$ or $\tilde{R}_i$. Since $|\eta_S - A|$ has an irreducible divisor, by the Vanishing Theorem and Riemann-Roch, we have $(\eta_S - A)^2 = \eta_S^2 = 4$. One can easily list all possibilities for the nodal cycle $R_i$ and verify that $(\eta_S - A)^2 > 4$ for all sums of possible irreducible components of $F_i$ and $R_i$. 

### 7.9 Nodal Enriques surfaces and smooth congruences of lines in $\mathbb{P}^3$

Suppose $S$ is an Enriques surface with $K_S \neq 0$ embedded in $G_1(\mathbb{P}^3)$ as a smooth congruence of bidegree $(7,3)$. Let $H$ be the class of a hyperplane section of $S$ in the Plücker embedding. It follows from Lemma 7.8.1 that $R = H - F_i - F_{-i}$ is a nodal cycle and $|F_i|$ is a half-fiber of a genus one pencil and $R \cdot F_i = 3$.

**Theorem 7.9.1.** Let $S$ be an Enriques surface with $K_S \neq 0$. Assume that there exists a genus one curve $F$ not moving in a pencil and a nodal cycle $R$ with $F \cdot R = n \geq 3$ such that $|F + R + K_S|$ has no fixed components. Then there exists a birational map $f : S \to S'$, where $S'$ is a surface in the Grassmann variety $G = G_1(\mathbb{P}^n)$ with at most rational double points as singularities. If $H = 2F + R + K_S$ is an ample divisor, then $S \cong S'$. The map is given by the vector bundle $\mathcal{V} = f^*(i^*S_G)$ that fits in a non-split extension

$$0 \to \mathcal{O}(F) \to \mathcal{V} \to \mathcal{O}_S(F + R + K_S) \to 0. \tag{7.9.1}$$

**Proof.** Let $F, R$ be as in the assertion of the theorem. Since $h^0(R) \neq 0$, by Riemann-Roch and Serre Duality,

$$\dim \text{Ext}^1(\mathcal{O}_S(R + K_S), \mathcal{O}_S) = \dim H^1(S, \mathcal{O}_S(-R + K_S)) = \dim H^1(S, \mathcal{O}_S(R)) = 1.$$  

This implies that there exists a non-split extension (7.9.1). We will show that the vector bundle $\mathcal{V}$ defines the asserted map to the Grassmannian.

We have $(F + R + K_S)^2 = 2F \cdot R_2 = 2n - 2$. Since $h^1(\mathcal{O}_S(F)) = 0$ because $h^0(\mathcal{O}_S) = 1$, taking cohomology and using Riemann-Roch, we obtain

$$h^0(\mathcal{V}) = n + 1.$$  

Let us show that $\mathcal{V}$ is spanned by its global sections. Let $s_F$ be a non-zero section of $\mathcal{O}_S(F)$. For every $s \in H^0(S, \mathcal{V})$ the section $s_F \wedge s$ is either zero, or vanishes on a curve $F + D(s) \in |\mathcal{O}_S(H)|$ for some $D(s) \in |H - F|$. Since the map

$$H^0(S, \mathcal{V}) \to H^0(S, \mathcal{O}_S(H - F)) = H^0(S, \mathcal{O}_S(F + R + K_S))$$

is surjective, we find that $\mathcal{V}$ is generated by its global sections outside the curve $F$ unless $|H - F|$ has base points. If the latter happens, then, by Corollary 2.6.8, there exists a genus one curve $F'$
with \((H - F) \cdot F' = (F + R) \cdot F' = 1\). Obviously, \(F \cdot F' \leq 1\). If \(F \cdot F' = 0\), \(F \equiv F'\) and \(R \cdot F' = n > 1\). If \(F \cdot F' = 1\), then \(F' \cdot R = 0\), and \(R\) is a component of the pencil \(|2F'|\). But then \(2F' \cdot F' > F' \cdot R = n > 2\) that implies \(F \cdot F' > 1\). So, we have shown that \(|H - F|\) is base-point-free, and \(V\) is spanned by its global sections outside \(F\). We shall show that the same \(V\) can be represented by an extension:

\[
0 \to \mathcal{O}(F') \to V \to \mathcal{O}_S(F' + R + K_S) \to 0. \tag{7.9.2}
\]

where \(F' \in |F + K_S|\). Then repeating the argument we find that \(V\) is spanned by its global sections outside \(F'\). Since \(F \cap F' = \emptyset\), this would imply that \(V\) is spanned by its global sections everywhere. Tensoring (7.9.2) by the sheaf \(\mathcal{O}_S(-F')\), we obtain an exact sequence:

\[
0 \to \mathcal{O}_S(F - F') \to \mathcal{V}(-F') \to \mathcal{O}_S(R) \to 0. \tag{7.9.3}
\]

Since \(h^1(F' - F') = h^1(K_S) = 0\), this implies that \(h^0(\mathcal{V}(-F')) \neq 0\). Let \(\mathcal{O}_S(F') \to \mathcal{V}\) be a non-trivial map of sheaves defined by a non-zero section of \(\mathcal{V}(-F')\). Let \(\mathcal{L}\) be the maximal rank 1 subsheaf of \(\mathcal{V}\) which contains \(\mathcal{O}_S(F')\) (identified with its image) and such that the quotient sheaf \(\mathcal{V}/\mathcal{L}\) is torsion-free. Then, it is easy to see that \(\mathcal{L}\) is a reflexive sheaf. Since \(S\) is a smooth surface, \(\mathcal{L}\) is an invertible sheaf. We have an exact sequence:

\[
0 \to \mathcal{L} \to \mathcal{V} \to \mathcal{I}_Z \otimes \mathcal{L}' \to 0, \tag{7.9.4}
\]

where \(\mathcal{I}_Z\) is the ideal sheaf of some 0-dimensional subscheme of \(S\) and \(\mathcal{L}'\) is an invertible sheaf. This easily follows from the structure of rank 1 torsion free sheaves on a regular two-dimensional scheme. Counting the Chern classes of \(V\), we obtain:

\[
c_1(\mathcal{V}) = c_1(\mathcal{L}) + c_1(\mathcal{L}'),
\]

\[
c_2(\mathcal{V}) = c_1(\mathcal{L}) \cdot c_1(\mathcal{L}') + h^0(\mathcal{O}_Z).
\]

On the other hand, we count the Chern classes of \(V\) by using (7.9.2) to get:

\[
c_1(\mathcal{V}) = h, \quad c_2(\mathcal{V}) = F \cdot (H - F) = n.
\]

If \(\mathcal{L} = \mathcal{O}_S(E)\), then we find that \(\mathcal{L} \cong \mathcal{O}_S(H - E)\) and \(Z = \emptyset\). This gives (7.9.3). Assume now that \(\mathcal{L} \neq \mathcal{O}_S(F')\). Since \(h^1(\mathcal{O}_S(F')) = 0\), we have \(h^0(\mathcal{L}) > h^0(\mathcal{O}_S(F')) = 1\). Let \(\phi: \mathcal{L} \to \mathcal{O}_S(H - F)\) be the composition of the inclusion \(\mathcal{L} \hookrightarrow \mathcal{V}\) and the projection \(\mathcal{V} \to \mathcal{O}_S(H - F)\). If \(\phi\) is trivial, \(\mathcal{L}\) is a subsheaf of \(\mathcal{O}_S(F)\) which has a one-dimensional space of sections. This contradiction shows that \(\phi\) is non-trivial, hence \(\mathcal{L} \cong \mathcal{O}_S(D)\), where \(D = F' + R_1\) for some effective divisor \(R_1 \neq 0\), and \(|H - F - D| = |H - F - F' - R| = |R - R_1| \neq \emptyset\). We have

\[
n = c_2(\mathcal{V}) = D \cdot (H - D) + h^0(\mathcal{O}_Z) \geq D \cdot (H - D) = (F' + R_1) \cdot (h - F' - R_1) = (F' + R_1) \cdot (F + R + R_1) = n + R_1 \cdot (R - R_1).
\]

This implies that \(R_1 \cdot (R - R_1) = 0\), \(Z = \emptyset\). Since \(R\) is connected, \(R = R_1\), and \(D = F' + R = H - F\). So, (7.9.4) becomes

\[
0 \to \mathcal{O}_S(F' + R) \to \mathcal{V} \to \mathcal{O}_S(F) \to 0.
\]

However, we have

\[
\text{Ext}^1(\mathcal{O}_S(F), \mathcal{O}_S(F' + R)) \cong H^1(S, \mathcal{O}_S(F' - F + R)) \cong H^1(\mathcal{O}_S(R + K_F)) \cong 0.
\]

This shows that \(\mathcal{V}\) splits into the sum \(\mathcal{O}_S(h - F) + \mathcal{O}_S(F)\) in conflict with its construction.

Let \(f: F \to \mathbb{G} = G(1, \mathbb{P}^n)\) be the regular map given by the bundle \(\mathcal{V}\). Recall that it assigns to a
Thus, the composition of \( f \) and the Plücker embedding of \( S' = f(S) \subset G \) is given by the linear system \([H]\). So, the theorem will follow if we verify that \([H]\) defines the map \( S \to S' \) with the asserted properties. By Theorem 2.4.16, it suffices to check that \( \Phi(H) \geq 3 \), i.e. for any genus one curve \( E' \) one has \( H \cdot E' \geq 3 \). We have seen already that \( \Phi(H - F) \geq 2 \) and \( (H - F') \cdot F' = 2 \) implies \( F' \cdot F = 1 \). This obviously gives that \( \Phi(H) \geq 3 \). This concludes the proof of the theorem.

**Corollary 7.9.2.** Let \( S \) be an Enriques surface with \( K_S \neq 0 \). Assume that there exists an ample divisor \( H \) with \( H^2 = 4n - 2 \) and a genus one curve \( F \) with \( h \cdot F = n \geq 3 \) such that \( h^0(H - 2F - K_S) \neq 0 \). Then \( S \) is isomorphic to a smooth surface \( S \) in the Grassmann variety \( G(1, \mathbb{P}^n) \).

**Remark 7.9.3.** We refer to [125], where it is shown that there exists a linear system \( W \) of quadrics in \( \mathbb{P}^n \) of dimension \( \binom{n}{2} \) such that the image of \( S \) in \( G(1, \mathbb{P}^n) \) is contained in the subvariety of \( G(1, \mathbb{P}^n) \) parameterizing lines in \( \mathbb{P}^n \) contained in a codimension 2 linear subspace of \( W \). Note that contrary to the case \( n = 3 \), if \( n > 3 \) the linear system of quadrics must be very special in order that it defines a generalized Reye congruence.

**Remark 7.9.4.** Let \( h_{18} = \frac{1}{7}(g_1 + \cdots + g_9) \) be a Mukai (numerical) polarization with nef vectors \( g_i \). Suppose that it is obtained from a non-degenerate isotropic 10-sequence \( (f_1, \ldots, f_{10}) \) by formula

\[
  g_i = h_{10} - f_i - f_{10}, \quad i = 1, \ldots, 9,
\]

where \( h_{10} \) is a (numerical) Fano polarization. Let us lift the numerical classes to obtain a representative \( H_{18} \) of \( h_{18} \) and representatives \( F_i \) of \( f_i \). Since \( f_i \cdot h_{18} = 5 \) if \( i \neq 10 \). The image of \( F_i \) under a map \( \phi_{H_{18}} \) given by the linear system \([H_{18}]\) is a genus one curve of degree 5 spanning a \( \mathbb{P}^4 \). Let \( F_{-i} \in [F_i + K_S] \), where we assume that \( K_S \neq 0 \). Then \( H_{18} - 2F_i - K_S = H_{18} - F_i - F_{-i} \), hence \( h^0(H_{18} - 2F_i - K_S) > 0 \) if and only if \( F_i + F_{-i} \) span a hyperplane in \( \mathbb{P}^9 \). If this happens, then the Mukai model of \( S \) lies in the subvariety of the Grassmannian variety \( G(1, \mathbb{P}^5) \) parameterizing lines in a 10-dimensional linear system \( W \) of quadrics in \( \mathbb{P}^5 \) that are contained in a codimension 2 linear subspace of \( W \). If we compose the embedding \( S \to G(1, \mathbb{P}^5) \) with the Plücker embedding \( G(1, \mathbb{P}^5) \to \mathbb{P}^{14} \), then we obtain that the image of \( S \) is contained in the intersection of \( G(1, \mathbb{P}^5) \) with a linear subspace of dimension 9. This is a 3-dimensional Fano variety.

Now we assume that \( K_S = 0 \) and \( n = 3 \). Recall that \( S \) must be either a \( \mu_2 \)-surface or \( \alpha_2 \)-surface.

**Lemma 7.9.5.** Let \( (F_1, F_2, F_3) \) be a non-degenerate \( U_{[3]} \)-sequence. Suppose that \( |F_2 + F_3 - F_1 + K_S| = 0 \). Then, \( F_1 \cap F_2 \cap F_3 = \emptyset \).

**Proof.** Recall that, by Proposition 4.10.5, two half-fibers have no common irreducible component. Thus \( F_1 \cap F_2 \) consists of one point. Consider the natural exact sequence coming from restriction of the sheaf \( \mathcal{O}_S(F_1 - F_2) \) to \( F_3 \):

\[
0 \to \mathcal{O}_S(F_1 - F_2 - F_3) \to \mathcal{O}_S(F_1 - F_2) \to \mathcal{O}_{F_3}(F_1 - F_2) \to 0.
\]

We have \( (F_1 - F_2 - F_3) \cdot F_1 = -2 \). Since \( F_1 \) is nef, the divisor class \( F_1 - F_2 - F_3 \) is not effective. Thus, by Riemann-Roch and Serre’s Duality, \( h^1(\mathcal{O}_S(F_1 - F_2 - F_3)) = 0 \) since \( h^0(\mathcal{O}_S(K_S + F_3 + F_2 - F_1)) = 0 \) by assumption. Now, \( h^0(\mathcal{O}_S(F_1 - F_2)) = 0 \), because \( (F_1 - F_2) \cdot F_1 = -1 \) and
$F_1$ is nef. Suppose $F_1 \cap F_2 \cap F_3 \neq \emptyset$, then $\mathcal{O}_{F_3}(F_1 - F_2) \cong \mathcal{O}_{F_3}$ and $h^0(\mathcal{O}_{F_3}(F_1 - F_2)) = 1$. It remains to consider the exact sequence of cohomology and get a contradiction. \hfill \Box

**Remark 7.9.6.** Note that for any $D \in |F_2 + F_3 - F_1 + K_S|$, we have $D^2 = -2$ and $D \cdot F_2 = D \cdot F_3 = 0$, so $D$ consists of $(-2)$-curves contained in fibers of $|2F_2|$ and $|2F_3|$.

**Theorem 7.9.7.** Let $S$ be an Enriques surface with $K_S = 0$. Assume that $S$ contains a genus one curve $F$ and a $(-2)$-curve $R$ with $F \cdot R = 3$ such that $H = 2F + R$ is ample. Then there exists an isomorphism $f : S \rightarrow S'$, where $S'$ is a surface in the Grassmann variety $G(1, \mathbb{P}^3)$. The map from $S$ to the Plücker space is given by the complete linear system $|H|$.

**Proof.** We use the same argument as in the proof of Theorem 7.9.1, only this time, instead of taking $F' \in |F + K_S|$ we take $F$. Let

$$3H_{10} \sim F_1 + \cdots + F_{10} \quad (7.9.5)$$

with $F_i = F$. Since $H_{10}$ is ample, each $F_i$ is a half-fiber. We claim that, using the previous lemma, one can choose two $F_i$ and $F_j$ different from $F_1$ such that $F_1 \cap F_i \cap F_j = \emptyset$. Suppose, $F_i \cap F_j \cap F_k \neq \emptyset$ for some $i < j < k$. Then $|F_a + F_b - F_c| \neq \emptyset$ for $\{a, b, c\} = \{i, j, k\}$. Since $H_{10} \cdot (F_a + F_b - F_c) = 3$ and $(F_a + F_b - F_c)^2 = -2$, $|F_a + F_b - F_c| = \{R_i\}$, where $R_i$ is a $(-2)$-curve with $h \cdot R_i = 3$, or $|F_a + F_b - F_c| = \{R_1, R_2\}$, where $R_1, R_2$ are two $(-2)$-curves with $H_{10} \cdot R_1 = 2$ and $H_{10} \cdot R_1 = 1$. In both cases each $F_i, F_j, F_k$ contains $R_1$ or $R_1 + R_2$ in one of their reducible fibers. Since $(F_i + F_j - F_k) \cdot (F_i + F_j - F_k) = -1$ if $l \not\in \{i, j, k\}$, we see that $|F_i + F_j - F_k|$ or $|F_i + F_j - F_k|$ contain a $(−2)$-curve $R'$ with $H_{10} \cdot R' = 1$, we call such $R'$ a line. Now take $F_i = F_1$, then $1 = H_{10} \cdot L = (2F + R) \cdot L$ for any line $L$ implies that $F \cdot L = 0$. Since $H_{10}$ is ample, there exists $F_i$ such that $F_i \cdot L \neq 0$. Thus $|F_1 + F_i - F_j| = \{R_j\}$ for any $j \neq i$, where $R_j$ is a $(−2)$-curve with $H_{10} \cdot R_j = 3$. Since $(F_1 + F_i - F_j) \cdot (F_1 + F_i - F_k) = R_j \cdot R_k = -1$ for different $1, i, j, k$, we get a contradiction. Thus, we can always find $i \neq j \neq 1$ such that $F_1 \cap F_i \cap F_j = \emptyset$. Without loss of generality we assume that $i = 2, j = 3$.

Now we shall show that the non-split vector bundle $\mathcal{V}$ given by the exact sequence:

$$0 \rightarrow \mathcal{O}_S(F) \rightarrow \mathcal{V} \rightarrow \mathcal{O}_S(H - F) \rightarrow 0 \quad (7.9.6)$$

can be also represented by an extension

$$0 \rightarrow \mathcal{O}_S(F_i) \rightarrow \mathcal{V} \rightarrow \mathcal{O}_S(H - F_i) \rightarrow 0 \quad (7.9.7)$$

for $i = 2$ and $3$. Then, applying Lemma 7.8.1 (vii), and repeating the argument from the proof of Theorem 7.9.1 with the curve $F'$ replaced by $F_i$, we find that $\mathcal{V}$ is generated by its global sections. The rest of the proof proceeds in the same way.

Twisting (7.9.6) by $\mathcal{O}_S(-F_2)$, and using that $h^0(H - F - F_2) = 1$ because $(H - F - F_2)^2 = 0$ and $(H - F - F_2) \cdot F_3 = 1$, we find a non-trivial map $\mathcal{O}_S(F_2) \rightarrow \mathcal{V}$. Let $\mathcal{L} = \mathcal{O}_S(D)$ be an invertible subsheaf of $\mathcal{V}$ which contains the image of $\mathcal{O}_S(F_2)$ such that the quotient sheaf $\mathcal{V}/\mathcal{L} \cong \mathcal{L}_Z(D')$ for some zero-cycle $Z$ and an effective divisor $D'$.

Consider the map $\phi : \mathcal{O}_S(D) \rightarrow \mathcal{O}_S(H - F)$ obtained as the composition of the inclusion $\mathcal{O}_S(D)$ in $\mathcal{V}$ and the projection map $\mathcal{V} \rightarrow \mathcal{O}_S(H - F)$. If $\phi$ is trivial, then $\mathcal{O}_S(D)$ is isomorphic to a subsheaf of $\mathcal{O}_S(F)$, hence $F - F_3 > 0$, a contradiction. Thus $\phi$ is not trivial, hence $h \cdot D - D$ is an effective divisor. Thus $H - F - D \sim F + R - D \geq 0$, and we obtain $h \cdot D \leq 7$. If $h \cdot D = 7$, we have $h \cdot (F + R - D) = 0$, hence...
hence $H - F \sim F + R \sim D$. Thus $\mathcal{V}$ fits in the extension
\[ 0 \to \mathcal{O}(H - F) \to \mathcal{V} \to \mathcal{O}(F) \to 0. \]  
(7.9.8)

Since $\mathcal{O}_S(H - F)$ is not a subsheaf of $\mathcal{O}_S(F_1)$, we see that the composition of $\mathcal{O}(H - F) \to \mathcal{V}$ and the projection to $\mathcal{O}_S(F_1)$ in extension (7.9.7) is an isomorphism. Thus the extension splits contrary to its construction. We have $3 = c_2(\mathcal{V}) \geq D \cdot (H - D)$ that implies that $D^2 \geq 4$ if $h \cdot D = 6$. The Hodge Index Theorem $10D^2 < (H \cdot D)^2$ gives a contradiction. Thus $H \cdot D \leq 5$ and $D^2 \leq 2$.

We have $h^0(D) = 1$, otherwise using Proposition 2.6.1 we obtain that $\Phi(H) \leq 2$, a contradiction. If $L = \mathcal{O}_S(D) \neq \mathcal{O}_S(F_1)$, the quotient sheaf is a torsion sheaf, and we obtain $h^0(L) \geq 2$ contradicting the previous remark. So we have an exact sequence
\[ 0 \to L \to \mathcal{V} \to \mathcal{I}_Z(D') \to 0, \]
where $L = \mathcal{O}_S(F_1)$ or $L = \mathcal{O}_S(H - F_1)$. Computing the Chern classes we obtain, as in the proof of Theorem 7.9.1, that $Z = \emptyset$ and $D' \sim H - F_1$ or $D' \sim F_1$, respectively. In the latter case, the extension must split contradicting our construction; in the former case we obtain (7.9.7). This ends the proof of the theorem.

**Corollary 7.9.8.** Let $S$ be a general nodal Enriques surface in the sense of definition from Section 6.5. Then $S$ is isomorphic to a congruence of lines in $\mathbb{P}^3$ of bidegree $(7, 3)$.

**Proof.** By Theorem 6.5.4, $S$ admits an ample Fano polarization $h_{10}$ such that there exists a $(-2)$-curve $R$ with $h_{10} \cdot R = 4$. It follows from Corollary 6.3.11 that it represents the numerical class $h_{10} - 2f_1$, where $(f_1, \ldots, f_{10})$ is a non-degenerate canonical isotropic sequence such that $3h_{10} = f_1 + \cdots + f_{10}$. We have $f_1 \cdot R = h_{10} \cdot f_i = 3$. Since $f_i$ is nef, it represents a half-fiber $F_i$ of a genus one fibration. Now we can apply Theorem 7.9.1 and Theorem 7.9.7.

Applying Proposition 7.7.4, we obtain

**Corollary 7.9.9.** Assume $p \neq 2$. The Picard lattice of the canonical cover of a general nodal Enriques surface is isomorphic to $\mathbb{U} \oplus \mathbb{E}_8(2) \oplus \mathbb{A}_1(2)$.

Let $\mathcal{E}$ be a rank 2 vector bundle on $S$ with $c_1(\mathcal{E}) = H_{10}$, where $H_{10}$ is an ample Fano polarization and $c_2(\mathcal{E}) = 3$. Assume that $\mathcal{E}$ has a section $s$ with only isolated zeros. The section $s$ defines an exact sequence
\[ 0 \to \mathcal{O}_S \to \mathcal{E} \to \mathcal{I}_Z(H) \to 0, \]
where $Z$ is a 0-dimensional closed subscheme of $S$ with $h^0(\mathcal{O}_Z) = 3$. Suppose $\mathcal{E} = \mathcal{V}$, where $\mathcal{V}$ gives an embedding $\phi$ of $S$ into the Grassmannian $G = G(1, \mathbb{P}^3)$. Suppose $K_S \neq 0$. Taking global sections, we find that $h^0(S, \mathcal{I}_Z(H_{10} + K_S)) = 4$. This shows that the image of $Z$ in $G$ spans a line, a trisecant line of $\phi(S)$. Since $H^0(S, \mathcal{V}) = 4$, we have a $\mathbb{P}^3$ of trisecant lines. Note that a smooth Fano model admits 20 planes of trisecant lines, the lines in 20 planes generated by the images of the half-fibers $F_i$ from (7.9.5). See more about this in the next section.

The following theorem was proven in [171] under the assumption that $K_S \neq 0$. We can modify it along the lines of the proof of Theorem 7.9.7 to extend it to the case $K_S = 0$. 

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**CHAPTER 7. REYE CONGRUENCES**
Theorem 7.9.10. Let $E$ be a rank 2 vector bundle on an Enriques surface $S$ which admits a section with only isolated zeros. Assume $c_1(E)$ is an ample Fano polarization $H_{10}$ and $c_2(E) = 3$. Then either $E$ decomposes into the direct sum of invertible sheaves $\mathcal{O}_S(F_i) \oplus \mathcal{O}_S(H_{10} - F_i)$, or $\mathcal{O}_S(F_i + K_S) \oplus \mathcal{O}_S(H_{10} - F_i + K_S)$, or it is isomorphic to one of the following non-split extensions

(i)  
$$0 \to \mathcal{O}_S(H_{10} - F_i) \to E \to \mathcal{O}_S(F_i) \to 0$$

or

$$0 \to \mathcal{O}_S(H_{10} - F_i + K_S) \to E \to \mathcal{O}_S(F_i + K_S) \to 0.$$ 

(ii)  
$$0 \to \mathcal{O}_S(F_i) \to E \to \mathcal{O}_S(H_{10} - F_i) \to 0$$

or

$$0 \to \mathcal{O}_S(F_i + K_S) \to E \to \mathcal{O}_S(H_{10} - F_i + K_S) \to 0.$$ 

In case (i) the vector bundle $E$ defines an isomorphism from $S$ to a congruence of bidegree $(7, 3)$ in $G(1, \mathbb{P}^3)$.

Remark 7.9.11. If $K_S \neq 0$, in case (i) the bundle $E$ represents the unique isomorphism class of a stable rank 2 vector bundle with $c_1(E) = H_{10}$ and $c_2(E) = 3$. It is also an exceptional vector bundle in the sense that $\text{Ext}^1(E, E) = \{0\}$.

More generally, it follows from [375], [376] that any exceptional rank 2 vector bundle on an Enriques surface with $c_1(E)^2 = 4n - 2$, $c_2(E) = n \geq 3$ is isomorphic to a vector bundle $E$ given by a non-split extension

$$0 \to \mathcal{O}_S(F) \to E \to \mathcal{O}_S(F + R) \to 0,$$

where $R$ is a nodal cycle and $F$ is a half-fiber of a genus one fibration with $F \cdot R = n$. The bundle $E$ is stable and maps $S$ onto a generalized Reye congruence.

Definition 7.9.12. A Fano polarization $H_{10}$ is called a Reye polarization if the linear system $|H|$ maps $S$ into a nonsingular quadric $Q$ in $\mathbb{P}^5$.

Recall from Theorem 6.3.10 that, for any Fano polarization on a nodal Enriques surface $S$ there exists a $(-2)$-curve $R$ with $R \cdot H_{10} \leq 4$.

Corollary 7.9.13. Let $H_{10}$ be a Fano polarization on a general nodal Enriques surface. Then $H_{10}$ is a Reye polarization if and only if there exists a smooth rational curve $R$ on $S$ such that $H_{10} \cdot R = 4$.

Proof. By Corollary 6.3.11, $[R] = h_{10} - 2f$, where $f$ is a nef primitive isotropic class. Thus we can choose its representative $F$ such that $H_{10} - 2F \sim R$. Now we apply the previous results. 

We do not know whether any nodal Enriques surface admits a Reye polarization.
7.10 Fano-Reye polarizations

Let $H_{10}$ be a Fano polarization on $S$ which we do not assume to be ample.

Let $\mathbf{f} = (f_1, \ldots, f_{10})$ be a canonical isotropic sequence such that

$$3h_{10} = f_1 + \cdots + f_{10}.$$ 

Let $c$ be the degeneracy invariant of $\mathbf{f}$. Let $f_{i_1}, \ldots, f_{i_c}$ be its nef members. Choose representatives $F_{i_1}, \ldots, F_{i_c}$ of $f_{i_1}, \ldots, f_{i_c}$ such that

$$3H_{10} \sim F_1 + \cdots + F_{10},$$

where $R_{i_k} = F_{i_k+1} - F_{i_k}$ is a nodal cycle of type $A_{i_k+1-i_k-1}$ such that $F_{i_k} \cdot R_{i_k} = 1$. The linear system $|H_{10}|$ maps $S$ birationally onto a surface $S'$ in $\mathbb{P}^5$ with singular points of type $A_{i_k+1-i_k-1}$, the images of the nodal cycles $R_{i_k}$. The surface $S'$ is called a Fano model of $S$. The polarization $H_{10}$ is ample if and only if $S \cong S'$, or, equivalently, the isotropic sequence $\mathbf{f}$ is non-degenerate. The images of the curves $F_{i_1}, \ldots, F_{i_c}$ are plane curves of degree 3. They span the planes $\Lambda_{i_1}, \ldots, \Lambda_{i_c}$ in $\mathbb{P}^5$. If $K_S \neq 0$, we denote by $F_{-i_j}$ the second half-fiber in the genus one pencil $|2F_{i_j}|$. Let $\Lambda_{-i_j}$ be the plane spanning the image of $F_{-i_j}$. If $K_S = 0$, by definition, $F_{i_j} = F_{-i_j}, \Lambda_{i_j} = \Lambda_{-i_j}$.

**Definition 7.10.1.** A choice of representatives $F_i, H_{10}$ of $f_i, h_{10}$ in $\text{Pic}(S)$, together with an order prescribed by the definition of a canonical isotropic sequence is called a supermarking of $S$. The planes $\Lambda_{i_1}, \ldots, \Lambda_{i_c}$ are called the Fano planes of the supermarking.

**Lemma 7.10.2.** The singular point $x_{i_j}$ lies in $\Lambda_{i_j}$. Two Fano planes $\Lambda_{i_j}$ and $\Lambda_{i_k}$ intersect at a unique point which is a nonsingular point of $S'$.

**Proof.** The only assertion that does not follow immediately from the definition of a canonical isotropic sequence is the assertion that $\dim \Lambda_{i_j} \cap \Lambda_{i_k} < 1$. Suppose the intersection contains a line $\ell$. Then the preimage of the linear system of hyperplanes containing the two planes defines a linear system $|H_{10} - F_{i_j} - F_{i_k}|$ on $S$ of dimension $\geq 1$. We have $(H_{10} - F_{i_j} - F_{i_k})^2 = 0$ and $(H_{10} - F_{i_j} - F_{i_k}) \cdot H_{10} = 4$. The fixed part of $|H_{10} - F_{i_j} - F_{i_k}|$ consists of the union of nodal cycles and the moving part $|M|$ must satisfy $M^2 = 0$. Since $M \cdot H_{10} < 4$, $|M|$ cannot be equal to $|2F|$ for a genus one curve $F$. This shows that $\dim |M| = \dim |H_{10} - F_{i_j} - F_{i_k}| = 0$. 

**Proposition 7.10.3.** Let $S' \subset \mathbb{P}^5$ be a Fano model of an Enriques Enriques surface with $K_S \neq 0$. The following properties of $S'$ are equivalent.

(i) $S'$ lies in a quadric;

(ii) $S'$ lies in a quadric of corank $\leq 1$;

(iii) the Fano planes $\Lambda_{-i}$ and $\Lambda_i$ intersect;

(iv) for any Fano plane $\Lambda_i$ there exists a hyperplane $H$ that cuts out $S'$ along $(\Lambda_i \cap S') \cup (\Lambda_{-i} \cup S') \cup C$, where $\deg C = 4$ and one of the following cases occurs

- $C$ is a connected curve of degree 4 and arithmetic genus 1 spanning a $\mathbb{P}^3$ and one of the singular points of $S'$ lies on $C$ and coincides with $\Lambda_i \cap \Lambda_{-i}$;
7.10. FANO-REYE POLARIZATIONS

(iv) " $C$ is a connected curve of degree 4 and arithmetic genus 0 that spans a hyperplane. If $C$ contains a singular point of $S'$, then one of them is the point $\Lambda_i \cap \Lambda_{-i}$."

(v) if $S = S'$, then $S'$ is isomorphic to a Reye congruence.

Proof. (i) $\Leftrightarrow$ (ii) If $S$ lies on a quadric $Q$, then every Fano plane $\Lambda_i$ is contained in $Q$ (since it cuts out a cubic curve in $S$). If $Q$ contains a line in its singular locus, then two planes intersect along a line. This contradicts Lemma 7.10.2.

(ii) $\Rightarrow$ (iii) Since all planes in a singular quadric intersect, we may assume that the quadric is nonsingular. Then the proof follows from the proof of property (v) in Lemma 7.8.1. To carry on the proof we need only that $c \geq 2$. It is covered by our assumption.

(iii) $\Rightarrow$ (iv). If $S = S'$, this was proven in Lemma 7.8.1 (vii). In this case the second possibility is not realized. In the notation of this proof, assume that $R \in |H_{10} - F_i - F_{-i}|$ contains a nef part $R_1$ with $R_1^2 \geq 0$. As in the proof of Lemma 7.8.1 we find that $R_1^2 = 0$ and $H_{10} \cdot (R - R_1) \leq 1$. It is maybe equal to zero instead of 1 because $H_{10}$ is not ample anymore.

Suppose $H_{10} \cdot (R - R_1) = 1$. If $S'$ is smooth, we deduced from this that $H_{10} \cdot R_1 = 3, (R - R_1)^2 = -2$ and got a contradiction. If $S'$ is singular this only implies that $R_1$ coincides with one of $F_j$ and if $R_1 \equiv F_i$ then $(R - R_1)^2 = -8, R_1 \cdot (R - R_1) = 3$. If $R_1 \neq F_i$, then $(R - R_1)^2 = -4, R_1 \cdot (R - R_1) = 1$. In both cases the image of $R - R_1$ is a line $\ell$ on $S'$ passing through some singular points of $S'$ and the image of $R_1$ spans one of the planes $\Lambda_j$. In the first case $\ell \subset \Lambda_j$ and in the second case $\ell \cap \Lambda_j \neq \emptyset$. In both cases, the image of $R$ spans a projective subspace of dimension $\leq 3$. But $h^0(H_{10} - R) = h^0(F_i + F_{-i})$ hence the image of $R$ spans a hyperplane in $\mathbb{P}^5$, a contradiction.

Suppose $H_{10} \cdot (R - R_1) = 0$. Then $H_{10} \cdot R_1 = 4$ and $(H_{10} - 2F_i) \cdot R_1 = R_1 \cdot (R - R_1)$ implies that either $F_i \cdot R_1 = 1, F_i \cdot (R - R_1) = 2$ and $R_1 \cdot (R - R_1) = 2$, or $F_i \cdot R_1 = 2, F_i \cdot (R - R_1) = 1$ and $R_1 \cdot (R - R_1) = 0$. It follows that $R - R_1$ is blown down to one singular point equal to $\Lambda_i \cap \Lambda_{-i}$ and this point lies on the image of $R_1$. The image of $R_1$ is a curve of degree 4 and arithmetic genus 1.

If each non-empty part of $R$ is negative definite, the argument applies without change and shows that $R = R_1 + R_2$ is a connected curve, where $R_1$ is a nodal chain with $R_1^2 = -2$ mapped to a curve of degree 4 of arithmetic genus 0 and $R_2$ is empty or a nodal chain blown down to singular points $s_1', ..., s_k'$ on $S'$. In the latter case, we have $-2 = (R_1 + R_2)^2$ implies $R_2^2 = -2R_1 \cdot R_2 < 0$, hence $R_1 \cdot R_2 > 0$. Also, $(H_{10} - 2F_i) \cdot R_2 = -2F_i \cdot R_2 = R_1 \cdot R_2 + R_2^2 = -R_1 \cdot R_2$ implies that $R_2 \cdot F_i > 0$. Thus $\Lambda_i \cap \Lambda_{-i}$ is one of the singular points $s_i'$ and the image of $R_1$ contains all the singular points $s_1', ..., s_k'$.

(iv) $\Rightarrow$ (v) This follows from Corollary 7.9.2.

(v) $\Rightarrow$ (i) Obvious. 

Any conditions in the previous Proposition characterizes a Reye polarization among Fano polarizations.
Suppose $H_{10}$ is an ample Reye polarization, so that $S' = \text{Rey}(W)$ for a regular web of quadrics $W$. Let $\mathcal{U}$ be the universal family of lines contained in quadrics from $W$. We cited a result from [412] that a choice of a plane component in each of the ten reducible quadrics $Q_i$ in $W$ defines a small resolution (in the category of algebraic spaces) of the double cover of $W$ branched along the quartic symmetroid $D(W)$. Since the Reye lines in each plane form the cubic curve $F_i$ or $F_{-i}$ from Lemma 7.8.1, we see that this choice is equivalent to a choice of a supermarking of the Reye polarization.

Note that, if $H_{10}$ is a Reye polarization and $K_S \neq 0$, then $|H_{10} + K_S|$ is not a Reye polarization. In fact, $|H_{10} - F_i - F_{-i}| \neq \emptyset$ implies that $|H_{10} + K_S - F_i - F_{-i}| = |H_{10} - 2F_i| = \emptyset$.

Remark 7.10.4. We do not know any examples when a Fano polarization maps $S$ into a singular quadric.

Recall that a line $l$ in $\mathbb{P}^n$ is called a trisecant line of a subvariety $V \subset \mathbb{P}^n$ if it intersects $V$ in at least three points taken with appropriately defined multiplicities. If $S'$ is a Fano model of an Enriques surface $S$ in $\mathbb{P}^5$ and $\Lambda_i$ is a Fano plane containing a cubic curve $F_i$, then obviously any general line in $\Lambda_i$ is a trisecant line of $S'$.

**Proposition 7.10.5.** Assume $K_S \neq 0$. Let $H_{10}$ be an ample Reye polarization and let $S'$ be the image of $S$ under $\phi_{|H_{10} + K_S|}$. Then there is a three-dimensional family of trisecant lines of $S'$.

**Proof.** Let $\mathcal{V}$ be the vector bundle on $S$ defined by an extension (7.9.7), where the image of $F_i$ on $S'$ spans a Fano plane. We have $c_1(\mathcal{V}) = h_{10}, c_2(\mathcal{V}) = 3$. Let $s$ be a section of $\mathcal{V}$ with reduced zero scheme $Z$ of length 3. It defines an exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{V} \to \mathcal{I}_Z(H_{10}) \to 0.$$ 

Tensoring the exact sequence by $\mathcal{O}_S(K_S)$ and applying Corollary 7.8.4, we obtain an exact sequence

$$0 \to \mathcal{O}_S(K_S) \to \mathcal{V} \to \mathcal{I}_Z(H_{10} + K_S) \to 0.$$ 

We know that $h^0(\mathcal{V}) = 4$. Taking cohomology, we find that

$$h^0(\mathcal{I}_Z(H_{10} + K_S)) = h^0(\mathcal{V}) = 4.$$ 

This shows that the image $Z'$ of $Z$ in $S'$ is contained in a 3-dimensional linear system of hyperplane section. The base locus of this linear system is a line containing $Z'$. It is a trisecant of $S'$. One can reverse the argument to show that any trisecant line $t$ of $S'$ not passing through singular points of $S'$ defines a nonzero section $s$ with 0-dimensional scheme $Z(s)$. Taking $t$ to be a general line in a Fano plane we see that $Z(s)$ is reduced. Thus we obtain a three-dimensional family of lines parametrized by an open Zariski subset of the space $|\Gamma(\mathcal{V})| \cong \mathbb{P}^3$. \hfill \Box

Now we shall prove the converse.

**Theorem 7.10.6.** Assume $K_S \neq 0$. Let $H_{10}$ be a Fano polarization of an Enriques surface $S$ and $f : S \to S' \subset \mathbb{P}^5$ be a birational map given by the linear system $|H_{10}|$. Assume that there exists a line $t$ in $\mathbb{P}^5$ which intersects $S'$ at three nonsingular points and which does not lie in one of the Fano planes $\Lambda_i$. Then $H_{10} + K_S$ is a Reye polarization.
7.10. FANO-REYE POLARIZATIONS

Proof. The preimage in $S$ of $t \cap S'$ is a 0-cycle $Z$ of length 3. Consider the exact sequence

$$0 \to \mathcal{I}_Z(H_{10}) \to \mathcal{O}_S(H_{10}) \to \mathcal{O}_Z(H_{10}) \to 0.$$  

Since $|H^0(\mathcal{I}_Z(H_{10}))|$ is isomorphic to the web of hyperplanes in $\mathbb{P}^5$ containing the line $t$, and $h^0(\mathcal{O}_Z(H_{10})) = 3$, we obtain that $h^1(\mathcal{I}_Z(H_{10})) = 1$. By Serre Duality,  

$$\dim H^1(S, \mathcal{I}_Z(H_{10}) \otimes \omega_S^{-1}) = \dim \text{Ext}^1(\mathcal{I}_Z(H_{10}), \mathcal{O}_S),$$

hence there exists a non-split extension

$$0 \to \mathcal{O}_S \to \mathcal{V} \to \mathcal{I}_Z(H_{10} + K_S) \to 0. \quad (7.10.1)$$

Choose a hyperplane in $\mathbb{P}^5$ which contains $t$ and one of the planes $\Lambda_{-i}$. Then it defines an effective divisor $F_{-i} + A \sim H_{10}$, where $Z \subset A$. Adding to both sides $K_S$, we get a divisor $F_i + A \in |H_{10} + K_S|$. Let $s$ be the section of $\mathcal{I}_Z(H_{10} + K_S - F_i)$ defined by the divisor $A$. Tensoring $(7.10.1)$ by $\mathcal{O}_S(-F_i)$ and taking cohomology, we get an exact sequence

$$0 \to H^0(\mathcal{V} \otimes \mathcal{O}_S(-F_i)) \to H^0(\mathcal{I}_Z(H_{10} + K_S - F_i)) \to H^1(\mathcal{O}_S(-F_i)).$$

Since $h^1(F_i + K_S) = h^1(F_{-i}) = 0$, $H^0(\mathcal{V} \otimes \mathcal{O}_S(-F_i)) \neq 0$ and there is a nonzero section $s$ of $\mathcal{V}(-F_i)$ that extends s. Let

$$0 \to \mathcal{O}_S(F_i) \to \mathcal{V} \to \mathcal{F} \to 0$$

be the corresponding exact sequence. Suppose $\mathcal{O}_S(F_i)$ is not saturated in $\mathcal{V}$, i.e. $\mathcal{F}$ is not torsion-free. This implies that $\mathcal{V}$ contains an invertible sheaf isomorphic to $\mathcal{O}_S(F_i + C)$ for some effective divisor $C$ such that $\mathcal{O}_S(C)$ is isomorphic to a subsheaf of $\mathcal{F}$. We get an exact sequence

$$0 \to \mathcal{O}_S(F_i + C) \to \mathcal{V} \to \mathcal{I}_{Z'}(H_{10} + K_S - F_i - C) \to 0, \quad (7.10.2)$$

where $Z \subset Z'$ and $h^0(H_{10} + K_S - F_i - C) > 0$. Replacing $F_i$ with some other nef $F_j$ in the isotropic sequence, we obtain a section of $\mathcal{V}(-F_j)$. We have $h^0(\mathcal{V}(-F_j)) = h^0(H_{10} + K_S - F_j) = 4$. Since $h^0(F_i + C) \leq h^0(H_{10} + K_S - F_i) = 3$, we get $h^0(F_i + C - F_j) \leq 3$. Tensoring $(7.10.2)$ by $\mathcal{O}_S(-F_j)$ and taking cohomology, we obtain that $h^0(H_{10} + K_S - F_i - F_j - C) \neq 0$. Thus $C$ is a part of the unique divisor $F_{ij} \in |H_{10} + K_S - F_i - F_j|$. This implies

$$C \cdot H_{10} \leq C \cdot F_{ij} = 4, C \cdot F_k \leq F_{ij} \cdot F_k = 1, k = i, j. \quad (7.10.3)$$

Computing $c_2(\mathcal{V})$ using $(7.10.2)$, we find

$$c_2(\mathcal{V}) = 3 = -(F_i + C)^2 + H_{10} \cdot C + h^0(\mathcal{O}_{Z'}) \geq 3 - (F_i + C)^2 + H_{10} \cdot C.$$  

This gives $(F_i + C)^2 = 2F_i \cdot C + C^2 \geq H_{10} \cdot C$. Since $F_i \cdot C \leq 1$ and $C$ does not move, hence $C^2 \leq 0$, we obtain $H_{10} \cdot C \leq 2$. In particular, the image of $C$ on $S'$ is a line or a conic, or it is contained in the singular locus of $S'$. In any case, no part of $C$ has self-intersection equal to 0. This implies that $C^2 \leq -2$, and $H_{10} \cdot C \leq 0$. Since $H_{10}$ is nef, we obtain $H_{10} \cdot C = 0, C^2 = -2$. Thus $C$ is a nodal cycle and it is blown down to a singular point of $S'$. Since $C$ contains $Z$, and, by assumption, $Z$ does not contain singular points, we get a contradiction.

So, we have proved that the sheaf $\mathcal{F}$ is torsion-free, hence it is isomorphic to a sheaf of the form $\mathcal{I}_{Z'} \otimes \mathcal{O}_S(D)$ for some zero cycle $Z'$ and divisor $D$. Computing the Chern classes of the bundle $\mathcal{V}$, we find:

$$c_1(\mathcal{V}) = H_{10} + K_S = D + F_i, 3 = c_2(\mathcal{V}) = (H_{10} + K_S - F_i) \cdot F_i + h^0(\mathcal{O}_{Z'}) = 3 + h^0(\mathcal{O}_{Z'}).$$
This implies that
\[ Z' = \emptyset, \ D \sim H_{10} + K_S - F_i, \]
and we have an exact sequence:
\[ 0 \to \mathcal{O}_S(F_i) \to \mathcal{V} \to \mathcal{O}_S(H_{10} + K_S - F_i) \to 0. \] (7.10.4)

If this extension splits, every section of \( \mathcal{V} \) vanishes on a subset of \( F_i \). In particular, \( t \) is contained in the image of \( F_i \) on \( S' \), hence the trisecant line \( t \) lies in \( \Lambda_i \). By assumption, this does not happen. Hence (7.10.4) does not split. Therefore
\[ \text{Ext}^1(\mathcal{O}_S(H_{10} + K_S - F_i), \mathcal{O}_S(F_i)) \cong H^1(\mathcal{O}_S(2F_i - H_{10} - K_S)) \cong H^1(\mathcal{O}_S(H_{10} - 2F_i)) \neq \{0\}, \]
and, by Riemann-Roch, \( |H_{10} - 2F_i| \neq \emptyset \). Since \((H_{10} - 2F_i)^2 = -2, (H_{10} - 2F_i) \cdot H_{10} = 4\), it is easy to see that \( |H_{10} - 2F_i| = |h_S + K_S + F_i + F_{-i}| \) consists of a nodal cycle \( R \) with \( R \cdot E_i = 3 \). Now we can apply Theorem 7.9.1 to conclude the proof. \( \square \)

Let us study in more detail a Fano model \( S' \) of an Enriques surface \( S \). We choose a Fano polarization \( H_{10} \) and a supermarking \((F_1, \ldots, F_{10})\) such that \( 3H_{10} \sim F_1 + \cdots + F_{10} \). Consider the linear system \( |3H_{10}| \). By Riemann-Roch, \( \dim |3H_{10}| = 45 \). On the other hand, \( \dim |\mathcal{O}_{\mathbb{P}^5}(3)| = 55 \). Thus we see that \( S' \) is contained in the base locus of a linear system of cubic hypersurfaces of dimension \( \geq 9 \).

The proof of the following proposition can be found in [242].

**Proposition 7.10.7.** Let \( S \subset \mathbb{P}^5 \) be a Fano model of an Enriques surface with \( K_S \neq 0 \) corresponding to an ample Fano polarization \( H_{10} \). Then \( S \) is 3-normal, i.e. the restriction map \( |\mathcal{O}_{\mathbb{P}^5}(3)| \to |\mathcal{O}_S(3H_{10})| \) is surjective. The homogeneous ideal of \( S \) is generated by quadrics and cubics.

It follows from Proposition 7.10.3 that the homogeneous ideal of \( S \) is generated by cubics if the Fano polarization is not a Reye polarization. Thus \( S \) is the base locus scheme of a 9-dimensional linear system of cubic hypersurfaces in \( \mathbb{P}^5 \).

We can say a little more about the ideal of \( S \) when \( H_{10} \) is a Reye or a Cayley polarization.

Let \( W = |L| \) be a regular web of quadrics in \( \mathbb{P}^3 = |E| \) and let \( \text{Rey}(W) \) be the Reye congruence contained in the Grassmann variety \( G(2, E) \) of lines in \( |E| \). We first give a resolution of \( \text{Rey}(W) \) in \( G(2, E) \).

**Proposition 7.10.8.** The following sequence is a locally free resolution of the ideal sheaf \( \mathcal{I}_{\text{Rey}(W)} \) of the Reye congruence \( \text{Rey}(W) \) as a closed subvariety of \( G(2, E) \).
\[ 0 \to S^2(S_{G(2,E)})(-3) \to L^\vee \otimes \mathcal{O}_{G(2,E)}(-3) \to \mathcal{I}_{\text{Rey}(W)} \to 0. \]

**Proof.** Restricting quadrics from \( W = |L| \) to lines in \( \mathbb{P}^3 \) defines a surjective map of locally free sheaves \( L \otimes \mathcal{O}_{G(2,E)} \to S^2(S_{G(2,E)}^\vee) \). Dualizing, we get a linear map \( S^2(S_{G(2,E)}^\vee) \to L^\vee \otimes \mathcal{O}_{G(2,E)} \). A Reye line \( \ell \) is contained in a pencil of quadrics from \( W \). Since this pencil intersects any net of quadrics, we see that \( \ell \) is contained in the Montesano cubic complex of any net. In other words, the Reye congruence lies in the intersection of all cubic hypersurfaces defined by points in the dual
space $\tilde{W} = |L^\vee|$. This defines a linear map $L^\vee \otimes O_{G(2,E)}(3) \rightarrow I_{\text{Rey}(W)}(3)$ that gives an exact sequence

$$0 \rightarrow S^2(S_G(2,E))(-3) \rightarrow L^\vee \otimes O_{G(2,E)}(-3) \rightarrow I_{\text{Rey}(W)} \rightarrow 0.$$ 

By Proposition 7.10.7, the homogeneous ideal $I$ of $\text{Rey}(W)$ in $\mathbb{P}^5 = \mathbb{P}(\mathbb{A}^2 \oplus E^\vee)$ is generated by 10 cubics. We have a 6-dimensional linear space of cubics of the form $V(ql)$, where $q = 0$ is the quadric $G(2,E)$ and $l \in E^\vee$. Together with the 4-dimensional space of Montesano cubics they generate $I_3$. This shows that the map $O_{G(2,E)}(3) \rightarrow I_{\text{Rey}(W)}(3)$ is surjective and we get an exact sequence

$$0 \rightarrow S^2(S_G(2,E)) \rightarrow L^\vee \otimes O_{G(2,E)} \rightarrow I_{\text{Rey}(W)} \rightarrow 0.$$ 

It remains to tensor it with $O_{G(2,E)}(3)$. \hfill \Box

Now let us look at the Cayley model. Consider the variety of singular quadrics in $\mathbb{P}^3$. It is a discriminant quartic hypersurface $D$ in $\mathbb{P}^9$. The variety $D_3(2)$ of quadrics of corank 2 is known to be its singular locus and hence equal to the intersection of 10 cubic hypersurfaces defined by the partials of the discriminant quartic. The Cayley model is the intersection of $D_3(2)$ with a 5-dimensional subspace in $\mathbb{P}^9$. So it is contained in the base locus of a linear system of cubics of dimension $\leq 9$. Taking the minimal resolution of the ideal of $D_3(2)$ and restricting it to a transversal $\mathbb{P}^5$, we obtain the resolution of the Cayley model $S = \text{Cay}(W)$:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^5}(-5)^{\oplus 6} \rightarrow \mathcal{O}_{\mathbb{P}^5}(-4)^{\oplus 15} \rightarrow \mathcal{O}_{\mathbb{P}^5}(-3)^{\oplus 10} \rightarrow I_S \rightarrow 0. \quad (7.10.5)$$

It shows that $S$ is projectively normal and that the ten partials of the discriminant cubic generate the sheaf of ideals $I_S$ of $S$ in $\mathbb{P}^5$.

**Remark 7.10.9.** The Cayley model is scheme-theoretically defined by 6 cubics, the partials of the quartic symmetroid in $\mathbb{P}^5$. The 6 cubics generate a non-saturated ideal of $S(V^\vee)$, whose saturate is generated by the 10 partials of $D \subset \mathbb{P}^9$ restricted to $\mathbb{P}^5$.

Let $S$ be embedded in $\mathbb{P}^5 = |V|$ by an ample Fano polarization $H_{10}$ and let $(\Lambda_1, \ldots, \Lambda_{10})$ be the Fano planes defined by choice of a supermarking $(F_1, \ldots, F_{10})$. Fix a basis of $\bigwedge^6 V \cong \mathbb{k}$ and consider the wedge-product pairing

$$\omega : \bigwedge^3 V \times \bigwedge^3 V \rightarrow \bigwedge^6 V \cong \mathbb{k}, \quad (7.10.6)$$

Choose a representative $v_i \in \bigwedge^3 V$ of a Fano plane $\Lambda_i \in G(3,V)$. Since $\Lambda_i \cap \Lambda_j \neq \emptyset$, we obtain that the 3-vectors $(v_1, \ldots, v_{10})$ satisfy $v_i \wedge v_j = 0$. Suppose $K_S \neq 0$ and $H_{10}$ is not a Reye polarization. Then, we have an opposite supermarking defined by the Fano planes $(\Lambda_{-1}, \ldots, \Lambda_{-10})$. Let $(v_{-1}, \ldots, v_{-10})$ be the corresponding lifts to $\bigwedge^3 V$. We have $v_i \wedge v_{-j} = 0$, $i \neq j$, and $v_i \wedge v_{-i} \neq 0$. Suppose we have a linear dependence $\lambda_1 v_1 + \cdots + \lambda_{10} v_{10} = 0$. Taking the wedge-product with $v_{-i}$, we get $\lambda_i = 0$. Thus the 3-vectors $v_1, \ldots, v_{10}$ are linearly independent. They generate a maximal isotropic subspace (a Lagrangian subspace) of $\bigwedge^3 V$, where $\bigwedge^3 V$ is equipped with a structure of a symplectic space by means of the wedge-product pairing. Thus a supermarking of a non-Reye Fano polarization of an Enriques surface defines a Lagrangian subspace $A$ in the Plücker space of the Grassmannian $G(3,6)$.

Let $V = \mathbb{C}^6$ and let $A$ be a Lagrangian subspace of $\bigwedge^3 V$. To any $[x] = \mathbb{C}x \in \mathbb{P}(V)$, we can associate another Lagrangian subspace $x \wedge \bigwedge^2 V$ of $\bigwedge^3 V$. Following [200], one defines an $EPW$-
sextic hypersurface as the degeneracy locus of points \([x]\) for which these two Lagrangian subspaces are not in general position:

\[ X_A = \{ [x] \in \mathbb{P}(V) : (x \wedge \bigwedge^2 V) \cap A \neq 0 \}. \]

It is proven in [201] that \(X_A\) has a scheme structure of a sextic hypersurface in \(\mathbb{P}(V)\) unless it is equal to the whole space. For \(A\) generic, its singular locus is the set

\[ \text{Sing } X_A = \{ x \in X_A : \dim ((x \wedge \bigwedge^2 V) \cap A) \geq 2 \}. \]

It is a smooth surface \(S_A\) of general type of degree 40 with Hilbert polynomial

\[ 126P_0 - 120P_1 + 40P_2, \]

where \(P_n\) stands for the Hilbert polynomial of \(\mathbb{P}^n\). In [546] K. O’Grady proved that there exists a natural double cover

\[ \pi : \hat{X}_A \to X_A \]

ramified along \(S_A\) which is an irreducible symplectic 4-fold, deformation equivalent to the Hilbert square of a K3 surface.

**Remark 7.10.10.** Note that the exact sequence

\[ 0 \to \Omega^2_{|V|} \to \bigwedge^3 V \otimes \mathcal{O}_{|V|} \to \Omega^3_{|V|} \to 0 \]

allows one to identify the subspace \(v \wedge \bigwedge^2 V\) with the fiber of \(\Omega^2_{|V|}\) at the point \([v]\). Thus one can redefine the EPW sextic as

\[ X_A = \{ x \in \mathbb{P}^5 : \dim \Omega^2_{|V|}(x) \cap A \geq 1 \}. \]

Let

\[ X_A[k] := \{ x \in \mathbb{P}^5 : \dim \Omega^2_{|V|}(x) \cap A \geq k \}. \]

If \(A\) is not generic, we have the following proposition proved in [547].

**Proposition 7.10.11.** Assume that \(X_A\) is not the whole \(\mathbb{P}(V)\). Let \([v]\) \(\in X_A\). Then \(X_A\) is smooth at \([v]\) if and only if \([v]\) \(\not\in X_A[2]\) and \(A\) does not contain any decomposable form \(v \wedge \omega\). In other words, \(\text{Sing } X_A\) is the union of \(X_A[2]\) and the planes \(\mathbb{P}(W)\), where \(W\) varies through all 3-planes of \(V\) such that \(\bigwedge^3 W \subset A\).

**Proposition 7.10.12.** Let \(A\) be defined, as above, by a set of linearly independent Fano planes \(\Delta = (\Lambda_1, \ldots, \Lambda_{10})\). Then the singular locus of \(X_A\) is the union of the ten planes \(\Lambda_i\) from \(\Delta\) and the degree 40 surface \(Y := X_A[2]\).

We refer to [184] for the proof and many other geometric constructions related to this construction.

Finally we consider an ample Fano polarization \(H_{10}\) of an Enriques surface with \(K_S = 0\). We know that any \(\mu_2\)-surface that contains a nodal cycle \(R\) and a half-fiber \(F\) such that \(F \cdot R = 3\) defines a Reye polarization \(h = 2E + R\). Viewed as a smooth surface in \(G_1(\mathbb{P}^3)\) it is a congruence of bidegree \((7,3)\). Unfortunately, we do not know any geometric construction of such congruence. Recall that the construction from Section 7.8 of such congruence using webs of quadrics in characteristic 2 leads to classical Enriques surfaces. To give explicit examples, we use that the Cayley and
Reye polarizations coincide. Intersecting the variety $D_3(2)$ of reducible quadrics in $\mathbb{P}^3$ by a general 5-dimensional subspace $P$, we obtain an Enriques surface of degree 10 in $\mathbb{P}^5$. It is contained in a nonsingular quadric in $P$, the intersection of $P$ with the pfaffian hypersurface $D_3(1)$ of singular quadrics in $\mathbb{P}^3$. It has an obvious étale canonical cover $X \to S$, the restriction of the double cover $\mathbb{P}^5 \times \mathbb{P}^5 \to D_3(2)$.

We assume that $H_{10}$ is not a Reye polarization. As above we consider $S$ embedded in $\mathbb{P}^5 = |V|$. Since Pic($S$) = Num($S$), a maximal isotropic 10-sequence defined by $h_{10}$ defines, uniquely up to an order, a sequence $\Lambda = (\Lambda_1, \ldots, \Lambda_{10})$ of Fano planes. The wedge-product pairing (7.10.6) becomes a symmetric bilinear pairing. We equip $\wedge^3 V$ with a non-degenerate quadratic form defined, with respect to a basis $e_1, \ldots, e_{10}$ by

$$q(\sum x_{i_1i_2i_3}e_{i_1} \land e_{i_2} \land e_{i_3}) = \sum x_{i_1i_2i_3}x_{j_1j_2j_3},$$

where the summation is taken along the set of indices $i_1 < i_2 < i_3, j_1 < j_2 < j_3, i_1 < j_1, \{i_1, i_2, i_3\} \cap \{j_1, j_2, j_3\} = \emptyset$ (see [201, Lemma 8]).

Let $A$ be a maximal isotropic subspace of the quadratic form $q$, or, equivalently, a 9-dimensional subspace of $|\wedge^3 V|$ contained in the quadric $Q = V(q)$. We define $Z_A$ as in the definition of the EPW sextic from above.

The following beautiful result can be found in [201].

**Theorem 7.10.13.** Suppose $A$ is general enough. Then

$$Z_A := \{[v] \in |V| : \dim(v \land V) \cap A) \geq 3\}$$

is a smooth Fano model of a non-classical Enriques surface with symmetrically quasi-isomorphic resolutions

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-6) \to \mathcal{O}_{\mathbb{P}^5}^3 \to A^\vee \otimes \mathcal{O}_{\mathbb{P}^5}(-3) \to \mathcal{O}_{\mathbb{P}^5} \to \mathcal{O}_{Z_A} \quad (7.10.8)$$

There is also the converse.

**Theorem 7.10.14.** Assume $p = 2$. Let $S \subset \mathbb{P}^5 = |V|$ be a locally Gorenstein subscheme of dimension 2 and degree 10, with $\omega_S \cong \mathcal{O}_S$ and $h^i(\mathcal{O}_S) = 1$ for $i = 0, 1, 2$. Assume $S$ is linearly normal and not contained in a quadric. Then there exists a unique Lagrangian subspace $A \subset \wedge^3 V$ with respect to the quadratic form (7.10.7) such that $S = Z_A$ with locally free resolution (7.10.8).
Assuming that $S$ is unnodal, we find a degree 4 polarization $H$ and a genus one curve $F$ such that $H_{10} = H + F$ is an ample Fano polarization. Suppose $H_{10}$ is a Reye polarization. Then we may assume that $S$ lies on a smooth quadric and $F$ lies on a Fano plane $\Lambda$. It is easy to see that there exists a hyperplane that is tangent to the quadric along $\Lambda$. This implies that $|H_{10} - 2F| \neq \emptyset$. Let $\phi : X \to S$ be the K3-cover, then $\tilde{H} = \phi^*(H)$ is a polarization of degree 8 embedding $X$ into $\mathbb{P}^5$. The pre-image of $F$ is embedded in $\mathbb{P}^5$ as an irreducible curve $\tilde{F}$ of degree 6 and arithmetic genus one. It is known that such a curve is not contained in a hyperplane. Thus $|\tilde{h} - \tilde{F}| = \emptyset$, contradicting the assumption that $|H_{10} - 2F| \neq \emptyset$.

Bibliographical notes

General facts about the Grassmannians of lines can be found in [233], [282], and [180]. Unfortunately, no modern text-books that discuss the theory of line congruences have been written yet. We refer to classical monographs of C. Jessop [343] and R. Sturm [662] for rather complete expositions of the theory of complexes and congruences of lines in $\mathbb{P}^3$. Some modern survey on congruences of lines can be found in [15].

The Reye congruences were first discovered by G. Darboux [143] and were studied more extensively by T. Reye in [589, NeunzehnterVortrag]. In his terminology, a Reye line is a Haupstrahl of a Gebühr (web) of quadrics. It is curious that A. Cayley who introduced the Cayley quartic symmetroid surface in [113] did not notice the construction of a congruence of lines defined by the symmetroid. The first modern exposition of the theory of Reye congruences can be found in [133]. Some general facts about the hyperwebs of quadrics can be found in [180] or [674]. Cayley symmetroids were used in Artin-Mumford’s construction of the first examples of unirational but not rational threefolds [36]. The examples are birationally isomorphic to the double covers of $\mathbb{P}^3$ branched along a quartic symmetroid.

The extension of Reye’s construction to characteristic 2 seems to be new.

The proof of the fact that a smooth congruence of order 7 and class 3 is isomorphic to an Enriques surface can be found in [255]. Although the fact that a general, in sense of moduli, nodal Enriques surface is isomorphic to a Reye congruence follows from counting constants, the results from section 7.9 are much more explicit. In particular, they show that a general nodal surface in the sense of the definition from section 6.5 is isomorphic to a Reye congruence. Theorem 7.9.1, in slightly weaker form, was proven in the case $p \neq 2$, by Cossec [133, Theorem 3.3.1]. In this chapter we prove stronger forms of his result in arbitrary characteristic. The relationship between Reye congruences and exceptional rank 2 vector bundles on Enriques surfaces was studied in [171]. Here one also finds a characterization of a Reye polarization among Fano polarization in terms of the variety of trisecants of the embedded surface.

The relationship between Fano polarizations and Lagrangian subspaces in the Plücker space of planes in $\mathbb{P}^5$ traces back to a paper of D. Eisenbud, S. Popescu and C. Walters [201] who have found the equations of the Fano model of an Enriques surface $S$ with $K_S = 0$. Over fields of characteristic different from 2 this involves a choice of a supermarking and leads to a relationship between supermarked Enriques surfaces and irreducible symplectic 4-folds studied by K. O’Grady [201]. We refer to this relationship to [184].
Chapter 8

Automorphisms of Enriques surfaces

8.1 General facts

Let $X$ be a proper scheme over a field $k$. For any $k$-scheme $T$ we consider the group of automorphisms $\text{Aut}_T(X \times T)$ of $X_T := X \times_k T$, considered as a scheme over $T$. It is easy to see that $T \to \text{Aut}_T(X)$ defines a contravariant functor on the category of schemes over $k$. It is known that this functor is representable by a group scheme $\text{Aut}_{X/k}$ of locally finite type over $k$ [476]. The tangent space of $\text{Aut}_{X/k}$ is canonically identified with $\text{Aut}_{I}(X_I)$, where $I = \text{Spec} \ k[t]/(t^2)$ is the algebra of dual numbers. There is also a canonical isomorphism of vector spaces $H^0(X, \Theta_X) \cong \text{Aut}_{I}(X_I)$, where $\Theta$ is the tangent sheaf of $X$. When $k$ is perfect, the restriction of the functor $T \to \text{Aut}_T(X_T)$ to the category of reduced $k$-schemes is representable by $(\text{Aut}_{X/k})_{\text{red}}$. If $\text{char}(k) = 0$, all group schemes are reduced, so the automorphism scheme is reduced. Let $\text{Aut}_{0}X/k$ be the connected component of identity. The group $\text{Aut}_{X/k}/\text{Aut}_{0}X/k$ is at most countable, because $X$ can be defined over a countable field.

Theorem 8.1.1. Let $S$ be an Enriques surface $S$. Then $\dim \text{Aut}_{S/k}^0 = 0$. If $p \neq 2$, or $p = 2$ and $H^0(S, \Theta_{S/k}) = 0$ (e.g. $S$ is a $\mu_2$-surface), then $\text{Aut}_X/k$ is reduced and $\text{Aut}_{X/k}^0$ is trivial.

Proof. The second assertion follows immediately from the discussion in above. In any case we have $h^0(\Theta_{S/k}) \leq 1$. Suppose $H^0(S, \Theta_{S/k}) \neq 0$. If $\text{Aut}_{S/k}$ is reduced then $\text{Aut}_{S/k}^0$ is a one-dimensional connected algebraic group $G$ over $k$. There are three possibilities: $G = \mathbb{G}_m, \mathbb{G}_a$, or $G$ is an elliptic curve. A connected algebraic group acts trivially on the Néron-Severi group of $S$. Since $S$ has a non-trivial genus one fibration with some rational fibers, the group $G$ preserves the set of singular fibers, and being connected, preserves any singular fiber. If $G$ is an elliptic curve, then $G$ must fix any point $x$ on a singular fiber. Thus $G$ acts linearly on any $m_{S,x}^k/m_{S,x}^{k+1}$, and being complete, it acts trivially. This implies that $G$ acts trivially on the completion of the local ring $O_{S,x}$, hence on the ring itself, hence on its fraction ring, hence on $S$.

Suppose $G$ is a linear algebraic group acting on an irreducible algebraic variety $X$. Then, by Rosenlicht’s Theorem (see, for example, [176], Theorem 6.2), there exists a $G$-invariant open subset $U$ of $X$ such that the geometric quotient $U \to U/G$ exists and its fibers are orbits of $G$. In particular,
when $X$ is a surface, $U$, and hence $X$, must be a ruled surface of Kodaira dimension $-\infty$. Applying this to $S$, we find a contradiction.

Next we assume that $H^0(S, \Theta_{S/k}) \neq \{0\}$. This happens when $S$ is an $\alpha_2$-surface or an exceptional classical surface. In this case $\dim H^0(S, \Theta_{S/k}) = 1$. Then $S$ has an action of $\mu_2$ or $\alpha_2$ dependent on whether a non-zero vector field is of $\mu_2$ or $\alpha_2$. We refer to [209] and [472] for the study of this group.

**Theorem 8.1.2.** Suppose $S$ is a general $\alpha_2$-surface. Then $\text{Aut}_{S/k}^0 \cong \mu_2$.

Let $\partial$ be a nonzero regular vector field on $S$ and let $G$ be the corresponding group scheme of order 2. The quotient $S/G = S^0$ is a normal surface and the Frobenius morphism $S \to S^{(2)}$ factors through the quotient morphism $\tau : S \to S/G$. Since the Frobenius morphism is finite, the morphism $\tau' : S/G \to S^{(2)}$ is a finite inseparable morphism of degree 2.

**Proposition 8.1.3.** Suppose that $S$ is an $\alpha_2$-surface and $\partial$ has only isolated zeros. Then $(S^0)^{(1/2)} \to S$ is the canonical cover of $S$.

**Proof.** Let $Y = S^0$. Applying Proposition 0.3.14 to the morphism $\tau : S \to S^0$, we obtain that $\tau^*(\omega_Y) \cong \mathcal{O}_S$. On the other hand, the cover $\pi : Y \to S^{(2)} \cong S$ is an inseparable cover of degree 2 and $\omega_Y \cong \pi^*(\mathcal{L})$ for some invertible sheaf $\mathcal{L}$ which is a part of the data defining the cover. We have $(\pi \circ \tau)^*(\mathcal{L}) \cong \mathcal{O}_S$. Since $\text{Pic}(S)$ has no torsion, applying Proposition 0.2.10, we obtain that $\mathcal{L} \cong \mathcal{O}_S$. Thus, $\pi$ is defined by an element of $H^1(S, \alpha_2)$, and hence it is a principal $\alpha_2$-cover. Since it is unique, up to isomorphism, $\pi$ must be the canonical cover.

Let $\partial$ be a nonzero regular vector field on $S$ of multiplicative type. Then $S$ must be one of the exceptional Enriques surface which we discussed in Section 4.4. Suppose $S$ is not exceptional, then it is an $\alpha_2$-surface. Under the isomorphism $\Theta_{S/k} \to \Omega^1_{S/k}$, the scheme $Z(\partial)$ is isomorphic to the scheme of zeros $Z(\omega)$ of a regular 1-form $\omega$ on $S$. By Proposition 1.3.8, the divisorial part of $Z(\omega)$ is not reduced. Thus $\partial$ has only isolated simple zeros. Applying the previous Proposition, we obtain that the canonical cover has only ordinary double points. Since $c_2(S) = 12$, there are 12 of them, counting with multiplicities. Let us record this information.

**Corollary 8.1.4.** Let $S$ be an Enriques surface with $\text{Aut}_{S/k}^0 \cong \mu_2$. Suppose it is not an exceptional Enriques surface. Then $S$ is an $\alpha_2$-surface and its canonical cover is a normal surface with 12 ordinary double points, counting with multiplicities.

**Remark 8.1.5.** In the case of exceptional Enriques surfaces, the group $\text{Aut}_{S/k}^0$ has been recently computed by G. Martin.

1. If $S$ is of type $E_6$, then $\text{Aut}_{S/k}^0 \cong \mu_2$.
2. If $S$ is of type $E_7$, then $\text{Aut}_{S/k}^0 \cong \alpha_2$.
3. If $S$ is of type $E_8$, then $\text{Aut}_{S/k}^0 \cong \mu_2$ or $\alpha_4$. 
8.1. GENERAL FACTS

Since \( \text{Aut}_{S/k}(S)(k) \cong (\text{Aut}_{S/k}(S)/\text{Aut}_{S/k}^0(k))(k) = \text{Aut}(S) \) we will concentrate now on the structure of the automorphism group of \( S \). Since \( S \) is a minimal surface, this group is isomorphic to the group of birational automorphisms of \( S \). The standard tool to study the automorphism group is to look at its representations in the orthogonal groups of \( \text{Pic}(S) \) and \( \text{Num}(S) \)

\[
\rho : \text{Aut}(S) \rightarrow \text{O}(\text{Num}(S)), \quad (8.1.1) \\
\tilde{\rho} : \text{Aut}(S) \rightarrow \text{O}(\text{Pic}(S)) \quad (8.1.2)
\]

We denote by \( \text{Aut}(S)^\ast \) the image of \( \rho \). The natural questions are the questions about the structure of \( \text{Aut}(S)^\ast \) and the kernel of \( \rho \) and \( \tilde{\rho} \). We deal with the former question in the next section, and concentrate the rest of the chapter on the latter one.

We know that any Enriques surface has a genus one fibration \( f : S \rightarrow \mathbb{P}^1 \) whose jacobian fibration \( j : J \rightarrow \mathbb{P}^1 \) is a rational genus one surface. Let \( \text{MW}(j) \) be the Mordell-Weil group of \( j \). It acts by translations on the generic fiber of \( f \), and this action extends to a biregular action on \( S \) that preserves all fibers of the fibration.

We will be extending some results about finite automorphism groups of Enriques surfaces in characteristic 0 to the case of arbitrary characteristic of the ground field. The following result of J.-P. Serre [633] shows that nothing new appears if \( p \neq 2 \) and the order of the group is coprime to \( p \).

**Theorem 8.1.6.** Let \( W(k) \) be the ring of Witt vectors with algebraically closed residue field \( k \), and let \( X \) be a smooth projective variety over \( k \), and let \( G \) be a finite automorphism group of \( X \). Assume

- \( \#G \) is prime to \( \text{char}(k) \);
- \( H^2(X, \mathcal{O}_X) = 0 \);
- \( H^2(X, \Theta_X) = 0 \), where \( \Theta_X \) is the tangent sheaf of \( X \).

Then the pair \( (X, G) \) can be lifted to \( W(k) \), i.e. there exists a smooth projective scheme \( \mathcal{X} \rightarrow \text{Spec } W(k) \) with special fiber isomorphic to \( X \) and an action of \( G \) on \( \mathcal{X} \) over \( W(k) \) such that the induced action of \( G \) in \( X \) coincides with the action of \( G \) on \( X \).

If \( X = S \) is a classical Enriques surface, we have

\[
H^2(S, \Theta_S) \cong H^0(S, \Omega^1_S(K_S)) \cong H^0(S, \Theta_S)
\]

the theorem also applies to this case when \( p = 2 \) and \( H^0(S, \Theta_S) = 0 \).

The following theorem extends Theorem 5.5.1 to any characteristic \( p \neq 2 \).

**Theorem 8.1.7.** Assume that \( p \neq 2 \) or \( S \) is a \( \mu_2 \)-surface. Also assume that the K3 cover \( X \) of \( S \) is not supersingular. Then \( W_S^\text{mod} \times \text{Aut}(S) \) is a subgroup of finite index in \( W(\text{Num}(S)) \).

**Proof.** We follow an argument from [439] applied to K3 surfaces. First we assume that the canonical cover \( X \) of \( S \) is not supersingular. By Corollary 4.2 from loc. cit. a non-supersingular K3 surface \( X \) over an algebraically closed field \( k \) of positive characteristic \( p \) can be lifted to characteristic zero in such a way that the homomorphism of specialization of the Picard groups is an isomorphism. More precisely, this means that there exists a smooth projective morphism \( f : \mathcal{X} \rightarrow \text{Spec } A \), where
A is a complete discrete valuation ring of characteristic 0 with residue field $\mathbb{k}$ and the quotient field $K$, such that its fiber over the closed point is isomorphic to $X$. The geometric generic fiber $X_{\overline{K}}$ is a K3 surface over the algebraic closure $\overline{K}$ of $K$. Recall that the specialization homomorphism (see [269], Expose X, Appendix 7)

$$sp : \text{Pic}(\mathcal{X}_K) \to \text{Pic}(X)$$

(8.1.3)

preserves the numerical equivalence and the intersection form.

We take $X$ to be the canonical cover $\pi : X \to S$. The Picard lattice $\text{Pic}(X)$ contains the sublattice $\text{Pic}(X)_+ = \pi^*(\text{Pic}(S))$ of invariant divisor classes with respect to the Enriques involution $\tau$. It is isomorphic to $E_{10}(2)$. Its pre-image under the specialization map is a sublattice of $\text{Pic}(\mathcal{X}_K)$ isomorphic to $E_{10}(2)$. It is proven in [439, Corollary 2.4] that $sp$ preserves the nef and ample cones. Also, since the quotient of $X$ by $\tau$ is an Enriques surface, the involution $\tau$ acts non-identically on crystalline cohomology $H^2(X/W)$ and hence we can apply [339, Theorem 3.2] to obtain that $\tau$ can be lifted to an involution $\tau_K$ of $\mathcal{X}_K$ [339, Theorem 3.2]. It is obviously fixed-point-free since it is fixed-point-free on the closed fiber.

After embedding $\overline{K}$ into $\mathbb{C}$, we obtain that $\mathcal{X}_K$ is a complex K3 surface that admits an ample lattice $E_{10}(2)$ polarization. Replacing $K$ by its finite extension $L$ and $A$ by its normal closure $A_L$ in $L$, we may assume that $\tau_K$ is defined over $\overline{K}$. Let $X_K$ denote the generic fiber of $f$. Any non-trivial $g \in \text{Aut}(\mathcal{X}_K)$ extends to a birational automorphism of $\mathcal{X}$ over $A$. Since $p \neq 2$, we know that $H^0(S, \Theta_S) = \{0\}$. Under this assumption, it is proven in [439, Theorem 2.1] that the restriction homomorphism $\text{Aut}(\mathcal{X}) \to \text{Aut}(X)$ defines an injective homomorphism

$$\sigma : \text{Aut}(X_K) \to \text{Aut}(X)$$

which is $\sigma$-invariant with respect to the natural action on the Picard lattices. By definition of the lift, $\sigma(\tau_K) = \tau$. Passing to the quotient schemes and the sublattices of $\tau_K$- and $\tau$-invariant divisor classes (isomorphic to $E_{10}(2)$) we obtain a lift $\tilde{f} : S \to \text{Spec}(W)$ of our Enriques surface $S$. The group $\text{Aut}(S_K)$ (resp. $\text{Aut}(S)$) coincides with the centralizer subgroup of $\tau_K$ (resp. $\tau$). Hence we obtain an injective homomorphism

$$\tilde{\sigma} : \text{Aut}(S_K) \hookrightarrow \text{Aut}(S).$$

It is known that the specialization homomorphism (8.1.3) preserves the nef cones [439, Corollary 2.4] and hence defines an isomorphism of the reflection groups $W^n_{\mathcal{X}_K} \to W^n_X$. Since any reflection $s_r$ in the class of a $(-2)$-curve on $S$ (resp. $S_K$) lifts to the product of two reflections $s_{r'} \circ s_{r''}(r')$ (resp. $s_{r'} \circ s_{r''}(r')$) on its K3-cover, we obtain that the specialization isomorphism $\text{Num}(S_K) \to \text{Num}(S)$ defines an isomorphism $W^n_{S_K} \to W^n_S$. After we embed $\overline{K}$ into $\mathbb{C}$ and apply Theorem 5.5.1 we obtain that $W^n_{S_K} \rtimes \text{Aut}(S_K)$ is a subgroup of finite index in $W_{S_K}$, and therefore $W^n_S \rtimes \text{Aut}(S)$ is a subgroup of finite index in $W(\text{Num}(S))$.

The supersingular case was treated in [698]. It uses the Global Torelli Theorem for supersingular K3 surfaces Theorem 10.1.17. The following statement is deduced from this theorem in the same manner as its analogue in the case $\mathbb{k} = \mathbb{C}$ is deduced from the Global Torelli Theorem of Piyatetskii-Shapiro and Shafarevich (see [551, Corollary to Theorem II']).
**Theorem 8.1.8.** Assume \( p \neq 2 \). Let \( Y \) be a supersingular K3 surface over a field \( k \) of characteristic \( p \neq 2 \). Then \( W^\text{nod}_Y \rtimes \text{Aut}(Y) \) is a subgroup of finite index in \( W(\text{Num}(Y)) \).

**Corollary 8.1.9.** Assume \( p \neq 2 \) and the K3 cover \( X \) of \( S \) is a supersingular K3 surface. Then \( W^\text{nod}_S \rtimes \text{Aut}(S)^* \) is a group of finite index in \( O(\text{Num}(S)) \).

**Proof.** Let \( \pi : X \to S \) be the canonical cover and let \( \tau \) be the Enriques involution. Let \( \text{Nef}(X) \) be the nef cone of \( X \). By Theorem 8.1.8, the fundamental domain of the group \( \text{Aut}(X) \) acting in \( \text{Nef}(X) \) is a finite polyhedral cone. Let \( \text{Nef}(X,\tau) := \text{Nef}(X) \cap \text{Nef}(X)^\tau \). Let \( x = \pi^*(y) \), for any \((-2)\)-curve \( R \) on \( X \), we have \( \pi^*(y) \cdot R = y \cdot \pi_*(R) \). The image of \( R \) on \( S \) is either a \((-2)\)-curve on \( S \) or the class of some irreducible curve with non-negative self-intersection. Since we may assume that \( x = \pi^*(y) \in \text{Nef}(X,\tau) \) is the class of an irreducible curve on \( X \), \( y \) is the class of an irreducible curve on \( S \), and hence \( y \cdot \pi_*(R) \geq 0 \) always. This shows that \( \text{Nef}(X,\tau) = \pi^*(\text{Nef}(S)) \).

The centralizer subgroup \( \text{Cent}(\tau) \) in \( \text{Aut}(X) \) is isomorphic to \( \text{Aut}(S) \) and acts on \( \text{Nef}(S) \) with a finite polyhedral fundamental domain. Hence the action \( \text{Aut}(S) \) on \( \text{Nef}(S) \) has the same property. This property is equivalent to that in the assertion. \( \square \)

Arguing as in the case \( k = \mathbb{C} \) we extend Corollaries 5.5.2, 5.5.3, 5.5.4 to the case of Enriques surfaces over a field of arbitrary characteristic \( p \neq 2 \).

**Corollary 8.1.10.** Assume \( p \neq 2 \) or \( S \) is a \( \mu_2 \)-surface. Then the following assertions are true.

(i) \( \text{Aut}(S) \) is a finitely generated group.

(ii) In its action on the nef cone of \( S \), the group \( \text{Aut}(S) \) admits a rational convex polyhedral fundamental domain.

(iii) The group \( \text{Aut}(S) \) has only finitely many orbits on the sets of smooth rational curves, elliptic fibrations and nef divisor classes \( D \) with fixed \( D^2 > 0 \).

In the next theorem we do not need any assumption on the characteristic. Note that for the proof we use the classification of Enriques surfaces with finite automorphism group given in Sections 8.9 and 8.10.

**Theorem 8.1.11.** The following assertions are equivalent.

(i) The group \( \text{Aut}(S) \) is finite.

(ii) \( W^\text{nod}_S \) is a subgroup of finite index in \( W(\text{Num}(S)) \).

(iii) \( \text{Num}(S) \) contains a crystallographic basis formed by the classes of \((-2)\)-curves.

(iv) The set of smooth rational curves on \( S \) is finite.

(v) Any genus one fibration on \( S \) is an extremal fibration, i.e. the Mordell-Weil group of its jacobian fibration is finite.

(vi) There set of genus one fibrations on \( S \) is finite and all of them consists of extremal fibrations.
Proof. Let $\mathbb{H}^9$ be the hyperbolic space associated with $\text{Num}(S)_\mathbb{R}$. We know from Section 0.8 that $W(\text{Num}(S)) = W^\text{nod}_S \times A(P)$, where $P$ is a fundamental domain for the reflection group $W^\text{nod}_S$. By Proposition 0.8.17, the subgroup $W^\text{nod}_S$ of $W(\text{Num}(S))$ is of finite index if and only if the set of $(-2)$-curves on $S$ form a crystallographic basis. This proves the equivalence of assertions (ii) and (iii).

The assertion (iii) follows from (iv) immediately since the reflection polytope has finitely many faces and hence it is of finite volume. By definition, if (ii) is true, the reflection polytope has only finitely many faces, and hence there exists a finite crystallographic basis of $(-2)$-curves. Any other such curve intersects any element of this basis non-negatively, hence belongs to the fundamental polytope (in other words it belongs to the nef cone $\text{Nef}(S)$). But this implies that its self-intersection is non-negative, a contradiction.

By Vinberg’s Criterion 0.8.22, a surface with finitely many $(-2)$-curves contains only finitely many connected parabolic subdiagrams formed by $(-2)$-curves and all of them are parts of a parabolic subdiagram of maximal possible rank. This implies that such a surface contains only finitely many genus one fibrations and all these fibrations are extremal. Thus (iii) implies (vi).

So, we have proved that the assertions (ii), (iii), (iv) are equivalent. Suppose (ii) is true. The group $\text{Aut}(S)^*$ leaves invariant the set of $(-2)$-curves on $S$ and hence it is contained in $A(P)$. Since $W^\text{nod}_S$ is of finite index in $W(\text{Num}(S))$, we see that $\text{Aut}(S)^*$ is a finite group. Unfortunately, we do not know a direct proof of the converse. We used the Global Tofrelli theorem to prove the converse in the complex case and we can also proved this under the assumption that $p \neq 2$ in Corollary 8.1.9. We prove the converse by using the classification of surfaces with finite automorphism group which we discuss in Sections 8.9 and 8.10 reveals that the converse is true. It uses the obvious necessary condition (v) for finiteness of $\text{Aut}(S)$ and starting from this finds all possible surfaces which may contain a crystallographic basis of $(-2)$-curves. So, it proves that (v) implies (iii) and hence (i) implies (iii). Obviously, (v) implies (vi).

8.2 Numerically and cohomologically trivial automorphisms

An automorphism from $\text{Ker}(\bar{\rho})$ (resp. $\text{Ker}(\rho)$) is called cohomologically trivial (resp. numerically trivial). The reason for this terminology is the fact that $\text{NS}(S) \otimes \mathbb{Z}_\ell = H^2_\text{ét}(S, \mathbb{Z}_\ell)$ (resp. $\text{NS}(S) \otimes \mathbb{Z}_p = H^2_\text{fl}(S, \mathbb{Z}_p[1])$). We let

$$\text{Aut}_{\text{ct}}(S) = \text{Ker}(\rho), \quad \text{Aut}_{\text{nt}}(S) = \text{Ker}(\rho_n).$$

We start with the following.

Proposition 8.2.1. The groups $\text{Aut}_{\text{ct}}(S)$ and $\text{Aut}_{\text{nt}}(S)$ are finite groups.

Proof. This applies to any surface $S$ with trivial $\text{Aut}^0_{S/\mathbb{k}}(k)$ and trivial $\text{Pic}^0(S)$. In fact, the latter condition implies that $\text{NS}(S) = \text{Pic}(S)$ and $\text{Num}(S)$ is the quotient of $\text{NS}(S)$ by a finite group $A$. It follows from the theory of abelian groups that

$$O(\text{NS}(S)) \cong \text{Hom}(\text{Num}(S), \text{Tors}(\text{NS}(S))) \cong O(\text{Num}(S)).$$
This implies that
\[ \text{Aut}_{\text{int}}(S) / \text{Aut}_{\text{ct}}(S) \subset \text{Tors}(\text{NS}(S))^{\oplus \rho(S)}. \] (8.2.1)
So, it is enough to prove that \( \text{Aut}_{\text{ct}}(S) \) is a finite group. The group acts trivially on \( \text{Pic}(S) \), hence leaves invariant the isomorphism class of any very ample invertible sheaf \( \mathcal{L} \). For any \( g \in G \) let \( \alpha_g : \mathcal{L}^* \to \mathcal{L} \) be an isomorphism. Define a structure of a group on the set \( \tilde{G} \) of pairs \( (g, \alpha_g) \) by
\[ (g, \alpha_g) \circ (g', \alpha_{g'}) = (g \circ g', \alpha_{g'} \circ g^*(\alpha_g)). \]
The homomorphism \( (g, \alpha_g) \to g \) defines an isomorphism \( \tilde{G} \cong \mathbb{K}^* \times G \). The sheaf \( \mathcal{L} \) admits a natural \( \tilde{G} \)-linearization, and hence the group \( \tilde{G} \) acts linearly on the space \( H^0(S, \mathcal{L}) \) and the action defines a homomorphism \( G \to \text{Aut}(\mathbb{P}(H^0(S, \mathcal{L}))) \). The group of projective transformations of \( S \), embedded by \( |\mathcal{L}| \), is a linear algebraic group that has finitely many connected components. We know that the connected component of the identity is trivial. Thus the group \( G \) is finite.

In our case when \( S \) is an Enriques surface, we have \( \text{Tors}(\text{NS}(S)) \) is of order \( \leq 2 \), hence we get the following

**Corollary 8.2.2.** The quotient group \( \text{Aut}_{\text{int}}(S) / \text{Aut}_{\text{ct}}(S) \) is a 2-torsion group.

Let \( f : S \to D \) be a bielliptic map of degree 2 defined by a non-degenerate or degenerate \( U \)-pair as defined in 3.3. Assume that \( f \) is separable. Then the birational deck involution extends to a biregular involution of \( S \). We call it a **bielliptic involution**.

Assume that \( f \) is given by a non-degenerate \( U \)-pair \( (F_1, F_2) \) so that \( D \) is one of the three possible anti-canonical quartic del Pezzo surfaces
\[
\begin{align*}
D_1 : & x_0^2 + x_1x_2 = x_0^2 + x_3x_4 = 0, \\
D_2 : & x_0^2 + x_1x_2 = x_3x_1 + x_4(x_0 + x_2 + x_4) = 0, \\
D_3 : & x_0^2 + x_1x_2 = x_3x_1 + x_4(x_2 + x_4) = 0. 
\end{align*}
\]
that depend on whether \( S \) is a classical, or \( \mu_2 \)-, or an \( \alpha_2 \)-surface.

Recall that Proposition 0.6.26 and its proof give an explicit description of the connected component of the group of automorphisms of the surface \( D \). For convenience sake let us remind it.

- **Action of \( \text{Aut}(D_1)^0 \):**
  \[ [x_0, x_1, x_2, x_3, x_4] \mapsto [x_0, \lambda x_1, \lambda^{-1} x_2, \mu x_3, \mu^{-1} x_4] \]
- **Action of \( \text{Aut}(D_2)^0 \):**
  \[ [x_0, x_1, x_2, x_3, x_4] \mapsto [x_0 + \alpha x_1, x_1, \alpha^2 x_1 + x_2, \beta x_0 + (\alpha \beta + \alpha^2 \beta + \beta^2)x_1 + \beta x_2 + (\alpha + \beta^2)x_4, \beta x_1 + x_4] \]
- **Action of \( \text{Aut}(D_3)^0 \):**
  \[
  \begin{align*}
  [x_0, x_1, x_2, x_3, x_4] & \mapsto [x_0 + \alpha x_1, x_1, \alpha^2 x_1 + x_2, (\alpha^2 \beta + \beta^2)x_1 + \beta x_2 + x_3 + \alpha^2 x_4, \beta x_1 + x_4] \\
  [x_0, x_1, x_2, x_3, x_4] & \mapsto [x_0, \lambda^{-1} x_1, \lambda x_2, \lambda^3 x_3, \lambda x_4] 
  \end{align*}
\]
By definition, any \( g \in \text{Aut}_\text{int}(S) \) leaves the divisor classes \([2F_1], [2F_2]\) invariant, and hence leaves the linear system \([2F_1 + 2F_2]\) invariant. Thus it descends to a projective automorphism \( \bar{g} \) of the surface \( D \) leaving invariant the pencils of conics. Also, if \( g \in \text{Aut}_\text{ct}(S) \), then it preserves \([F_1], [F_2]\), hence \( \bar{g} \) leaves invariant the lines on \( D \) that give rise to the double fibers \( F_1, F_2 \), and hence belongs to the connected component \( \text{Aut}(D)^0 \).

The known information about the automorphism group of the surfaces \( D \) allows us to give a criterion for an automorphism to be a bielliptic involution.

**Proposition 8.2.3.** Let \((F_1, F_2)\) be a non-degenerate \(U\)-pair and let \( g \) be a non-trivial automorphism of \( S \). Assume that \( g \) preserves \( F_1, F_2 \) and a \((-2)\)-curve \( E \) with \( E \cdot F_1 = E \cdot F_2 = 0 \), which is not a component of one of the half-fibers \( F_1, F_2, F'_1, F'_2 \). If \( S \) is an \( \alpha_2 \)-surface, assume additionally that \( g \) has order \( 2^n \). Then, \( g \) is the bielliptic involution associated to the linear system \([2F_1 + 2F_2]\).

**Proof.** Let \( \phi : S \to D \) be a bielliptic map defined by the linear system \([2F_1 + 2F_2]\). Since \( g \) leaves \([2F_1 + 2F_2]\) invariant, it descends to an automorphism of \( \mathbb{P}^2 = [2F_1 + 2F_2]^* \) that leaves \( D \) invariant. Moreover, the induced automorphism preserves the lines on \( D \) by assumption. Recall that \( E \cdot F_1 = E \cdot F_2 = 0 \), hence \( \phi(E) \) is a point \( P \). Since \( E \) is not a component of one of the half-fibers, \( P \) does not lie on any of the lines of \( D \). If \( D = D_1 \), this means that \( P \) is not on the hypersurface \( x_0 = 0 \) and if \( D \in \{D_2, D_3\} \), it means that \( P \) is not on the hypersurface \( x_1 = 0 \).

If \( D = D_1 \), the \( x_0 \) coordinate \( x_0(P) \) of \( P \) is non-zero, the equations of \( D_1 \) from Corollary 0.6.11 show that all \( x_i(P) \) are non-zero. The explicit description of \( \text{Aut}(D_1)^0 \) shows that the group has no fixed points outside the union of the four lines. Therefore, \( g \) coincides with the covering involution of \( \phi \).

If \( D \in \{D_2, D_3\} \), we have \( x_1(P) \neq 0 \). Again, using the explicit description of \( \text{Aut}(D_2)^0 \), there is no automorphism of \( D_2 \) fixing \( P \) and preserving the lines except the identity. For \( D_3 \), we use the additional assumption to exclude the case that \( g \) acts on \( D_3 \) via \( \mathbb{G}_m \).

**Remark 8.2.4.** In fact, as we will see later, the failure of this criterion without the additional assumption in the \( \alpha_2 \)-case leads to the existence of cohomologically trivial automorphisms of odd order.

**Lemma 8.2.5.** Let \( \tau \) be the bielliptic involution associated to a linear system \([2F_1 + 2F_2]\). Suppose \( \tau \) is numerically trivial. Then, \( \text{Num}(S)_\mathbb{Q} \) is spanned by the numerical classes \([F_1], [F_2]\) and eight smooth rational curves that are contained in fibers of both \([2F_1]\) and \([2F_2]\).

**Proof.** We have a finite degree 2 cover \( S' = S - E \to D' = D - P \), where \( E \) is the union of \((-2)\)-curves blown down to a finite set of points \( P \) on \( D \). We have \( \text{Pic}(D')_\mathbb{Q} = \text{Pic}(D)_\mathbb{Q} \) and \( \text{Pic}(S')_\mathbb{Q} \) (the invariant part) = \( f^*(\text{Pic}(D')_\mathbb{Q}) \) is spanned by the restriction of \( F_1, F_2 \) to \( S' \). Since \( \text{Pic}(S) \) is spanned by \( \text{Pic}(S') \) and the classes of components of \( E \), we can write any invariant divisor class as a linear combination of \([F_1], [F_2]\) and invariant components of \( E \). In our case all divisors classes are invariant. Since \( \text{dim}(\text{Pic}(S)_\mathbb{Q}) - \text{dim}([F_1,F_2])_\mathbb{Q} = 8 \), \( E \) consists of eight \((-2)\)-curves.

**Corollary 8.2.6.** Let \((F_1, F_2)\) be a non-degenerate \(U\)-pair such that the bielliptic involution \( \tau \) associated to \([2F_1 + 2F_2]\) is numerically trivial. Then, \([2F_1]\) and \([2F_2]\) are extremal genus one pencils on \( S \).
Moreover, the following hold:

1. For every fiber $D$ of $|2F_1|$, all but one component $C$ of $D$ is contained in fibers of $|2F_2|$.

2. $C$ is a component in the fiber of multiplicity at most 2.

3. Neither $|2F_1|$ nor $|2F_2|$ have a singular fiber of multiplicative type with more than two components.

**Proof.** By the previous lemma, there are eight $(-2)$-curves contained in fibers of both $|2F_1|$ and $|2F_2|$. Since a fiber of $|2F_1|$ cannot contain a full fiber of $|2F_2|$, this implies $8 \leq \sum_{D \in |2F_1|} b_2(D) - 1 \leq 8$, where $b_2(D)$ is the number of irreducible component of $D$. Hence, by Shioda-Tate formula (4.3.2), $|2F_1|$ is extremal and so is $|2F_2|$. Moreover, if, for some fiber $D$ of $|2F_1|$, two components of $D$ are not contained in fibers of $|2F_2|$, then, by the same formula, $|2F_1|$ and $|2F_2|$ share less than eight $(-2)$-curves. This contradicts Lemma 8.2.5.

For (2), note that the remaining component $C$ of multiplicity $m$ in $D$ satisfies $2 = D \cdot F_2 = mC \cdot F_2$. Since $C \cdot F_2 > 0$, this yields (2).

As for (3), assume that $D$ is a singular fiber of multiplicative type with more than 2 components. Note that $C$ meets distinct points on distinct components of $D$. The connected divisor $D' = D - C$ satisfies $D' \cdot (2F_1 + 2F_2) = 0$, hence it is contained in the exceptional locus of the bielliptic map $\phi$. Since $\tau$ preserves the components of $D'$, $\phi(C)$ is an irreducible curve with a node. But $C$ is contained in the pencil of conics induced by $|2F_1|$. This is a contradiction. \qed

After these preliminary results we are ready to prove our main results. We start with cohomologically trivial automorphisms of even order.

**Theorem 8.2.7.** Let $S$ be an Enriques surface which is not extra-special.

1. If $S$ is classical or $\mu_2$-surface, then $|Aut_{ct}(S)| \leq 2$. If $S$ is also unnodal, then $Aut_{ct}(S) = \{1\}$.

2. If $S$ is an $\alpha_2$-surface, then the statements of (1) hold for the $2$-Sylow subgroup $G$ of $Aut_{ct}(S)$.

Moreover, if a non-trivial $g \in Aut_{ct}(S)$ (resp. $g \in G$) exists, then $g$ is a bielliptic involution.

**Proof.** Let $g \in Aut_{ct}(S)$ and assume that $g$ has order $2^n$ if $S$ is an $\alpha_2$-surface. We will show that there is a $U$-pair such that $g$ satisfies the conditions of Proposition 8.2.3 or all half-fibers are irreducible. Note that, by definition, $g$ fixes all half-fibers on $S$ and all $(-2)$-curves, so it suffices to find a $(-2)$-curve, which is contained in two simple fibers of genus one fibrations forming a $U$-pair.

Take a $c$-degenerate 10-sequence on $S$ with $c$ maximal. If $3 \leq c \leq 9$, then there is a $(-2)$-curve $R$ in this sequence such that $F \cdot R = 0$ for at least 3 half-fibers $F$ in the sequence. Now, Lemma 6.2.2 shows that $R$ is contained in a simple fiber of two pencils $|2F_1|$ and $|2F_2|$. By Corollary 8.2.3, $g$ is the bielliptic involution associated to $|2F_1 + 2F_2|$. In particular, $g$ is unique.
If \( c = 10 \), assume that one of the half-fibers, say \( F_1 \), is reducible. Then, by Lemma 6.2.2, for every \( F_i \) in the sequence, all but one component of \( F_1 \) is contained in simple fibers of \(|2F_1|\). Hence, we find some component \( R \) with \( R \cdot F_i = 0 \) for at least 3 half-fibers and the same argument as before applies.

If \(|F_i + F_j - F_k| \neq 0\) or \(|F_i + F_j - F_k + K_S| \neq 0\) for some half-fibers \( F_i, F_j, F_k \) that occurs in the sequence, then by Remark 7.9.6 after Lemma 7.9.5, there is an effective divisor \( D \) with \( D \cdot F_i = D \cdot F_j = 0 \) and \( D^2 = -2 \). Since \( F_i \) and \( F_j \) can be assumed to be irreducible, \( D \) contains a \((-2)\)-curve which is contained in a simple fiber of both \(|2F_i|\) and \(|2F_j|\). Again, Corollary 8.2.3 applies.

Therefore, we can assume that all half-fibers are irreducible and \( F_i \cap F_j \cap F_k = \emptyset \) by Lemma 7.9.5. This is immediate if \( S \) is unnodal. Then, \( g \) fixes all \( F_i \) pointwisely. This could happen only if \( p = 2 \) since the locus of fixed points is smooth if \( p \neq 2 \). But in the case \( p = 2 \), the generic fiber of the genus one fibration defined by the pencil \(|2F_1|\) has four fixed points, two coming from \( F_2 \) and two coming from \( F_3 \). It follows from the description of an automorphism group of an elliptic curve or a cuspidal curve that it does not happen (see the proof of Proposition 4.4.7).

In the case of classical Enriques surface in characteristic 2, we can say more, using the classification of Enriques surfaces with finite automorphism group which we will give later in this Chapter.

In [363], the cohomologically trivial and numerically trivial automorphism groups of extra-special surfaces have been calculated. For their examples, the groups are given in Table 8.1 (see also Tables 8.11, 8.12).

<table>
<thead>
<tr>
<th>Type</th>
<th>( \text{Aut}_{ct}(S) )</th>
<th>( \text{Aut}_{nt}(S) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>classical of type ( E_8 )</td>
<td>{1}</td>
<td>{1}</td>
</tr>
<tr>
<td>( \alpha_2 )-surface of type ( E_8 )</td>
<td>( \mathbb{Z}/11\mathbb{Z} )</td>
<td>( \mathbb{Z}/11\mathbb{Z} )</td>
</tr>
<tr>
<td>classical of type ( D_8 )</td>
<td>( \mathbb{Z}/2\mathbb{Z} )</td>
<td>( \mathbb{Z}/2\mathbb{Z} )</td>
</tr>
<tr>
<td>( \alpha_2 )-surface of type ( D_8 )</td>
<td>( Q_8 )</td>
<td>( Q_8 )</td>
</tr>
<tr>
<td>classical of type ( E_7 )</td>
<td>{1}</td>
<td>( \mathbb{Z}/2\mathbb{Z} )</td>
</tr>
</tbody>
</table>

Table 8.1: Numerically trivial automorphisms of extra-special surfaces

**Corollary 8.2.8.** Let \( S \) be a classical Enriques surface in characteristic 2 which is not \( E_8 \)-extra-special. Then, \( \text{Aut}_{ct}(S) \cong \mathbb{Z}/2\mathbb{Z} \) if and only if \( S \) is \( D_8 \)-extra-special.

**Proof.** Let \( F_1 \) be a half-fiber on \( S \). By Theorem 6.1.13, we can extend \( F_1 \) to a non-degenerate 2-sequence. Assume that there exists a non-trivial \( g \in \text{Aut}_{ct}(S) \). Then, \( g \) acts on \( D_1 \) via its action on \(|2F_1 + 2F_2|\). By our description of the automorphism group \( \text{Aut}(D_1)^0 \), \( g \) acts via \( \mathbb{G}_m^2 \) on \( D_1 \). But \( g \) has order 2 by Theorem 8.2.7, hence it acts trivially on \( D_1 \). Therefore, \( g \) is the covering involution of the bielliptic map and by Corollary 8.2.6, \(|2F_1|\) is extremal. Therefore, every genus one fibration on \( S \) is extremal. In particular, by [363] Section 4, \( S \) has finite automorphism group. The groups \( \text{Aut}_{ct}(S) \) of these surfaces have been calculated in [363] and the only surfaces for which
the calculation of the groups depends on the specific example given in [363] are the ones of type \( \tilde{D}_8 \) and \( \tilde{D}_4 + \tilde{D}_4 \) (see Table 8.1 and Remark 8.2.19). In the latter case, there is a \( U \)-pair of fibrations with simple \( \tilde{D}_8 \) fibers, which share only 7 components. By Corollary 8.2.6, the corresponding bielliptic involution is not numerically trivial. Therefore, the calculation of the groups in [363] shows that the \( D_8 \)-extra-special surface is the only classical Enriques surface which is not \( E_8 \)-extra-special and has a non-trivial cohomologically trivial automorphism.

Before we start with the treatment of cohomologically trivial automorphisms of odd order of \( \alpha_2 \)-Enriques surfaces, let us collect the known examples. These surfaces have finite automorphism groups and a detailed study can be found in [363]. In Table 8.2, we give the group of cohomologically trivial automorphisms of these examples. Again, it is not known whether there are more examples of these surfaces than the ones given in [363]. We will give examples later (see Tables 8.11, 8.12).

<table>
<thead>
<tr>
<th>Type</th>
<th>( \text{Aut}_{\text{ct}}(S) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_8 )</td>
<td>( \mathbb{Z}/11\mathbb{Z} )</td>
</tr>
<tr>
<td>( E_7 )</td>
<td>( \mathbb{Z}/7\mathbb{Z} ) or ( {1} )</td>
</tr>
<tr>
<td>( E_6 )</td>
<td>( \mathbb{Z}/5\mathbb{Z} )</td>
</tr>
</tbody>
</table>

Table 8.2: Examples of cohomologically trivial automorphisms of odd order

The surface of type \( \tilde{E}_6 \) is the exceptional Enriques surface of type \( \tilde{E}_6 \) which we discussed in Section 4.10. The dual graph of \((-2)\)-curves on this surface is given in (6.2.17).

**Lemma 8.2.9.** Let \( S \) be an \( \alpha_2 \)-Enriques surface which is not \( E_8 \)-extra-special and let \( G \subseteq \text{Aut}_{\text{ct}}(S) \) be a non-trivial subgroup of odd order. Then, \( G \) is cyclic and acts non-trivially on the base of every genus one fibration of \( S \).

**Proof.** Take any half-fiber \( F_1 \) and extend it to a non-degenerate 2-sequence \((F_1, F_2)\) on \( S \). Since \( G \) has odd order, it acts on \( D_3 \) via a finite subgroup of \( \mathbb{G}_m \), hence \( G \) is cyclic. By explicit computation of \( \text{Aut}(D_3)^0 \), we know that a generator \( g \) of \( G \) acts on the image \( D_3 \) of the bielliptic map as

\[
(x_0 : x_1 : x_2 : x_3 : x_4) \mapsto (x_0 : \lambda^{-1}x_1 : \lambda x_2 : \lambda^3 x_3, \lambda x_4).
\]

The pencils of conics that give rise to our pencils are given by the equations

\[
ax_3 + b(ex_0 + x_2 + x_4) = ax_4 + bx_1 = 0, \quad a(ex_0 + x_2 + x_4) + bx_1 = ax_3 + bx_4 = 0. \tag{8.2.2}
\]

Such an automorphism acts non-trivially on these pencils of conics, hence \( g \) acts non-trivially on \( |2F_1| \).

**Lemma 8.2.10.** Let \( F \) be a fiber of a genus one fibration and let \( g \) be a tame automorphism of finite order that fixes the irreducible components of \( F \). Then, the Lefschetz fixed-point formula

\[
e(F^g) = \sum_{i=0}^{2} (-1)^i \text{Tr}(g^*|H^i_{\text{et}}(F, \mathbb{Q}_l))). \tag{8.2.3}
\]
holds for $F$. If $F$ is reducible and not of type $\tilde{A}_1$, then $e(F^g) = e(F)$. If $F$ is of type $\tilde{A}_1$, then $e(F^g) = e(F) = 2$ or $e(F^g) = 4$. The latter case can only occur if $g$ has even order.

Proof. In the case the order is equal to 2, this is proven in [183] by a case-by-case direct verification. The proof uses only the fact that a tame non-trivial automorphism of finite order of $\mathbb{P}^1$ has two fixed points. Also note that the verification in case $F$ is of type $\tilde{A}_1$ and $g$ interchanges the two singular points of $F$ was missed, but it still agrees with the Lefschetz fixed-point formula.

Proposition 8.2.11. Let $g \in \text{Aut}_{ct}(S)$ be an automorphism of odd order. Then, every genus one pencil $|D|$ of $S$ has one of the following combinations of singular fibers

$$D_4 + D_4, \; D_8 + A_6^*, \; E_6 + A_2^*, \; E_7 + A_1^*, \; E_8 + A_0^*, \; A_8 + A_0 + A_0 + A_0, \; D_7, \; E_7 \quad (8.2.4)$$

The last three configurations can only occur if $g$ has order 3.

Proof. The claim is clear if $S$ is $E_8$-extra-special, hence we can apply Lemma 8.2.9 and find that $g$ acts non-trivially on all genus one pencils. Since the order of $g$ is odd, it fixes two members $F_1, F_2$ of the pencil, one of which is a double fiber. Since all other fibers are moved, the set of fixed points $S^g$ is contained in $F_1 \cup F_2$. Applying the Lefschetz fixed-point formula, we obtain

$$e(S) = 12 = e(S^g) = e(F_1^g) + e(F_2^g), \quad (8.2.5)$$

where $e(\cdot)$ denotes the $l$-adic topological Euler-Poincaré characteristic.

If one of the fibers, say $F_1$ is smooth, then, since $g$ has odd order and $e(F_2^g) \leq 10$, $g$ acts as an automorphism of order 3 on $F_1$. It is known that an automorphism of order 3 of an elliptic curve has three fixed points. Therefore, $F_2$ is of type $\tilde{A}_8, \tilde{D}_7$ or $\tilde{E}_7$ and $g$ has order 3. By [417], we get the last three configurations of the Proposition.

If both fibers or the corresponding half-fibers are singular curves, then $e(F_1) = e(F_2^g)$. Indeed, for irreducible and singular curves, this follows from $e(F_2^g) \leq 10$ and for reducible fibers, this is Lemma 8.2.10 for automorphisms of odd order. The formula for the Euler-Poincaré characteristic of an elliptic surface from [134], Proposition 5.1.6 implies that $F_1$ and $F_2$ are the only singular fibers of $|D|$. In this case, denoting the number of irreducible components of $F_i$ by $m_i$, we have $m_1 + m_2 \geq 8$, hence $|2F|$ is extremal and both fibers are of additive type. The classification of singular fibers of extremal rational genus one fibrations is known [417], [418], [329]. Since the types of singular fibers of a genus one fibration and of its Jacobian fibration are the same, it is straightforward to check that the list given in the Proposition is complete.

Corollary 8.2.12. If $S$ admits an automorphism $g \in \text{Aut}_{ct}(S)$ of odd order at least 5, then $S$ is one of the surfaces in Table 8.2.

Proof. By Proposition 8.2.11, every genus one fibration on $S$ is extremal. It is shown in [363] Section 4, that such an Enriques surface has finite automorphism group. Using the list of Proposition 8.2.11, the claim follows from the classification of $\alpha_2$-Enriques surfaces with finite automorphism group.

Proposition 8.2.13. Assume that $S$ is not one of the surfaces in Table 8.2. If $S$ admits an automorphism $g \in \text{Aut}_{ct}(S)$ of order 3, then $S$ contains the following diagram of $(-2)$-curves
In this case, \( \text{Aut}_\text{ct}(S) = \mathbb{Z}/3\mathbb{Z} \).

**Proof.** If every special genus one fibration on \( S \) is extremal, then \( S \) has finite automorphism group by [363] Section 4. Therefore, we observe that, by Proposition 8.2.11, \( S \) has to admit a special genus one fibration with special bisection \( N \) such that \( g \) fixes two fibers \( F_1 \) and \( F_2 \), where \( F_1 \) is a smooth supersingular elliptic curve and \( F_2 \) is of type \( \tilde{E}_7 \) or \( \tilde{D}_7 \). If \( F_1 \) is a simple fiber, then \( N \) meets two distinct points of \( F_1 \), since \( g \) does not fix the tangent line at any point of \( F_1 \). But then, \( g \) fixes three points on \( N \), hence it fixes \( N \) pointwise, which contradicts Corollary 8.2.9.

Therefore, \( F_1 \) is a double fiber and an argument similar to the above also shows that \( N \) meets a component of multiplicity 2 of \( F_2 \). Now, depending on the intersection behaviour of \( N \) with \( F_2 \), we see that \( N \) and components of \( F_2 \) form a half-fiber of type \( \tilde{D}_n \) or \( \tilde{E}_6 \) of some other genus one fibration. Using the list of Proposition 8.2.11, we conclude that \( F_2 \) is of type \( \tilde{D}_7 \) and \( N \) intersects \( F_2 \) as follows:

The five leftmost vertices form a diagram of type \( \tilde{D}_4 \). By Proposition 8.2.11, this diagram is a half-fiber of a fibration with singular fibers \( \tilde{D}_4 + \tilde{D}_4 \). Adding the second fiber to the diagram, we arrive at the diagram of the Proposition.

Now, observe that the fibration we started with has three \((-2)\)-curves as bisections. They are the curves \( N, N_1, N_2 \) in the diagram from the assertion of the proposition. All of them are fixed pointwise by any cohomologically trivial automorphism of order 2, since such an automorphism fixes their intersection with \( F_1 \) and \( F_2 \). The known structure of the set of fixed points on an elliptic curve shows that no such automorphism can exist. Since no cohomologically trivial automorphisms of higher order can occur on \( S \) by Corollary 8.2.12, we obtain \( \text{Aut}_\text{ct}(S) = \mathbb{Z}/3\mathbb{Z} \). \( \square \)

**Remark 8.2.14.** In fact, one can show that the only genus one fibrations on the \( \alpha_2 \)-Enriques surface of Proposition 8.2.13 are quasi-elliptic fibrations with singular fibers of types \( \tilde{D}_4 + \tilde{D}_4 \) or elliptic fibrations with a unique singular fiber of type \( \tilde{D}_7 \).

**Theorem 8.2.15.** Assume that the automorphism groups of surfaces of type \( \tilde{E}_8, \tilde{D}_8, \tilde{E}_7 \) and \( \tilde{E}_6 \), are as in Table 8.1 and Table 8.2. Then, for any \( \alpha_2 \)-Enriques surface \( S \) in characteristic 2, we have \( \text{Aut}_\text{ct}(S) \in \{ 1, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/5\mathbb{Z}, \mathbb{Z}/7\mathbb{Z}, \mathbb{Z}/11\mathbb{Z}, \mathbb{Q}_8 \} \).

Finally let us investigate numerically trivial automorphisms. If \( K_S = 0, \text{Aut}_\text{nt}(S) = \text{Aut}_\text{ct}(S) \), so we only have to treat the case that \( K_S \neq 0 \), i.e. \( S \) is classical.
By definition, any \( g \in \text{Aut}_{nt}(S) \) leaves invariant any genus one fibration, however, it may act non-trivially on its base, or equivalently, it may act non-trivially on the corresponding pencil \([D]\). Also, by definition, any \( g \in \text{Aut}_{ct}(S) \) fixes the half-fibers of a genus one fibration (their difference in \( \text{Pic}(S) \) is equal to \( K_S \)). The following lemma proves the converse.

**Lemma 8.2.16.** A numerically trivial automorphism \( g \) that fixes all half-fibers on \( S \) is cohomologically trivial.

**Proof.** Since \( g \) is numerically trivial, it fixes any smooth rational curve, because they are the unique representatives in \( \text{Pic}(S) \) of their classes in \( \text{Num}(S) \). By assumption, it fixes the linear equivalence class of all irreducible genus one curves. Applying Enriques Reducibility Lemma from [134], Corollary 3.2.3 we obtain that \( g \) fixes the linear equivalence classes of all curves on \( S \). □

**Lemma 8.2.17.** Let \( G \) be a finite, tame group of automorphisms of an irreducible curve \( C \) fixing a nonsingular point \( x \). Then, \( G \) is cyclic.

**Proof.** Since \( G \) is finite and tame, one can linearize the action in the formal neighborhood of the point \( x \). It follows that the action of \( G \) on the tangent space of \( C \) at \( x \) is faithful. Since \( x \) is nonsingular, the tangent space is one-dimensional and therefore the group is cyclic. □

**Theorem 8.2.18.** Let \( S \) be an Enriques surface and assume that \( p \neq 2 \). Then, \( \text{Aut}_{nt}(S) \cong \mathbb{Z}/2^a\mathbb{Z} \) with \( a \leq 2 \). Moreover, if \( S \) is unnodal, then \( \text{Aut}_{nt}(S) = \{1\} \).

**Proof.** By Theorem 8.2.7 and Lemma 8.2.16, any \( g \in \text{Aut}_{nt}(S) \) has order 2 or 4, so it suffices to show that \( \text{Aut}_{nt}(S) \) is cyclic. Since \( \text{Aut}_{nt}(S) \) is tame, every numerically trivial automorphism has smooth fixed locus.

Assume that there is some \( g \in \text{Aut}_{nt}(S) \setminus \text{Aut}_{ct}(S) \). Then, \( g \) switches the half-fibers of some elliptic fibration \([2F_1]\) on \( S \) by Lemma 8.2.16. The argument with the Euler-Poincaré characteristics from the proof of Proposition 8.2.11 applies and shows that one of the two fibers \( F', F'' \) of \([2F_1]\) fixed by \( g \), say \( F' \), has at least 5 components. Hence, if \( S \) is unnodal, then \( \text{Aut}_{nt}(S) = \{1\} \) by Theorem 8.2.7.

If \( F' \) is additive, then it has some component \( R \), which is fixed pointwise by \( \text{Aut}_{nt}(S) \), because it is adjacent to at least three other components. Since the fixed loci are smooth, any automorphism fixing a \((-2)\)-curve adjacent to \( R \) is trivial. Hence, the claim follows from Lemma 8.2.17.

If \( F' \) is multiplicative, the fixed point formula shows that \( F' \) is of type \( \tilde{A}_7 \) and \( g \) has four fixed points on \( F'' \). Extend \( F_1 \) to a non-degenerate \( U_{[2]} \)-sequence \((F_1, F_2)\). Since \( F'' \cdot F_2 = 2 \), \( F' \) contains a cycle of 3 \((-2)\)-curves contained in a fiber \( D \) of \([2F_2]\). Now, as in the additive case, we find a \((-2)\)-curve, which is fixed pointwise by \( \text{Aut}_{nt}(S) \). Indeed, if \( D \) is additive, we use the same argument as before and if \( D \) is multiplicative, then some component of \( D \) meets a component of \( F' \) exactly once in a nonsingular point of \( F' \). This component is fixed pointwise by \( \text{Aut}_{nt}(S) \). □

**Remark 8.2.19.** The previous Theorem is not true if \( p = 2 \). Indeed, there is an Enriques surface \( S \) of type \( \tilde{D}_4 + \tilde{D}_4 \) with the dual graph of \((-2)\) curves.
that satisfies $\text{Aut}_{\text{int}}(S) = (\mathbb{Z}/2\mathbb{Z})^2$ (see [363]). Moreover, we have seen in the proof of Corollary 8.2.8 that $\text{Aut}_{\text{ct}}(S) = \{1\}$.

If $p = 2$, even though we still have the same bound on the size of $\text{Aut}_{\text{int}}(S)$, the cyclic group of order 4 cannot occur.

**Theorem 8.2.20.** Let $S$ be a classical Enriques surface in characteristic 2 which is not $E_8$-extra-special. Then, $\text{Aut}_{\text{int}}(S) \cong (\mathbb{Z}/2\mathbb{Z})^b$ with $b \leq 2$.

**Proof.** By Corollary 8.2.8, $\text{Aut}_{\text{ct}}(S) \neq \{1\}$ if and only if $S$ is $D_8$-extra-special and for such a surface we have $\text{Aut}_{\text{int}}(S) = \text{Aut}_{\text{ct}}(S) = \mathbb{Z}/2\mathbb{Z}$. Hence, we can assume $\text{Aut}_{\text{ct}}(S) = \{1\}$. By Lemma 8.2.16, we have $\text{Aut}_{\text{int}}(S) = (\mathbb{Z}/2\mathbb{Z})^b$ and we have to show $b \leq 2$. Suppose that $b \geq 3$ and take some half-fiber $F_1$. By Theorem 6.1.13, we can extend $F_1$ to a non-degenerate 2-sequence $(F_1, F_2)$. Since $|\text{Aut}_{\text{int}}(S)| > 4$, there is some numerically trivial involution $g$ that preserves $F_1$ and $F_2$. Using the structure of $\text{Aut}(D_1)^0$, we find that such an automorphism acts trivially on $D_1$, hence it has to coincide with the bielliptic involution associated to $|2F_1 + 2F_2|$. Both fibrations have a unique reducible fiber $F$ (resp. $F'$) which has to be simple, since there is some numerically trivial involution which does not preserve $F_i$. By Corollary 8.2.6, $F$ and $F'$ are singular fibers of additive type and share 8 components. This is only possible if they are of type $\widetilde{D}_8$ or $\widetilde{E}_8$. Note that $F : F' = 4$ is impossible if both of them are of type $\widetilde{D}_8$. In the remaining cases, it is easy to check that the surface is $D_8$-extra-special. We have already treated this surface. \[\square\]

**Theorem 8.2.21.** Assume $p \neq 2$. Let $S$ be an Enriques surface such that $\text{Aut}_{\text{int}}(S) \neq \{1\}$. Then $S$ contains one of the following three configurations of $(-2)$-curves with the dual intersection graphs as follows.

\[\begin{align*}
(a) & \quad (b) & \quad (c)
\end{align*}\]

**Proof.** By Theorem 8.2.18, $\text{Aut}(S)$ contains a numerically trivial involution. In Section 8.7 we will classify all involutions on $S$ and their action on $\text{Num}(S)$. We prove that any involution is a bielliptic involution and classify possible actions by looking at the branch curve $W$ of the bielliptic involution. The numerically trivial involutions are listed in the first three rows of Table 8.6. It follows from Table 8.7 that $S^g$ is equal to the union of four isolated points, four $(-2)$-curves and maybe one elliptic curve.
There are different bielliptic maps defining the same involution, so their branch curves \( W \) may be different.

We start with Type 1 which can be obtained by taking \( W = W_0 \cup \ell \), where \( W_0 \) is an elliptic plane quintic curve with tacnodes at \( p_2, p_4 \) that passes through \( p_1 \) and has an additional cusp \( q \). The line \( \ell \) is the cuspidal tangent line passing through the point \( p_1 \). This is case (10) in Table 8.7. The pre-image of \( \ell \) is a fiber \( F_7 \) of the elliptic fibration \( |2F_1| \). The proper transform of \( \ell \) is its end component that enters with multiplicity 2. Together with the proper transform of the conic from the pencil \( |2e_0 - e_2 - \cdots - e_5| \) passing though \( q \) defines our diagram (a).

To realize Type 2 case, we take \( W \) to be the union of four conics, two from each pencil \( |2e_0 - e_2 - e_3 - e_4 - e_5| \) and \( |e_0 - e_1| \). The singular locus of \( W \) consists of 8 isolated ordinary double points. They define eight disjoint \((-2)\)-curves on \( S \). This is case (30) from Table 8.7. The proper transform of each conic gives a fiber of the elliptic fibration \( |2F_1| \) of type \( D_4 \). This gives us picture (b) of curves on \( S \) from the assertion of the theorem.

Finally to realize Type 3 of a cohomologically trivial involution from Table 8.6, we use the double plane construction from case (14) of Table 8.7. The branch curve \( W \) is an irreducible rational sextic with a singular point \( q \) of type \( e_7 \) and a singular point \( q' \) of type \( a_1 \). The pre-image of the line from the pencil \( |e_0 - e_1| \) passing through \( q \) gives a fiber \( F \) of type \( E_7 \) of the elliptic fibration \( |2F_1| \). The proper transform of \( \ell \) intersects the component \( e_2 \) (we use the pictures from Lemma 8.7.11). The proper transform \( \tilde{W} \) of \( W \) intersects the fiber at the components \( e_7 \) and \( e_1 \) and intersects \( \ell \) at one point. The exceptional curve \( R \) over \( q' \) intersects \( \tilde{W} \) at two points. Next we consider a cubic \( C \in |3e_0 - e_1 - \cdots - e_5| \) with an ordinary double point at \( q \) that intersects \( W \) at this point with multiplicity 8. Its proper transform \( \tilde{C} \) intersects the components \( e_2 \) and \( e_6 \). Together \( F, \tilde{W}, \tilde{C} \) form a diagram as in picture (c).

Remark 8.2.22. In each case of Theorem 8.2.21, one can find a maximal root sublattice of rank 8: \( E_8 \) (case (a)), \( D_8 \) (case (b)), \( E_7 \oplus A_1 \) (case (c)). Over the complex numbers, these root lattices give Nikulin \( R \)-invariants \( (E_8, \{0\}), (D_8, \mathbb{Z}/2\mathbb{Z}), (E_7 \oplus A_1, \mathbb{Z}/2\mathbb{Z}) \) of Enriques surfaces with numerically trivial automorphisms. To classify \( R \)-invariants of such Enriques surfaces was the main idea of Mukai and Namikawa [503].

The following theorem gives a classification of Enriques surfaces in characteristic \( \neq 2 \) that admit a numerically trivial involutions.

**Theorem 8.2.23.** Assume \( p \neq 2 \) and let \( S \) be an Enriques surface with non-trivial \( \text{Aut}_{\text{int}}(S) \). Then the isomorphism class of \( S \) belongs to one of the following three 2-dimensional irreducible families:

(A) \( S \) is the double cover of a non-degenerate 4-nodal quartic surface \( D_4 \) with the branch curve \( W \) equal to the union of two conics intersecting at 2 points and an irreducible hyperplane section through the intersection points of the two conics. The surface contains the diagram (a) of \((-2)\)-curves. The deck transformation is cohomologically trivial. One of the members of the family contains a numerically trivial automorphism whose square is equal to the deck transformation.

(B) \( S \) is the double cover of a non-degenerate 4-nodal quartic surface \( D_4 \) with the branch curve \( W \) equal to the union of two pairs of disjoint conics. The deck transformation is numerically...
trivial but not cohomologically trivial. The surface contains the diagram of \((-2)\)-curves of type (b).

(C) \(S\) is the double cover of a non-degenerate 4-nodal quartic surface \(D_1\) with the branch curve \(W\) equal to the union of two conics and a nodal irreducible hyperplane section. The deck transformation is numerically trivial but not cohomologically trivial. The surface contains the diagram (c) of \((-2)\)-curves.

Any pair \((S, \sigma)\) consisting of an Enriques surface \(S\) and its numerically but not cohomologically trivial involution is isomorphic to a surface from family (C).

**Proof.** Applying Theorem 8.2.21, we obtain that all surfaces that admit a bielliptic map with numerically trivial deck involution are divided into 3 classes (A), (B), (C) according to the type one of the diagrams (a), (b), (c) of \((-2)\)-curves lying on it.

**Type (A):**

We give another realization of diagram (a) by choosing construction (28) from Table 8.7. It corresponds to a choice of a \(U\)-pair of elliptic fibrations with which contain a fibers of type \(\tilde{D}_8\) which we can easily locate in the diagram. In this the octic branch curve on \(D_1\) is the union of two conics and a hyperplane section that passes through the intersection points of the conics. In the plane model they correspond to a cubic \(C\) passing through \(p_1, \ldots, p_5\), a line \(\ell\) from \([e_0 - e_1]\), and a conic \(K\) from the pencil \([2e_0 - e_1 - \cdots - e_5]\). The cubic \(C\) passes through two intersection points \(q_1, q_2\) of the line and the conic. The curve \(W\) has two simple singular points at \(q_1, q_2\) of type \(d_4\).

In appropriate projective coordinates in the plane, the equation of the double plane is given by

\[
x_3^2 + x_1 x_2 (x_1 - x_2)(x_0^2 - x_1 x_2)(ax_1^2 x_2 + bx_0^2 x_1 + cx_0^2 x_2 + dx_1^2 x_1) = 0,
\]

where \(a + b + c + d = 0\). The points \(q_1, q_2\) have the coordinates \([\pm 1, 1, 1]\). For example, if we take the parameters \((a, b, c, d) = (1, -1, 1, -1)\), we obtain the standard Cremona involution \(T: (x_0, x_1, x_2) \mapsto (x_1 x_2, x_0 x_2, x_0 x_1)\) that transforms \(F\) to \(-(x_0^2 x_1 x_2)^2 F\). Consider a birational model of \(S\) as a surface of degree 8 in the weighted projective space \(\mathbb{P}(1, 1, 1, 4)\) given by the equations \(x_3^2 + F(x_0, x_1, x_2) = 0\). Then the formula \((x_0, x_1, x_2, x_3) \mapsto (x_1 x_2, x_0 x_2, x_0 x_1, i x_3 x_0^2 x_1 x_2)\) defines a birational automorphism of order 4 of \(S\) whose square is equal to the deck transformation \(\sigma\). Since it fixes the points \(q_1, q_2\) and the curve branch curve invariant, it is easy to see that it is numerically trivial.

Let us show that the bielliptic involution is cohomologically trivial. Let \(N\) be the sublattice of \(\text{Num}(S)\) spanned by the \((-2)\)-curves represented by all vertices of the diagram (a). We compute the discriminant of this lattice and find that it is equal to \(-1\). Thus the span of the curves generate \(\text{Pic}(S)\) and since the involution fixes these curves, it is cohomologically trivial.

**Type (B):**

We use the bielliptic involution from the previous theorem. A choice of two lines and two conics depend on 2 parameters. We fix one line and a conic as in the previous case to assume that their equations are \(x_1 - x_2 = 0\) and \(x_1 x_2 - x_0^2 = 0\). Then the second line has an equation \(x_1 + ax_2 = 0\).
and the second conic has an equation \( x_1x_2 + bx_0^2 = 0 \). Thus the equation of the double plane is
\[
x_3^2 + x_1x_2(x_1 - x_2)(x_1 + ax_2)(x_1x_2 - x_0^2)(x_1x_2 + bx_0^2) = 0,
\]
where \( a \neq 0, -1, b \neq -1 \).

Note that in this case, the bielliptic involution is not cohomologically trivial but only numerically trivial. To see this we consider the Halphen pencil of curves of degree 6 with double points at \( p_1, \ldots, p_5 \) and four of the double points \( q_1, \ldots, q_4 \) of \( W \) no three of which lie on \( \ell \) or \( \ell' \). It is the pencil \( \lambda A + \mu B^2 \), where \( A = 0 \) is the equation of the curves \( A = \ell + \ell' + K + K' \) and \( B = 0 \) is the equation of the unique cubic \( B \) through the nine points \( p_i, q_i \). The pre-image of this pencil on \( S \) is locally equal to \( \lambda A^2 + \mu B^2 = 0 \), and it general member splits into the union of two disjoint elliptic curves. It defines an elliptic fibration over the double cover of the line parameterizing the Halphen pencil. It is an elliptic fibration on \( S \) with a reducible fiber of type \( \tilde{A}_7 \) which we can locate on picture (b). The deck involution does not act trivially on its base, so it is not cohomologically trivial.

Type (C):

We could use the construction of the double plane described in the previous theorem, but for the diversity reason let us give another construction which is case (29) from Table 8.7. We take \( W \) to be the union of a line \( \ell \) from the pencil \( |e_0 - e_1| \), a conic from the pencil \( |2e_0 - e_1 - \cdots - e_5| \) and a cubic from \( |3e_0 - e_1 - \cdots - e_5| \) with a node \( q \) that passes through one of the intersection points \( q_1, q_2 \) of the line and the conic. This corresponds to the branch curve on \( D_1 \) equal to the union of two conics and a hyperplane section that passes through one of the intersection points and tangent to the surface at some point. The curve \( W \) has one simple singularity of type \( d_4 \) and four ordinary double points. The pre-images of the exceptional curves of resolution of singularities of \( W \), the proper transforms of the components of \( W \) form the diagram (c). The vertex connected with the double edge is the pre-image of the exceptional curve over the node \( q \).

The equation of the double plane is
\[
x_3^2 + x_1x_2(x_1 - x_2)(x_1x_2 + x_0^2)(ax_1^2x_0 + bx_2^2x_0 + cx_0^2x_1 + dx_0^2x_2 + ex_0x_1x_2) = 0
\]
where \( a + b + c + d + e = 0 \) and we have to add one more condition that the cubic is singular.

We claim that the deck transformation of the bielliptic map is numerically trivial but not cohomologically trivial. The argument is the same as in the case (A). We consider an Halphen pencil \( \lambda A + \mu B^2 = 0 \), where \( A = 0 \) is the equation of \( \ell + K + C \) and \( B = 0 \) is the equation of a unique cubic \( c' \in \mathcal{P}_3 \) passing through the four ordinary double points of \( \ell + K + C \). \( \square \)

Remark 8.2.24. Consider a surface \( S \) from family (B). The surface contains 8 disjoint \((-2)\)-curves. Their pre-image on the canonical cover \( X \) is a set of 16 disjoint \((-2)\)-curves. If \( k = \mathbb{C} \), by Nikulin’s Theorem [530], \( X \) must be birationally isomorphic to the Kummer surface associated to an abelian surface. In fact, for any \( k \) of characteristic \( \neq 2 \), one can see it directly by considering the base change of the bielliptic map \( f : S \to D_1 \) under the cover \( F_0 \to D_1 \). It defines a degree 2 map \( \tilde{f} : \tilde{X} \to \tilde{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1 \) branched along the union of 8 lines, for in each ruling. It is well-known and easy to see that it shows that \( X \) is birationally isomorphic to the Kummer surface \( \text{Kum}(E_1 \times E_2) \) of the product \( E_1 \times E_2 \) of two elliptic curves. Moreover, one sees that the Enriques involution \( \tau \) on \( X \) is the descent to the Kummer surface of the involution \( (a, b) \mapsto (a + \epsilon, -b + \eta) \), where
\( \epsilon \in E_1, \eta \in E_2 \) are non-trivial 2-torsion points.

Consider the family of type (C) constructed in the proof of the previous theorem. We can locate additional four \((-2)\)-curves \(C_1, C_2, C_3\) on \(S\) equal to the proper inverse transform of a line \(\ell'\) and the conic \(K'\) from the same pencils that pass through the node \(q\) and the conic that passes through the intersection point of the line and the cubic component of the branch sextic. We get the following diagram of \((-2)\)-curves on \(S\):

Here \(C_0\) corresponds to the vertex of diagram (c) connected by the double edge.

Let \(\pi : X \to S\) be the canonical cover of \(S\). The pre-images of the curves \(R_4, R_5, R_6, R_7, R_8, R_9\) are 12 disjoint \((-2)\)-curves on \(X\). The pre-images of the curves \(C_0, C_1, C_2, C_3\) are eight \((-2)\)-curves, among which we can find four mutually disjoint ones. Together with the previous set of twelve \((-2)\)-curves they give 16 disjoint curves on \(X\). In the case \(k = \mathbb{C}\), we can apply Nikulin’s Theorem do deduce that \(X\) must be birationally isomorphic to a Kummer surface associated to some abelian surface. It is shown in [503] that \(X\) is birationally isomorphic to \(\text{Kum}(E_1 \times E_2)\) again, but the Enriques involution is different. We refer to its description to section 2 of loc.cit..

In Section 8.9 we will classify all Enriques surfaces with finite automorphism groups over a field of characteristic \(\neq 2\). We will see that all three diagrams are realized on surfaces of types I, III and V, respectively.

If \(p = 2\), the classification of Enriques surfaces with a numerically trivial automorphism is unknown at present. Obviously, if \(K_S = 0\), a numerically trivial automorphism is cohomologically trivial.

We start with the following.

**Lemma 8.2.25.** Let \(f : S \to \mathbb{P}^1\) be a genus one fibration on an Enriques surface. Suppose it admits a bisection \(B\) such that \(f|B : B \to \mathbb{P}^1\) is a separable cover (we call it a separable bisection). Then \(S\) admits an automorphism of order 2 that preserves the fibration and the bisection \(B\). Conversely, any automorphism of order 2 of \(S\) that preserves a genus one fibration, acts identically on the base, and leaves invariant its bisection arises in this way.

**Proof.** A bisection is a point \(x\) of degree 2 on the generic fiber \(S_\eta\) of \(f\). The assumption implies that the residue field of this point is a separable extension of \(k(\eta)\). The linear system \([x]\) defines a separable degree 2 map \(S_\eta \to \mathbb{P}^1_\eta\). Its deck transformation is a birational involution on \(S\) that extends to a biregular involution. The point \(x\) is the pre-image of a rational point on \(\mathbb{P}^1_\eta\) and hence is preserved by the deck transformation.
Conversely, an automorphism $\sigma$ of order 2 of $S$ that preserves a genus one fibration $f : S \rightarrow \mathbb{P}^1$ and its bisection restricts to $S_\eta$ to define a separable map $\phi : S_\eta \rightarrow C = S_\eta/(\sigma)$. Since $\sigma$ leaves invariant the generic point $x$ of the bisection, we obtain that $x = \phi^{-1}(y)$ for some point of degree 1 on $C$. Hence $C \cong \mathbb{P}^1_\eta$, and $\sigma$ arises in the way described in above.

**Corollary 8.2.26.** Suppose $S$ admits an elliptic fibration $f : S \rightarrow \mathbb{P}^1$. Then the automorphism group $\text{Aut}(S)$ contains an element of order 2.

**Proof.** We know that $f : S_\eta \rightarrow \mathbb{P}^1$ is a torsor over its jacobian elliptic curve of period 2. Applying Theorem 4.6.5 and Lemma ??, we obtain that $S_\eta$ contains a point $x$ of degree 2 with a separable residue field. Then the assertion follows from the previous lemma.

### 8.3 Automorphisms of an unnodal Enriques surface

In the previous section we discussed the kernel of the homomorphism

$$\rho : \text{Aut}(S) \rightarrow \text{O}(\text{Num}(S)) \cong \text{O}(E_{10}).$$

**Proposition 8.3.1.** Suppose $S$ is an unnodal Enriques surface, then

$$\text{Ker}(\rho) = \{1\},$$

unless $p = 2$ and $S$ is a classical Enriques surface in which case $\# \text{Ker}(\rho) = 2$ and a generator of this group acts non-identically on the base of each elliptic fibration.

**Proof.** If $p \neq 2$, it follows from Theorem 8.2.18 or Theorem 8.2.21 that $\text{Aut}_{\text{nt}}(S) = \text{Ker}(\rho)$ is trivial. If $p = 2$, Theorem 8.2.7 implies that $\text{Aut}_{\text{nt}}(S)$ is trivial, in particular $\text{Aut}_{\text{nt}}(S) = \{1\}$ is trivial if $K_S = 0$. By Theorem 8.2.20, we have $\# \text{Aut}_{\text{nt}}(S)$ is an elementary 2-group of rank $a \leq 2$. Suppose $a = 2$. Fix a genus one fibration. Since its half-fibers are invariant with respect to $\text{Aut}_{\text{nt}}(S)$, there exists $g$ of order 2 such that it fixes two half-fibers. Since $S$ is classical, the set $S^g$ of fixed points of $g$ is connected [373]. This implies that there exists only one singular fiber. It must be of type 9C in classification of fibers of elliptic fibrations in characteristic 2 [418]. We will see later in Section 10.2 that the canonical cover of $S$ has only one singular point, a double rational point of type $D_1^{(0)}$ or a minimal elliptic singularity of Arnold’s type $E_{12}$. By Theorem [474, Theorem 1.4] this implies that $S$ is a $\alpha_2$-surface, contradicting our assumption.

We do not know any examples of unnodal Enriques surfaces with non-trivial $\text{Ker}(\rho)$.

Now we need to discuss its image. From now on, we fix an isomorphism $\text{Num}(S) \cong E_{10}$ and identify these two lattices.

Applying Theorem 8.1.7 and its Corollary 8.1.9, we obtain the following.

**Proposition 8.3.2.** Let $S$ be an unnodal Enriques surface. Assume that $p \neq 2$. Then $\text{Aut}(S)^* \text{ is a subgroup of finite index in } W(E_{10})$. 

\[\text{Proposition 8.3.2.}\]
Applying Theorem 0.8.5, we see that the reduction homomorphism $E_{10} \to E_{10} = E_{10}/2E_{10}$ defines a surjective homomorphism $W(E_{10}) \to O^+(10, \mathbb{F}_2) \cong D_{E_{10}(2)}$. Let

$$W(E_{10})(2) := \ker(W(E_{10}) \to O^+(10, \mathbb{F}_2)).$$

It is called the 2-level congruence subgroup of $W(E_{10})$. It is clear that under any marking $\phi : \text{Num}(S) \to E_{10}$, the subgroup $\phi \circ W(E_{10})(2) \circ \phi^{-1}$ is equal to the subgroup $W_S(2) := \{\sigma \in W(\text{Num}(S)) : \sigma \equiv \text{id}_{\text{Num}(S)} \mod 2 \text{Num}(S)\}$. We call it the 2-level congruence subgroup of the Weyl group $W(\text{Num}(S))$.

Over $\mathbb{C}$, it follows Theorem 5.5.1 that $\text{Aut}(S)^* \text{ contains } W(\text{Num}(S))(2)$. In the following we will prove this result in the case of a field of arbitrary characteristic.

**Proposition 8.3.3** (A. Coble). The subgroup $W(E_{10})(2)$ is the smallest normal subgroup containing the involution $\sigma = 1_U \oplus (-1)_{E_8}$ for some (hence any) orthogonal decomposition $E_{10} = U \oplus E_8$.

**Proof.** Coble’s proof is computational. It is reproduced in [134], Chapter 2, §10. The following nice short proof is due to Allcock [8].

Let $\Gamma = \langle \langle \sigma \rangle \rangle$ be the minimal normal subgroup containing $\sigma$. It is generated by the conjugates of $\sigma$ in $W = W(E_{10})$. Let $\{f, g\}$ be the standard basis of the hyperbolic plane $U$. If $\{\alpha_0, \ldots, \alpha_7\}$ is the basis of the sublattice $E_8$ corresponding to the subdiagram of type $E_8$ of the Dynkin diagram of the Enriques lattice $E_{10}$, then we may take

$$f = 3\alpha_0 + 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8, \quad g = f + \alpha_9.$$

The stabilizer $W_f$ of $f$ in $W$ is the semi-product $E_8 \rtimes W(E_8) \cong W(E_9)$. The image of $v \in E_8 = U^\perp$ in $W_f$ is the transformation

$$\phi_v : x \mapsto x + (x \cdot v - \frac{1}{2}(x \cdot f)v^2)f - (x \cdot f)v.$$

The inclusion of $W(E_8)$ in $W(E_9)$ is natural, it consists of compositions of the reflections in the roots $\alpha_0, \ldots, \alpha_7$. In particular, any $w \in W(E_8)$ acts identically on $E_8^\perp \cong U$. The image of $\sigma$ in $W(E_9)$ is equal to $(-1)_{E_8} \in W(E_8) \subset W(E_9)$. Let us compute the $\phi_v$-conjugates of $\sigma$. If $x \in E_8$, we have

$$\phi_v \circ \sigma \circ \phi_{-v}(x) = \phi_v(\sigma(x - (v \cdot x)f))$$

$$= \phi_v(-x - (v \cdot x)f) = -(v \cdot x)f = -x - 2(v \cdot x)f.$$

Thus the intersection of $\Gamma$ with $W_f$ is equal to $\phi(2E_8) \ltimes \mathbb{Z}/2\mathbb{Z}$. The quotient $W_f/(\Gamma \cap W_f)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^8 \rtimes O(8, \mathbb{F}_2)^+$. It injects into $O(E_{10}) \cong O(10, \mathbb{F}_2)^+$.

Let us consider the subgroup $H$ generated by $W_f$ and $\Gamma$. Since $W_f/(\Gamma \cap W_f)$ injects in $O(E_{10})$, the kernel of the homomorphism $H \to O(E_{10})$ coincides with $\Gamma$. To finish the proof it suffices to show that $H$ coincides with the preimage $W_f$ in $W$ of the stabilizer subgroup of the image $\tilde{f}$ of $f$ in $O(E_{10})$. Indeed, the kernel of $W_f \to O(E_{10})$ is equal to $W(E_{10})(2)$ and hence coincides with $\Gamma$.

Consider the sublattice $L$ of $E_{10}$ generated by the roots $\alpha_0, \ldots, \alpha_8$ and $\alpha'_0 = \alpha_8 + 2g - f$. The Dynkin diagram is the following.
Here all the roots, except $\alpha'_9$, are orthogonal to $f$. So, $H$ contains the reflections defined by these roots. Also the root $\alpha_8 - f$ is orthogonal to $f$, and $\alpha'_9$ is transformed to it under the group $2E_8 \subset \Gamma$ stabilizing $g$. So $H$ contains $s_{\alpha'_9}$ too. The Dynkin diagram contains 3 subdiagrams of affine types $E_7, E_8$ and $D_8$. The Weyl group is a crystallographic group with a Weyl chamber being a simplex of finite volume with 3 vertices at the boundary. This implies that $H$ has at most 3 orbits of $(\pm$ pairs) of primitive isotropic vectors in $L$. On the other hand, $W_f$ contains $H$ and has at least three orbits of them, because the stabilizer of $\tilde{f}$ in $O(E_{10})$ has three orbits of isotropic vectors (namely, $\{\tilde{f}\}$, the set of isotropic vectors distinct from $\tilde{f}$ and orthogonal to $\tilde{f}$, and the set of isotropic vectors not orthogonal to $\tilde{f}$). This implies that the set of orbits of $H$ is the same as the set of orbits of $W_f$. Since the stabilizers of $\tilde{f}$ in these two groups are both equal to $W_f$, it follows that $H = W_f$.

\[ \square \]

**Theorem 8.3.4.** Let $S$ be an unnodal Enriques surface. Then the image $\text{Aut}(S)^* \text{ of } \rho$ contains $W(\text{Num}(S))(2)$.

**Proof.** Consider a bielliptic linear system $|D| = |2F_1 + 2F_2|$ with $D^2 = 8$. It defines a degree 2 bielliptic map $\phi : S \to D \subset \mathbb{P}^4$, where $D$ is one of the anti-canonical surfaces $D_1, D_2, D_3$. Since $S$ has no smooth rational curves, the cover is separable map. Let $\sigma$ be the deck transformation of the cover and $\sigma^* = \rho(\sigma) \in W(\text{Num}(S))$. For any irreducible curve $C$ on $S$, the divisor class $c = [C + \sigma^*(C)]$ is $\sigma^*$-invariant. Since Pic$(D)_{\mathbb{Q}} \cong \mathbb{Z}^2$ and $\phi$ is a finite map of degree 2, we have $c = (c \cdot [F_2])[F_1] + (c \cdot [F_1])[F_2]$ in $\text{Num}(S)_{\mathbb{Q}}$. This shows that $\sigma^*$ acts as the identity on $\langle [F_1], [F_2] \rangle \cong \mathbb{U}$ and as the minus identity on $\langle [F_1], [F_2] \rangle^+ \cong E_8$.

Now any conjugate of $\sigma^*$ in $W(\text{Num}(S))$ is also realized by some automorphisms. In fact, $w \cdot \sigma^* \cdot w^{-1}$ leaves invariant $w(\langle F_1, F_2 \rangle)$, and the deck transformation corresponding to the linear system $|2w(F_1) + 2w(F_2)|$ realizes $w \cdot \sigma^* \cdot w^{-1}$.

Now we invoke the previous proposition that says that $W(E_{10})(2)$ is the minimal normal subgroup of $W(E_{10})$ containing $\sigma^*$.

\[ \square \]

Recall from Section 5.2 that, over $\mathbb{C}$, for a moduli general Enriques surface the group $\text{Aut}(X)^*$ coincides with $W(\text{Num}(S))(2)$. Since we do not know how to check explicitly whether a surface is unnodal it is hard to give an example of an unnodal surface with a larger automorphism group.

We refer to [471] for the proof of the following result.

**Theorem 8.3.5.** Let $S$ be an unnodal Enriques surface. Suppose $p \neq 2$, or $S$ is a $\mu_2$-surface. Let $\tilde{\text{Aut}}(S)$ be the image of $\text{Aut}(S)$ in $W(E_{10})/W(E_{10})(2) \cong O^+(10, \mathbb{F}_2)$. Then $\#\tilde{\text{Aut}}(S) \in \{1, 2, 4\}$. The group is trivial if $p = 2$.

We refer to loc.cit. for examples of unnodal surfaces with extra automorphisms.
8.4 Automorphisms of a general nodal surface

In this section we assume that $S$ is a general nodal Enriques surface in the sense of the definition from section 6.5. Recall that we have defined the Reye lattice in (6.4.6)

$$\text{Rey}(S) = \{ x \in \text{Num}(S) : x \cdot R \equiv 0 \mod 2 \text{ for any } (-2)-\text{curve } R \}.$$ 

Obviously, the action of $\text{Aut}(S)$ on $S$ preserves the set of $(-2)$-curves, hence preserves the Reye lattice. Thus we have a homomorphism

$$\rho : \text{Aut}(S) \to O(\text{Rey}(S)).$$

By (0.8.12), $O(\text{Rey}(S))' = W(\text{Rey}(S))$. Thus we may assume that the image of $\rho$ is contained in the Weyl group $W(\text{Rey}(S))$.

**Proposition 8.4.1.** Assume that $S$ is a general nodal Enriques surface. The homomorphism $\rho$ is injective if $p \neq 2$ or $p = 2$ and $K_S = 0$. Otherwise, $\text{Ker}(\rho)$ is of order $\leq 2$, and does not act trivially on the base of any elliptic fibration and any reducible fiber of an elliptic fibration is of type $A_1^*$.

**Proof.** By Theorem 8.2.21, the assertion is true if $p \neq 2$. Let us assume that $p = 2$. First let us prove that $\text{Aut}_{\text{int}}(S)$ is trivial. Let $R$ be a $(-2)$-curve on $S$. By Theorem 6.3.3, it is a component of some fiber $|2F|$ of an elliptic pencil on $S$. We extend the pair $(F, F + R)$ to a canonical isotropic sequence $(F, F + R, F_2, F_3)$ and find that $R \cdot F_2 = R \cdot F_3 = 0$. Since $g \in \text{Aut}_{\text{int}}(S)$ fixes $F_1, F_2, R$, we find a $g$-invariant bielliptic map that blows down $R$ to a point. It follows from Proposition 8.2.3 that $g$ coincides with the bielliptic involution defined by this map. But now the assertion follows from Lemma 8.2.5.

It remains to prove that $\text{Aut}_{\text{int}}(S)$ is trivial. So, we may assume that $S$ is a classical Enriques surface. By Theorem 8.2.20, $\text{Aut}_{\text{int}}(S) \cong (\mathbb{Z}/2\mathbb{Z})^a$, $a \leq 2$. Suppose $a = 2$. Fix an elliptic fibration with only irreducible fibers, for example, the fibration that admits a special bisection. Since any $g \in \text{Aut}_{\text{int}}(S)$ leaves invariant the set of two double fibers, one of elements of $\text{Aut}_{\text{int}}(S)$ fixes the two fibers and hence fixes all fibers because $p = 2$. Since $S$ is classical, the set $S^9$ of fixed points of $g$ is connected [373]. This shows that there is only one irreducible singular fiber or a half-fiber and all other fibers and half-fibers are smooth. As we will learn in Section 10.2, this implies that the canonical cover of $S$ has only one singular point, a double rational point of type $D_{12}^{(0)}$ or a minimal elliptic singularity of type $E_{12}$. However, by [474, Theorem 1.4] this could happen only $K_S = 0$.

Thus $a = 1$. We know that any $(-2)$-curve $R$ is realized as a component of a fiber $F$ of type $\tilde{A}_1$ or $A_1^*$ of an elliptic fibration. It is fixed by $g$. If $g$ fixed all fibers, the same argument as above shows that $F$ is of type $\tilde{A}_1^*$ and $S^9$ consists of its unique singular point. We also see that $g$ acts non-trivially on the base of any elliptic fibration.

We do not know an example of a general nodal Enriques surface with non-trivial $\text{Ker}(\rho)$.

Fix a canonical root basis in $E_{10}$ defined by an isotropic 10-sequence $(f_1, \ldots, f_{10})$. It defines a sublattice of $E_{10}$ isomorphic to the Reye lattice $E_{2,4,6}$. Its canonical root basis is formed by the
vectors $(\beta_0, \ldots, \beta_9)$ with

\[
\begin{align*}
\beta_0 &= f_1 + f_2 + f_3 + f_4 - h_{10}, \\
\beta_i &= f_{i+1} - f_i, \quad i > 0.
\end{align*}
\]

(8.4.1)

The vectors $\beta_0, \ldots, \beta_9$ form a root basis of $E_{2,4,6}$ with Dynkin diagram of type $T_{2,4,6}$

\[
\begin{array}{ccccccccc}
\beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 & \beta_8 & \beta_9 \\
\beta_0
\end{array}
\]

Figure 8.1: Reye lattice

We fix this embedding $E_{2,4,6} \hookrightarrow E_{10}$. It satisfies the property that $E_{2,4,6} = \{ x \in E_{10} : x \cdot r \in 2\mathbb{Z} \}$, where $r = h_{10} - 2f_{10}$. The root basis $(\beta_0, \ldots, \beta_9, r)$ is a crystallographic root basis of $E_{2,4,6}$ with Dynkin diagram (8.2). Recall that we have already encountered this crystallographic basis when we considered extra-special Enriques surfaces in characteristic 2.

\[
\begin{array}{cccccccccc}
\beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 & \beta_8 & \beta_9 & r \\
\beta_0
\end{array}
\]

Figure 8.2: Crystallographic root basis in $E_{2,4,6}$

**Definition 8.4.2.** A Reye marking of a general nodal Enriques surface is an isomorphism $\phi : \text{Num}(S) \to E_{10}$ such that $\phi(\text{Rey}(S)) = E_{2,4,6}$ and $r = \phi^{-1}(r)$ is the class of a $(-2)$-curve $R$.

By Proposition 0.8.9, $W(E_{2,4,6}) = \langle \langle s_r \rangle \rangle \rtimes W_{2,4,6}$, where $W_{2,4,6}$ is the Coxeter group corresponding to the standard root basis of the lattice $E_{2,4,6}$ with Dynkin diagram $T_{2,4,6}$. Using a Reye marking $\phi : \text{Rey}(S) \to E_{2,4,6}$, we obtain

$$W(\text{Rey}(S)) = \langle \langle s_r \rangle \rangle \rtimes W(\text{Rey}(S))_0,$$

where $W(\text{Rey}(S))_0 = \phi^{-1}(W_{2,4,6})$.

**Proposition 8.4.3.**

\[W_S^{\text{nod}} = \langle \langle s_r \rangle \rangle,\]

i.e. $W_S^{\text{nod}}$ is a minimal normal subgroup containing $s_r$.

**Proof.** Fix a Reye marking $\phi : \text{Rey}(S) \to E_{2,4,6}$. Let $\beta_i = \phi^{-1}(\beta_i)$. By Corollary 6.5.10, the canonical cover of $S$ is birationally isomorphic to a K3 surface. Since $\beta_1, \ldots, \beta_{10}$ intersect $r$ with even multiplicity, we can apply Lemma 6.3.14 to obtain that any $W(E_{2,4,6})$-conjugate of $r$ is the class of a $(-2)$-curve. By Corollary 0.8.10, $(E_{2,4,6})_{-2}$ has two orbits with respect to $W(E_{2,4,6})$. We have shown that the orbit of $r$ is contained in the set of effective roots (the classes of $(-2)$-curves). Suppose an effective root $r'$ belongs to another orbit. Then $r' = w(\beta_1)$ for some $w \in W_{2,4,6}$. Since
Proposition 8.4.4. Let $W(\text{Rey}(S))_0$ be the image of $W_{2,4,6}$ for some Reye marking. Then the subgroup $W(\text{Rey}(S))_0$ consists of elements from $W(\text{Rey}(S))$ that leave invariant the nef cone $\text{Nef}(S)$ of $S$.

Proof. Suppose $w(\text{Nef}(S)) \neq \text{Nef}(S)$ for some $w \in W(\text{Rey}(S))_0$. Since $W(\text{Rey}(S))_0$ normalizes $W_S^{\text{mod}}$, its elements permute $w'(\text{Nef}(S))$, $w' \in W_S^{\text{mod}}$. Thus $w$ must coincide with some element from $W_S^{\text{mod}}$. Since $W_{2,4,6} \cap \langle \langle s_r \rangle \rangle = \{1\}$, we get a contradiction. Since no element of $W_S^{\text{mod}}$ leaves $\text{Nef}(S)$ invariant, we obtain that $W(\text{Rey}(S))_0$ is the group of isometries of $E_{2,4,6}$ that leaves $\text{Nef}(S)$ invariant.

Corollary 8.4.5. Under a Reye marking $\phi$ that sends $r$ to the class of a $(-2)$-curve, the nef cone of $S$ is equal to the union of $W_{2,4,6}$-translates of the set

$$C = \{ x \in (E_{2,4,6})_\mathbb{R} : x \cdot \beta_i \geq 0, x \cdot r \geq 0 \}.$$ 

Recall from Section 6.4 that

$$\bar{E}_{2,4,6} = E_{2,4,6}/2E_{2,4,6}$$

is a 10-dimensional quadratic space $V$ with the quadratic form $q(x + 2E_{2,4,6}) = \frac{1}{2}x^2$. Its radical is spanned by the coset $r$ of $r$ and the coset $\bar{f}_{E_8}$ of the generator $f_{E_8}$ of the radical of the affine root sublattice of type $\bar{E}_8$ in $E_{2,4,6}$. Note that $\frac{1}{2}f_{E_8} - f_1 \in \bar{E}_{10}$. Also note that $\frac{1}{2}r$ and $\frac{1}{2}f_{E_8}$ generate the discriminant group of the lattice $E_{2,4,6}$.

It is known that the orthogonal group $O(E_{2,4,6})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus 8} \rtimes \text{Sp}(8, \mathbb{F}_2)$. The 2-elementary abelian group can be identified with $(V/V^\perp)^{\vee}$ that acts on $V$ by

$$e_l : v \mapsto v + l(v + V^\perp)\bar{f}_{E_8}.$$ 

The subgroup $\text{Sp}(8, \mathbb{F}_2) = \text{Sp}(V/V^\perp)$ can be identified with the orthogonal group of the 9-dimensional subspace equal to the orthogonal complement of $\bar{f}_{E_8}$ in $V$.

Lemma 8.4.6. The reduction homomorphism

$$r : W_{2,4,6} \to O(\bar{E}_{2,4,6}) \cong 2^8 \rtimes \text{Sp}(8, \mathbb{F}_2).$$

is surjective. In particular,

$$W_{2,4,6}/W_{2,4,6}(2) \cong 2^8 \rtimes \text{Sp}(8, \mathbb{F}_2).$$

Proof. It is a special case of a more general result about reduction mod 2 of Coxeter groups with Coxeter diagram of type $T_{p,q,r}$ that can be found in [249]. However, for completeness sake, we provide a proof. First, the normal subgroup $\langle \langle s_\alpha \rangle \rangle$ is contained in the kernel of the action of $O(E_{2,4,6})$ on $\bar{E}_{2,4,6}$. Thus, it suffices to prove that the reduction homomorphism $O(E_{2,4,6})' = W(E_{2,4,6}) \to O(\bar{E}_{2,4,6})$ is surjective. We can identify $\bar{E}_8$ with the affine root sublattice of $E_{2,4,6}$ corresponding to the parabolic subdiagram of type $\bar{E}_8$. We know from (0.8.17) from Section 0.8 that

$$W_{2,3,6} = W(\bar{E}_8) = E_8 \rtimes W(E_8) \cong \mathbb{Z}^8 \rtimes W_{2,3,5}.$$
Formula (0.8.8) shows that under the reduction modulo 2 map the image of the subgroup $W(E_8)$ is equal to $2^8 \rtimes W(E_8) \cong 2^8 \rtimes O(8, \mathbb{F}_2)^+$. It is known that $O(8, \mathbb{F}_2)^+$ is a maximal subgroup of $\text{Sp}(8, \mathbb{F}_2)$ of index 136 (see [129]). Thus the image of the whole group $O(E_{2,4,6})$ is equal to $2^8 \rtimes \text{Sp}(8, \mathbb{F}_2)$. \hfill \Box

Let $r : W(E_{2,4,6}) = O(E_{2,4,6})' \to 2^8 \rtimes \text{Sp}(8, \mathbb{F}_2)$ be the reduction homomorphism. By Theorem 0.8.5, the homomorphism is surjective. We know that $W(E_{2,4,6}) = \langle \langle s_r \rangle \rangle \rtimes W_{2,4,6}$. Since $r$ intersects any vector from $E_{2,4,6}$ with even multiplicity, its image under the reduction homomorphism is 0. Thus, we have a surjective homomorphism

$$r : W_{2,4,6} \to 2^8 \rtimes \text{Sp}(8, \mathbb{F}_2).$$

Let $W_{2,4,6}(2) = \ker(r)$ be the 2-level congruence subgroup of $W_{2,4,6}$. We have

$$W_{2,4,6}/W_{2,4,6}(2) \cong 2^8 \rtimes \text{Sp}(8, \mathbb{F}_2) \quad (8.4.2)$$

Let

$$W'_{2,4,6} \subset W_{2,4,6} \quad (8.4.3)$$

be the normal subgroup of $W_{2,4,6}$ that is equal to the pre-image of the normal subgroup $2^8$ of $O(E_{2,4,6}/2E_{2,4,6}) \cong 2^8 \rtimes \text{Sp}(8, \mathbb{F}_2)$.

**Theorem 8.4.7.** Assume $p \neq 2$ or $S$ is a $\mu_2$-surface. Under a Reye marking, $\text{Aut}(S)$ contains a subgroup isomorphic to $W'_{2,4,6} = W_{2,4,6} \cap W_{2,3,7}(2)$.

**Proof.** By Proposition 8.4.1, the homomorphism $\text{Aut}(S) \to \text{Aut}(S)^*$ is bijective, so we can identify $\text{Aut}(X)$ with a subgroup of $W(\text{Rey}(S))$ and via a marking with a subgroup of $W_{2,4,6}$, where we identify $E_{2,4,6}$ with the Reye sublattice of $E_{10}$ of vectors intersecting the fixed root $r$ with even multiplicity. Since $W_{2,4,6}$ is generated by reflections in vectors from $E_{2,4,6} \subset E_{10}$, it extends to a subgroup of $W(E_{10})$. Thus $W(E_{10})(2) \cap W_{2,4,6}$ makes sense.

Applying Corollary 8.4.4 and Theorem 8.1.7, it suffices to show that this intersection coincides with $W'_{2,4,6}$. Let

$$0 \to 2E_{10} \cap E_{2,4,6}/2E_{2,4,6} \to E_{2,4,6}/2E_{2,4,6} \to E_{2,4,6}/2E_{10} \cap E_{2,4,6} \to 0$$

be the natural exact sequence of quadratic spaces over $\mathbb{F}_2$. We know from Section 6.4 that $E_{2,4,6}/2E_{10} \cap E_{2,4,6}$ is a 9-dimensional subspace of $E_{10} = E_{10}/2E_{10}$ equal to the orthogonal complement to a non-isotropic vector. The subspace $2E_{10} \cap E_{2,4,6}/2E_{2,4,6}$ is generated by an isotropic vector $v_0$ in the 2-dimensional radical of $E_{2,4,6}/2E_{2,4,6}$. We have

$$W_{2,4,6}/W_{2,4,6}(2) \cong O(E_{2,4,6}/2E_{2,4,6})$$

and, since any element of the right-hand-side group preserves the unique isotropic vector in its radical, we have a natural homomorphism

$$W_{2,4,6}/W_{2,4,6}(2) \to O(E_{2,4,6}/E_{2,4,6} \cap 2E_{10}) = \text{Sp}(8, \mathbb{F}_2).$$

By Witt’s theorem it is surjective. The kernel of this homomorphism is equal to the quotient group $W_{2,4,6} \cap W(E_{10})(2)/W_{2,4,6}(2)$. This shows that $W_{2,4,6} \cap W(E_{10})(2) = W'_{2,4,6}$. \hfill \Box

**Corollary 8.4.8.** Suppose $k = \mathbb{C}$. Then there exists an open subset of the moduli space $\mathcal{M}_{\text{Enr}}^{\text{mod}}$ of nodal Enriques surfaces such that for any $S$ with the isomorphism class in $U$, $\text{Aut}(S) \cong W_{2,4,6}$. 
8.4. AUTOMORPHISMS OF A GENERAL NODAL SURFACE

Proof. We know that $\text{Aut}(S)^* = \text{Aut}(S)$ for any nodal surface and hence $\text{Aut}(S)$ contains $W_{2,4,6}$. Since $\text{Aut}(S)^*$ acts identically on $\mathcal{M}_{\text{Enr}}^{m,\text{nod}}$, the kernel of the action of $W(E_{10})$ on $\mathcal{M}_{\text{Enr}}^{m,\text{nod}}$ contains the group $W_{2,4,6}$. Since the kernel of the action is a normal subgroup of $W(E_{10})$ and the normal closure of $W_{2,4,6}$ contains $W_{2,4,6}$ with quotient $\text{Sp}(8,F_2)$, we obtain that either the kernel of the action coincides with $W_{2,4,6}$ or contains $W_{2,4,6}$. Since $W_{S,\text{nod}}^{\text{mod}} \subset W_{2,4,6}$, this is impossible. Since the set of points in $U$ such that some point in its pre-image $\tilde{U}$ in $\mathcal{M}_{\text{Enr}}^{m}$ has non-trivial finite stabiliser is a closed proper subset, we see that a general point in $\tilde{U}$ has trivial stabilizer, and hence $\text{Aut}(S) \cong W_{2,4,6}$ for such points. □

For any involution $\sigma$ of a lattice $M$ we set

$$M^+ = \{ m \in M : \sigma(m) = m \}, \quad M^- = \{ m \in M : \sigma(m) = -m \}.$$ 

It is obvious, that $(M^+)^\perp = M^-$. 

We choose a standard root basis $(\beta_0, \ldots, \beta_9)$ of $E_{2,4,6}$ with Coxeter-Dynkin diagram of type $T_{2,4,6}$ as above, we immediately observe one affine subdiagram of types $\widetilde{E}_7$ and one affine subdiagram of type $\widetilde{E}_8$. Let

$$\tau_1 = 2\beta_0 + \beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4 + 3\beta_5 + 2\beta_6 + \beta_7, \quad (8.4.4)$$

$$\tau_2 = 3\beta_0 + 2\beta_2 + 4\beta_3 + 6\beta_4 + 5\beta_5 + 4\beta_6 + 3\beta_7 + 2\beta_8 + \beta_9, \quad (8.4.5)$$

be their primitive isotropic vectors.

Lemma 8.4.9. The group $W(E_{2,4,6})$ acts transitively on the set of involutions $\sigma$ of $E_{2,4,6}$ such that $E_{2,4,6} \cong E_8$ and has two orbits on the set of involutions with $E_{2,4,6} \cong E_7$ or $E_{2,4,6} \cong E_7 \oplus A_1$.

Proof. It suffices to show that the group $W(E_{2,4,6})$ acts transitively (resp. with two orbits) on the set of primitive sublattices $M$ of the lattice $E_{2,4,6}$ isomorphic to $E_8$ (resp. $E_7$, resp. $E_7 \oplus A_1$). The lattice $E_8$ embeds as the direct summand $\langle \beta_0, \beta_2, \ldots, \beta_8 \rangle$ of $E_{2,4,6}$ with the orthogonal complement $\langle \tau_2, \tau_1 - \beta_9 \rangle$ isomorphic to $1^{11}(2) \cong A_1 \oplus A_1(-1)$. It follows from [532, Proposition 1.15.1] that a primitive embedding of $E_8$ is unique.

Since $D(E_7) \cong \mathbb{Z}/2\mathbb{Z}$ and $D(E_{2,4,6}) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$, the loc. cit result of Nikulin shows that there are two primitive embedding, up to isometry, of $E_{2,4,6}$. They can be represented by the sublattices $E_7$ and $\langle \beta_0, \beta_1, \ldots, \beta_6 \rangle$ and $\langle \beta_0, \beta_2, \ldots, \beta_7 \rangle$. Their orthogonal complements are $\langle \beta_8, \beta_9, \tau_1 \rangle \cong U \oplus A_1$ and $\langle \beta_9, \tau_1, \tau_2 \rangle \cong U(2) \oplus A_1$, respectively.

It follows that there are two primitive embedding of $E_7 \oplus A_1$, one with orthogonal complement isomorphic to $U$ and another with the orthogonal complement $U(2)$. □

Definition 8.4.10. A Kantor involution $K$ (resp. Bertini involution $B$, resp. Geiser involution $G$) of the lattice $E_{2,4,6}$ is an involution with $(E_{2,4,6}^-, E_{2,4,6}^+) \cong (E_7, U \oplus A_1)$ (resp. $(E_{2,4,6}, E_{2,4,6}^+) \cong (E_8, A_1 \oplus A_1(-1))$, resp. $(E_{2,4,6}, E_{2,4,6}^+) \cong (E_7 \oplus A_1, U(2))$.

Consider the following sublattices of $E_{2,4,6}$.

- $L_1 = \langle \beta_0, \beta_1, \ldots, \beta_6 \rangle \cong E_7$ with $L_1^+ = \langle \tau_1, \beta_8, \beta_9 \rangle \cong U \oplus A_1$. 

• \(L_2 = \langle \beta_0, \beta_2, \ldots, \beta_8 \rangle \cong E_8\) with \(L_2^+ = \langle r_2, r_1 - \beta_9 \rangle \cong A_1 \oplus A_1(-1)\).

• \(L_3 = \langle \beta_0, \beta_2, \ldots, \beta_7, \beta_9 \rangle \cong E_7 \oplus A_1\) with \(L_3^+ = \langle r_1, r_2 \rangle \cong U(2)\).

Note that we have
\[L_1 \perp L_1^+ \cong L_2 \perp L_2^+ \cong E_{2,4,6},\]
and \(L_3 \perp L_3^+\) is a sublattice of index 2 in \(E_{2,4,6}\).

The involution \(-id_{L_i} \oplus id_{L_i^+}\) is a Kantor (resp. Bertini, resp. Geiser) if \(i = 1\) (resp. 2, resp. 3).

It follows from the definition that \(B, K, G\) act trivially on the discriminant groups of \(L_i \perp L_i^+\), hence can be extended to involutions of the Enriques lattice \(E_{10}\). We will often identify these involutions with the corresponding extended involutions of \(E_{10}\).

The main ingredient of the proof of our main result is the following analogue of Proposition 8.3.3.

**Theorem 8.4.11.** The group \(W_{2,4,6}(2)\) (resp. \(W_{2,4,6}(2)\)) as defined in (8.4.3) is equal to the smallest normal subgroup of \(W_{2,4,6}\) containing the involutions \(B\) and \(K\) (resp. \(B\), \(K\) and \(G\)).

A computational proof can be found in [134]. We refer for a conceptual proof to [8].

We will show now that the involutions \(B\), \(K\) and \(G\) can be realized by automorphisms of a general nodal Enriques surface.

Fix a Reye marking \(\phi : \text{Rey}(S) \to E_{2,4,6}\) and consider a geometric canonical basis \((\beta_0, \ldots, \beta_9)\) with \(\phi(\beta_i) = \beta_i\). By definition of a Reye marking, \(\phi^{-1}(r)\) is the class of a \((-2)\)-curve \(R\) with \([R] = r\). Using (8.4.1), we can extend the Reye marking to a marking \(\tilde{\phi} : \text{Num}(S) \to E_{10}\) that defines an isotropic sequence \((f_1, \ldots, f_{10})\) such that \(R = R_{10} := h_{10} - 2f_{10}\), where \(h_{10}\) is the numerical Fano polarization defined by the isotropic sequence. Let \(f_{ij} = h_{10} - f_i - f_j, 1 < i < j \leq 10\) and \(R_i = h_{10} - 2f_i, i = 1, \ldots, 10\).

**Proposition 8.4.12.** Let \(K\) be a Kantor involution in \(W(E_{2,4,6})\). Then there exists a marking \(\phi : \text{Num}(S) \to E_{10}\) with the associated Reye marking \(\phi_r : \text{Rey}(S) \to E_{2,4,6}\) and an automorphism \(g_K\) of \(S\) such that \(K = \phi_r \circ g_K^* \circ \phi_r^{-1}\).

**Proof.** In the notation from above, we have \(f_{8,9} \cdot f_{9,10} = 1\) and \(f_{8,9} \cdot R_9 = f_{9,10} \cdot R_9 = 0\). Let \(|2F_{i,j}|\) be an elliptic pencil with \([F_{i,j}] = f_{i,j}\). Let us consider the bielliptic map defined by the linear system \(|D| = |2F_{8,9} + 2F_{9,10}|\) of degree \(D^2 = 8\). This is a polarization of type (i) in Proposition 3.3.1 (note that \([F_{8,9} + F_{9,10} - R_9]\) can be taken as the class of an isotropic nef class). The bielliptic map \(S \to D\) blows down \(R_9\) to a point \(x_0 \in D\), and the composition of this map with the projection from \(x_0\) defines a regular degree 2 map \(f : S \to C\), where \(C\) is the Cayley symmetroid cubic surface in \(\mathbb{P}^3\). Recall from section 0.7 that it is obtained from a 4-nodal anti-canonical quartic del Pezzo surface \(D_4\) by blowing up one point \(x_0\) (corresponding to the point \(p_0\) in the notation of Propositions 0.7.2, 0.7.3, 0.7.4). Since \(R\) is not a component of a double fiber, the point \(x_0\) does not lie on a line on \(D\). It follows from Theorem 6.5.4 that no other \((-2)\)-curve is blown down, so \(f\) is a finite separable cover of degree 2. The pencils \(|2F_1|\) and \(|2F_2|\) are the pre-images of pencils of conics on \(\text{CM}(V)\) and the image of \(R\) is a line on \(C\). It is easy to see that \(f^*(\text{Pic}(C)) = \text{Pic}(S)^{g_K}_Q\).
8.4. AUTOMORPHISMS OF A GENERAL NODAL SURFACE

is generated over \( \mathbb{Q} \) by the classes of \( F_{8,9}, F_{9,10} \) and \( R_9 \). Since \( 2F_{ij} \sim R_i + R_j \), it is also generated by \( R_8, R_9, R_{10} \). Let \( g_K \) be the deck transformation of this cover. The Picard group of \( C \) is of rank 3 and is generated over \( \mathbb{Q} \) by the classes of \( f(F_{8,9}), f(F_{9,10}), f(R_9) \). For any divisor class \( A \), there exist rational numbers \( m, n, l \) such that

\[
g_K^*(A) + A = mR_8 + nR_9 + lR_{10}.
\]

Suppose \( A \cdot R_9 = A \cdot R_9 = A \cdot R_{10} = 0 \). Intersecting both sides with \( R_8, R_9, R_{10} \), we obtain \( m = n = l = 0 \). This shows that \( g_K^*(A) = -A \). Observe, that the sublattice of \( \text{Num}(S) \) generated by \( R_8, R_9, R_{10} \) is isomorphic to \( \mathbb{U}(2) \oplus A_1 \) and its orthogonal complement is isomorphic to \( L_1 \). More precisely, we have the following formula for \( g_K^* \):

\[
g_K^*(x) = -x + (x \cdot f_{9,10})R_8 + (x \cdot f_{8,10})R_9 + (x \cdot f_{8,9})R_{10}.
\]

(8.4.6)

It is clear that \( \phi_r \circ g_B^* \circ \phi_r^{-1} \) is the Kantor involution \( K \).

**Proposition 8.4.13.** Let \( B \) be a Bertini involution in \( W(\mathbb{E}_{2,4,6}) \). Then there exists a marking \( \phi : \text{Num}(S) \to \mathbb{E}_{10} \) with the associated Reye marking \( \phi_r : \text{Rey}(S) \to \mathbb{E}_{2,4,6} \) and an automorphism \( g_B \) of \( S \) such that \( B = \phi_r \circ g_B^* \circ \phi_r^{-1} \).

**Proof.** This time we choose a Reye marking as in the previous case and choose the linear system \( |4F_9 + 2R_{10}| \). It defines a bielliptic map \( f : S \to D' \) to an anti-canonical del Pezzo surface \( D' = D'_1, D'_2, \) or \( D'_3 \). The curve \( R \) is blown down to the singular point of \( D' \) and the pencil \( |2F_1| \) is the pre-image of a pencil of conics on \( D' \) that contains a double line. It follows from Propositions 0.7.2, 0.7.3 and 0.7.4 that \( f \) defines a finite map of degree 2 \( f' : S \to C' \) to a degenerate cubic symmetroid surface. The image \( f'(R_{10}) \) is a line on \( C' \) (blown up from the point \( p_0 \) in the notation of the Propositions). The involution \( B \) is realized by the deck transformation \( g_B \) of the double cover \( f' : S \to D' \). In fact, as above, we have \( g_B^*(F_9) = F_9, g_B^*(R_{10}) = R_{10} \) and the numerical class of any \( g_B \)-invariant curve is contained in \( \mathbb{Q}[F_9] + \mathbb{Q}[R_{10}] = f^*(\text{Pic}(C') \mathbb{Q}) \cong \mathbb{Q}^2 \). We have \( g_B^*(x) = -x \) for any numerical class orthogonal to the sublattice \( \langle [F_9], [R_{10}] \rangle = \langle f_9, h_{10} - 2f_{10} \rangle \). Intersection of this sublattice with \( \text{Rey}(S) \) is equal to \( \langle 2f_9, h - 2f_{10} \rangle \cong A_1 \oplus A_1(-1) \). It is easy to see that its orthogonal complement contains the sublattice \( \langle \beta_0, \beta_2, \ldots, \beta_8 \rangle \cong \mathbb{E}_8 \). Since the discriminant of \( A_1 \oplus A_1(-1) \oplus \mathbb{E}_8 \) is equal to the discriminant of the whole lattice \( \mathbb{E}_{2,4,6} \), we obtain that \( \langle 2f_9, h - 2f_{10} \rangle \cong \mathbb{E}_8 \) and \( \mathbb{E}_{2,4,6} \cong \mathbb{E}_8 \oplus A_1 \oplus A_1(-1) \cong A_1 \oplus \mathbb{E}_7 \oplus \mathbb{U} \). The involution \( g_B \) acts on \( \text{Num}(S) \) by the formula:

\[
g_B^*(x) = -x + 2(x \cdot R_{10})f_9 + 2(x \cdot f_9)(2f_9 + R_{10}).
\]

(8.4.7)

Thus \( g_B^* \) realizes the Bertini involution \( B \).

**Proposition 8.4.14.** Let \( G \) be a Geiser involution in \( W(\mathbb{E}_{2,4,6}) \). Then there exists a marking \( \phi : \text{Num}(S) \to \mathbb{E}_{10} \) with the associated Reye marking \( \phi_r : \text{Rey}(S) \to \mathbb{E}_{2,4,6} \) and an automorphism \( g_G \) of \( S \) such that \( G = \phi_r \circ g_G^* \circ \phi_r^{-1} \).

**Proof.** We have \( s_{\beta_9}(R_{10}) = s_{\beta_9}(h_{10} - 2f_{10}) = h_{10} - 2f_9 \). By Lemma 6.3.14, \( h_{10} - 2f_9 \) is effective. Since \( h_{10} \cdot (h_{10} - 2f_9) = 4 \), it must be the class of the \((-2)\)-curve \( R_9 \). We have \( R_9 + R_{10} \) is a member of the elliptic pencil \( |2(H_{10} - F_9 - F_{10})| \) with a half-fiber fiber \( F_{9,10} \sim H_{10} - F_9 - F_{10} \). Consider the bielliptic linear system \( |2F_1 + 2F_{9,10}| \). It defines a bielliptic map \( f : S \to D_1 \), where \( D_1 \) is a 4-nodal anti-canonical quartic del Pezzo surface. We have \( (F_1 + F_{9,10}) \cdot R = 1 \), hence any
(--2)-curve intersects $F_1 + F_{9,10}$ with odd multiplicity. Thus the linear system $|2F_1 + 2F_{9,10}|$ is ample, and the map $f : S \to D_1$ is a degree 2 finite map. Let $g_G$ be its deck transformation. Then we argue as in the previous cases to obtain that $g_G$ acts as the identity on $\langle f_1, f_{9,10} \rangle$ and as the minus identity on $\langle f_1, f_{9,10} \rangle$. The intersection of $\langle f_1, f_{9,10} \rangle$ with Rey$(S)$ is equal to $(2f_1, F_{9,10}) \cong U(2)$. Its orthogonal complement in Rey$(S)$ is equal to the sublattice $\langle \beta_0, \beta_2, \ldots, \beta_6, \beta_8 \rangle \cong E_7 \oplus A_1$. Let $f_{9,10} = [F_{9,10}]$. We have the formula for $g_G^2$

$$g_G^2(x) = -x + 2(x \cdot f_{9,10})f_1 + 2(x \cdot f_1)f_{9,10}. \quad (8.4.8)$$

Thus $g_G^2$ realizes the Geiser involution $G$.

**Corollary 8.4.15.** Let $W(E_{2,4,6})^+ = \langle (K, B, G) \rangle$ be the minimal normal subgroup of $W(E_{2,4,6})$ that contains a Kantor, a Bertini and a Geiser involution. Then there exists a Reye marking $\phi_r : \text{Rey}(S) \to E_{2,4,6}$ such that $\phi_r^{-1} \cdot W_{2,4,6}^+ \cdot \phi \subset \text{Aut}(S)^*$.

**Proof.** The previous three propositions realize some representatives of the conjugacy classes of Kantor, Bertini and Geiser involution in $W(E_{2,4,6})$. Since $W_{2,4,6}$ transforms the root $\alpha$ to a root $\beta$ congruent to $\alpha$ modulo $2E_{10}$, we can apply Lemma 6.3.14 to see that under the new Reye marking $\beta$ goes to the class of a $(-2)$-curve on $S$. Thus, using the above propositions, we will be able to realize all such involutions and, of course, their products.

Applying Theorem 8.4.11, we obtain the main result of this section.

**Theorem 8.4.16.** Let $S$ be a general nodal Enriques surface defined over a field of arbitrary characteristic. Then there is a homomorphism $\text{Aut}(S) \to W_{2,4,6}$ whose kernel is trivial if $p \neq 2$ or $K_S = 0$ (see Proposition 8.3.1), and its image contains the normal subgroup $W(E_{2,4,6})(2)'$ with the quotient isomorphic to $\text{Sp}(8, \mathbb{F}_2)$.

**Corollary 8.4.17.** All smooth rational curves on a general nodal Enriques surface form one orbit with respect to the automorphism group.

**Proof.** By Lemma 6.5.1, for any two $(-2)$-curves $R_1, R_2$, we have $R_1 \cdot R_2 \geq 2$. This shows that $(R_1 + R_2)^2 \geq 0$, and hence the hyperplanes $H_{[B_1]}$ and $H_{[B_2]}$ in the hyperbolic space $\mathbb{H}^9$ do not intersect in $\mathbb{H}^9$. Since $W_{S}^{\text{nod}} \rtimes W_{2,4,6}$ is a subgroup of finite index of $W_{S}^{\text{nod}} \rtimes W_{2,4,6} = W(E_{2,4,6})$, it has a rational polyhedral fundamental domain in the nef cone. Since the representatives of the orbits of $W_{2,4,6}$ on the set of hyperplanes $H_S$ are boundary faces of the polyhedron, no two of them must diverge. This contradiction proves the assertion.

Let us compute the fibers of the forgetting map $\mathcal{M}_{\text{Enr},v} \to \mathcal{M}_{\text{Enr}}$ over the locus of general nodal Enriques surfaces as promised in Section 5.4.

We use Corollary 8.4.5 and its notation that describes the nef cone $\text{Nef}(S)$. Let $\beta_i^*$ be the dual vectors of the standard root basis $(\beta_1, \ldots, \beta_{10})$ of the Reye lattice $E_{2,4,6}$. The inverse matrix $C^{-1} = (\beta_i^*, \beta_j)$ of the Cartan matrix $C = (\beta_i, \beta_j)$ is equal to the matrix
8.5 Automorphisms of a Cayley quartic symmetroid surface

In this section we explain the names Kantor, Bertini, and Geiser attached to the involutions studied in the previous section. We will also give another proof of Theorem 8.4.11 that follows closely the ideas of A. Coble. We assume that \( p \neq 2 \) and \( S \) is a general nodal surface and consider a Fano-Reye polarization \( H = H_{10} \) that defines an isomorphism from \( S \) onto a Reye congruence
Rey(W) ⊂ G1(⟨H⟩∗) of an excellent web of quadrics in P3. We know from Corollary 7.9.8 that
the canonical cover X of S is isomorphic to a minimal nonsingular model D(W) of a quartic
symmetroid surface D(W). We have 10 elliptic fibrations |2F| with |F| = f1 and 10 smooth
canonical cover \( \overline{W} \) such that \( R_i \in |2F| \), where \( F_{i,j} = H - F_i - F_j \)
is a double fiber of the elliptic pencil \( |2F| \). The pre-images of the curves \( R_i \) on \( X \) are 20 smooth
canonical cover \( \Theta_0, \tau^*(\Theta_i) \), where \( \tau \) is the deck Enriques involution of \( X \). By Proposition 7.7.1, we
have divisor classes \( \eta_S, \eta_H \) on \( X \) such that

\[
2\eta_S = 3\eta_H - \Theta_1 - \cdots - \Theta_{10}.
\]

The linear system \( |\eta_H| \) maps \( X \) to \( P^3 \) onto the quartic symmetroid \( D(W) \) and the linear system
\( |\eta_S| \) maps \( X \) onto a smooth quartic surface in another \( P^3 \). The curves \( \Theta_i \) are identified with the
cexceptional curves of \( D(W) \to D(W) \).

We make a further assumption that rank Pic(\( X \)) = 11, i.e. \( \eta_H, \eta_S, \Theta_i \) generate Pic(\( X \))\( \big|_{Q} \). Applying
Proposition 7.7.4, we obtain that \( \eta_S, \eta_H, \Theta_1, \ldots, \Theta_9 \) form a basis of Pic(\( X \)).

Let \( \mathcal{P} = \{p_1, \ldots, p_{10}\} \) be the ten nodes of the quartic symmetroid \( D(W) \) and let \( \text{Bl}_\mathcal{P} \to \mathbb{P}^3 \)
be the blow-up of \( \mathbb{P}^3 \) with center at the closed reduced subscheme defined by the set \( \mathcal{P} \). Let \( E_i = \sigma^{-1}(p_i) \)
be the exceptional divisors of the blow-up. Let \( e_0 \) be the divisor class corresponding to \( \sigma^*O_{\mathbb{P}^3}(1) \)
and \( e_i \) be the divisor classes of the exceptional divisors \( E_i \). We have

\[
K_{\text{Bl}_\mathcal{P}} = -4e_0 + e_1 + \cdots + e_{10}.
\]

The proper transform of \( D(W) \) under the blow-up is isomorphic to \( X \). Let

\[
r : \text{Pic}(\text{Bl}_\mathcal{P}) \to \text{Pic}(X)
\]
be the restriction homomorphism. We have
\[ t = r(\frac{1}{2} K_{\text{Bl} P}) = -2\eta_H + \Theta_1 + \cdots + \Theta_{10} = -2\eta_S + \eta_H. \]

**Lemma 8.5.1.** Let \( \tau \) be the Enriques involution of \( X \). Then \( \tau^* \) acts on \( \text{Pic}(X) \) as the reflection
\[ s_t : x \mapsto x + \frac{1}{2}(x \cdot t)t. \]

**Proof.** It is enough to check this formula on the classes \( \Theta_i, \eta_S, \eta_H \). We have \( t^2 = -4, t \cdot \Theta_i = -2, t \cdot \eta_S = -2, t \cdot \eta_H = -8 \). Thus \( \frac{1}{2} t \in \text{Pic}(X)^{\vee} \), hence we can consider the reflection
\[ s_t : x \mapsto x + \frac{1}{2}(x \cdot t)t. \]

Applying Proposition 7.7.2, we check that
\[
\begin{align*}
\tau^*(\eta_S) &= 3\eta_S - \eta_H = s_t(\eta_S), \\
\tau^*(\eta_H) &= 8\eta_S - 3\eta_H = s_t(\eta_H), \\
\tau^*(\Theta_i) &= \Theta_i - t = s_t(\Theta_i).
\end{align*}
\]

\[ \square \]

Now we have to invoke the theory of Cremona actions of Weyl groups on the configuration spaces of points in the projective space (see [121], [170], [179]). Let \( e^0 \) be the class of the preimage of a line in \( \text{Bl} P \) in the Chow ring \( A^*(\text{Bl} P) \), and let \( e^i \) be the class of a line in the exceptional divisor \( E_i \). The intersection bilinear form \( A^1(\text{Bl} P) \times A^2(\text{Bl} P) \rightarrow \mathbb{Z} \), gives us that
\[
(e_i, e^j) = 0, \quad i \neq j, \quad (e_0, e^0) = 1, \quad (e_i, e^i) = -1.
\]
Consider the map \( \Psi : A^1(\text{Bl} P) \rightarrow A^2(\text{Bl} P) \) defined by sending \( e_0 \) to \( 2e^0 \) and \( e_i \) to \( e^i \) and define a quadratic lattice structure on \( A^1(\text{Bl} P) \) by
\[ x \cdot y = (x, \Psi(y)). \]

Obviously, it is isomorphic to the odd lattice \( \langle 2 \rangle \perp \langle -1 \rangle^{\oplus 10} \). Let
\[ \kappa = \Psi(\frac{1}{2}K_{\text{Bl} P}) = 4e_0 - e_1 - \cdots - e_{10}. \]

Then
\[ N_P = K_{\text{Bl} P}^+ = \{ x \in A^1(\text{Bl} P) : (x, \kappa) = 0 \} = \langle e_0 - e_1 - e_2 - e_3 - e_4, e_1 - e_2, \ldots, e_9 - e_{10} \rangle. \]
Consider the classes
\[ \alpha_0 = e_0 - e_1 - e_2 - e_3 - e_4, \quad \alpha_i = e_i - e_{i+1}, \quad i = 1, \ldots, 9. \]

We have \( \alpha_i^2 = -2 \) and \( (\alpha_i, \alpha_j) = -2I_{10} + \Gamma \), where \( \Gamma \) is the incidence matrix of the graph \( T_{2,4,6} \). Thus \( B = (\alpha_0, \ldots, \alpha_9) \) is a standard root basis of the lattice \( K_{\text{Bl} P}^+ \cong E_{2,4,6} \). The reflection group generated by the reflections \( s_{\alpha_i} \) is isomorphic to \( W_{2,4,6} \). Let \( \mathcal{P} \) be the set of roots of \( E_{2,4,6} \), i.e. elements of the orbit of \( W_B(E_{2,4,6}) \) of any simple root \( \alpha_i \). We say that a root \( \alpha \) is effective if \( \Psi(\alpha) \) can be represented by the class of a curve on \( \text{Bl} P \). For example, \( \alpha_i \) is always effective since we assuming that \( p_i \neq p_j \). The root \( \alpha_0 \) is effective if and only if \( p_1, p_2, p_3, p_4 \) do not lie on a conic, in particular, they are not coplanar. We say that \( \mathcal{P} \) is an unmodal set if all roots are not effective. One
can show that the subset of $(\mathbb{P}^3)^{10}$ which consists of unnodal sets is equal to the complement of a countable set of closed proper subsets ([179], Lemma 2.2).

Let $P_3^{10}$ be the GIT-quotient of $(\mathbb{P}^3)^{10}$ by the group $\text{SL}(4)$ acting diagonally with respect to the linearization defined by the invertible sheaf $\mathcal{O}_{\mathbb{P}^3}^{\otimes 10}$. Let $\text{un} P_3^{10}$ be the subset of orbits of unnodal sets. We denote by $|\mathcal{P}|$ the orbit of an unnodal set. The group $W_{2,4,6}$ acts on $\text{un} P_3^{10}$ as follows. Its subgroup generated by simple roots $s_i$, $i \neq 0$, acts by permuting the points. The element $s_0$ acts as follows. We find a representative of $\mathcal{P}$ such that the first 4 points have the projective coordinates $[1,0,0,0],[0,0,1,0],[0,0,1,0],[0,0,0,1]$. Then we consider the standard quadratic Cremona transformation $T_0$ that sends $[t_0,\ldots,t_3]$ to $[t_0^{-1},t_1^{-1},t_2^{-1},t_3^{-1}]$ and then send $(p_1,\ldots,p_{10})$ to $(p_1,p_2,p_3,p_4,T_0(p_5),\ldots,T_0(p_{10}))$.

We use the basis $(e_0,\ldots,e_{10})$ to define an isomorphism of lattices

$$\phi_{\mathcal{P}} : (2) \oplus \langle -1 \rangle^{\otimes 10} \rightarrow A^1(\text{Bl}_\mathcal{P}).$$

We call it a geometric marking. Its restriction to $K_{\mathcal{P}}^{\perp}$ defines an isomorphism of lattices $\phi'_{\mathcal{P}} : E_{2,4,6} \rightarrow K_{\mathcal{P}}^{\perp}$.

**Proposition 8.5.2.** For any $w \in W_{2,4,6}$ and a geometric marking $\phi_{\mathcal{P}}$, the isomorphism

$$\phi_{\mathcal{P}} \circ w^{-1} : (2) \oplus \langle -1 \rangle^{\otimes 10} \rightarrow A^1(\text{Bl}_\mathcal{P})$$

is a geometric marking defined by some unnodal set of points $Q$ such that $w(|\mathcal{P}|) = |Q|$.

Suppose that $w(|\mathcal{P}|) = |\mathcal{P}|$, i.e. $w$ belongs to the stabilizer of $W_{2,4,6}$ in its action on $\text{un} P_3^{10}$. Then Proposition 2 from loc.cit. tells us that there exists a birational automorphism $f : \text{Bl}_\mathcal{P} \dasharrow \text{Bl}_\mathcal{P}$ which is an isomorphism outside of a closed subset of dimension 1 (a pseudo-automorphism) such that $w = \phi_{\mathcal{P}}^{-1} \circ f^* \circ \phi_{\mathcal{P}}$.

Observe that under the restriction homomorphism $r : A^1(\text{Bl}_\mathcal{P}) = \text{Pic}(\text{Bl}_\mathcal{P}) \rightarrow \text{Pic}(X)$, the image of the lattice $K_{\text{Bl}_\mathcal{P}}^{\perp}$ is equal to the sublattice

$$\pi^*(\text{Rey}(S)) = \langle \eta_1 - \Theta_1 - \Theta_2 - \Theta_3 - \Theta_4, \Theta_1 - \Theta_2, \ldots, \Theta_9 - \Theta_{10} \rangle.$$

However, it is not an isomorphism of lattices, but becomes isomorphism of the lattices

$$K_{\text{Bl}_\mathcal{P}}^{\perp}(2) \rightarrow \pi^*(\text{Rey}(S)) \cong \text{Rey}(S)(2) \cong E_{2,4,6}(2).$$

Suppose $w \in W_{2,4,6}$ satisfies $w(|\mathcal{P}|) = |\mathcal{P}|$ and hence defines a pseudo-automorphism $f_w : \text{Bl}_\mathcal{P} \dasharrow \text{Bl}_\mathcal{P}$. It follows from the definition that $w$ leaves invariant the linear system $|-K_{\mathcal{P}}^{\perp}|$. Now observe that $\mathcal{P}$ consists of the nodes of a quartic symmetroid $D(W)$, so $|-K_{\mathcal{P}}^{\perp}| = \{D(W)\}$. This shows that $f_w|X$ is a birational transformation. Since $X$ is a minimal surface of non-negative Kodaira dimension, it extends to an automorphism of $X$.

Coble shows that the Kantor, Bertini, or Geiser involution $w$ of $E_{2,4,6}$ belongs to the stabilizer of $|\mathcal{P}|$ and hence defines automorphism of $X$ that commutes with $\tau$ and descends to the automorphisms $g_K, g_B, g_G$ which we used before.

The Kantor involution is defined by the 6-dimensional linear system $|Q|$ of quartics with double points at the nodes $p_1,\ldots,p_7$ of the symmetroid. This linear system defines a degree 2 rational map $\mathbb{P}^3 \dasharrow V \subset \mathbb{P}^6$, where $V$ is a projective cone over a Veronese surface in $\mathbb{P}^5$. Consider the net $|L|$ of quadrics through the seven points and, automatically, through the eighth point.
8.5. AUTOMORPHISMS OF A CAYLEY QUARTIC SYMMETROID SURFACE

It defines a rational map \( \mathbb{P}^3 \rightarrow \mathbb{P}^2 \) with eight indeterminacy points. Blowing up these points, we obtain an elliptic fibration \( f : Y \rightarrow |L^\vee| \cong \mathbb{P}^2 \) with a section defined by the exceptional divisor \( E(q) \) over the point \( q \). Let \( \sigma_K \) be the rational involution of \( Y \) that is defined by the negation automorphism \( y \mapsto -y \) on the general fiber. It extends to a pseudo-automorphism of \( Y \), called the Kantor involution (see [121]). The base scheme of a general pencil \( P \) in \( |L| \), considered as a point in \( |L^\vee| \), is a quartic elliptic curve \( E(P) \) isomorphic to the fiber of \( f \) over the point \( P \). Take a point \( p \) on a quartic elliptic curve \( E(P) \) through \( p_1, \ldots, p_7, q \). A quartic surface with seven nodes at \( p_1, \ldots, p_7 \) passing through the point \( p \) intersects \( E(P) \) at one more point \( p' \). It cuts out the divisor \( 2(p_1 + \cdots + p_7) + p + p' \) on \( E(P) \). On the other hand a quartic with 8 nodes at \( p_1, \ldots, p_7, q \) cuts out the divisor \( 2(p_1 + \cdots + p_7) + 2q \). This shows that \( p + p' \sim 2q \), so that \( p' = -p \) in the group law on \( E(P) \). Thus the involution \( p \mapsto p' \) on \( E(P) \) is the restriction of the Kantor involution to \( E(P) \). The set of fixed points of \( \sigma_K \) restricted to \( E(P) \) is the set of four 2-torsion points. The set of fixed points of \( \sigma_K \) in \( Y \) is the closure of the set of 2-torsion points on a generic fiber of \( f \). It is equal to the union of the exceptional divisor \( E(q) \) and the proper inverse transform of a certain surface of degree 6 with triple points at \( p_1, \ldots, p_7 \), the Cayley dianome surface (see [121], [170]).

Our quartic symmetroid surface is a quartic with 10 nodes, including the nodes at \( p_1, \ldots, p_7 \). It does not pass through the point \( q \) (otherwise the quadrics through \( p_1, \ldots, p_8 \) will cut out the net \( |2\eta_H - \Theta_1 - \cdots - \Theta_8| \), however, by Riemann-Roch, this is just a pencil). Thus the three remaining nodes of the symmetroid are fixed and this makes the symmetroid invariant under \( \sigma_K \).

For any general plane \( \Pi \) in \( \mathbb{P}^3 \), the linear system \( |Q| \) maps \( \Pi \) to a surface of degree 16 which is cut out in \( V \) by a quartic hypersurface. This shows that \( \sigma_K^*(\Pi) \) \( \sim f^*(4H) \), where \( H \) is a hyperplane section in \( \mathbb{P}^6 \). Let \( e_0, e_1, \ldots, e_7 \) be the geometric basis of \( \text{Pic}(\text{Bl}_{p_1, \ldots, p_7}) \). Then

\[
\sigma_K^*(e_0) = 4(4e_0 - 2(e_1 + \cdots + e_7)) - e_0 = 15e_0 - 8(e_1 + \cdots + e_7).
\]

Similarly, we see that the proper transform of the linear system \( |Q| \) to \( Y \) maps each exceptional divisor \( E(p_i) \) to a surface of degree 2 cut out from \( V \) by a quadric. This gives

\[
\sigma_K^*(e_i) = -e_i + 2(2e_0 - e_1 - \cdots - e_7), \quad i = 1, \ldots, 7.
\]

Restricting to \( X \), we get

\[
\begin{align*}
\sigma_K^*(\eta_H) &= 15\eta_H - 8\Theta_1 - \cdots - 8\Theta_7, \\
\sigma_K^*(\Theta_i) &= -\Theta_i + 2(2\eta_H - \Theta_1 - \cdots - \Theta_7), \quad i = 1, \ldots, 7, \\
\sigma_K^*(\Theta_j) &= \Theta_j, \quad j = 8, 9, 10.
\end{align*}
\]

One rewrite these formulas in the following way

\[
\sigma_K^*(x) = -x + (x \cdot \tilde{f}_{9,10})\Theta_8 + (x \cdot \tilde{f}_{8,10})\Theta_9 + (x \cdot \tilde{f}_{8,9})\Theta_{10},
\]

where \( \tilde{f}_{i,j} = \pi^*((F_{i,j})) \). Comparing this with formula (8.4.6), we find that \( \sigma_K \) is the lift to \( X \) of a Kantor involution on \( S \). The involution \( \sigma_K \) commutes with the Enriques involution \( \tau \) and descends to an involution of \( S \) whose action on \( \text{Num}(S) \) coincides with the action of a Kantor involution \( g_K \). Since no automorphism of \( S \) acts identically on \( \text{Num}(S) \), we obtain that it coincides with \( g_K \).

Next we consider the lift of a Bertini involution. Recall that a Bertini involution of \( \mathbb{P}^2 \) is defined by a choice of 8 points \( q_1, \ldots, q_8 \) such that its blow-up \( \text{Bl}_{q_1, \ldots, q_8} \) is a weak del Pezzo surface \( D \) of degree 1. The linear system \( |-2K_P| \) defines a degree 2 finite map onto a singular quadric \( Q \) in \( \mathbb{P}^3 \) with a smooth canonical curve of degree 6 as the branch locus (see [180, 8.8]). The deck
transformation of the cover $\text{Bl}_{q_1,\ldots,q_8} \to \mathbf{Q}$ defines a birational involution of the plane, called a \textit{Bertini involution}. One can also define it by choosing the pencil of cubic curves through $p_1,\ldots,p_8$ and, automatically, an additional point $q_9$. For any nonsingular member $E$ of the pencil, the point $q_9$ defines the group law on $E$ with 0 equal to $q_9$. For a general point $x$ in the plane, we find a member $E(x)$ of the pencil passing through $x$ and define $x' = -x \in E(x)$. This is the Bertini involution associated to 8 points in the plane. We see that the Kantor involution is its 3-dimensional analogue.

For any birational transformation $T : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ given by homogeneous polynomials $G_0, \ldots, G_n$ of degree $d$ in variables $t_0, \ldots, t_n$, one defines a \textit{dilated transformation} of $\mathbb{P}^{n+1}$ as follows. Choose homogeneous polynomial $F_{n+1}(t_0, \ldots, t_n, t_{n+1}) = t_{n+1}A_1(t_0, \ldots, t_n) + A_2(t_0, \ldots, t_n)$ of degree $d + r$ and homogeneous polynomials $Q(t_0, \ldots, t_n, t_{n+1}) = t_{n+1}B_1(t_0, \ldots, t_n) + B_2(t_0, \ldots, t_n)$ of degree $d + r$ such that $A_1B_2 - A_2B_1 \neq 0$. Then the transformation defined by the polynomials $(QG_0, QG_1, \ldots, QG_n, F_{n+1})$ is a birational transformation $\widetilde{T}$ of $\mathbb{P}^{n+1}$ such that

$$\text{pr}_o \circ \widetilde{T} = T \circ \text{pr}_o,$$

where $o = [0, \ldots, 0, 1] \in \mathbb{P}^{n+1}$ and $\text{pr}_o : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^n$ is the projection from $o$ to the hyperplane $t_{n+1} = 0$ (see [121], [560]). The base scheme of the linear system defining the dilated transformation is the cone with the vertex at $o$ over the base scheme $B$ of the linear system defining the transformation $T$. It follows from the definition of a dilated transformation that the multiplicity of a general member of the linear system defining $\widetilde{T}$ at the point $o$ is equal to $d + r - 1$. Let $n_i$ (resp. $k_i$) be the minimal of multiplicities of $F_{n+1}$ (resp. $Q$) at the line $\overline{o, q}$, where $q \in B$. Then a general member of the linear system defining the base locus $\overline{B}$ of $\widetilde{T}$ has multiplicity at this line equal to $\min\{n_i, k_i + m_i\}$, where $m_i$ is the multiplicity of $q$ in $B$.

For example, if we take for $T$ the standard quadratic transformation $T_0 : [t_0, t_1, t_2] \mapsto [t_1t_2, t_0t_2, t_0t_1]$ and take $F_3 = t_0t_1t_2$ and $Q = t_3$, we obtain the standard cubic transformation of $\mathbb{P}^3$ defined by the formula

$$\widetilde{T}_0 : [t_0, t_1, t_2, t_3] \mapsto [t_1t_2t_3, t_0t_2t_3, t_0t_1t_3, t_0t_1t_2].$$

It is known that the planar Bertini transformation $\beta$ acts on the Pic(Bl$_{q_1,\ldots,q_8}$) by the formula

$$\beta^*(e_0) = 17e_0 - 6(e_1 + \cdots + e_8),$$
$$\beta^*(e_i) = -e_i + 6e_0 - 2(e_1 + \cdots + e_8),$$

where $(e_0, e_1, \ldots, e_8)$ is the natural basis in Pic(Bl$_{q_1,\ldots,q_8}$) (see [180], 8.8.2). We hope that no confusion arises because we are using the same notations for a geometric basis of the blow-up of $\mathbb{P}^2$ and $\mathbb{P}^3$.

Let $p_1, \ldots, p_9$ be general points in $\mathbb{P}^3$, and $q_1, \ldots, q_8$ be their projections to a general plane in $\mathbb{P}^3$ from the point $p_9$. Let $\beta$ be the Bertini involution in the plane defined by the points $q_1, \ldots, q_8$ and $\overline{\beta}$ be its dilation defined by the point $o = p_9$. Coble shows that one can choose polynomials $F_3$ and $Q$ such that the dilated transformation $\overline{\beta}$ is given by the linear system of hypersurfaces of degree 33 vanishing at the point $p_9$ with multiplicity 32 and vanishing on the lines $\overline{p_9, p_i}$ with multiplicity 12. In the geometric basis of the blow-up $\text{Bl}_{p_1,\ldots,p_9}$, the transformation $\overline{\beta}$ acts by the formula

$$\overline{\beta}^*(e_0) = 33e_0 - 32e_9 - 12(e_1 + \cdots + e_8),$$
$$\overline{\beta}^*(e_i) = 6e_0 - 2(e_1 + \cdots + e_8) - e_i - 6e_9, \quad i = 1, \ldots, 8,$n$$
$$\overline{\beta}^*(e_9) = 16e_0 - 2(e_1 + \cdots + e_8) - 15e_9.$$
Let $D(W)$ be a quartic symmetroid with nodes at $p_1, \ldots, p_{10}$. One checks that $\tilde{\beta}$ leaves the linear system $|4e_0 - 2(e_1 + \ldots + e_9)|$ invariant. In particular, the image of the symmetroid under $\tilde{\beta}$ is a quartic surface with nodes at $p_1, \ldots, p_9, \tilde{\beta}(p_{10})$. One can show that the nine nodes of a quartic symmetroid determine the tenth one. Thus $\tilde{\beta}(D(W))$ is a quartic symmetroid with the same set of nodes, so it must coincide with $D(W)$. This shows that $\tilde{\beta}$ defines an automorphism $\sigma_B$ of $D(W)$ that acts on the natural basis of the Picard lattice of a minimal resolution $X$ of $D(W)$ as follows.

$$
\begin{align*}
\sigma_B(\eta_H) &= 33\eta_H - 32\Theta_9 - 12(\Theta_1 + \cdots + \Theta_8), \\
\sigma_B^*(\Theta_i) &= 6\eta_H - 2(\Theta_1 + \cdots + \Theta_8) - \Theta_i - 6\Theta_9, \ i = 1, \ldots, 8, \\
\sigma_B^*(\Theta_9) &= 16\eta_H - 6(\Theta_1 + \cdots + \Theta_8) - 15\Theta_9, \\
\sigma_B^*(\Theta_{10}) &= \Theta_{10}.
\end{align*}
$$

One checks that this transformation of the Picard group is of order 2, so it defines an involution $\sigma_B$ of $X$. One also checks that $\sigma_B$ commutes with the Enriques involution, hence $\sigma_B^*$ descends to an involution $g_B$ of the Reye congruence $S = \text{Rey}(W)$. Using Proposition 7.7.2, we easily find the action of $g_B$ on $\text{Num}(S)$:

$$
\begin{align*}
g_B^*(h) &= 33h - 16R_9 - 6(R_1 + \cdots + R_8), \\
g_B^*(R_i) &= 12h - 2(R_1 + \cdots + R_8) - R_i - 6R_9, \ i = 1, \ldots, 8, \\
g_B^*(R_9) &= 32h - 6(R_1 + \cdots + R_8) - 15R_9, \\
g_B^*(R_{10}) &= R_{10}.
\end{align*}
$$

It follows that $g_B^*(2f_9) = g_B^*(h - R_9) = h - R_9$. Thus $g_B^*$ acts identically on the sublattice $\langle 2f_9 + R_{10}, R_{10}\rangle \cong A_1 \oplus A_1(-1)$. We also check that it acts as the minus identity on its orthogonal complement. Therefore $g_B^*$ coincides with a Bertini involution.

Finally, we consider the lift of a Geiser involution to the K3-cover. Let $B_{q_1, \ldots, q_7}$ be a weak del Pezzo surface of degree 2 obtained by blowing up 7 points $q_1, \ldots, q_7$ in $\mathbb{P}^2$. Recall that the anticanonical map is a degree 2 map onto $\mathbb{P}^2$ and its deck transformation $\gamma$ is called a planar Geiser involution associated to 7 points $q_1, \ldots, q_7$ on $\mathbb{P}^2$. For a general point $q$ in the plane, $\gamma(q)$ is the ninth base point of the pencil of cubic curves passing through the points $q_1, \ldots, q_7, q$. It is known (see [180, 8.7.2]) that the planar Geiser transformation $\beta$ acts on the $\text{Pic}(B_{q_1, \ldots, q_7})$ by the formula

$$
\begin{align*}
\gamma^*(e_0) &= 8e_0 - 3(e_1 + \cdots + e_7), \\
\gamma^*(e_i) &= 3e_0 - (e_1 + \cdots + e_7) - e_i, \ i = 1, \ldots, 8,
\end{align*}
$$

where $(e_0, e_1, \ldots, e_7)$ is the natural basis in $\text{Pic}(B_{q_1, \ldots, q_7})$. Let $p_1, \ldots, p_7, p_8$ be the first eight nodes of the quartic symmetroid. One can define a dilated Geiser transformation $\sigma_G$ of $\mathbb{P}^3$ that acts on the geometric basis of $B_{p_1, \ldots, p_8}$ as follows

$$
\begin{align*}
\tilde{\gamma}^*(e_0) &= 15e_0 - 14e_8 - 6(e_1 + \cdots + e_7), \\
\tilde{\gamma}^*(e_i) &= 3e_0 - (e_1 + \cdots + e_7) - e_i - 3e_8, \ i = 1, \ldots, 8, \\
\tilde{\gamma}^*(e_9) &= 7e_0 - 3(e_1 + \cdots + e_7) - 6e_8.
\end{align*}
$$

We immediately check that it leaves the linear system $|4e_0 - 2(e_1 + \ldots + e_8)|$ invariant and sends our symmetroid $D(W)$ to a quartic symmetroid with nodes at $p_1, \ldots, p_9, \sigma_G(p_{10})$. One can show that the nine nodes of a quartic symmetroid determines the tenth one. Thus $\sigma_G(D(W))$ is a quartic symmetroid with the same set of nodes, so it must coincide with $D(W)$. This shows that $\tilde{\gamma}$
defines an automorphism $\sigma_G$ of $D(W)$.

$$\sigma^*_G(\eta_H) = 33\eta_H - 32\Theta_9 - 12(\Theta_1 + \cdots + \Theta_8),$$  
(8.5.10)  
$$\sigma^*_G(\Theta_i) = 6\eta_H - 2(\Theta_1 + \cdots + \Theta_8) - \Theta_i - 6\Theta_9, \ i = 1, \ldots, 8,$$

$$\sigma^*_G(\Theta_9) = 16\eta_H - 6(\Theta_1 + \cdots + \Theta_8) - 15\Theta_9,$$

$$\sigma^*_G(\Theta_{10}) = \Theta_{10}.$$  

One checks that this transformation of the Picard group is of order 2, so it defines an involution $\sigma_G$ of $X$. We also check that $\sigma_G$ commutes with the Enriques involution, hence $\sigma_G$ descends to an involution $g_G$ of the Enriques surface $S(W)$. Using Proposition 7.7.2, we easily find the action of $g_G$ on $\text{Num}(S)$:

$$g^*_G(h) = 33h - 16R_9 - 6(R_1 + \cdots + R_8),$$  
(8.5.11)  
$$g^*_G(R_i) = 12h - 2(R_1 + \cdots + R_8) - R_i - 6R_9, \ i = 1, \ldots, 8,$$

$$g^*_G(R_9) = 32h - 6(R_1 + \cdots + R_8) - 15R_9,$$

$$g^*_G(R_{10}) = R_{10}.$$  

It follows that $g^*_G(2f_9) = g^*_G(h - R_9) = h - R_9$. Thus $g^*_G$ acts identically on the sublattice $\langle f_9, R_{10} \rangle$. We also check that it acts as the minus identity on its orthogonal complement. Thus $g^*_G$ coincides with a Geiser involution.

### 8.6 Cyclic groups of automorphisms of an Enriques surface

Let $G$ be a finite group of automorphisms of an Enriques surface and let $G^*$ be its image in $W(\text{Num}(S)) \cong W(E_{10})$. We have already studied the possible kernel of the homomorphism $G \rightarrow W(\text{Num}(S))$. Let us identify $W(\text{Num}(S))$ with $W(E_{10})$ and consider the reduction homomorphism

$$r : W(E_{10}) \rightarrow O^+(10, \mathbb{F}_2).$$

Let $G^*_0 = G^* \cap W(E_{10})(2)$ be the kernel of the restriction of this homomorphism to $G^*$. The following Proposition is due to D. Allcock.

**Proposition 8.6.1.** Let $H$ be a finite non-trivial subgroup contained in $W(E_{10})(2)$. Then it is a group of order 2, and all such subgroups are conjugate in $W(E_{10})$.

**Proof.** We identify $\text{Num}(S)$ with the lattice $E_{10}$. Suppose $H$ contains an element $\sigma$ of order 2. Then $V = (E_{10})_\mathbb{Q}$ splits into the orthogonal direct sum of eigensubspaces $V_+$ and $V_-$ with eigenvalues 1 and $-1$. For any $x = x_+ + x_- \in E_{10}$, $x_+ \in V_+$, $x_- \in V_-$, we have

$$\sigma(x) = (x_+ + x_-) \pm (x_+ + x_-) \in 2E_{10}.$$  

This implies $2x_+ \in 2E_{10}$, hence $x_+ \in E_{10}$ and the lattice $E_{10}$ splits into the orthogonal sum of sublattices $V_+ \cap E_{10}$ and $V_- \cap E_{10}$. Since $E_{10}$ is unimodular, the sublattices must be unimodular. This gives $V_+ \cap E_{10} \cong \mathbb{Z}$ or $E_8$ and $V_- \cap E_{10} \cong E_8$ or $U$, respectively. Since $O(E_{10}) = W(E_{10}) \times \{ \pm \text{id}_{E_{10}} \}$, only one of these possibilities occurs, say the latter one. Thus all elements of order 2 in $W(E_{10})(2)$ are conjugate to the element $\text{id}_U \pm \text{id}_{E_8}$. 


Suppose $H$ contains an element $\sigma$ of odd order $m$. Then $\sigma^m - 1 = (\sigma - 1)(1 + \sigma + \cdots + \sigma^{m-1}) = 0$, hence, for any $x \not\in 2E_{10}$ which is not $\sigma$-invariant, we have
\[
x + \sigma(x) + \cdots + \sigma^{m-1}(x) \equiv mx \mod 2E_{10}.
\]
Since $m$ is odd, this gives $x \in 2E_{10}$, a contradiction.

Finally, we may assume that $H$ contains an element of order $2^k$, $k > 1$. Then it contains an element $\sigma$ of order 4. Let $M = \ker(\sigma^2 + 1) \subset E_{10}$. Since $\sigma^2 = -\id_{E_8} \oplus \id_U$ for some direct sum decomposition $E_{10} = E_8 \oplus U$, we obtain $M \cong E_8$. The equality $(\sigma^2 + 1)(\sigma(x)) = \sigma^3(x) + \sigma(x) = \sigma(\sigma^2 + 1)(x)$, implies that $\sigma(M) = M$. Consider $M$ as a module over the principal ideal domain $R = \mathbb{Z}[t]/(t^2 + 1)$. Since $M$ has no torsion, it is isomorphic to $R^{\oplus 4}$. This implies that there exists $v, w \in M$ such that $\sigma(v) = w$ and $\sigma(w) = -v$. However, this obviously contradicts our assumption that $\sigma \in W(E_{10})/(2)$.

**Corollary 8.6.2.** Suppose that an Enriques surface $S$ has an element $g$ of finite order whose image $g^*$ in $W(\text{Num}(S))$ belongs to the 2-congruence subgroup $W(\text{Num}(S))(2)$. Then $g$ is the deck transformation of a degree 2 separable bielliptic map $f : S \to D$ to a quartic symmetroid surface in $\mathbb{P}^4$.

**Proof.** It follows from the Proposition 8.6.1 that $g^*$ is of order 2 and decomposes $\text{Num}(S)$ into a direct sum of sublattices $U$ and $E_8$ isomorphic to $U$ and $E_8$. We know that there exists a unique $w \in W^\text{mod} S$ such that $w(U)$ contains a nef isotropic class. Obviously, $g^*$ leaves $w(U)$ invariant and hence also its orthogonal complement, so we may assume that $U$ defines a nondegenerate $U$-pair. Since $g^*$ cannot act as $-\id_U$ on $U$, hence it must be conjugate to $\id_U \oplus -\id_{E_8}$. We know that $U$ defines a bielliptic map $f : X \to D_1$ to an anti-canonical quartic del Pezzo surface. The sublattice $U$ generates over $\mathbb{Q}$ the sublattice $f^*(\text{Pic}(D_1))$. It follows from this that the map $f$ is a finite map (since otherwise some of the exceptional curves will be invariant under $g$).

**Example 8.6.3.** Suppose $S$ is an unnodal surface without extra automorphisms, then, by Proposition 8.3.1, $\ker(\text{Aut}(S) \to \text{Aut}(S)^*)$ is trivial and hence $\text{Aut}(S) \cong W(\text{Num}(S))(2)$. So we may apply the previous Corollary to obtain that the elements of finite order in $\text{Aut}(S)$ are bielliptic involutions. Also, if $S$ is a general nodal surface, then again, by Proposition ??, we have $\text{Aut}(S) = \text{Aut}(S)^*$ and the proof of Theorem 8.4.7 shows that $\text{Aut}(S)^*$ contains the subgroup $W(\text{Rey}(S))^\prime \subset W(\text{Num}(S))(2)$. So, if $S$ has no extra automorphisms, then again the only elements of finite order in $\text{Aut}(S)$ are bielliptic involutions.

We say that a group of automorphisms of an Enriques surface is of translation type if it leaves invariant a genus one fibration and is realized by a subgroup of the Mordell-Weil group of the jacobian fibration that acts on $S$ by translation automorphisms.

The proof of the following result can be found in [549].

**Theorem 8.6.4.** Let $J \to \mathbb{P}^1$ be a rational jacobian elliptic surface. Then its Mordell-Weil group is isomorphic to one of the following groups:
\[
\mathbb{Z}^r \ (1 \leq r \leq 8), \ \mathbb{Z}^r \oplus \mathbb{Z}/2\mathbb{Z} \ (1 \leq r \leq 4), \ \mathbb{Z}^r \oplus \mathbb{Z}/3\mathbb{Z} \ (1 \leq r \leq 2), \ \mathbb{Z}^r \oplus (\mathbb{Z}/2\mathbb{Z})^2 \ (1 \leq r \leq 2),
\]
\[
\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}, \ \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \ \mathbb{Z}/3\mathbb{Z}^2, \ (\mathbb{Z}/2\mathbb{Z})^2, \ \mathbb{Z}/5\mathbb{Z}, \ \mathbb{Z}/6\mathbb{Z}, \ \mathbb{Z}/4\mathbb{Z}, \ \mathbb{Z}/3\mathbb{Z}, \ \mathbb{Z}/2\mathbb{Z}, \ {1}.
\]
In particular, all these groups are realized as groups of automorphisms of Enriques surfaces of translation type.

Suppose $G$ is a group of translation type of a quasi-elliptic surface. The Mordell-Weil group of a jacobian quasi-elliptic pencil is a $p$-torsion group and its structure can be found in Table (4.9). An Enriques surface with a quasi-elliptic fibration is a torsor of such a surface only if $p = 2$. This gives us the following possible groups $\mathbb{Z}/2\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^2$, or $(\mathbb{Z}/2\mathbb{Z})^4$. They already occur in the previous list.

In the following we will try to classify possible cyclic group actions on an Enriques surface. If $p \neq 2$, we can apply Theorem 8.1.6 to assume that $\mathbb{k} = \mathbb{C}$ and use the theory of periods of the K3-covers. However, we would like to avoid this non-geometric approach.

Let $G$ be a finite subgroup of $\text{Aut}(S)$. Applying Lemma 6.4.7, we obtain, after identifying $\text{Num}(S)$ with $E_{10}$, that the image $G^*$ of $G$ in $\text{Aut}(S)^*$ is contained in some parabolic subgroup $W_J$ of $W(E_{10})$. As we explained after this Lemma, $W_J$ is a subgroup of the Weyl groups of one of the following diagrams:

$$A_9, D_9, E_8 + A_1, A_1 + A_8, A_6 + A_2 + A_1, A_4 + A_5, E_6 + A_3, E_7 + A_2, D_5 + A_4.$$ (8.6.1)

**Lemma 8.6.5.** Let $g \in \text{Aut}(S)$ and let $x \in \text{Num}(X)$ be the numerical divisor class with $x^2 \geq 0$ such that $g^*(x) = x$. Then there exists $w \in W_S^{\text{nod}}$ such that $y = w(x)$ is nef and $g^*(y) = y$.

**Proof.** Since $\text{Nef}(S)$ is a fundamental domain for $W_S^{\text{nod}}$ there exists a unique $w \in W_S^{\text{nod}}$ such that $w(x) \in \text{Nef}(S)$. We have

$$g^* \circ w \circ (g^*)^{-1}(x) = g^*(w(x)) = g^*(y) \in \text{Nef}(S).$$

This shows that $g^* \circ w \circ (g^*)^{-1} = w$ and hence $y = w(x) = g^*(y)$. \qed

Computing the orthogonal complement of these lattices in $E_{10}$, we obtain the following.

**Corollary 8.6.6.** Let $G$ be a finite group of automorphisms of $S$, then it preserves a nef numerical class $h$ with $h^2 = 10, 4, 2, 18, 42, 30, 12, 6, 20$.

We will use the known classification of conjugacy classes of elements in the Weyl groups of root systems of finite type given in Table 8.6. According to [104] they are indexed by certain graphs. We call them Cartesian graphs. One writes each element $w \in W$ as the product of two involutions $w_1 w_2$, where each involution is the product of reflections with respect to orthogonal roots. Let $R_1, R_2$ be the corresponding sets of such roots. Then the graph has vertices identified with elements of the set $R_1 \cup R_2$ and two vertices $\alpha, \beta$ are joined by an edge if and only if $(\alpha, \beta) \neq 0$. A connected Carter graph with no cycles is a Dynkin diagram. It represents the conjugacy class of the Coxeter element of the corresponding Weyl group. The (first) subscript $n$ in the notation $A_n, D_n, E_n, A_n(a_k), D_n(a_k), E_n(a_k)$ of a Carter graph indicates the number of vertices. The notation also indicates that the conjugacy class is realized by an element of the Weyl group of the corresponding type. It may or may not be the conjugacy class of a Coxeter element of this group, if
not, it has an additional notation like $E_6(a_1)$. The subscript $n$ is also equal to the difference between the rank of the root lattice $Q$ and the rank of its fixed sublattice $Q^w$.

The Carter graph determines the characteristic polynomial of $w$. In particular, it gives the trace $\text{tr}_2(g^*)$ of $g^*$ on the $l$-adic cohomology space $H^2(S, \mathbb{Q}_l)$. If the order of $g$ is prime to the characteristic, we can apply the Lefschetz fixed-point formula to obtain

$$\text{tr}(g^*) := \text{tr}(g^*|H^*(S, \mathbb{Q}_l)) = 2 + \text{tr}_2(g^*) = e(S^g). \quad (8.6.2)$$

The following Table gives the conjugacy classes of elements defined by connected graphs.

<table>
<thead>
<tr>
<th>Graph</th>
<th>Order</th>
<th>Characteristic polynomial</th>
<th>Trace</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_k$</td>
<td>$k + 1$</td>
<td>$t^k + t^{k-1} + \cdots + 1$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$D_k$</td>
<td>$2k - 2$</td>
<td>$(t^{k-2} + 1)(t^2 + 1)$</td>
<td>$0$</td>
</tr>
<tr>
<td>$D_k(a_1)$</td>
<td>$1.\text{c.m.}(2k - 4, 4)$</td>
<td>$(t^{k-3} + 1)(t^3 + 1)$</td>
<td>$0$</td>
</tr>
<tr>
<td>$D_k(a_2)$</td>
<td>$1.\text{c.m.}(2k - 6, 6)$</td>
<td>$(t^{k-3} + 1)(t^3 + 1)$</td>
<td>$0$</td>
</tr>
<tr>
<td>:</td>
<td>:</td>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td>$D_k(a_{k-1})$</td>
<td>even $k$</td>
<td>$(t^{\frac{k}{2}} + 1)^2$</td>
<td>$0$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$12$</td>
<td>$(t^4 - t^2 + 1)(t^4 + t + 1)$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$E_6(a_1)$</td>
<td>$9$</td>
<td>$t^9 + t^3 + 1$</td>
<td>$0$</td>
</tr>
<tr>
<td>$E_6(a_2)$</td>
<td>$6$</td>
<td>$(t^2 - t + 1)^2(t^2 + t + 1)$</td>
<td>$1$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$18$</td>
<td>$(t^6 - t^3 + 1)(t + 1)$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$E_7(a_1)$</td>
<td>$14$</td>
<td>$t^4 + t^1 + 1$</td>
<td>$0$</td>
</tr>
<tr>
<td>$E_7(a_2)$</td>
<td>$12$</td>
<td>$(t^4 - t^2 + 1)(t^4 + 1)$</td>
<td>$0$</td>
</tr>
<tr>
<td>$E_7(a_3)$</td>
<td>$30$</td>
<td>$(t^3 + 1)(t^3 - t + 1)$</td>
<td>$1$</td>
</tr>
<tr>
<td>$E_7(a_4)$</td>
<td>$6$</td>
<td>$(t^2 - t + 1)^2(t^2 + 1)$</td>
<td>$2$</td>
</tr>
<tr>
<td>$E_8$</td>
<td>$30$</td>
<td>$t^8 + t^4 - t^2 - t^4 + t^6 + t + 1$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$E_8(a_1)$</td>
<td>$24$</td>
<td>$t^8 - t^4 + 1$</td>
<td>$0$</td>
</tr>
<tr>
<td>$E_8(a_2)$</td>
<td>$20$</td>
<td>$t^8 - t^6 + t^2 - t^4 + 1$</td>
<td>$0$</td>
</tr>
<tr>
<td>$E_8(a_3)$</td>
<td>$12$</td>
<td>$(t^4 - t^2 + 1)^2$</td>
<td>$0$</td>
</tr>
<tr>
<td>$E_8(a_4)$</td>
<td>$18$</td>
<td>$(t^6 - t^4 + 1)(t^2 - t + 1)$</td>
<td>$1$</td>
</tr>
<tr>
<td>$E_8(a_5)$</td>
<td>$15$</td>
<td>$t^8 - t^4 + t^6 - t^4 + t^4 + t + 1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$E_8(a_6)$</td>
<td>$10$</td>
<td>$(t^4 - t^3 + t^2 - 1)^2$</td>
<td>$2$</td>
</tr>
<tr>
<td>$E_8(a_7)$</td>
<td>$12$</td>
<td>$(t^4 - t^2 + 1)(t^2 - t + 1)^2$</td>
<td>$2$</td>
</tr>
<tr>
<td>$E_8(a_8)$</td>
<td>$6$</td>
<td>$(t^2 - t + 1)^4$</td>
<td>$4$</td>
</tr>
</tbody>
</table>

Table 8.4: Carter graphs and characteristic polynomials.

Let $H$ be a finite subgroup of $W(E_{10})$ (or its element $w$). We say that $H$ (or $w$) is of $E_8$-type if it is conjugate to a subgroup of the parabolic subgroup $W_J$ defined by the set $J$ of vertices of the subdiagram of $T_{2,3,7}$ of type $T_{2,3,5}$ (of type $E_8$). It is easy to see that it must coincide with the parabolic subgroup $W_J$, where $J$ is the set of vertices of a subdiagram of type $E_8$. A finite group $G$ of automorphisms (resp. an automorphism $g$) of $S$ is said to be of $E_8$-type if the image $G^*$ of $G$
(resp. \(g^*\) in \(W(\text{Num}(S))\) is of \(E_8\)-type. All conjugacy classes of elements in \(W(E_8)\) are listed in [104]. Note that two elements of the same type are not necessary conjugate, for example, there are two conjugacy classes of elements of type \(4A_1\) or \(2A_3\).

**Lemma 8.6.7.** Let \(G\) be a finite group of automorphisms of \(S\). Assume that its image \(G^*\) in \(W(\text{Num}(S))\) is of \(E_8\)-type. Then there exists a \(G\)-equivariant bielliptic map \(\phi : S \to D\) such that \(G\) is a lift of a group \(G'\) of automorphisms of \(D\). The group \(G\) leaves invariant the genus one fibrations defined by the pre-images of the pencils of conics on \(D\) and a bisection defined by another genus one fibration (if \(D = D_1', D_2', D_3')\) or a smooth rational curve (if \(D = D_1, D_2, D_3\)).

**Proof.** We know that \(\text{E}_{10} \cong E_8 \oplus U\). Thus \(G\) leaves invariant a hyperbolic plane \(U\) and acts identically on it. It follows from Proposition 6.1.5 that there exists a unique \(w \in W_8^{\text{gal}}\) such that \(w(U)\) is generated by a canonical isotropic 2-sequence \((f_1, f_2)\) that defines a bielliptic linear system \(|D|^*\). Thus we may assume that \(G\) leaves \(U\) invariant and acts identically on it. Let \(\phi : S \to D\) be the corresponding bielliptic map. Then \(G\) acts in \(|D|^*\) leaving \(D\) invariant. Thus \(G\) is a lift of a subgroup of automorphisms of \(|D|^*\) which leaves \(D\) invariant. If \((f_1, f_2)\) are both nef, \(D = D_1, D_2, D_3\), and we have two genus one pencils \([2F_1]\) and \([2F_2]\) with \([F_1] = f_1\), both fixed by \(G\). Otherwise, \(f_1\) is nef, and \(f_2 = f_1 + r\), where \(r\) is the class of a smooth rational curve \(R\). Then \(D = D_1', D_2', D_3'\), and we have one genus one fibration \([2F_1]\) with \([F_1] = f_1\) and a bisection \(R\). \(\square\)

The following useful lemma was communicated to the first author by J.-P. Serre (letter September 2, 2008).

**Lemma 8.6.8.** Let \(V\) be a smooth proper connected variety of dimension \(n\) over an algebraically closed field with \(H^i(V, \mathcal{O}_V) = 0, i > 0\). Then any endomorphism \(g\) of \(V\) has a fixed point.

**Proof.** If \(g\) is of finite order prime to \(p\) this follows from the Woods-Hole formula [319, Corollary 6.12]:

\[
\sum_{i=0}^{n} (-1)^i \text{tr}(g^*|H^i(V, \mathcal{O}_V)) = \sum_{x \in V^g} \frac{1}{\det(1 - dg_x)}
\]

(8.6.3)

where \(dg\) is the differential of \(g\) at a fixed point \(x\) (under the assumption that the set \(V^g\) is finite). So, if we assume that \(V^g\) is empty, we obtain that the right-hand side is 0 but the left-hand side is positive, a contradiction.

Without any assumption on the order of \(g\) but assuming that \(V^g\) is a finite set, we use the same argument and the following formula [319, Remark 6.12.1]:

\[
\sum_{i=0}^{n} (-1)^i \text{tr}(g^*|H^i(V, \mathcal{O}_V)) = \sum_{x \in V^g} \text{Res}_x \frac{du_1 \wedge \ldots \wedge du_n}{(u_1 - g^*(u_1)) \cdots (u_n - g^*(u_n))},
\]

(8.6.4)

where \((u_1, \ldots, u_n)\) are local coordinates at a point \(x \in V^g\). \(\square\)

Recall that an automorphism of a K3 surface \(X\) is called symplectic if it acts identically on \(H^0(X, \omega_X)\).

**Lemma 8.6.9.** Assume \(p \neq 2\) or \(S\) is a \(\mu_2\)-surface. Let \(g\) be an automorphism of \(S\) of odd order \(n\). Then its lift \(\bar{g}\) of order \(n\) to the canonical cover \(X\) is a symplectic automorphism of \(X\).
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Proof. We follow a nice argument from [509]. Suppose $\tilde{g}$ is not symplectic. Let $\phi : X \to Y = X/(\tilde{g})$ be the quotient map, it extends to a separable morphism $\phi' : X' \to Y'$, where $X'$ is birational to $X$ and $Y'$ is a nonsingular model of $Y$. Since $\phi'$ is separable, the map $\phi'^* : H^i(Y', \omega_{Y'}) \to H^i(X', \omega_{X'})$, $i = 0, 1$, is injective. Its image is contained in $H^1(X', \omega_{X'}) = \{0\}$. Thus $\text{ker}(Y') \leq 0$ and $q(Y') = p_\sigma(Y') = 0$. Since an automorphism of order equal to the characteristic is symplectic, we may assume that $n$ is prime to $p$. It follows from Lemma 8.6.8 that the descent of the Enriques involution $\tau$ to $Y'$ and then its lift to $Y''$ has a fixed point. Its image on $Y'$ is a fixed point $y \in Y$. The fiber over this point has odd cardinality and it is invariant with respect to $\tau$. Hence $\tau$ has a fixed point on the fiber, contradicting the fact that it acts on $X$ freely.

Proposition 8.6.10. Assume $p \neq 2$. Let $g$ be a non-trivial automorphism of an Enriques surface of odd order $n$. Then $n = 3$ or 5.

Proof. Assume first that $p \neq 2$ or $S$ is a $\mu_2$-surface. The odd order $n$ of a symplectic automorphism of a K3 surface takes possible values $1, 3, 5, 7$ and $11$ [178]. The latter case occurs only if $p = 11$ [173]. The classification of elements of finite order in $W(E_{10})$ shows that there are no elements of order 11.

Let us exclude the case $n = 7$. An element of order 7 in $W(E_{10})$ is of $E_8$-type. Hence it fixes a genus one fibration and its bisection. In particular, it cannot be of translation type (the latter follows also from the fact that the Mordell-Weil group of the jacobian fibration cannot be of order 7). Thus $g$ acts on the base of the fibration, and since $n$ is odd, it fixes the two double fibers. Since an element of order $p$ fixes only one point in $\mathbb{P}^1$, we must have $p \neq 7$. It is known that in this case a symplectic automorphism of order 7 has three fixed points. This applies to the lift $\tilde{G}$, and since the Enriques involution leaves this set invariant, we obtain that it has a fixed point, a contradiction.

The following result in the case $k = \mathbb{C}$ can be found in [554, Proposition 3.1].

Theorem 8.6.11. Assume $p \neq 2$. Let $n$ be the order of an element $\sigma \in G^*$. Then

$$ n \in \{1, 2, 3, 4, 5, 6, 8\}. $$

Proof. We already know that an odd $n$ must be equal to 3 or 5. Thus $n = 2^a \cdot 3$ or $2^a \cdot 5$. The classification of possible orders of elements in a finite parabolic subgroup of $W(E_{10})$ shows that $a \leq 3$. Since there are no elements of order 15, the only possibilities are listed in the assertion and $n = 10, 12$.

To finish the proof it suffices to exclude the last two cases.

Assume $n = 10$. We will show later in Proposition 8.6.18 that an element of order 10 lifts to a symplectic automorphism of the canonical cover $X$. If $p \neq 5$, there are no such automorphisms of $X$. The same is true if $p = 5$ [374].

Let us assume that $n = 12$. Applying the Woods-Hole formula (8.6.3) to $g^3$ we obtain that $\#S^g^3 = 2$. Since $S^g \subset S^g^3$, this implies that $\#S^g \leq 2$. 
Assume \( p \neq 3 \), then the same formula gives us \( \#S^{g^4} = 3 \) and the differential of \( g^4 \) acts at a fixed point with eigenvalues \( \epsilon_3, \epsilon_3^2 \) (this is the only possibility to make the left-hand side to be an integer). Since \( g \) leaves the set \( S^{g^4} \) invariant and fixes at most two points, we obtain \( \#S^g = 1 \) and differential of \( g \) acts at a fixed point with eigenvalues \( \epsilon_{12}, \epsilon_{12}^2 \). Now we apply again the Woods-Hole formula, to get a contradiction.

Next we will assume that \( p = 3 \). First, we use that \( h = g^4 \) is an element of order 3. It follows from [373] that \( g \) has only one fixed point. This implies that \( S^g \) consists of one point. But now, it follows from the Woods-Hole formula (8.6.3) that \( \#S^{g^3} = 2 \), since \( S^g \subset S^{g^3} \) we see that \( g \) must fix two points, a contradiction.

**Remark 8.6.12.** If \( p = 2 \), we do not know whether there exists an Enriques surface with an automorphism of order 10 or 12.

**Remark 8.6.13.** Suppose \( g \) is of order \( 3 \neq p \). We will prove later in Proposition 8.6.16 that one of its two lifts to the canonical cover is a symplectic automorphism of order 3. Table 8.6 below shows that it has 6 fixed points, they are switched by pairs by the Enriques involution so that \( g \) has three fixed points. Applying the Lefschetz fixed-point formula, we obtain that \( \text{tr}_2(g) = 1 \). The Carter classification shows that \( g^a \) acting on \( \text{Num}(S) \) must be of type \( 3A_2 \). It follows from [104] that all such elements \( g \) contained in \( W(E_8) \) are conjugate. If \( g \) is contained in \( W(A_9) \cong S_{10} \), then it is the product of three commuting cyclic permutations, and there is only one conjugacy class of such permutations. If \( g \) is contained in \( W(D_9) \cong 2^8 \rtimes S_9 \), then it is conjugate to an element in a subgroup isomorphic to \( S_9 \) and hence again all elements of type \( 3A_2 \) are conjugate. Other possible maximal parabolic subgroups containing \( g \) are of types \( E_6 + A_3 \) or \( E_6 + A_2 \). One checks again using Carter’s Table 7 from [104] that any element \( g \) of type \( 3A_2 \) is conjugate to an element of this type in \( W(E_8) \).

If \( p = 3 \), an order 3 automorphism has one fixed point \( x \). It cannot be of translation type. It follows from [373] that the quotient surface has one rational singularity. According to [567, Remark after Corollary 5.15], it is a rational double point of type \( E_6 \). Also, we can use the following formula of Kato and Saito [353]

\[
e_c(U) = pe_c(V) + p - 1 - (p - 1)l_g \tag{8.6.5}
\]

where \( e_c \) denotes the \( l \)-adic Euler-Poincaré characteristic with compact support, \( U = S \setminus \{x\}, V = U/(g) \) and \( l_g \) is the index of intersection of the graph of \( g \) with the diagonal at the point \( x \). The computation from [567] show that \( l_g = 3 \) in our case, and we get that \( e_c(V) = 5 \). This gives \( e(S/(g)) = 12 \) and \( \text{rank} \ H^2(S, \mathbb{Q})^g = e_c(V) - 1 = 4 \). It follows that the type of \( g \) must be \( 3A_2 \) as in the case when \( p \neq 3 \).

Suppose \( g \) is of order \( 5 \neq p \). Similar argument shows that \( g \) fixes 2 points and \( g^a \) is of type \( 2A_4 \). Again, Carter’s Table 7 shows that there is only one conjugacy class of such elements in \( W(E_8) \). However, in this case \( W(A_9) \) contains an element of type \( 2A_4 \), a product of two cyclic permutations of order 5. It can be represented by the parabolic subgroup \( W_J \), where \( J = \{ \alpha_i, i \neq 0, \alpha_5 \} \). However this parabolic subgroup is conjugate to a subgroup \( W_{J'} \), where \( J' = \{ \alpha_i, i \neq 4, \alpha_9 \} \).

Since it is contained in \( W(E_9) \), it is also conjugate to a subgroup of \( W(E_8) \). Any other other maximal parabolic that contains such an element must be of type \( A_4 + A_5 \) or \( D_5 + A_4 \). It is easy to see that it is conjugate to an element of type \( 2A_4 \) in \( W(E_8) \).
If \( p = 5 \), again we have only one fixed point and the quotient singularity is a rational singularity. Playing again with formula (8.6.5), we find that \( e_c(V) = 7 \). This implies that \( g \) must be of type \( A_4 \). We do not know what kind of singularity the quotient has.

Following S. Mukai, we introduce the following.

**Definition 8.6.14.** An automorphism \( g \) of an Enriques surface \( S \) is called semi-symplectic if it acts trivially on \( H^0(S, \mathcal{O}_S(2K_S)) \cong k \).

Note that, although \( \mathcal{O}_S(2K_S) \cong \mathcal{O}_S \), the isomorphism is not canonical, so the action does not coincide with the trivial action on the constants. By duality, we have a canonical isomorphism \( H^0(S, \mathcal{O}_S(2K_S)) \cong H^2(S, \mathcal{O}_S(-K_S))^\vee \). Therefore, if \( K_S \cong \mathcal{O}_S \), the action on \( H^0(S, \mathcal{O}_S(2K_S)) \) is isomorphic to the action on \( H^2(S, \mathcal{O}_S) \).

The following two propositions show that the semi-simplicity is an analogue of the condition for an automorphism of a K3 surface \( X \) to be symplectic.

**Proposition 8.6.15.** Assume that \( p \neq 2 \), or \( S \) is a \( \mu_2 \)-surface. Let \( \pi : X \to S \) be the K3-cover. Then an automorphism \( g \) of \( S \) is semi-symplectic if and only if one of its lifts to an automorphism of \( X \) is symplectic.

*Proof.* Since \( \pi^*(\omega_S) \cong \mathcal{O}_X \), we have a canonical isomorphism
\[
H^0(S, \mathcal{O}_S(2K_S)) \cong H^2(S, \mathcal{O}_S(-K_S))^\vee \cong H^2(X, \mathcal{O}_X)^\vee \cong H_0(X, \omega_X).
\]
So, the action of two lifts differ by the action of the canonical cover involution. If \( p = 2 \), both of the lifts could be symplectic. \( \square \)

**Proposition 8.6.16.** Suppose \( g \) is of odd order \( n \) prime to \( p \). Then it is semi-symplectic if and only if \( S^g \) consists of isolated fixed points and the quotient surface \( Y = S/\langle g \rangle \) has only rational double points of type \( A_{n-1} \).

*Proof.* A section of \( H^0(S, \mathcal{O}_S(2K_S)) \) is locally, in an affine neighborhood \( U \), of a fixed point \( x \in S \) can be represented by \( \phi(dx \wedge dy)^2 \), where \( \phi \) is an invertible function on \( U \) and \( x, y \) are local coordinates. Since \( g \) is semi-symplectic \( \det(df_x)^2 = 1 \) and, since \( g \) is of odd order, we have \( \det(dg_x) = 1 \). Since \( \langle p, n \rangle = 1 \), we may find local coordinates such that \( g \) acts on them by a diagonal matrix \( \text{diag}(\epsilon_n, \epsilon_n^{-1}) \). The image of the point \( x \) in the quotient is locally isomorphic to the double rational point \( A_{n-1} \). \( \square \)

**Remark 8.6.17.** The assumption that \( \langle n, p \rangle = 1 \) is essential. It follows from [373] that the locus \( S^g \) of fixed point of an automorphism of \( S \) of order \( p \) could be a connected curve or an isolated point and, in the latter case the quotient may have a singular point which is not a rational double point.

It is known that the possible order of a symplectic automorphism of a K3 surface of order \( n \) prime to \( p \) satisfies \( n \leq 8 \), and all such values are realized (see [178], Theorem 3.3). Moreover, if \( n = p \), then \( p \leq 11 \) (loc.cit., Theorem 2.1). The symplectic lifts of automorphisms of an Enriques surfaces satisfy a stricter condition.
Proposition 8.6.18. Assume $p \neq 2$. Any automorphism $g$ of order $n$ not divisible by 4 is semi-symplectic. An element of order 8 cannot be semi-symplectic.

Proof. Let $\pi : X \to S$ be the K3-cover. It follows from Lemma 8.6.9 that any element of odd order is semi-symplectic. A lift of an automorphism of order 2 of $X$ is either symplectic or its composition with the covering involution $\tau$ is symplectic.

Suppose $n = 2k$, where $k > 1$ is odd. Let $\tilde{g}$ be a lift of $g$ such that $\tilde{g}^2$ is a symplectic lift of $g^2$. If $\tilde{g}$ is not symplectic, then it acts as $-1$ on $H^0(X, \omega_X)$, hence $\tilde{g} \circ \tau$ acts identically and defines a symplectic lift of $g$.

It remains to exclude the case $n = 8$.

Suppose a semi-symplectic automorphism $g$ has order $n = 8$. Let $\tilde{g}$ be its symplectic lift to the canonical cover $X$. It is known that $\#X^g = 2$. But then $S^g$ consists of one point. Applying the Woods-Hole formula, we find a contradiction. Note that there is no contradiction with a symplectic lift of an element of order 4 since it gives us that $\#S^g = 2$ and the differential at each point has eigenvalues $\pm \sqrt{-1}$. The formula confirms that $\#S^g = 2$.

Let $\tilde{g}$ be a symplectic lift of an automorphism $f$ of $S$. A point in $X^g/(\tau)$ is called a symplectic fixed point, and a point in $X^{g^0}\tau/(\tau)$ is called an anti-symplectic fixed point. We denote the set of symplectic (resp. anti-symplectic) fixed points by $S^g_+$ (resp. $S^g_-$). Note that, $S^g_+ \cap S^-_-$ is empty because the differential $dg_{\pi(x)}$ and $d\tilde{g}_x$ are isomorphic linear representations of $g$ and $\tilde{g}$.

Let $g$ be a symplectic automorphism of finite order $n > 1$ coprime to $p$ of a K3 surface $X$. Then $X^g$ is finite and we have the following Table for possible number $f$ of fixed points (see [178, Theorem 3.3]).

<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$</td>
<td>8</td>
<td>6</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 8.5:

Corollary 8.6.19. Let $g$ be a semi-symplectic automorphism of order $n$ of $S$. If $p \nmid n$, then $\#S^g = 1$ or $S^g$ is a connected curve. Otherwise

$\quad$ (n=2) $\#S^g_+ = 4$ and $e(S^g_+) = \text{tr}(g^*) - 4$;

$\quad$ (n=3) $\#S^g_+ = \#S^g = 3$;

$\quad$ (n=4) $\#S^g_+ = \#S^g_- = 2$;

$\quad$ (n=5) $\#S^g_+ = \#S^g = 2$;

$\quad$ (n=6) $\#S^g_+ = 1$ and $\#(S^g_-) \in \{1, 2\}$.
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Proof. Assume \( p | n \). If \( n = p \), then it is known that \( S^g \) is connected [373]. Write \( n = p^kn' \), then \( S^g \subset S^{p^{k-1}n'} \) and the assertion follows. Now we assume that \( (n, p) = 1 \). If \( n = 4 \), then \( X^g \circ \tau \subset X^g = X^g \), so \( S^g = S^g_+ \). If \( n = 6 \), then \( \#X^g = 2 \). Also \( X^g \circ \tau \) and \( X^g \) are subsets of the set \( \#X^g_+ \) of cardinality 6. This implies \( \#S^g_+ = 1 \) and \( \#S^g_- \leq 2 \).

Example 8.6.20. All possible orders of semi-symplectic automorphisms can be realized by translation automorphisms of the Mordell-Weil group of the jacobian elliptic fibration. We know that these numbers are among possible orders of such groups. It remains to see that the translation automorphisms are semi-symplectic. In fact, one of the two lifts of such an automorphism is a translation automorphism of an elliptic fibration on the K3-cover, the pre-image of the elliptic fibration on \( S \). It is well-known that such an automorphism is symplectic. One can see it, for example, by considering the exact sequence

\[
0 \to \omega_X(-F) \to \omega_X \to \mathcal{O}_F(\omega_X) \to 0,
\]

where \( F \) is a general fiber of the elliptic fibration. Via the adjunction formula \( \omega_F \cong \mathcal{O}_F(\omega_X) \), it defines a canonical isomorphism \( H^0(X, \omega_X) \to H^0(F, \omega_F) \). Since the translation automorphism of an elliptic curve acts identically on \( H^0(F, \omega_F) \), we obtain that it acts identically on \( H^0(X, \omega_X) \).

The conjugacy classes of elements of finite order \( n \) can be classified according to the types of the corresponding root sublattice of \( E_{10} \).

8.7 Involution of Enriques surfaces

Next we classify possible types of involutions assuming that \( p \neq 2 \). All involutions on Enriques surfaces over the field of complex numbers have been classified by H. Ito and H. Ohashi in [330]. Their classification relies heavily on the Global Torelli Theorem for K3 surfaces and results of Nikulin on quadratic lattices together with an isometry of order 2. There are 18 types of involutions in the case \( k = \mathbb{C} \). We will show that the same result is true over a field of arbitrary characteristic \( p \neq 2 \).

We start with the following.

Lemma 8.7.1. Let \( f : S \to \mathbb{P}^1 \) be a genus one fibration on an Enriques surface. Suppose it admits a bisection \( B \) such that \( f|B : B \to \mathbb{P}^1 \) is a separable cover (we call it a separable bisection). Then \( S \) admits an automorphism of order 2 that preserves the fibration and the bisection \( B \). Conversely, any automorphism of order 2 of \( S \) that preserves a genus one fibration, acts identically on the base, and leaves invariant its bisection arises in this way.

Proof. A bisection is a point \( x \) of degree 2 on the generic fiber \( S_\eta \) of \( f \). The assumption implies that the residue field of this point is a separable extension of \( \mathbb{k}(\eta) \). The linear system \( |x| \) defines a separable degree 2 map \( S_\eta \to \mathbb{P}^1_\eta \). Its deck transformation is a birational involution on \( S \) that extends to a biregular involution. The point \( x \) is the pre-image of a rational point on \( \mathbb{P}^1_\eta \) and hence is preserved by the deck transformation.
Conversely, an automorphism $\sigma$ of order 2 of $S$ that preserves a genus one fibration $f : S \to \mathbb{P}^1$ and its bisection restricts to $S_\eta$ to define a separable map $\phi : S_\eta \to C = S_\eta/(\sigma)$. Since $\sigma$ leaves invariant the generic point $x$ of the bisection, we obtain that $x = \phi^{-1}(y)$ for some point of degree 1 on $C$. Hence $C \cong \mathbb{P}^1_{\mathbb{F}_r}$ and $\sigma$ arises in the way described in above.

**Corollary 8.7.2.** Suppose $S$ admits an elliptic fibration $f : S \to \mathbb{P}^1$. Then the automorphism group $\text{Aut}(S)$ contains an element of order 2.

**Proof.** We know that $f : S_\eta \to \mathbb{P}^1$ is a torsor over its jacobian elliptic curve of period 2. Applying Theorem 4.6.5 and Lemma ??, we obtain that $S_\eta$ contains a point $x$ of degree 2 with a separable residue field. Then the assertion follows from the previous lemma. □

Let us look how an involution $g$ can act on $\text{Num}(S)$.

According to R. Richardson [592], one classifies the conjugacy classes of involutions in a Coxeter group as follows. We say that a subset $J$ of the set of Coxeter generators $s_{\alpha_i}$ satisfies the $(-1)$-condition if the Coxeter group $W_J$ contains an element $\sigma_J$ that acts as $-\text{id}$ on the corresponding root sublattice (finite Coxeter groups of this type must be the Weyl groups of orthogonal sums of root lattices types $A_1, D_{2n}, E_7, E_8$). In our case, we have the following types of irreducible root sublattices generated by a subset satisfying the $(-1)$-condition:

$$A_1, D_4, D_6, D_8, E_7, E_8.$$ Other root sublattices are direct sums of these sublattices. The following proposition is Theorem A from [592].

**Proposition 8.7.3.** Any involution in a Coxeter group is conjugate to the involution $\sigma_J$ for some subset $J$ defining a finite parabolic subgroup that satisfies the $(-1)$-condition. Two involutions $\sigma_J$ and $\sigma_{J'}$ are conjugate if and only if the subsets $J$ and $J'$ are equivalent (as explained in Section 6.4 after Lemma 6.4.7).

Applying this proposition we find 15 different diagrams that correspond to an involution in $W(E_{10})$. Let $\alpha_1, 2A_1, 3A_1, 4A_1, 5A_1, D_4, A_1 + A_1, D_4 + A_1, D_4 + 2A_1, D_6, D_6 + A_1, D_8, E_7, E_7 + A_1, E_8, E_8 + A_1.$ (8.7.1)

Note that not all of these lattices are primitively embedding in $E_{10}$ (although they may admit another primitive embedding). Namely, the sublattices $M = 5A_1, D_4 + 2A_1, D_6 + A_1, E_7 + A_1, D_8$ do not admit such embedding. In fact, applying an element from $W(E_{10})$ we will be able to re-embed these lattices in the parabolic sublattice isomorphic to $E_9$ spanned by the roots $\alpha_0, \alpha_2, \ldots, \alpha_9$. Let $\mathfrak{f}$ spans the radical of this lattice. The image of $M$ in $\mathfrak{f}^\perp/\mathfrak{f} \cong E_8$ must be a primitive embedding. Obviously, if $M$ is of rank 8, this is impossible. This excludes the lattices $E_7 + A_1$ and $D_8$. Let $l$ be the smallest number of generators of the discriminant group $D(M)$. Then the discriminant group of the orthogonal complement $M^\perp$ of the image $M$ in $E_8$ is generated by $l$ elements. However, for the lattice $M = 5A_1$ (resp. $M = D_4 + 2A_1, D_6 + A_1$), we have $l = 5$ (resp. $l = 4, 3$) but rank $M^\perp = 3 < l$ (resp. rank $M^\perp = 2$ or 1, both less than 4).

To summarize we find the following Table of the following possible lattices $\text{Num}(S)_+$ and $\text{Num}(S)_-$ are found in the following Table 8.6.
Lemma 8.7.4. Let $g$ be an involution of $S$. Then $g$ preserves a numerical polarization of degree 2.

Proof. We observe from Table 8.6 that the orthogonal complement of each anti-invariant sublattice contains a vector of square norm 2. So, in all other cases we can apply Lemma 8.6.5 to obtain the assertion.  

It follows from this lemma that an involution $g$ leaves invariant a bielliptic linear system $|2F_1 + 2F_2|$, where $F_2$ is nef, or $F_2 = F_1 + R$ for some $(-2)$-curve $R$. It follows that $g$ is either the bielliptic involution associated with a bielliptic map $\phi : S \rightarrow D_1$ or a lift of an involution of a 4-nodal anti-canonical quartic del Pezzo surface $D$. In any case it leaves invariant the sublattice generated by $[F_1], [F_2]$ and acts on it identically, or permutes $[F_1]$ with $[F_2]$. It follows that the primitive sublattice $\text{Num} (S)_+$ of $\text{Num} (S)$ on which $\sigma$ acts identically contains a sublattice isomorphic to $U$ or $(2)$ as its direct summand with orthogonal complement generated by the classes of irreducible components of curves blown down to points by the morphism $\phi$.

Proposition 8.7.5. Assume $p \neq 2$. Let $g$ be an involution of an Enriques surface $S$ with isolated fixed points. Then the number of fixed points is equal to 4 and the quotient surface is birationally isomorphic to an Enriques surface.

Proof. Let $f : S' \rightarrow S$ be the blow-up of $S^g$ and $Y = S'/(g')$, where $g'$ is the lift of $g$ to $S'$. The quotient cover $f' : S' \rightarrow Y$ is ramified over the union $B$ of $k$ disjoint $(-2)$-curves on $Y$. Then $e(S') = 12 + k$, where $k = \#S^g$. The usual Hurwitz type formulas gives $e(S') = 2e(Y) - 2k$ and $K^2_Y = -4 = 2(K^2_F - \frac{1}{2}k)$ that gives $e(Y) = \frac{1}{2}(12 + 3k)$, $K^2_Y = 0$. It follows that $Y$ is a surface with $q = p_g = 0$ and $K^2_Y = 0$. Suppose $Y$ is a rational surface, then $K^2_Y = 0$ and Lemma 9.1.1 implies (since $Y$ has no rational curves with self-intersection $<-2$) that $Y$ is obtained by blowing up 9 points in $\mathbb{P}^2$, hence $e(Y) = 12$ and $k = 4$. We have $|K_Y| \neq 0$ and contains the proper transform $C$ of a plane cubic through the nine points. It does not intersect the branch curve $B$. We have $K_{S'} = f'^*(K_Y + R)$, where $f'^*(R)$ is the disjoint union of four $(-1)$-curves on $S'$. Blowing them down we obtain that $|K_S|$ contains an effective divisor, the pre-image on $S$ of a divisor from $|K_Y|$. Since $S$ is an Enriques surface we get a contradiction. Other possibility is that $Y$ is a minimal non-ruled surface. It follows from the classification of algebraic surfaces that $Y$ must be an Enriques surface and again $k = 4$.  

Example 8.7.6. Let $f : S \rightarrow \mathbb{P}^1$ be an elliptic fibration on an Enriques surface $S$ with a fiber $F = 2R_0 + R_1 + \cdots + R_4$ of type $D_4$. We have $R_1 + \cdots + R_4 = F - 2R_0 \sim 2F_1 - 2R_0$, where $F_1$ is one of half-fibers of $f$. Let $\pi : \tilde{S} \rightarrow S$ be the double cover of $S$ defined by the line bundle $O_S(F_1 - R_0)$ and its section $s$ with the zero divisor $R_1 + \cdots + R_4$. The pre-image of $R_1 + \cdots + R_4$ on $\tilde{S}$ is the disjoint union $E$ of four $(-1)$-curves. The pre-image of $R_0$ is an elliptic curve $\tilde{R}_0$ with self-intersection $-4$. We also have $K_{\tilde{S}} = \pi^*O_S(K_S + F_1 - R_0) \cong \pi^*O_S(F'_1 - R_0)$, where $F'_1$ is another half-fiber of $|2F_1|$. Let $\tilde{S} \rightarrow S'$ be the blowing down of $E$. As in the proof of the previous Lemma, we check that $S'$ is a surface with $e(S') = 12$, $K^2_{S'} = 0$ that admits an elliptic fibration, the pre-image of the elliptic fibration on $S$. Since it birationally dominates $S$, it cannot be a ruled surface. The only other possibility is that $S'$ is an Enriques surface. The covering involution $S' \rightarrow S$ defines the deck involution $g$ with 4 fixed points, the images of $E$ on $S'$.

Another example is given in [330, 5.2]
Example 8.7.7. Let $X$ be an Enriques sextic surface, a birational model of an Enriques surface $S$. Assume that it is invariant with respect to the projective involution $\tau : [x, y, z, w] \mapsto [y, x, w, z]$. Its equation must be of the form

$$
(a_1(x^2 + y^2) + a_2(z^2 + w^2) + a_3xy + a_4zw + a_5(xz + yw) + a_6(xw + yz))xyzw
+ (x^2y^2z^2 + x^2y^2w^2 + x^2z^2w^2 + y^2z^2w^2) = 0.
$$

The set of fixed points of the involution $\tau$ in $\mathbb{P}^3$ is the union of two lines $x + y = z + w = 0$ and $x - y = z - w = 0$. Plugging in these equation in the equation of $X$, we find that the lines intersect $X$ at four isolated points not lying on the coordinate tetrahedron $xyzw = 0$. Their pre-images on $S$ are four isolated fixed points of the involution $\tau$ lifted to $S$. Note that a pair of opposite edges defines on $S$ a choice of a bielliptic linear system $|2F_1 + 2F_2|$. Our involution preserves any of them. It leaves invariant the pencils $|2F_1|$ and $|2F_2|$, but either permutes both pairs of half-fibers $(F_1, F'_1)$ and $(F_2, F'_2)$ or permutes only one pair of half-fibers. This shows that it is a lift of an involution of a quartic surface $D$ under the bielliptic map.

Remark 8.7.8. We leave to the reader to see that any involution on an Enriques surface with 4 isolated fixed points arises from a double cover of an Enriques surface with the branch curve equal to the union of four isolated fixed points. These can be only fibers of type $D_n, n \leq 8$, or $E_7$, or $E_8$. We do not know whether any set of four different fibers occurs in this way.

Before we proceed, let us remind the structure of the group of automorphisms of a 4-ndual anticanonical quartic del Pezzo surface $D = D_1$ which we described in Section 0.6 from Volume 1.

We chose the equations of $D_1$ to be

$$
x_0^2 + x_1x_2 = x_3^2 + x_3x_4 = 0.
$$

The connected component of the identity $\text{Aut}(D_1)^0$ is the 2-dimensional torus that acts by formulas

$$
t_{\lambda, \mu} : [x_0, \ldots, x_4] \mapsto [x_0, \lambda x_1, \lambda^{-1}x_2, \mu x_3, \mu^{-1}x_4].
$$

The group of connected components is generated by transformations

$$
g_{1324} : [x_0, x_1, x_2, x_3, x_4] \mapsto [x_0, x_3, x_4, x_2, x_1],
g_{12} : [x_0, x_1, x_2, x_3, x_4] \mapsto [x_0, x_2, x_1, x_3, x_4].
$$

They generate the dihedral group $D_8 = (\mathbb{Z}/4\mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z})$ with the normal subgroup of order 4 generated by $g_{1324}$ and the quotient group generated by $g_{12}$. They satisfy $g_{12} \circ g_{1324} \circ g_{12} = g_{1324}^{-1}$. The group $D_8$ acts by permuting the four lines in $D_1$ given by equations $x_0 = x_i = x_j = 0, i \in \{1, 2\}, j \in \{3, 4\}$.

The anti-canonical rational map $f : \mathbb{P}^2 \to D_1$ is given by the linear system of plane cubics

$$
[x, y, z] \mapsto [\sqrt{-1}xyz, x^2y, yz^2, xy^2, xz^2].
$$

The elements of $\text{Aut}(D_1)$ correspond to to the following Cremona transformations of the plane

$$
t_{\lambda, \mu} : [x, y, z] \mapsto [\lambda x, \mu y, z],
g_{1324} : [x, y, z] \mapsto [xy, z^2, xz],
g_{12} : [x, y, z] \mapsto [z^2, xy, xz].
$$
8.7. INVOLUTIONS OF ENRIQUES SURFACES

Note that \( g_{1234}^2 \) is the standard Cremona involution \([x, y, z] \mapsto [yz, xz, xy] \). Now let us see which involutions in \( \text{Aut}(D) \) can be lifted to an involution of \( S \).

**Lemma 8.7.9.** Assume \( p \neq 2 \). Let \( g \) be an involution of \( S \) that does not coincide with the deck transformation of a bielliptic map given by a linear system \([2F_1 + 2F_2]\). Then \( S^g \) consists of four isolated fixed points.

**Proof.** It follows from Lemma 8.7.4 that, without any assumption, \( g \) leaves invariant a bielliptic map \( \phi : S \to D \) and hence arises as the deck transformation or as a lift of an involution \( \tau \) of \( D \) leaving the branch curve invariant. Suppose the latter case occurs. Then \( S^g \) consists of the pre-image of \( D^\tau \) with the union of the branch curve \( W \) of \( \phi \) and the singular points of \( D \).

We consider here only the case when \( D = D_1 \) and leave the case \( D = D_1' \) to the reader. The involutions in \( \text{Aut}(D)^0 \) are \( t_{1,-1}, t_{-1,1} \) and \( t_{-1,-1} \). Suppose \( t_{1,-1} : (x, y, z) \mapsto (x, -y, z) \) lifts to a birational involution of the double plane \( w^2 = xyf_6(x, y, z) \). If \( f_6(x, -y, z) = f_6(x, y, z) \), then it lifts to an automorphism of order 4 defined by \( (x, y, w) \mapsto (x, -y, iw) \) whose square is the deck transformation \( (x, y, w) \mapsto (x, y, -w) \). If \( f_6(x, -y) = -f_6(x, y) \), then it follows from the formula for the Enriques octic (3.3.9) in Section 3.3 that \( xy \) divides \( f_6(x, y) \) and the double plane is non-normal. Similarly we consider the involution \( t_{-1,1} \).

The involution \( t_{-1,-1} \) lifts to a biregular involution \( g \) of \( S \). Its set of fixed points in \( D \) is equal to the union of the four lines. Since \( S^g \) is smooth, the only possible scenario is that \( S^g \) consists of two half-fibers of one of the pencils \([2F_1]\), the pre-images of a pair of skew lines on \( D \), and four isolated points, the pre-images of the intersection points of the branch curve \( W \) with another pair of skew lines. The restriction of \( t_{-1,-1} \) to \( W \) has eight fixed points, the intersection points with the quadrangle of lines. The argument from the proof of Proposition 8.7.5 shows that the quotient surface \( V \) is a surface with \( p_g = q = 0 \) and \( K_V^2 = 0 \). It contains a pencil of elliptic curves whose pre-image on \( S \) is the pencil generated by one of the pencils \([2F_1], [2F_2]\) such \( S^g \) contains two its half-fibers. The image of two half-fibers of another pencil are \((-1)\)-curves on \( V \). This implies that \( V \) is a rational elliptic surface that contains 4 disjoint \((-2)\)-curves whose some is divisible by 2 in \( \text{Pic}(V) \). It follows from Proposition 0.7.1 in Section 0.7 in Volume 1 that \( V \) is obtained by blowing up 4 points on a quartic symmetroid 4-nodal surface \( D \). Thus we obtain that the quotient map \( S \to S/(g) \) coincides with a bielliptic map and \( g \) coincides with a bielliptic involution.

The involutions in \( \text{Aut}(D)/ \text{Aut}(D)^0 \) are non-zero elements in the group \((\mathbb{Z}/2\mathbb{Z})^2\) generated by \( g_{1234}^2 = g_{1234} \) and \( g_{12} \). As we noted before the involution \( g_{1234} \) corresponds to the standard Cremona involution in the plane. The curve \( f_6(x, y, z) = 0 \) must be invariant, hence \( f_6(yz, xz, xy) = x^2y^2z^2f_6(x, y, z) \) and \( xzf_6(x, y, z) \) is transformed to \( x^3y^3z^4f_6(x, y, z) \). This lifts to an involution \( [x, y, z, w] \mapsto [yz, xz, xy, wz^2] \). One checks that it indeed lifts to an involution \( g \) of \( S \). The fixed points of the standard Cremona involutions are \([\pm1, \pm1, 1]\). The points \([1, 1, 1]\) and \([-1, -1, 1]\) lift to the four fixed points of \( g \). They lie on the fiber of the elliptic fibration over the line \( y - x = 0 \). This is the involution from Example 8.7.6. The other pair of points lift to 4 points that are permuted by pairs by \( g \). The composition with the deck involution is an involution of \( S \) with four fixed points equal to the pre-images of \([1, -1, 1]\) and \([-1, 1, 1]\).

The Cremona transformation \( g_{12} \) transforms the Enriques octic \( xyf_6(x, y, z) \) to \( x^3yz^6f_6(x, y, z) \). It lifts to an involution \( g \) of the double plane that acts by \( [x, y, z, w] \mapsto [z^2, xy, xz, wxz^3] \). The
fixed locus of $g_{12}$ in the plane consists of two lines $x \pm z = 0$. The pre-image $C_1$ of the line $x - z = 0$ and the four intersection points of the line $x + z = 0$ with the branch curve $f_0 = 0$ is the locus of fixed points of $g$ in $S$. Let $C_2$ be the pre-image of the line $x + z = 0$. The curves $C_1$ and $C_2$ span a pencil of curves of genus 2 on $S$. The fixed locus $S^g$ consists of a smooth curve $C_1$ of genus 2 and four isolated points $q_1, \ldots, q_4$. The pre-image of the pencil $|e_0 - e_2|$ on $D$ is a $g$-invariant pencil of curves of genus 2 with two base points permuted by $g$. In the quotient surface $Y = S/(g)$ it defines a pencil of elliptic curves with one base point. We have $e(Y) = 9$, and we find that a minimal resolution $Y'$ of singularities of $Y$ is a weak del Pezzo surface with 4 disjoint $(-2)$-curves, the images of the exceptional curves of the blow-up of $S$ at four isolated fixed points. Since $\text{rank } \text{Pic}(Y') = 9$, we find that $\text{rank } \text{Num}(S)_+ = 5$. There are two possible weak del Pezzo surfaces $D$ of degree 1 with 4 disjoint $(-2)$-curves. They correspond to non-conjugate root bases of type $4A_1$ in $E_8$. One possibility is to obtain $D$ as the blow up of 4 points $p_1, \ldots, p_4$ and 4 infinitely near point $p_i' \succ p_i$.

Another possibility is to obtain $D$ as the blow-up of two points on a 4-nodal cubic surface, or, equivalently to blow up six vertices $p_{ij} = \ell_i \cap \ell_j$ of a complete quadrilateral with sides $\ell_1, \ldots, \ell_4$ and two more points $p_1, p_2$. It is easy to see that the image $\tilde{C}_2$ of $C_2$ on $D$ is a $(-1)$-curve that intersects the four $(-2)$-curves. In the first scenario for the weak del Pezzo surface $D$ there are 4 disjoint $(-1)$ curves, each intersecting one of the $(-2)$-curves with multiplicity 1. After we blow down these curves and then the images of the $(-2)$-curves we obtain that the image of $\tilde{C}_2$ in the plane is a curve with self-intersection 3, a contradiction. Thus we are dealing with the weak del Pezzo surface of the second type. We find two lines $L_1, L'_1$ (resp. $L_2, L'_2$) passing trough $p_1$ (resp. $p_2$) and $p_{12}, p_{34}$ (resp. $p_{13}, p_{24}$). The pre-images of these lines on $D$ are four $(-1)$-curves $E_1, E'_1, E_2, E'_2$, each intersecting two of the $(-2)$-curves. The pre-images $G_1, G'_1, G_2, G'_2$ of these four $(-1)$-curves on $S$ are $g$-invariant elliptic curves such that $G_1 \cdot G_2 = 1$ and $G_1 \equiv G'_1, G_2 \equiv G'_2$. Thus, we have found a bielliptic linear system $|2G_1 + 2G_2|$ invariant with respect to the involution $g$. The bielliptic map blows down three disjoint $(-2)$-curves, the pre-images of the diagonal of the complete quadrilateral. So, we have proved that $g$ coincides with the bielliptic involution of a bielliptic map. The involution $g$ is of type (15) from Table 8.8 below.

\[\square\]

**Remark 8.7.10.** The Cremona transformation $g_{12}$ acting on $D_1$ interchanges the pencils of conics $|e_0 - e_1|$ and $|2e_0 - e_2 - e_3 - e_4 - e_5|$. Thus its lift interchanges the classes $[F_1], [F_2]$ and hence defines the involution of $\text{Num}(S)$ with $\text{Num}(S)_- \cong E_8 \oplus A_1, \text{Num}(S)_+ \cong \langle 2 \rangle$. It seems this case was omitted in [330].

Let us now assume that $g$ is the deck involution of a bielliptic map $\phi : S \to D$. We will use the following result from [641, Section 3].

**Lemma 8.7.11.** Assume $p \neq 2$ and let $X$ be a smooth minimal projective surface with $\text{kod}(X) \geq 0$. Let $f : X \to Y$ be a morphism of degree 2 onto a normal surface. Then any connected fiber $C = f^{-1}(y)$ over a nonsingular point of $Y$ is a point or the union of $(-2)$-curves whose dual graph is of type $A_n, D_n, E_n$ as in the following picture.
The deck transformation $\sigma$ of $f$ extends to a biregular automorphism of $X$. It acts on the components of $C$ as follows

- $\sigma(a_i) = a_{n+1-i}, i = 1, \ldots, n$;
- $\sigma(d_i) = d_i$ if $n$ is even;
- $\sigma(d_1) = d_2, \sigma(d_i) = d_i, i \neq 1, 2$ if $n$ is odd;
- $\sigma(e_1) = e_1, \sigma(e_i) = e_{8-i}, i \neq 1, 6$ if $n = 6$;
- $\sigma(e_i) = e_i$ if $n = 7, 8$.

Moreover, $C$ contains $k$ irreducible components fixed by $\sigma$ pointwise, where

$$k = \begin{cases} 
0, & a_n, \\
1, & d_{2m}, d_{2m+1}, \\
2, & e_6, \\
3, & e_7, \\
4, & e_8.
\end{cases}$$

**Proof.** Let $X \to X' \to Y$ be the Stein factorization of $f$, where $X \to X'$ is a birational morphism and $X' \to Y$ is a finite morphism of degree 2. Since $K_X$ is nef, any curve $R$ blown down to a point of $X'$ satisfies $R^2 < 0$ and $R \cdot K_X \geq 0$. By the adjunction formula, this implies that $R$ is a $(-2)$-curve. By Proposition 0.4.2, the intersection matrix of the irreducible components of $C$ is negative definite. Since $x = f(C)$ is a nonsingular point of $Y$, we obtain that $x$ is a singular point of the branch curve $B$ of the cover $f^{-1}(Y \setminus \text{Sing}(Y)) \to Y \setminus \text{Sing}(Y)$. Thus the fiber $f^{-1}(x)$ coincides with $C$. Let $\phi(u, v) = 0$ be a local equation of $B$ at $x$. Then the pre-image $x'$ of $x$ in $X'$ is a singular point locally isomorphic to $w^2 + \phi(u, v) = 0$ and $X \to X'$ is a minimal resolution of $x'$ over an open neighborhood of $x'$ equal to a minimal resolution of a rational double point of type $A_n, D_n, E_n$. Thus $x'$ is such a point and $x$ is a simple singularity of $B$ of the corresponding type. To see how $\sigma$ acts on the irreducible components of $C$, one resolves the singular point explicitly and observe the action of the involution $w \mapsto -w$ on the irreducible components. We leave this exercise to the reader.

Finally, we use that $C^\sigma$ does not contain isolated fixed points from $X^\sigma$ since $f(C)$ is a nonsingular point of $Y$. The intersection $X^\sigma \cap C$ consists of $k$ irreducible components and points where the proper inverse transform $\bar{B}$ of $B$ intersects $C$. 

\[\begin{array}{cccccccc}
 & \bullet & \bullet & \bullet & \cdots & \bullet & \bullet & \bullet \\
A_n & a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\
 & \bullet & \cdots & \bullet & \bullet & \bullet & \bullet & \bullet \\
D_n & d_2 & d_3 & d_4 & \cdots & d_{n-1} & d_n \\
 & \bullet & d_1 & \bullet & \bullet & \bullet & \bullet & \bullet \\
E_n & e_2 & e_3 & e_4 & e_5 & \cdots & e_{n-1} & e_n \\
 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array}\]
In case $A_{2k}$, we see that no component is invariant with respect to $\sigma$. In the case $A_{2k+1}$ we see that only the central component is invariant and it contains two points on it that belong to $B$. Thus $k = 0$.

In case $D_{2m}$, we see that $\bar{B}$ intersects the components $d_1, d_2, d_n$ at one point, and the components $d_3, d_5, \ldots, d_{n-1}$ must enter in $X^\sigma$. Thus $k = m - 1$.

In case $D_{2m+1}$, $\bar{B}$ intersects the components $d_3$ and $d_n$ at one point, and the components $d_4, d_6, \ldots, d_{n-1}$ must enter in $X^\sigma$. Thus $k = m - 1$, again.

In case $E_6$, $\bar{B}$ intersects the component $e_4$ at one point, and the second invariant component $e_1$ must enter in $X^\sigma$. Thus $k = 1$.

In case $E_7$, $\bar{B}$ intersects the components $e_7$ and $e_1$. The components $e_2, e_4, e_6$ enter in $X^\sigma$. Thus $k = 3$.

In case $E_8$, $\bar{B}$ intersects the components $e_1$ and the components $e_2, e_4, e_6, e_8$ enter in $X^\sigma$. Thus $k = 4$.

Let $E$ be the sublattice generated by irreducible curves blown down by $\phi$. They are common irreducible components of fibers of the pencils $|2F_1|$ and $|2F_2|$ (or common irreducible components of $|2F_1|$ not intersecting $R$). We denote by $E(x)$ the direct summand of $E$ generated by irreducible components blown down to a point $x \in D$. It follows from Lemma 8.2.5 and Lemma 8.7.11 that the $g^*$-invariant part $E(x)_+$ of $E(x)$ is isomorphic to the following lattice:

$$
\begin{cases}
A_1^{\oplus k} & \text{if } E(x) \text{ is of type } A_{2k-1}, A_{2k} \\
D_{2k} & \text{if } E(x) \text{ is of type } D_{2k+1}, \\
D_4 & \text{if } E(x) \text{ is of type } E_6, \\
E(x) & \text{otherwise}
\end{cases}
$$

For example if $E(x) \cong E_6$ then $E(x)_+$ has a root basis $e_1, e_4, e_3 + e_4 + e_5, e_2 + e_3 + e_4 + e_5 + e_6$ of type $D_4$. Since $g$ acts identically on the sublattice $U$ generated by $[F_1], [F_2]$, we have $E_+ \oplus U$ is a sublattice of finite index in $\text{Num}(S)_+$. It follows from above that $E$ is a 2-elementary sublattice. The known structure of $\text{Num}(S)_+$ gives all possible cases. We list them in Table 8.7. Here $A_k$ denotes a part of $E$ of type $A_{i_1} + \cdots + A_{i_s}$, where $\sum_{j=1}^s \left[ \frac{i_j + 1}{2} \right] = k = \delta$. Thus $A_k$ define many cases, which we do not want to list for typographical reason. For example, $A_1 = \{ A_1, A_2 \}$ but $A_2 = \{ 2A_1, A_2 + A_1, A_3 \}$.

Let $S'$ be the blow-up of $S$ at the four isolated fixed points of $g$. The quotient surface $D' = S'/\langle g \rangle$ is a nonsingular model of the 4-nodal quartic surface $D$. The double plane model $w^2 + xyzf_0(x, y, z) = 0$ of $S$ is obtained as the composition of the quotient map $S' \to D' \to \mathbb{P}^2$, where $\alpha : D' \to \mathbb{P}^2$ is a birational map that sends the proper transform in $D'$ of the branch curve of the bielliptic map $\phi : S \to D$ to the part $W : f_0(x, y, z) = 0$ of the branch curve of $\phi$. There is a choice of the blowing down map $\alpha : D' \to \mathbb{P}^2$ that gives a different double plane model of $S$ with curves $W$ Cremona equivalent. Note that the curve $W$ consists of $h$ irreducible components $W_k$ represented in $\text{Pic}(D)$ by the classes $d_0e_0 - m_1^{(i)}e_1 - \cdots - m_5^{(i)}e_5$ that add up to the class $6e_0 - 2e_1 - \cdots - 2e_5$. 

\[ \]
8.7. INVOLUTIONS OF ENRIQUES SURFACES

We let \((d_1, \ldots, d_h)\) be the degrees of these components and \((g_1, \ldots, g_h)\) their geometric genera. The usual formula

\[
5 = p_a(W) = \sum_{i=1}^{h} g_i + \delta - h + 1
\]
gives that the total \(\delta\)-invariant \(\sum_{p \in W} \delta_p(W)\) equal to \(\delta = 4 + h - \sum_{i=1}^{h} g_i\).

It is not difficult but tedious to list all possibilities for possible singularities of \(W\) that gives the information on the lattice \(E\), the locus of fixed points \(S^g\) and the action of \(g\) on \(\text{Num}(S)\). The following Table gives the representatives of Cremona equivalence classes of possible curves \(W\).

Now we use Lemma 8.7.11 and combine Table 8.6 with Table 8.7 to obtain the following Table 8.8.

Here \(C(g)\) denote a smooth curve of genus \(g\). In cases (1)-(3), we assume that \(p \neq 2\) and we use Corollary 8.2.23 to describe possible numerically trivial involutions of an Enriques surface.

Remark 8.7.12. Here are some comments and hints for the above classification.

1. The first three types (1)-(3) in Table 8.8 are numerically trivial involutions. They correspond to Cases (A),(B),(C) from Corollary 8.2.23.

2. Type \((13^*)\) is the only type of an involution that does arise as a bielliptic involution.

3. A plane sextic curve \(C\) has no simple singular point of type \(D_8\). In fact locally such singularity is given by equation \(x(y^2 + x^6) = 0\). The line \(y = 0\) intersects it with multiplicity 7 unless it is an irreducible component of \(C\). But then the curve has a singular point locally isomorphic to \(y(y^2 + x^6)\). This is not a simple singularity.

4. We often use the Cremona transformation given by the linear system \(|3e_0 - e_2 - e_3 - e_4 - e_5 - 2e_P|\) or by choosing appropriately an extra point \(P\). That allows sometimes to lower the degree of components of \(W\). If \(P\) is taken to be a triple point of \(W\) we acquire a conic as an additional component.

5. Another useful Cremona transformation is a quadratic involution given by the linear system of conics \(|2e_0 - e_2 - e_3 - e_4|\) or \(|2e_0 - e_2 - e_3 - e_4|\). It exchanges the linear system of quartic \(|4e_0 - 2e_1 - e_2 - e_3 - e_4 - e_5|\) with the linear system of quintics \(|5e_0 - e_1 - 2e_2 - 2e_3 - 2e_4 - 2e_5|\).

6. The degrees of possible irreducible components of the branch curve on \(D\) are \((8), (6, 2), (4, 4), (4, 2, 2), (2, 2, 2, 2)\). This correspond to our cases \((6), (5, 1), (3, 3), (3, 2, 1), (2, 2, 1, 1)\)

7. Type \(6^*\) occurs when 4 singular points lie on two conics from the same pencil \(|e_0 - e_1|\) of \(|2e_0 - e_2 - e_3 - e_4 - e - 5|\).

Using the classification of involutions of \(S\), we can also classify cyclic groups of order 4 and 8.

We can also construct an involution on \(S\) by using the rational quadratic twist construction from Proposition 4.8.6 (see [306]). Recall that it is given by a choice of a rational elliptic surface \(j : J \to \mathbb{P}^1\) with a fixed section \(R_0\), a double cover \(\phi : X \to J\) branched along two fibers \(J_{t_1}, J_{t_2}\) of multiplicative type and a section \(s : B \to X\) of the \(X \times_{\mathbb{P}^1} B \to B\), where \(B \to \mathbb{P}^1\) is the double
\begin{center}
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
$\text{Num}(S)_-$ & $A_1$ & $2A_1$ & $3A_1$ & $4A_1$ & $D_4$ \\
\hline
$\text{Num}(S)_+$ & $E_7 + U$ & $D_6 + U$ & $D_4 + A_1 + U$ & $U + 4A_1$ & $D_4 + U$ \\
\hline
$\text{Num}(S)_-$ & $D_6$ & $D_4 + A_1$ & $E_7$ & $E_8$ & $E_8 + A_1$ \\
\hline
$\text{Num}(S)_+$ & $2A_1 + U$ & $3A_1 + U$ & $A_1 + U$ & $U$ & (2) \\
\hline
\end{tabular}
\end{center}

Table 8.6: Possible action of an involution on $\text{Num}(S)$

\begin{center}
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
$\delta$ & $g_0$ & $d_\xi$ & $\mathcal{E}$ & $\mathcal{E}_+$ & $\text{Num}(S)_+$ & $m$ \\
\hline
0 & (6) & (5) & 0 & $\{0\}$ & $\{0\}$ & $U$ & 10 \\
1 & (6) & (4) & 1 & $A_1$ & $A_1$ & $U + A_1$ & 9 \\
2 & (6) & (3) & 2 & $A_2$ & $2A_1$ & $U + 2A_1$ & 8 \\
3 & (6) & (2) & 3 & $A_3$ & $3A_1$ & $U + 3A_1$ & 7 \\
4 & (6) & (1) & 4 & $A_4$ & $4A_1$ & $U + 4A_1$ & 6 \\
5 & (6) & (0) & 5 & $A_5(n)$ & $5A_1$ & $U + D_4 + A_1$ & 5 \\
6 & (5,1) & (1,0) & 4 & $A_4$ & $4A_1$ & $U + D_6$ & 6 \\
7 & (5,1) & (1,0) & 5 & $A_5$ & $5A_1$ & $U + D_4 + A_1$ & 4 \\
8 & (5,1) & (0,0) & 6 & $A_6$ & $6A_1$ & $U + D_4$ & 3 \\
9 & (3,3) & (1,1) & 4 & $A_4$ & $4A_1$ & $U + D_4$ & 6 \\
10 & (3,3) & (1,0) & 5 & $A_5$ & $5A_1$ & $U + D_4 + A_1$ & 4 \\
11 & (3,3) & (0,0) & 6 & $D_4 + A_1$ & $4A_1$ & $U + D_6$ & 3 \\
12 & (3,3) & (1,0) & 7 & $D_4 + A_1$ & $4A_1$ & $U + D_6$ & 2 \\
13 & (3,3) & (0,0) & 8 & $A_8$ & $8A_1$ & $U + E_8$ & 2 \\
\hline
\end{tabular}
\end{center}

Table 8.7: Singularities of the sextic branch curve $W$
cover branched over $t_1, t_2$. It must satisfy $s \oplus s' = 0$ in $\text{MW}(f)$, where $s'$ is the image of $R$ under the deck transformation $\alpha$ of $X \to J$. Here $X$ might be singular over the singular points of the fibers $J_{t_1}, J_{t_2}$, so we should replace $X$ by its minimal resolution of singularities. The involution $\tau$ on $X$ defined by the composition of the translation $t_s$ and the deck transformation $\gamma$ is fixed-point free and the quotient $S = X/(\tau)$ is an Enriques surface. Consider the involution $\alpha = \gamma \circ \iota$, where $\iota$ is the negation involution on fibers. Let $x \in X_\iota$, then

$$\alpha \circ \tau(x) = \alpha(\gamma(x + s(t))) = \iota(x + s(t)) = -x - s(t),$$

$$\tau \circ \alpha(x) = \tau(-\gamma(x)) = \gamma(-\gamma(x) + s(t)) = -x - s'(t).$$

Since $s(x) + s'(x) = 0$, we obtain that the two involutions commute. Thus $\alpha$ descends to $S$ and defines an involution $\alpha'$ on $S$ that preserves the elliptic fibration on $S$ which is a torsor of $J$. The involution $\alpha$ has 8 fixed points lying on the pre-images of the fibers $J_{t_1}, J_{t_2}$ on $X$. The quotient surface $X'$ has 8 singular points of type $A_1$ and its minimal resolution $\tilde{X}'$ is a K3 surface. On the other hand, the involution $\tau$ descends to the quotient $X' = X/(\alpha)$ and defines an involution $\tau'$ on $X'$. Both involutions $\tau$ and $\alpha$ commute with the involution $\beta = \tau \circ \alpha$. The involutions $\tau$ and $\alpha$ descend to the same involution $\beta'$ on $Y = X/(\beta)$. We have a commutative diagram of the quotient maps

\[
\begin{array}{ccc}
X & \overset{\alpha}{\longrightarrow} & X' \\
\tau' \downarrow & & \downarrow \beta' \\
Y & \overset{\beta}{\longrightarrow} & X' \\
\alpha' \downarrow & & \downarrow \tau \\
S & \overset{\iota}{\longrightarrow} & Y
\end{array}
\]

### Table 8.8: Types of deck transformations of bielliptic maps ($p \neq 2$)

<table>
<thead>
<tr>
<th>Type</th>
<th>Num($S_-$)</th>
<th>Num($S_+$)</th>
<th>$\text{tr}(g^*)$</th>
<th>$\mathcal{E}$</th>
<th>$S^9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>$E_8 + U$</td>
<td>12</td>
<td>4, proof of Lemma 8.7.9</td>
<td>$C^{(1)} + 4P^1 + 4\text{pts}$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>$E_8 + U$</td>
<td>12</td>
<td>4, proof of Lemma 8.7.9</td>
<td>$4P^1 + 4\text{pts}$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>$E_8 + U$</td>
<td>12</td>
<td>4, proof of Lemma 8.7.9</td>
<td>$4P^1 + 4\text{pts}$</td>
</tr>
<tr>
<td>4</td>
<td>$A_1$</td>
<td>$E_7 + U$</td>
<td>10</td>
<td>9</td>
<td>$3P^1 + C^{(1)} + 4\text{pts}$</td>
</tr>
<tr>
<td>5</td>
<td>$A_1$</td>
<td>$E_7 + U$</td>
<td>10</td>
<td>9</td>
<td>$3P^1 + C^{(1)} + 4\text{pts}$</td>
</tr>
<tr>
<td>6</td>
<td>$2A_1$</td>
<td>$D_6 + U$</td>
<td>8</td>
<td>13, 19, 26</td>
<td>$2P^1 + C^{(1)} + 4\text{pts}$</td>
</tr>
<tr>
<td>7</td>
<td>$2A_1$</td>
<td>$D_6 + U$</td>
<td>8</td>
<td>13, 19, 26</td>
<td>$2P^1 + C^{(1)} + 4\text{pts}$</td>
</tr>
<tr>
<td>8</td>
<td>$3A_1$</td>
<td>$D_4 + A_1 + U$</td>
<td>6</td>
<td>12, 15</td>
<td>$P^1 + C^{(1)} + 4\text{pts}$</td>
</tr>
<tr>
<td>9</td>
<td>$3A_1$</td>
<td>$D_4 + A_1 + U$</td>
<td>6</td>
<td>12, 15</td>
<td>$P^1 + C^{(1)} + 4\text{pts}$</td>
</tr>
<tr>
<td>10</td>
<td>$D_4$</td>
<td>$D_4 + U$</td>
<td>4</td>
<td>22</td>
<td>$2C^{(1)} + 4\text{pts}$</td>
</tr>
<tr>
<td>11</td>
<td>$D_4$</td>
<td>$D_4 + U$</td>
<td>4</td>
<td>22</td>
<td>$C^{(1)} + 4\text{pts}$</td>
</tr>
<tr>
<td>12</td>
<td>$D_4$</td>
<td>$D_4 + U$</td>
<td>4</td>
<td>22</td>
<td>$C^{(1)} + 4\text{pts}$</td>
</tr>
<tr>
<td>13*</td>
<td>$D_4$</td>
<td>$D_4 + U$</td>
<td>4</td>
<td>6*</td>
<td>$C^{(1)} + 4\text{pts}$</td>
</tr>
<tr>
<td>14</td>
<td>$4A_1$</td>
<td>$4A_1 + U$</td>
<td>4</td>
<td>6*</td>
<td>$C^{(1)} + 4\text{pts}$</td>
</tr>
<tr>
<td>15</td>
<td>$D_4 + A_1$</td>
<td>$3A_1 + U$</td>
<td>2</td>
<td>4, proof of Lemma 8.7.9</td>
<td>$C^{(2)} + 4\text{pts}$</td>
</tr>
<tr>
<td>16</td>
<td>$D_6$</td>
<td>$2A_1 + U$</td>
<td>0</td>
<td>3</td>
<td>$C^{(3)} + 4\text{pts}$</td>
</tr>
<tr>
<td>17</td>
<td>$E_7$</td>
<td>$A_1 + U$</td>
<td>-2</td>
<td>2</td>
<td>$C^{(4)} + 4\text{pts}$</td>
</tr>
<tr>
<td>18</td>
<td>$E_8$</td>
<td>$U$</td>
<td>-4</td>
<td>1</td>
<td>$C^{(5)} + 4\text{pts}$</td>
</tr>
</tbody>
</table>
The surfaces $Y$ and $Z$ are rational surfaces. An example of this situation is the case when the quadratic twist corresponds to a special bisection $R$. Then the map $S \rightarrow Z$ is a bielliptic map given by the special bielliptic linear system, the map $Y \rightarrow Z$ is the double cover of the quartic symmetroid $Z$ ramified at its singular points.

Example 8.7.13. Let $g$ be the covering involution of a separable bielliptic map $f : S \rightarrow D$. Assume $p \neq 2$. If the branch curve $W$ is smooth, we know that $g$ is of type $E_8$. When $W$ is irreducible with $k$ ordinary nodes, then $S^g$ consists of 4 isolated fixed points and a curve of genus $5 - k$. This gives that the trace of $g^*$ on the cohomology is equal to $4 + 2 - 2(5 - k) = 2k - 4$. We also know that $g^*$ leaves invariant a hyperbolic plane. This gives that $g$ is of type $E_7, D_6, D_4 + A_1, 4A_1, 3A_1$, respectively.

The diagrams $5A_1, D_4 + 2A_1, D_6 + 2A_1, D_8$ define involutions that do not preserve a hyperbolic plane, and thus do not originate from bielliptic maps. However, they preserve an isotropic vector, and hence originate from an involution of an Enriques surface that preserves a genus one fibration. For example, they can be realized by the translation automorphism of the jacobian genus one fibration.

### 8.8 Finite groups of automorphisms of Mathieu type

Recall from [160], 6.2 that a Steiner system $S(t, k, v)$ consists of a set $\Sigma$ of cardinality $v$ whose elements are called points and a set $B$ of subsets of $\Sigma$ of cardinality $k$ whose elements are called blocks such that any subset of $t$ points is contained in a unique block. The number $b$ of blocks and the number $r$ of blocks containing a fixed point satisfy:

$$bk = vr, \quad r = \frac{(v - 1)(v - 2) \cdots (v - t + 1)}{(k - 1)(k - 2) \cdots (k - t + 1)}.$$  \hspace{1cm} (8.8.1)

Many interesting subgroups of $\text{Sym}(\Sigma)$ are realized as the symmetry groups of a Steiner system. The most remarkable examples are the five Mathieu sporadic simple groups $M_{11}, M_{12}, M_{22}, M_{23}$ and $M_{24}$. The corresponding Steiner systems are $S(4, 5, 11), S(5, 6, 12), S(3, 6, 22), S(4, 7, 23)$ and $S(5, 8, 24)$, respectively.

The group $M_{11}$ can be realized as a subgroup of $M_{12}$ that fixes one point, and the subgroups $M_{22}$ (resp. $M_{23}$) are realized as subgroups of $M_{24}$ that fix two (resp. one) points. The Mathieu group $M_{12}$ is also isomorphic to a subgroup of $M_{24}$. It is a subgroup that preserves the symmetric difference of two blocks meeting two points in $S(5, 8, 24)$.

The subgroup of $M_{11}$ that fixes a point is a Mathieu group $M_{10}$. It is not a simple group but contains the group $\mathfrak{A}_6$ as a subgroup of index 2 (but not isomorphic to $\mathfrak{S}_6$). It is the automorphism group of the Steiner system $S(3, 4, 10)$.

The number $\epsilon(n)$ of fixed points of the action of $\sigma \in M_{23}$ on $\Sigma$ with $v = 24$ depends only on the order $n$ of $\sigma$. It is equal to 0.

According to S. Mukai [504], any finite group $G$ of symplectic automorphisms of a complex K3 surface $X$ is isomorphic to a subgroup of the $M_{23}$ with $\geq 5$ orbits in its natural action on
8.8. FINITE GROUPS OF AUTOMORPHISMS OF MATHIEU TYPE

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>11</th>
<th>14</th>
<th>15</th>
<th>23</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_{M_{23}}(n)$</td>
<td>24</td>
<td>8</td>
<td>6</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Figure 8.3: Mathieu character for $M_{23}$

a set $\Sigma$. This is achieved by analyzing the character $\chi$ of the action of $G$ on the cohomology $H^*(X, \mathbb{C}) \cong \mathbb{Q}^{24}$. By the Lefschetz fixed point formula, for any $g \in G$ of order $n$, we have

$$\chi(g) = |X^g| = e(n) := 24 \left( n \prod_{p \mid n} \left( 1 + \frac{1}{p} \right) \right)^{-1},$$

and this coincides with the character of $M_{23}$ in its permutation representation on $\Sigma$ given in Table 8.3. The set of one-dimensional sub-representations of $G$ in $H^*(X, \mathbb{C})$ corresponds to the orbits of $G$ in its action on $\Sigma$. They are spanned by the Hodge subspaces $H^{0,0}, H^{2,2}, H^{2,0}, H^{0,2}$ and the subspace of $H^{1,1}$ generated by a $G$-invariant Kähler class. In the case of positive characteristic $p > 0$, the same is true if we replace the complex cohomology with the $l$-adic ones, assume that the order of $G$ is prime to $p$ (this is always satisfied if $p > 11$), and $X$ is not supersingular K3 with Artin invariant 1 (see [178]).

The number of fixed points of an element $\sigma \in M_{11}$ on the subset $\Sigma^+ \subset \Sigma$ of cardinality 12 depends only on the order $n$ of $\sigma$. We have

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_{M_{11}}(n)$</td>
<td>12</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Figure 8.4: Mathieu character for $M_{11}$

Following S. Mukai, we say that a finite group $G$ of automorphisms of an Enriques surface is of Mathieu type if it acts semi-symplectically and the character of $\chi$ on $H^*(X, \mathbb{Q}_l) \cong \mathbb{Q}_l^{12}$ satisfies

$$\text{tr}(g^*) = \# S^g = \chi_{M_{11}}(\text{ord}(g)).$$

In our case, by Theorems 8.6.11 and Proposition 8.6.15, if $p \neq 2$, we have $n \neq 8, 11$.

**Proposition 8.8.1.** Assume $p \neq 2$. Let $G$ be a finite group of automorphisms of an Enriques surface. Then $G$ is of Mathieu type if and only if it does not contain elements of order 8, and the conjugacy class of $g^* \neq 1$ must be of the following types:

- $(n = 2)$ $D_4$ or $4A_1$ with $e(S^g) = 4$;
- $(n = 3)$ $3A_2$ with $e(S^g) = 3$;
- $(n = 4)$ $D_4(a_1) + 2A_1$ with $e(S^g) = 4$;
- $(n = 5)$ $2A_4$ with $e(S^g) = 2$;
- $(n = 6)$ $D_4 + 2A_2$ with $e(S^g) = 1$. 
Proof. It follows from Proposition 8.6.15 that an element of order 8 is not semi-symplectic. Applying Table 8.6, we find the conjugacy classes of elements of order 2 (resp. 4, resp. 6) with the trace equal to 4 (resp. 4, resp. 1). In the case \( n = 4 \), we also eliminate those of them for which the conjugacy class of the square is not of type \( 4A_1 \) or \( D_4 \). The conjugacy classes of elements of odd order satisfy the assumption on the trace.

Using the fact that a group of automorphisms \( G \) of Mathieu type acting on an Enriques surface in characteristic \( p \neq 2 \) does not contain elements of order 8 and 11 gives some information on a possible structure of a maximal group of this type. Using [127], Table 10.3, we find that a maximal subgroup of \( M_{11} \) must be isomorphic to one of the following group:

\[
M_{10} \cong \mathfrak{A}_6 \times 2 \rtimes L_2(11), \ M_9 : 2 \cong 3^2 : Q_8,2, \ \mathfrak{S}_5, \ M_8 : \mathfrak{S}_3 \cong 2 \times \mathfrak{S}_4.
\]

(we employ the notation from loc.cit.). By analyzing subgroups \( G \) of these maximal groups which have no elements of order 8 and 11, we find that \( G \) is a subgroup of one of the following groups:

\[
\mathfrak{A}_6, \ \mathfrak{S}_5, \ \mathfrak{N}_{72}, \ C_2 \times \mathfrak{A}_4, \ C_2 \times C_4.
\] (8.8.2)

Here \( C_n \) denotes a cyclic group of order \( n \) and \( \mathfrak{N}_{72} = 3^2 \rtimes D_8 \).

We know that any finite subgroup of \( W(E_{10}) \) is conjugate to a subgroup of a finite parabolic subgroup \( W_J \) of one of the types listed in (6.4.4). The groups from (8.8.2) are not of \( E_8 \)-type because they contain elements of types \( 3A_2 \) and \( 2A_4 \) which are not of \( E_8 \)-type. The group \( \mathfrak{A}_6 \) is a subgroup of the Mathieu group \( M_{10} \), it must be conjugate to a subgroup of a parabolic subgroup \( W_J \) of type \( A_9 \) with \( W(A_9) \cong \mathfrak{S}_{10} \). The group \( \mathfrak{S}_5 \) must be a subgroup of a parabolic subgroup of type \( A_4 + A_5 \) with \( W(A_4 + A_5) \cong \mathfrak{S}_5 \times \mathfrak{S}_6 \). Finally, the group \( \mathfrak{N}_{72} \) is one of the maximal subgroups of \( M_{11} \) and occurs as a stabilizer of a 2-point set. It must be conjugate to a subgroup of a parabolic subgroup of type \( A_1 + A_8 \) with \( W(A_1 + A_8) \cong \mathfrak{S}_2 \times \mathfrak{S}_9 \).

It is known that the permutation action of the subgroups \( \mathfrak{A}_6, \ \mathfrak{S}_5, \ \mathfrak{N}_{72} \) of \( M_{11} \) on \( \Sigma^+ \) has orbits \((1,1,10),(1,5,6),(1,2,9)\), respectively. It shows that \( \dim H^*(S,\mathbb{Q})^G = 3 \), and \( \text{rank } \text{Num}(S)^G = 1 \). Applying Corollary 8.6.6, we obtain that \( \mathfrak{A}_6, \ \mathfrak{S}_5, \ \mathfrak{N}_{72} \) must preserve a polarization of degree 10, 30, 18, respectively.

We have also the converse.

**Proposition 8.8.2.** Suppose one of the groups \( G = \mathfrak{A}_6, \ \mathfrak{S}_5, \ \mathfrak{N}_{72} \) acts on an Enriques surface \( S \) preserving an ample polarization of degree 10, 30, 18, respectively. Then the action of \( G \) on \( S \) is Mathieu.

Proof. It follows from Proposition 1.5.3 that an element in \( E_{10} \) of norm 10, 30, 18 with \( \Phi(\nu) \geq 3 \) is equal to the fundamental weight \( \omega_0, \omega_2, \omega_4 \), respectively. Its orthogonal complement is a sublattice of \( E_{10} \) isomorphic to \( A_9, A_1 \oplus A_8 \) and \( A_4 \oplus A_5 \), respectively. Thus \( G \) is embedded in the Weyl groups of these lattices.

Assume \( G = \mathfrak{A}_6 \). The analysis of maximal subgroups of \( \mathfrak{A}_{10} \) in [127] shows that the only possible maximal subgroups which may contain \( \mathfrak{A}_6 \) with this property are the following ones: \( \mathfrak{A}_9, \mathfrak{S}_8, (\mathfrak{A}_7 \times 3) : 2, (\mathfrak{A}_6 \times 4) : 2, \) and \( M_{10} \). Now we use that \( G \) is not conjugate to a subgroup \( W(A_9) \) of a parabolic group of type \( E_8 \). Suppose \( G \subset \mathfrak{S}_8 \), then \( \mathfrak{S}_8 \) acts on 10 letters by leaving 8 letters invariants and switching a dual in the complement. Then \( G \) is contained in \( \mathfrak{A}_8 \) which is contained
in \( W(E_8) \). This discards this subgroup. The second maximal group preserves a triad of letters, so \( G \) would be embedded in \( \mathfrak{A}_7 \). Similar argument discards the group \((\mathfrak{A}_6 \times 4) : 2 \). Suppose \( G \subset \mathfrak{A}_6 \). Then \( G \) is conjugate to a parabolic subgroup of type \( A_8 \). Its orthogonal complement in \( E_{10} \) contains an isotropic vector. This would imply that \( \mathfrak{A}_6 \) preserves a genus one pencil which is obviously impossible. The surviving option is the Mathieu group \( M_{10} \), this is what we need.

Assume \( G = \mathfrak{S}_5 \) or \( N_{72} \). It is conjugate to a subgroup of a parabolic subgroup of type \( A_4 \times A_5 \) or \( A_1 \times A_8 \) isomorphic to \( \mathfrak{S}_5 \times \mathfrak{S}_6 \) or \( \mathfrak{S}_2 \times \mathfrak{S}_9 \). We consider these groups as subgroups of \( \mathfrak{S}_{11} \). A maximal subgroup of this group either preserves a subset of cardinality 1, 2, 3, 4, 5 or is a subgroup of two non-conjugate subgroups isomorphic to the Mathieu group \( M_{11} \). It is easy to see that the first possibility in our case implies that \( G \) is of \( E_8 \)-type or preserves a genus one fibration. This leads to a contradiction. \( \square \)

The main result of [509] is the following.

**Theorem 8.8.3.** Assume \( \mathbb{k} = \mathbb{C} \) and let \( G \) be a finite group of Mathieu type acting on an Enriques surface \( S \). A finite group \( G \) admits a Mathieu action on \( S \) if and only if it is isomorphic to a subgroup of one of the five maximal groups from (8.8.2). Equivalently, \( G \) is isomorphic to a subgroup of \( \mathfrak{S}_6 \) and its order is not divisible by 16.

We are not going to provide a proof and restrict ourselves only with providing examples of Mathieu actions of the first three maximal groups from the list (8.8.2). We refer to the examples realizing the last two groups to [508] and [510].

In the following, following S. Mukai and H. Ohashi, we give examples of Enriques surfaces with a Mathieu type action of each of these groups.

**Example 8.8.4.** ([508]) Assume \( G \cong \mathfrak{S}_5 \). Let \( X' \) be a surface of degree 6 in \( \mathbb{P}^4 \) given by the equations

\[
\sum_{1 \leq i < j \leq 5} x_ix_j = \sum_{1 \leq i < j < k \leq 5} x_ix_jx_k = 0. \tag{8.8.3}
\]

The surface has obvious \( \mathfrak{S}_5 \)-symmetry. Also it admits an involution \( \sigma \) defined by the standard Cremona transformation \([x_1, \ldots, x_5] \to [1/x_1, \ldots, 1/x_5] \). Its fixed points in \( \mathbb{P}^4 \) are the points with coordinates \( \pm 1 \). They do not lie on \( X' \) unless \( p = 2 \) or 5.

The Cremona involution \( \sigma \) in \( \mathbb{P}^4 \) is defined by the linear system of quartic hypersurfaces passing through the points \( q_1 = [1, 0, \ldots, 0], \ldots, q_5 = [0, \ldots, 0, 1] \) with multiplicity \( \geq 3 \). The points are the only singular points of \( X' \), they are ordinary nodes. Let \( X \) be a minimal resolution of \( X' \). It is a K3 surface.

Assume that \( p \neq 2, 5 \). In this case the involution \( \sigma \) lifts to a fixed-point free involution \( \tau \) of \( X \) and the quotient \( S = X/\langle \tau \rangle \) is an Enriques surface. The Cremona involution \( \sigma \) restricted to \( X' \) is defined by quartics passing through the singular points. On \( X \), it is given by the linear system \([4h - 3(E_1 + \cdots + E_5)], \) where \( h \) is the pre-image of a hyperplane section class of \( X' \) on \( X \) and \( E_i \) are the exceptional curves of the resolution of singularities.

The linear system \([5h - 3(E_1 + \cdots + E_5)] \) is invariant with respect to the group generated by \( \mathfrak{S}_5 \) and \( \tau \). Thus \( X \) admits a polarization of degree \((5h - 3(E_1 + \cdots + E_5))^2 = 25 \cdot 6 - 9 \cdot 10 = 60. \)
It descends to a polarization of degree 30 on $S$ invariant with respect to the group $\mathfrak{S}_5$ acting on it. The action of $\mathfrak{S}_5$ descends to an action of this group on $S$. Applying Proposition 8.8.2, we obtain that the action of $\mathfrak{S}_5$ on $S$ is Mathieu.

Note that it is not true that any action of $\mathfrak{S}_5$ must be a Mathieu action. In the next section we will study a surface $S$ birationally isomorphic to the quotient of the surface

$$
\sum_{i=1}^{5} x_i = \sum_{i=1}^{5} \frac{1}{x_i} = 0
$$

by the Cremona involution. Its full group of automorphisms is isomorphic to $\mathfrak{S}_5$ but only its subgroup $\mathfrak{A}_5$ acts in a Mathieu fashion.

**Example 8.8.5.** ([509]) Assume $p \neq 2$. We construct $S$ as the quotient of a degree 8 K3 surface in $\mathbb{P}^5$ by a fixed-point free involution $\tau$. We use the notation from Section 3.4. Recall that $X$ is embedded in $\mathbb{P}^5$ by the linear system $|D'|$, where $h = [D']$ is a $\tau$-invariant polarization of degree 8, a lift of a polarization $h$ of degree 4 on $S$. The involution $\tau$ decomposes the linear space $E = H^0(X, \mathcal{O}_X(D'))$ (resp. its dual space $H^0(X, \mathcal{O}_X(D'))^\vee$) into eigensubspaces $E_+, E_-$ (resp. $E_\vee^\vee = E_+^\perp, E_\vee = E_-^\perp$). The net $|N|$ of quadrics vanishing on $X$ is defined by a triple $(N_+, N_-, \gamma)$, where $N_\pm$ is a 3-dimensional subspace of $S^2E_\pm$ of quadratic forms on $E_\pm^\gamma$, and $\gamma : N_+ \to N_-$ is an isomorphism of linear spaces. The K3 surface $X$ is given by equations

$$
X = \bigcap_{q \in N_+} V(q + \gamma(q)). \tag{8.8.4}
$$

The discriminant curve $\mathfrak{h} \subset |N|$ of $|N|$ is equal to the union $\mathfrak{h}_+ + \mathfrak{h}_-$ of the discriminant curves of the nets of conics $|N_\pm|$. We will deal with the case where the curves $\Delta_\pm$ are smooth cubics that intersect transversally at 9 points $p_1, \ldots, p_9$. It is known that the space $E_\pm$ is identified with the space $H^0(\Delta_\pm, \mathcal{O}_{\Delta_\pm}(\theta_\pm)(1))$, where $\theta_\pm$ is a non-trivial 2-torsion divisor class on $\Delta_\pm$ [180, 4.1.3].

To sum up, starting with any 3-dimensional linear space $N$, two nonsingular plane cubics $\Delta_\pm$ in $|N| = \mathbb{P}^2$ with non-trivial 2-torsion divisor classes $\theta_\pm$ on them, we can construct a K3 surface $X$ as follows. The pairs $(\Delta_\pm, \theta_\pm)$ define nets $N_\pm$ of conics in the planes $\mathbb{P}(E_\pm)$, where $E_\pm$ as above, then consider $E = E_+ \oplus E_-$, and take $X$ defined by equations (8.8.4). The isomorphism $\gamma$ is defined by identifying $N_+$ and $N_-$ with $N$.

We apply this construction, by taking the nets $|N_\pm|$ to be the nets of polar conics of two members of the Hesse pencil of plane cubic curves

$$
C_\lambda : x^3 + y^3 + z^3 + \lambda xyz = 0
$$

which we have already encountered in Section 4.7 and in Example 4.10.12.

Recall that the **Hessian determinant** $\text{Hess}(F)$ of a homogeneous polynomial $F$ in three variables is a homogeneous cubic polynomial. The correspondence $F \mapsto \text{Hess}(F)$ is an example of a **covariant** on the space of cubic ternary forms. The cubic curve $V(\text{Hess}(F))$ is denoted by $\text{Hess}(V(F))$ and is called the **Hessian curve** of $V(F)$. The Hessian curve $\text{Hess}(C)$ of a plane cubic $C$ is equal to the discriminant curve of the net of polar conics of $C$. The discriminant curve comes equipped with a non-trivial 2-torsion divisor class $\theta$. Thus, if $p \neq 2$ (resp. $p = 2$), there are essentially three (resp. one or none) different ways to represent a nonsingular cubic curve as the Hessian of another plane cubic curve.
The Hesse pencil is invariant with respect to the Hessian covariant $C_\lambda \to \text{Hess}(C_\lambda)$. In fact, we have $\text{Hess}(C_\lambda) = C_{\lambda'}$, where $\lambda' = -\frac{1 + 2\lambda^3}{6\lambda^2}$ (see [180], 3.2.2). Thus, the data $(\Delta_\pm, \theta_\pm)$ is determined by choosing two plane cubics $C_{\pm}$ in the Hesse pencil with $h_{\lambda_{\pm}} = \text{Hess}(C_{\lambda_{\pm}})$.

The Hesse pencil contains 6 anharmonic plane cubics (i.e. cubics whose Weierstrass equation has the form $y^2 + x^3 + ax = 0$). Using formula from Example 4.10.12, we find that the corresponding parameter $\lambda$ is a root of the equation $8\theta^6 + 20\lambda^3 - 1 = 0$. The six roots of this equation are 

$$\lambda_i = \frac{1}{2}(-1 \pm \sqrt{3})\omega,$$

where $\omega^3 = 1$. Take one of these roots, say $\lambda_+ = \frac{1}{2}(-1 + \sqrt{3})$. The Hesse curve $\text{Hess}(C_{\lambda_+})$ coincides with $C_{\lambda_-}$, where $\lambda_- = \frac{1}{2}(-1 - \sqrt{3})$. Also, $\text{Hess}(C_{\lambda_-}) = \text{Hess}(C_{\lambda_+})$.

The net of polar conics of a member $C_\lambda$ of the Hesse pencil is spanned by conics

$$x^2 + 2\lambda yz = 0, \; y^2 + 2\lambda xz = 0, \; z^2 + 2\lambda xy = 0.$$ 

Thus the surface $X$ can be given by equations

$$q_1 = x_0^2 + 2\lambda_+ x_1 x_2 + y_0^2 + 2\lambda_- y_1 y_2 = 0, \; \tag{8.8.5}$$

$$q_2 = x_1^2 + 2\lambda_+ x_0 x_2 + y_1^2 + 2\lambda_- y_0 y_2 = 0,$$

$$q_3 = x_2^2 + 2\lambda_+ x_0 x_1 + y_2^2 + 2\lambda_- y_0 y_1 = 0.$$ 

The Hessian group $G_{216}$ of projective automorphisms leaving invariant the Hesse pencil is a group of order 216 isomorphic to $3^2 : \text{SL}(2, \mathbb{F}_3)$. The blow-up of the base points of the pencil defines an extremal rational elliptic surface $f : J \to \mathbb{P}^1$ with the Mordell-Weil group isomorphic to $3^2 := (\mathbb{Z}/3\mathbb{Z})^2$. The subgroup $3^2$ acts as the group of translations (when we fix one section). The center of $\text{SL}(2, \mathbb{Z})$ acts as the negation transformation. It is given by the projective transformation $(x, y, z) \mapsto (x, z, y)$. The quotient by the center $\text{PSL}(2, \mathbb{F}_3) \cong \mathbb{A}_4$ acts on the base of the pencil as the tetrahedral group generated by the transformation $\sigma_1$ of order two defined by $\lambda \mapsto \frac{-\lambda + 1}{2\lambda + 1}$ and a transformation $\sigma_2$ of order three defined by $\lambda \mapsto \omega \lambda$. We check that $\sigma_1(\lambda_+) = \lambda_-$, thus a subgroup $3^2 : 4$ of order 36 leaves the discriminant pair invariant and acts on the surface $X$. In fact, one directly checks that the equations of $X$ are invariant with respect to these transformations (acting on the variables $x_i$ and $y_i$ diagonally).

We would like to extend this action to an action of the group $N_{72}$. To do this we consider an embedding

$$\Phi : E^\vee \hookrightarrow S^2E_+^\vee \oplus \bigwedge^2 E_+^\vee \subset E^\vee \otimes E^\vee = S^2E^\vee \oplus \bigwedge^2 E^\vee.$$ 

The projection map $\Phi(E^\vee) \to S^2E_+^\vee$ is equal to the composition $E^\vee \to E_+^\vee \to S^2E_+$, where the second map is given by the linear system of conics $N_+$. The second projection $\Phi(E^\vee) \to \bigwedge^2 E_-^\vee$ is equal to the composition $E^\vee \to E_-^\vee \to \bigwedge^2 E_-$, where the second map is given by canonical isomorphism $E_-^\vee \to \bigwedge^2 E_-$ defined by a choice of a volume form in $\bigwedge^3 E_-$. In coordinates $x_i, y_i$, if we identify the space $S^2E_+$ with the space of symmetric $(3 \times 3)$-matrices, and the space $\bigwedge^2 E_-$ with the space of skew symmetric $(3 \times 3)$-matrices, the map $\Phi$ is given by

$$(x_0, x_1, x_2, y_0, y_1, y_2) \mapsto A = \begin{pmatrix} \lambda_+ x_0 & x_2 + cy_2 & x_1 - cy_1 \\ x_2 - cy_2 & \lambda_+ x_1 & x_0 + cy_0 \\ x_1 + cy_1 & x_0 - cy_0 & \lambda_+ x_2 \end{pmatrix}, \tag{8.8.6}$$
where $c^2 = 1 - \lambda^2 = \frac{1}{2}\sqrt{3}$.

One checks immediately that the equations coincide with the equations

$$2a_{ii}^* = \lambda_+(a_{jk}^* + a_{kj}^*),$$

where $a_{ij}^*$ are the entries of the adjugate matrix $\text{adj}(A)$ and $\{i, j, k\} = \{1, 2, 3\}$. Also, note that the image $\Phi(E)$ is the subspace of matrices $A = (a_{ij})$ satisfying the similar equations, where $a_{ij}^*$ is replaced with $a_{ij}$. Consider the map $\text{adj} : A \mapsto \text{adj}(A)$ that defines a birational involution $\tilde{T}$ on the projective space $\mathbb{P}^8$ of $3 \times 3$-matrices. It is given by quadrics $V(a_{ij}^*)$. It follows that $T^{-1}(\Phi(\mathbb{P}(E))) \cap \Phi(\mathbb{P}(E)) = \mathbb{P}(E)$, hence

$$T(X) = \Phi(\mathbb{P}(E)) \cap T(\Phi(\mathbb{P}(E))) = \Phi(\mathbb{P}(E)) \cap T^{-1}(\Phi(\mathbb{P}(E))) = X.$$ 

Since $X$ is a minimal surface, the birational involution extends to a biregular involution $\bar{T}$ on $S$. By explicit computation, one checks that it commutes with $\tau$ and descends to an involution on $S$ that together with the subgroup $3^2 : 4$ generates a group isomorphic to $N_{T_2}$.

The intersection of $\Phi(\mathbb{P}(E))$ with the indeterminacy locus of $T$ is isomorphic to a complete intersection of three divisors of type $(1, 1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$. It is a curve $F_0$ of arithmetic genus one. Let $\bar{H}$ denote the class of a hyperplane section of $X$ embedded in $\mathbb{P}(E) \cong \mathbb{P}^5$ and let $H_4$ be the polarization of degree 4 on $S$ such that $\pi^*(H_4) = \bar{H}$. Since $T : X \to X$ is given by quadrics passing through $F_0$, we have $T^*(\bar{H}) \equiv 2\bar{H} - F_0$, where $F_0 = [F_0]$. Thus $\bar{H} + T^*(\bar{H}) = 3\bar{H} - F_0$ is a polarization on $X$ of degree $(3\bar{H} - F_0)^2 = 36$. Since $\tau$ preserves $\bar{H}$ and commutes with $T$, it descends to a polarization $H_{18} = 3H_4 - G_0$ of degree 18. Thus, the group $N_{T_2}$ of automorphisms of $X$ preserves a polarization of degree 18 on $X$. By Proposition 8.8.2, its action is Mathieu.

We will show in the next example that the same surface $S$ admits a Mathieu action of the group $\mathfrak{M}_6$. To do this we have to introduce more geometry of $S$ and its canonical cover $X$.

Let $xyz = 0$ be one of the four singular members of the Hesse pencil. Each of its line components contains three base points. Consider the pencil of quadrics parameterized by this line, say $x = 0$. Its Segre symbol is equal to $(2, 2, 2)$, and it follows from the classification of pencils of quadrics (see [180, 8.6.1]) that its base locus contains a plane. In our example, the pencil is generated by the quadrics $Q_2 = V(q_2)$ and $Q_3 = V(q_3)$, and we can choose a plane $\Pi_{23}$ given by equations

$$x_1 - y_1 = x_2 - y_2 = \lambda_+ x_0 - \lambda_- y_0 = 0. \quad (8.8.7)$$

The restriction of the net of quadrics to this plane is a conic. In this way we obtain three conics $K_{ij}$ in $X$, pairwise intersecting at one point corresponding to a singular point of the cubic $xyz = 0$. The union of these conics is a curve of arithmetic genus one and degree 6. One checks that this curve is disjoint from $F_0$, hence moves in the elliptic pencil $|F_0|$ on $X$. It is invariant with respect to the involution $\tau$ and descends to an elliptic pencil $|2G_0|$ on $S$ (it called the primary elliptic pencil in [509]). It contains four pairs of reducible fibers of type $A_2$.

Observe that the canonical cover $\bar{X}$ contains a line. A straightforward way to see this is to use the equation of such a line $\ell_1$ (see [509], 2.2):

$$[x_0, x_1, x_2, y_0, y_1, y_2] = [2\lambda_-, t + \sqrt{3}, t - \sqrt{3}, 2\sqrt{-1}\lambda_+, 1 + \sqrt{3}t, 1 - \sqrt{3}].$$

This line lies in the 3-plane $\Pi_1$ defined by equations

$$x_1 + x_2 + 2\lambda_+ x_0 = y_1 + y_2 + 2\lambda_- y_0 = 0.$$
8.8. Finite Groups of Automorphisms of Mathieu Type

It is contained in the quadric $Q_1$ of corank 2 corresponding to the base point $[0,1,-1]$ of the Hesse pencil. The 3-plane $\Pi_1$ also contains the plane $\Lambda_{23}$ with equations (8.8.7). It is contained in the base locus of the pencil generated by $Q_2, Q_3$ that coincides with the side $x = 0$ of the Hesse triangle $xyz = 0$. Thus the conic $K_{23} = X \cap \Lambda_{23}$ is contained in the quartic curve $\Pi_1 \cap X$. The latter becomes the union of two lines $\ell_1, \ell_1'$ and the conic $K_{23}$. Since the arithmetic genus of the intersection of two quadrics in $\mathbb{P}^3$ is equal to one, the curves $\ell_1, \ell_1', L_{23}$ intersect pairwise at one point forming a reducible fiber of type $A_2$ of the elliptic fibration $|F_1|$ corresponding to the quadric $Q_1$.

Note that the line $\tau(\ell_1)$ is skew to $\ell_1$ and also lies in the $\tau$-invariant 3-plane $\Pi_1$. The conic $\tau(K_{23})$ corresponds to the plane $\tau(\Lambda_{23})$ that also lies in $\Pi_1$. The curve $\tau(\ell_1) + \tau(\ell_1') + \tau(K_{23})$ is disjoint from $\ell_1 + \ell_1' + K_{23}$ and forms another reducible fiber of $|F_1|$. The image of this pair of fibers is a fiber of type $A_2$ of the elliptic fibration $|2G_1|$ on the Enriques surface $S$.

Replacing the quadric $Q_1$ with any other quadric $Q_i$ corresponding to different base point of the Hesse pencil, we obtain 36 lines $\ell_i, \ell_i', \tau(\ell_i), \tau(\ell_i')$ on $X$. Recall that the twelve sides of four Hesse triangles and 9 base points (or sections of $j : J \to \mathbb{P}^1$) of the Hesse pencil form the famous Hesse configuration $(12,9,4)$. This easily implies that each fibration $|2G_i|$ is of Hesse type, i.e. contains four reducible fibers of type $A_2$. Also each irreducible component of a reducible fiber of the primary fibration $|2G_0|$ (also of Hesse type) is contained in three reducible fibers of different fibrations $|2G_i|$.

Let $\ell_i^{(j)}, j = 1,2,3$, be the sides of the Hesse triangles $T_i, i = 1,2,3,4, \ldots$. Fix one side $\ell_i^{(j)}$ and let $Q_a, Q_b, Q_c$ be the quadrics corresponding to the base points $p_a, p_b, p_c$ lying on $\ell_i^{(j)}$. We can choose the plane $\Pi_i^{(j)}$ contained in the base locus of the pencil of quadrics defined by $\ell_i^{(j)}$ and a 3-plane $\Lambda_a, \Lambda_b, \Lambda_c$ in each $Q_a, Q_b, Q_c$ that contains this plane. Thus the conic $K_i^{(j)}$ corresponding to the plane $\Pi_i^{(j)}$ is a part of a reducible fiber of the elliptic fibrations $|F_a|, |F_b|, |F_c|$, the other two components are pairs of lines. The 3-plane $\Lambda_a$ intersects the two planes $\Pi_k^{(j)}, k \neq i$, at one point, this easily implies that $F_0 \cdot F_i = 2$.

Let $|2G_i|, i = 0, \ldots, 9$, be the descents of the elliptic fibrations $|F_i|$ to $S$. Let $g_i = [G_i] \in \text{Num}(S)$. Then $(g_0, \ldots, g_{10})$ is an isotropic 10-sequence. Let $g_i = h_i - g_{i+1}$ (they correspond to a different choice of a pencil of 3-planes on the quadric $Q_i$). At the end of Section 3.5, we have explained that $(g_{-1}, \ldots, g_{-9})$ form an isotropic 9-sequence that cannot be extended to an isotropic 10-sequence and $\frac{1}{2}(g_{-1} + \cdots + g_{-9})$ exists in $\text{Num}(S)$ and defines a Mukai polarization of degree 18 on $S$. Expressing $g_0$ in terms of $h_4, g_1, \ldots, g_9$, we find

$$2g_0 = -3h_4 + \sum_{i=1}^{9} g_i = -3h_4 + \sum_{i=1}^{9} (h_4 - g_{-i}) = 6h_4 - \sum_{i=1}^{9} g_{-i}.$$  

This shows that $3h_4 - g_0 = \frac{1}{2} \sum_{i=1}^{9} g_{-i}$ coincides with the Mukai polarization defined by the base points $p_1, \ldots, p_9$.

Now we are ready to explain the next example.

Example 8.8.6. Here we will show that the Enriques surface constructed in the previous example also realizes a Mathieu action of the group $\mathfrak{A}_9$.

We know that $S$ contains 10 elliptic fibrations $|2G_i|$ of Hesse type with $[G_i] = g_i$ such that
$h_{10} = \frac{1}{3}(g_0 + \cdots + g_9)$ is a numerical Fano polarization on $S$ and $h_{18} = \frac{1}{2}(g_1 + \cdots + g_9)$ is a Mukai polarization with $h_{18} \cdot g_i = 4, i \geq 1$.

The jacobian fibration of each fibration is an extremal rational elliptic surface of Hesse type, and has the Mordell-Weil group $\text{MW}_i$ isomorphic to the group $3^2 := (\mathbb{Z}/3\mathbb{Z})^\oplus 2$. So, we have ten such groups acting on $S$. Let $G$ be the subgroup of $\text{Aut}(S)$ generated by these groups. Let us see that $G$ is a finite group preserving the polarization $h_4$. To see this, we notice that, for any component $R_j$ of a reducible fiber of any of the ten pencils $|2G_i|$, we have $h_{18} \cdot R_j = h_{18} \cdot R_j = 2$. We know that $\text{MW}_i$ acts transitively on the set of irreducible components of fibers and these components together with some $\text{MW}_i$-invariant ample divisor generate $\text{Num}(S)_{\mathbb{Q}}$. By writing a divisor class as a linear combination of these generators, we easily find that it is $\text{MW}_i$-invariant if and only if it intersects equally all the irreducible fiber components.

The group $G$ leaves invariant an ample divisor $h$, and hence must be a finite group. Since the canonical non-degenerate isotropic 10-sequence $(g_0, \ldots, g_9)$ is uniquely defined by $h$, up to an order, we see that $G$ acts by permutations on the set of vectors $g_i$. Let us see that the image of $G$ in $\mathcal{S}_{10}$ is the subgroup of the Mathieu group $M_{10}$ isomorphic to $\mathfrak{A}_6$.

Recall that $M_{10}$ is isomorphic to the automorphism group of the Steiner system $\mathcal{S}(3, 4, 10)$ on a set $\Sigma$ of cardinality 10. It consists of 30 blocks of cardinality 4 and each point in $\Sigma$ is contained in 12 blocks.

We realize this Steiner system as follows. First we match $\Sigma$ with the set $\{g_0, \ldots, g_9\}$. Then we match blocks with the sets of irreducible components of reducible fibers of the elliptic fibrations $|2G_i|$. Recall that each fibration $|2G_i|$ contains 12 such curves. Each component is contained in fibers of four different fibrations $|2G_i|$. It corresponds to the property of the Hesse configuration that any base point is contained in four sides of the Hesse triangles. This we interpret by saying that each block consists of four points in $\Sigma$. Now each subset of three fibrations has a common irreducible component. This follows from the property of the Hesse configuration that each side of a Hesse triangle contains 3 base points. This realizes the Steiner system $\mathcal{S}(3, 4, 10)$ on which $G$ acts.

Let us consider the action homomorphism $\alpha : G \rightarrow M_{10}$. Its kernel acts identically on the set $\{h_{10}, g_0, \ldots, g_9\}$ and since this set generates $\text{Num}(S)_{\mathbb{Q}}$, it consists of cohomologically trivial automorphisms $g$. Such an automorphism leaves invariant components of different fibrations $|2G_i|$, for each component we have a component of another fibration that intersects it with multiplicity one at a nonsingular point of the fibers. This shows that we have at least 4 fixed points one each component, hence all of them are contained in $S^0$, an obvious contradiction. Thus $\alpha$ is injective. The image cannot be the whole $M_{10}$ because we know from Theorem 8.8.3 that $M_{10}$ cannot be realized as a group of automorphisms of $S$ of Mathieu type. Thus, the image of $\alpha$ is contained in $\mathfrak{A}_6$. Now we use that the image contains 10 subgroups $\alpha(\text{MW}_i)$ isomorphic to $3^2$. It is easy to see that such group must coincide with $\mathfrak{A}_6$. It follows from Proposition 8.8.2 that the action of $\mathfrak{A}_6$ is Mathieu.

Remark 8.8.7. Recall that $X$ contains a line. It follows from Remark 3.4.12 that this implies that the K3 surface $X$ is isomorphic to a minimal resolution $X'$ of the double cover of the plane $|N|$ branched along the union of the cubics $\Delta_{\pm}$. In Example 4.10.12 we explained how the data $(\Delta_{\pm}, \theta_{\pm})$ defines a quadratic twist $f : S' \rightarrow \mathbb{P}^1$ of the Hesse elliptic fibration $j : J \rightarrow \mathbb{P}^1$ with two
8.9. ENRIQUES SURFACES WITH FINITE AUTOMORPHISM GROUP: \( p \neq 2 \)

In this section we would like to classify Enriques surfaces \( S \) such that \( \text{Aut}(S) \) is a finite group. Over the field of complex numbers such classification was done by S. Kondo [397] and, via periods of their K3-covers, by V. Nikulin [535]. Recall from Corollary 8.1.10 that, if \( p \neq 2 \) or \( S \) is a \( \mu_2 \)-surface, such classification is equivalent to the classification of crystallographic root bases of the classes of \((-2)\)-curves in \( \text{Num}(S) \).

The key to the classification is the following obvious observation.

**Proposition 8.9.1.** Let \( S \) be an Enriques surface with finite automorphism group. Then the jacobian fibration of any genus one fibration on \( S \) has the finite Mordell-Weil group.

Since we know by Corollary 6.3.7 that a nodal Enriques surface has always a genus one fibration with a special bisection, we may start assuming that \( S \) has such a fibration with a special bisection \( R \). We use the classification of extremal genus one fibrations from Section 4.8. Instead of considering many possible cases how \( R \) could intersect a reducible fiber, we use the quadratic twist construction from Lemma 4.10.9 to create such a bisection from a section of the Mordell-Weil group. For each case we can find a new special elliptic fibration on \( S \) which should be extremal, and hence we find a new \((-2)\)-curve(s) on \( S \) which is a component of a singular fiber of a new genus one fibration. If the obtained dual graph of \((-2)\)-curves satisfies Vinberg’s condition (Theorem 0.8.22, 2), then we obtain a crystallographic basis. Otherwise we continue this process. Finally we get possible crystallographic basis of seven types, called of type \( I, \ldots, VII \).

We illustrate this procedure in the following example

**Example 8.9.2.** We start with the following diagram of type \( \tilde{E}_8 \)

![Diagram of \( \tilde{E}_8 \)](image)

We assume that a special bisection \( R \) intersects the component of the fiber of type \( \tilde{E}_8 \) that enters with multiplicity 1. Since \( p \neq 2 \), it intersects it with multiplicity 2. Thus we obtain the following diagram.

![Diagram of \( R \)](image)

Next consider the diagram of type \( \tilde{A}_1 \) which is a double fiber of a genus one fibration because it
meets a bisection $R_1$ with multiplicity 1. This fibration is of type $\tilde{E}_7 \oplus \tilde{A}_1$ because a subdiagram of type $E_7$ is disjoint from the diagram of type $\tilde{A}_1$. Since $p \neq 2$, we have the following:

Next we see another parabolic diagram of type $\tilde{A}_7$ which is a double fiber of a genus one fibration because it meets a bisection $R_2$ with multiplicity 1. The original bisection $R$ does not intersect any component of the curve of type $\tilde{A}_7$, and hence it must be a component of a parabolic subdiagram of type $\tilde{A}_7 \oplus \tilde{A}_1$. Since a bisection $R_3$ of this fibration meets $R$ with multiplicity 2, the fiber of type $\tilde{A}_1$ is not double. Therefore the new $(-2)$-curve should meet $R_2$ with multiplicity 2. Thus we obtain the diagram of a crystallographic basis of type I of $(-2)$-curves on Kondo’s surface of type I:

Recently, Martin [469] used the same idea to extend the classification for the cases of characteristic $p > 2$ or $\mu_2$-Enriques surfaces in $p = 2$. Moreover he presented a “critical” subdiagram of a crystallographic basis for each type of seven crystallographic basis such that any Enriques surface with a such subdiagram is one of the seven types.

In Nikulin’s approach, one translates Proposition 8.9.1 to the classification of possible $R$-invariants of $S$.

**Theorem 8.9.3.** Assume that $\mathbb{k} = \mathbb{C}$. Then $\text{Aut}(S)$ is finite if and only if Nikulin’s $R$-invariant $(K, H)$ is one of the following:

$$(E_8 \oplus A_1, \{0\}), \ (D_9, \{0\}), \ (D_8 \oplus A_1 \oplus A_1, (\mathbb{Z}/2\mathbb{Z})^\oplus 2),$$

$$(D_5 \oplus D_5, \mathbb{Z}/2\mathbb{Z}), \ (E_7 \oplus A_2 \oplus A_1, \mathbb{Z}/2\mathbb{Z}), \ (E_6 \oplus A_4, \{0\}), \ (A_9 \oplus A_1, \mathbb{Z}/2\mathbb{Z}).$$

**Proof.** We give a proof for the necessity. The sufficiency follows from the examples below. Let $S$ be an Enriques surface with the Nikulin $R$-invariant $(K, H)$. Recall that $K$ is a root lattice of rank $r = \text{rank}(K)$ and $H$ is an isotropic subgroup $H \subset K/2K$ with respect to the finite quadratic form $q_K : K/2K \to \mathbb{F}_2$. All such lattices are isomorphic to the direct sum $K_1 \oplus \cdots \oplus K_s$, where $K_i$ are root lattices of rank $r_i$ with $\sum r_i \leq 10$. So, by applying Lemma 6.4.5, we can list them all and possible subgroups $H$ and check whether the $R$-invariant $(K, H)$ satisfies the following conditions:

(I) $r = \text{rank}(K) \leq 10$. Moreover if $r = 10$, the number of minimal generators of the $p$-Sylow subgroup $(p \neq 2)$ of $D(K)$ is at most 2.

(II) The overlattice $K_H = \{ x \in K \otimes \mathbb{Q} : 2x \in H \}$ of $K$ does not contain $(-1)$-vectors because otherwise $S$ is not an Enriques surface, a Coble surface.

(III) $K/2K \mod H$ is isomorphic to $(\text{Nod}(S))$ as quadratic forms. In particular $K/2K \mod H$ can be embedded in the quadratic space $\bar{E}_{10}$ which is even, regular and non defective.
8.9. ENRIQUES SURFACES WITH FINITE AUTOMORPHISM GROUP: $P \neq 2$

(IV) (Proposition 8.9.1) For any non-zero isotropic vector $f \in \text{Nod}(S)$, the linear quadratic space generated by $\{x \in \text{Nod}(S) : b_{q_K}(x, f) = 0\}$ has the maximal dimension 8.

For the condition (I), recall that $K(2)$ is contained in the orthogonal complement of the transcendental lattice $T_X$ of the canonical cover $X$ of $S$ in $H^2(X, \mathbb{Z})^- \cong E_{10}(2) \oplus \mathbb{U}$. If rank $K = 10$, $K_H(2)$ and $T_X$ are the orthogonal complements to each other, and hence, by comparing their discriminant quadratic forms, we have the assertion.

First of all, we show that for each root lattice $K$ appeared in Theorem 8.9.3 the group $H$ is uniquely determined by the above conditions. In cases of $E_8 \oplus A_1$, $E_6 \oplus A_4$, the quadratic form $q_K$ is regular and hence $H = \{0\}$. In case of $D_9$, $q_K$ has a one-dimensional kernel $\ker(q)$ which is represented by a $(-4)$-vector in $K$. Thus we have $H = \{0\}$ by the condition (II). In cases of $E_7 \oplus A_2 \oplus A_1$, $A_9 \oplus A_1$, $q_K$ is defective and has a one-dimensional kernel. The condition (III) implies $H = \mathbb{Z}/2\mathbb{Z}$. In case of $D_8 \oplus A_1 \oplus A_1$, $q_K$ is defective and has a three-dimensional kernel, and hence $H = (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ by the conditions (II), (III). In case of $D_5 \oplus D_5$, $q_K$ is non defective and has a 2-dimensional kernel, and hence $H = \mathbb{Z}/2\mathbb{Z}$ by the conditions (II), (III).

Now we check the conditions (I), (II), (III) or (IV) for any pair $(K, H)$. In case that rank $K \leq 8$, the condition (IV) is not satisfied because the quadratic space $q_K$ has at most dimension 8. Thus it suffices to consider the cases rank $K = 9, 10$.

There are 88 possible lattices $K$ of rank 10.

(i) The following seven $K$ have regular even quadratic forms $q_K$ of rank 10 such that $K(2)$ can not be embedded in $E_{10}(2) \oplus \mathbb{U}$ (i.e. $K$ does not satisfy the condition (I) for $p = 3$):

$$E_6 \oplus A_2^{\oplus 2}, A_5 \oplus A_2^{\oplus 2} \oplus A_1, A_4 \oplus A_2^{\oplus 3}, A_3 \oplus A_2^{\oplus 3} \oplus A_1, A_2^{\oplus 5}, A_2^{\oplus 4} \oplus A_1^{\oplus 2}, A_2^{\oplus 3} \oplus A_1^{\oplus 4}.$$  

(ii) In the following five cases, $q_K$ is a regular quadratic form of odd type and of rank 10. In particular it does not satisfy the condition (III):

$$A_{10}, E_8 \oplus A_2, A_8 \oplus A_2, A_6 \oplus A_4, A_1^{\oplus 2} \oplus A_2.$$  

(iii) In the following nine cases, $q_K$ is not regular but of odd type and of dimension 10. In particular it does not satisfy the condition (III):

$$D_8 \oplus A_2, D_7 \oplus A_3, A_7 \oplus A_3, E_6 \oplus D_4, A_6 \oplus D_4, D_5 \oplus A_3 \oplus A_2, D_4^{\oplus 2} \oplus A_2, D_4 \oplus A_4 \oplus A_2, D_4 \oplus A_3^{\oplus 2}.$$  

(iv) In the following 50 cases, $H$ is non-trivial by the condition (III) but $K_H$ contains a $(-1)$-vector, i.e. it does not satisfy the condition (II):

$$D_{10}, D_9 \oplus A_1, E_8 \oplus A_1^{\oplus 2}, A_8 \oplus A_1^{\oplus 2}, E_7 \oplus A_3, E_7 \oplus A_3^{\oplus 2}, D_7 \oplus A_2 \oplus A_1, D_7 \oplus A_1^{\oplus 3}, A_7 \oplus A_1^{\oplus 3}, E_6 \oplus A_3 \oplus A_1, E_6 \oplus A_2 \oplus A_1^{\oplus 2}, E_6 \oplus A_1^{\oplus 4}, D_6 \oplus D_4, D_6 \oplus A_4, D_6 \oplus A_2^{\oplus 2}, D_6 \oplus A_1^{\oplus 4}, A_6 \oplus A_3 \oplus A_1, A_6 \oplus A_2 \oplus A_1^{\oplus 2}, A_6 \oplus A_1^{\oplus 4}, D_5 \oplus A_5, D_5 \oplus D_4 \oplus A_1, D_5 \oplus A_4 \oplus A_1, D_5 \oplus A_2^{\oplus 2} \oplus A_1, D_5 \oplus A_2 \oplus A_1^{\oplus 3}, D_5 \oplus A_1^{\oplus 5}, A_5 \oplus A_3 \oplus A_2, A_5 \oplus A_2 \oplus A_1^{\oplus 3}, A_5 \oplus A_1^{\oplus 5}, D_4 \oplus A_4 \oplus A_1^{\oplus 2}, D_4 \oplus A_3 \oplus A_2, D_4 \oplus A_2^{\oplus 2} \oplus A_1^{\oplus 2}, D_4 \oplus A_1^{\oplus 4}, D_4 \oplus A_1^{\oplus 6}, A_4^{\oplus 2} \oplus A_1^{\oplus 2}, A_4 \oplus A_3 \oplus A_2, A_4 \oplus A_3 \oplus A_1, A_4 \oplus A_3 \oplus A_1^{\oplus 3}, A_4 \oplus A_2^{\oplus 2} \oplus A_1^{\oplus 2}, A_4 \oplus A_2 \oplus A_1^{\oplus 4}, A_4 \oplus A_1^{\oplus 6}, A_3^{\oplus 2} \oplus A_1^{\oplus 6}, A_3 \oplus A_1^{\oplus 4}, A_3 \oplus A_2^{\oplus 2} \oplus A_1^{\oplus 3}, A_3 \oplus A_2 \oplus A_1^{\oplus 5}, A_3 \oplus A_1^{\oplus 7}, A_2^{\oplus 2} \oplus A_1^{\oplus 6}, A_2 \oplus A_1^{\oplus 8}, A_1^{\oplus 10}.$$  

(v) The following is the list of the remaining twelve cases of $K$ except the five types

\[ D_8 \oplus A_1 \oplus A_1, \ D_5 \oplus D_5, \ E_7 \oplus A_2 \oplus A_1, \ E_6 \oplus A_4, \ A_9 \oplus A_1 \]

in Theorem 8.9.3:

\[ A_7 \oplus A_2 \oplus A_1, \ D_6 \oplus A_3 \oplus A_1, \ D_6 \oplus A_2 \oplus A_1 \oplus A_1, \]
\[ A_6 \oplus A_2 \oplus A_1 \oplus A_1, \ D_5 \oplus A_3 \oplus A_1 \oplus A_1, \ A_5 \oplus A_5, \ A_5 \oplus A_4 \oplus A_1, \]
\[ A_5 \oplus A_3 \oplus A_1 \oplus A_1, \ D_4 \oplus A_1 \oplus A_1 \oplus A_1, \ D_4 \oplus A_3 \oplus A_1 \oplus A_1, \]
\[ A_3 \oplus A_2 \oplus A_1 \oplus A_1, \ D_5 \oplus A_3 \oplus A_1 \oplus A_1, \ A_5 \oplus A_4 \oplus A_1 \oplus A_1, \]
\[ A_5 \oplus A_3 \oplus A_1 \oplus A_1, \ D_4 \oplus A_2 \oplus A_1 \oplus A_1. \]

These cases do not satisfy the condition (IV). To show this we use the following Lemma (for the proof, e.g. [536, Corollary 4.4]).

Lemma 8.9.4. Let $R$ be an irreducible root lattice and let $A_1$ be a sublattice of $R$. Then the root sublattice $R_0$ of the orthogonal complement of $A_1$ in $R$ is given as follows.

\[ (i) \text{ In case } R = A_n, \ R_0 = A_{n-2} \text{ (} n \geq 3 \text{)}, \ R_0 = \emptyset \text{ (} n = 1, 2 \text{)}. \]

\[ (ii) \text{ In case } R = D_n, \ R_0 = A_1 \oplus D_{n-2} \text{ (} n \geq 5 \text{)}, \ R_0 = A_1 \oplus D_{n-2} \text{ (} n = 4 \text{)}. \]

\[ (iii) \text{ In case } R = E_n, \ R_0 = A_5 \text{ (} n = 6 \text{)}, \ R_0 = D_6 \text{ (} n = 7 \text{)}, \ R_0 = E_7 \text{ (} n = 8 \text{)}. \]

For example, in case $K = A_5 \oplus A_5$, $q_K$ has rank 9 and dimension 10. By the property (III), $H = \mathbb{Z}/2\mathbb{Z}$ and $q_K \mod H$ is defective and of dimension 9. Consider the sum of a root in each factor $A_5$ of $K$ which gives a non-zero isotropic vector in the quadratic form $q_K$. The root sublattice of its orthogonal complement in $K$ is isomorphic to $A_5 \oplus A_5 \oplus A_1$ (Lemma 8.9.4) whose image in $K/2K \mod H$ has dimension 7. Thus this case does not satisfy the condition (IV).

We remark that many cases as above do not satisfy not only one but also two or more conditions among the conditions (I)–(IV).

The proof for the case rank $K = 9$ is similar and is easier. Almost all cases do not satisfy the condition (IV). We leave the details to the reader.

\[ \square \]

It follows that there are seven types of such surfaces over $\mathbb{C}$. It is a surprising result of Martin’s classification that there are no new types of crystallographic bases in any characteristic $p \neq 2$, although some of Kondo’s construction do not live in arbitrary characteristic. The surfaces of the first two types form a one-dimensional family, the surfaces of other types are determined uniquely by the type of the basis. In this section we recall Kondo’s construction of these surfaces and analyze its possible extension to positive characteristic. But, to summarize, we exhibit the final result of classification in arbitrary characteristic $p \neq 2$. Here $D_8$ is the dihedral group of order 8.
### 8.9. ENRIQUES SURFACES WITH FINITE AUTOMORPHISM GROUP: $P \neq 2$

<table>
<thead>
<tr>
<th>Type</th>
<th>Dual Graph of $(-2)$-curves</th>
<th>Aut</th>
<th>$\text{Aut}_{nt}$</th>
<th>Char($k$)</th>
<th>Moduli</th>
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<tr>
<td>I</td>
<td><img src="image1" alt="Graph I" /></td>
<td>$D_8$</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
<td>any</td>
<td>$\mathbb{A}^1 - {0, -256}$</td>
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<tr>
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<td><img src="image2" alt="Graph II" /></td>
<td>$S_4$</td>
<td>${1}$</td>
<td>any</td>
<td>$\mathbb{A}^1 - {0, -64}$</td>
</tr>
<tr>
<td>III</td>
<td><img src="image3" alt="Graph III" /></td>
<td>$(\mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2) \rtimes D_8$</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
<td>$\neq 2$</td>
<td>unique</td>
</tr>
<tr>
<td>IV</td>
<td><img src="image4" alt="Graph IV" /></td>
<td>$(\mathbb{Z}/2\mathbb{Z})^4 \rtimes (\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z})$</td>
<td>${1}$</td>
<td>$\neq 2$</td>
<td>unique</td>
</tr>
<tr>
<td>V</td>
<td><img src="image5" alt="Graph V" /></td>
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<td>$\mathbb{Z}/2\mathbb{Z}$</td>
<td>$\neq 2, 3$</td>
<td>unique</td>
</tr>
</tbody>
</table>
CHAPTER 8. AUTOMORPHISMS OF ENRIQUES SURFACES

VI

VII

Table 8.9: Enriques surfaces with finite automorphism group ($p \neq 2$)

Type I

These surfaces realize the crystallographic basis whose Coxeter graph has 12 vertices and is given in the following picture

We know that these surfaces are a part of a 2-dimensional family of type (A) of surfaces which admit non-trivial numerically trivial automorphisms which we classified in Theorem 8.2.23. We constructed them as Enriques double planes with the branch curve of degree 6 equal to the union of a line, a conic and a cubic passing through the intersection points $q_1, q_2$ of the line and the conic. The curves $R_{10}$ and $R_{11}$ are the proper transforms of the lines $\ell_1 = \langle p_2, q_1 \rangle$ and $\ell_2 = \langle p_4, q_2 \rangle$, where $p_2, p_4$ are the tacnode singularities of the Enriques octic branch curve. The special property of our family is that one of the lines, say $\ell_1$ is tangent to the cubic at the point $q_1$. The other line will have the same property automatically.
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We know from the proof of Theorem 8.2.23 from which we borrow the notation that the equation of the 2-dimensional family of double plane is

$$x_3^2 + x_1x_2(x_1 - x_2)(x_0^2 - x_1x_2)(ax_0^2x_2 + bx_0^2x_1 + cx_0^2x_2 + dx_0^2x_1) = 0,$$  \hspace{1cm} (8.9.1)

where $a + b + c + d = 0$. The condition that the line $\ell_1 = V(x_0 - x_1)$ is tangent to the cubic at $q_1$ is $(a + c)^2 - 4bd = 0$. The same condition guarantees that the line $\ell_2 = V(x_0 + x_1)$ is tangent to $C$ at $q_2$. Thus we have a one-parameter family of branch curves $B$ satisfying our assumptions.

The canonical cover $\tilde{X}$ of the surface $S$ is birationally isomorphic to the double cover of $F_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ branched along the pre-image $\tilde{B}$ of the curve $B$. The curve $\tilde{B}$ is equal to the union of a quadrangle $T$ of lines, the pre-images of the conics, and a curve $C$ of bi-degree $(2, 2)$, the pre-image of a hyperplane section. It passes through the vertices of $T$. The pre-images of the lines $\ell_1, \ell_2$ are two conics $Q_1, Q_2$ on $\mathbb{P}^1 \times \mathbb{P}^1$ passing through the opposite vertices of $T$ and tangent to $C$ there. They intersect at two points fixed under the involution $\sigma$ such that $F_0/\{\sigma\} = D_1$.

![Figure 8.6: The branch curve for the canonical cover of type I surfaces](image)

**Proposition 8.9.5.** A crystallographic basis of $(-2)$-curves of type I can be realized in characteristic $p \neq 2$. Any Enriques surface realizing this basis is birationally isomorphic to the double plane given by equation (8.9.1), where $a + b + c + d = 0$, $(a + b)^2 = 4cd$. Moreover,

$$\text{Aut}(S) \cong D_8.$$

**Proof.** The only new assertion here is about the structure of the automorphism group. We already know that the surface admits a cohomologically trivial involution $g_0$. It can be realized by the deck transformation of the bielliptic cover $S \to D_1$ defined by the degenerate $U$-pair $(f, R)$, where $f$ is the class of a half-fiber in the elliptic fibration with a fiber of type $\tilde{E}_8$ and $R$ is its special bisection which we easily locate in the diagram.

Consider the action of $\text{Aut}(S)$ on the diagram. We will show that the image of $\text{Aut}(S)$ is the group of symmetries of the diagram isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Together with $g_0$, this will generate a group isomorphic to $D_8$.

The symmetry of the graph with respect to the middle line is realized by the Mordell-Weil group of the jacobian fibration of one of the two elliptic fibrations with a fiber of type $\tilde{E}_7$. The symmetry with respect to the line perpendicular to the middle line is realized by the Mordell-Weil group of the jacobian fibration of one of the two elliptic fibrations with a fiber of type $\tilde{D}_8$. \hfill \Box
Proposition 8.9.6. A surface of type I has genus one fibrations with reducible fibers of the following types:

\[ \tilde{E}_8 \ (4), \quad \tilde{D}_8 \ (2), \quad \tilde{E}_7 + 2 \tilde{A}_1 \ (2), \quad 2 \tilde{A}_7 + \tilde{A}_1 \ (1), \]

where the number in the brackets is the number of fibrations of given type. Let \([2F_i], i = 1, 2, 3, 4\) (resp. \([2F_5], [2F_6], [2F_7], [2F_8]\), resp. \([2F_9]\)) be the genus one pencils of the first (resp. the second, resp. the third, resp. the forth) type. The intersection matrix of the corresponding primitive isotropic vectors \(f_i\) is equal to

\[
\begin{pmatrix}
0 & 3 & 4 & 5 & 1 & 2 & 1 & 3 & 2 \\
3 & 0 & 5 & 4 & 2 & 1 & 1 & 3 & 2 \\
4 & 5 & 0 & 3 & 1 & 2 & 3 & 1 & 2 \\
5 & 4 & 3 & 0 & 2 & 1 & 3 & 1 & 2 \\
1 & 2 & 1 & 2 & 0 & 1 & 1 & 1 & 1 \\
2 & 1 & 2 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 3 & 3 & 1 & 1 & 0 & 2 & 1 \\
3 & 3 & 1 & 1 & 1 & 1 & 2 & 0 & 1 \\
2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 0
\end{pmatrix}
\]

In particular, \(nd(S) = 4\).

Proposition 8.9.7. The Fano root invariant of a surface with crystallographic basis of \((-2)\)-curves of type I is equal to \((E_8 \oplus U(4), \{0\})\). The Nikulin \(R\)-invariant is equal to \((E_8 \oplus A_1, \{0\})\). The quadratic space \(\langle \text{Nod}(S) \rangle\) is defective of rank 8 and dimension 9. The Reye lattice is isomorphic to \(E_8(4) \oplus A_1 \oplus A_1(-1)\).

Proof. Let \(f_1\) be the class of an elliptic fibration with half-fiber of type \(\tilde{A}_7\), \(f_2\) the class of a half-fiber of the elliptic fibration with fiber of type \(\tilde{D}_8\) which contains \(R_9, R_8, R_6, R_{12}\) as its components with multiplicity 1, and \(f_3\) the class of a half-fiber of the elliptic fibration with fiber of type \(\tilde{E}_7\) which contains \(R_4, R_6\) as its components with multiplicity 1. Let us consider the Fano polarization \(h\) defined by the following canonical isotropic sequence

\[
(f_1, f_1 + R_9, f_1 + R_9 + R_1, \ldots, f_1 + R_9 + R_1 + \ldots + R_4, f_2, f_2 + R_7, f_2 + R_7 + R_6, f_3).
\]

We find that \(h_{10} \cdot R_i = 0, i \neq 5, 8, 10, 11, 12\), \(h_{10} \cdot R_5 = h_{10} \cdot R_1 = 1, h_{10} \cdot R_8 = h_{10} \cdot R_{12} = 2, h_{10} \cdot R_{10} = 5\). Thus we see that \(N_{S_{h_{10}}}^h\) is spanned by all \(R_i\’s\) except \(R_{10}\). Computing the integral Smith normal form of the intersection matrix, we find that the discriminant group is isomorphic to \((\mathbb{Z}/4\mathbb{Z})^\oplus 2\). It is generated by \(\frac{1}{2}(\{R_6 + R_8 + R_9 + R_{12}\})\) and \(\frac{1}{2}(\{R_2 + R_4 + R_9 + R_{12}\})\). The discriminant quadratic form is \(u_2\). This implies that \(N_{S_{h_{10}}}^h \cong E_8 \oplus U(4)\).

If we add \(\delta_{R_{10}}\), we obtain that \(K\) contains a sublattice \(M\) isomorphic to \(E_8 \oplus A_1\). Since \(R_{11} + R_{10}\) is a simple fiber, \(\delta_{R_{11}} \in M\). Also we know that \(\delta_{R_{12}}, \delta_{R_{1}}, \ldots, \delta_{R_{8}}\) are linearly dependent, so \(M\) is of finite index in \(K\) and does not contain vectors of norm \(-1\). However, the discriminant group \(D(M)\) does not contain isotropic vectors. So, \(M = K\) and the Nikulin \(\tilde{R}\)-invariant is \((E_8 \oplus A_1, \{0\})\).

Applying Lemma 6.4.5, we obtain the assertion about the quadratic space \(\langle \text{Nod}(S) \rangle\). The Reye lattice is the pre-image of a one-dimensional defective quadratic space in \(\overline{\text{Num}}(S)\). This easily implies that \(\text{Rey}(S) \cong E_8(4) \oplus A_1 \oplus A_1(-1)\). □
8.9. **Enriques Surfaces with Finite Automorphism Group: P ≠ 2**

**Type II**

Here the crystallographic basis defines the following diagram of twelve \((-2)\)-curves on \(S\):

![Diagram of twelve (-2)-curves on S](image)

Figure 8.7: Crystallographic basis of type II

Let us construct such surfaces, they will depend on one parameter. Choose a non-degenerate \(U\)-pair formed by two half-fibers \(F_1 = R_1 + R_2 + R_9 + R_{10}\) and \(F_2 = R_3 + R_4 + R_5 + R_{11}\) of type \(A_3\). Each elliptic fibration \([2F_i]\) has an additional simple fiber \(R_4 + R_{11} + 2R_5 + 2R_6 + R_7 + R_{12}, R_1 + R_{10} + 2R_9 + 2R_8 + R_{12} + R_7\), respectively, of type \(D_5\). Let \(f : S \to D_1\) be the bielliptic map defined by the pair \((F_1, F_2)\). The two elliptic fibrations share 8 components \(R_1, R_4, R_5, R_7, R_9, R_{10}, R_{11}, R_{12}\) that span the lattice isomorphic to \(A_3^\oplus 2 \oplus A_1^\oplus 2\). Comparing with Table 8.7, we find the corresponding involution in row 26. Note that, since \(F_1\) and \(F_2\) are double fibers, they are mapped to two non-skew lines \(L_1\) and \(L_2\) on \(D_1\). The branch curve on \(D_1\) consists of two conics \(C_1, C_2\) from different pencils \(|e_0 - e_1|\) and \(|2e_0 - e_2 - e_3 - e_4 - e_5|\) and a hyperplane section \(C\) that are tangent to \(C_1\) and \(C_2\) at one of the intersection points of \(C_1\) and \(C_2\) with lines \(L_1, L_2\) on \(D_1\). In the plane model, \(C_1\) is a line \(\ell\), \(C_2\) is a conic \(K\), and \(C\) is a cubic curve \(C'\) tangent \(\ell\) at \(\ell \cap \langle p_2, p_4 \rangle\) and tangent to \(K\) at the exceptional curve over \(p_3\). The branch sextic \(\ell + K + C'\) has additionally two double points at the intersection points of \(\ell\) with \(K\) and two tacnodes. It is a rational curve. These points become the two tacnodes of \(W\). The ordinary nodes are the intersection points \(q, q'\) of the conics. The curves \(R_2\) and \(R_4\) are mapped to the lines \(L_1, L_2\). The conics are the images of \(R_6\) and \(R_8\). The exceptional points over their intersection points \(q, q'\) are the curves \(R_7\) and \(R_{12}\). The rest of the curves are the exceptional curves over the tacnodes of the branch curve on \(D_1\).

Here is the picture of the branch curve on \(D_1\). Here \(C_1, C_2\) are conics, and \(C\) is a hyperplane section. The conics intersect at two (different) points \(P, P'\).

The following picture describes the pre-image of the branch curve to the quadric \(Q\). It becomes the branch curve of the double cover \(X \to Q\). The pre-image of the plane section \(C\) is \(\bar{C}\). The pre-images of the conic \(C_1, C_2\) are the curves \(C_{1,2}^{\pm}\). The pre-images of the points \(q, q'\) are the points \(q_{\pm}, q'_{\pm}\). The pre-images of the two lines that contain the intersection points of \(C \cap C_1, C \cap C_2\) are the lines \(\bar{L}_1, \bar{L}_2\). They correspond to the reducible half-fibers on \(X\) of the two elliptic fibrations lifted from the elliptic fibrations on \(S\).

To find the equation of the double plane we find a line through \(p_1\) and a cubic through \(p_1, \ldots, p_5\) that are tangent at a point on the line \(V(x_0) = \langle p_2, p_4 \rangle\). They equation of the line must be \(ax_2 + dx_1 = 0\) and the equation of the cubic is \(ax_1^2x_2 + bx_0^2x_1 + cx_0^2x_2 + dx_1x_2^2 = 0\), where \(a^2 = d^2\). Now
we find a conic \( x_1x_2 + tx_0^2 = 0 \) through \( p_2, \ldots, p_4 \) that intersects the cubic with multiplicity 4 at \( p_2 \). It corresponds to the parameter \( t = b/a \). Replacing \( x_1 \) with \(-x_1\), we may assume \( a = -d = 1 \), so the equation of the double plane is

\[
x_3^2 + x_1x_2(x_1 - x_2)(x_1x_2 - x_0^2)(x_1x_2(x_1 - x_2) + x_0^2(bx_1 + cx_2)) = 0
\]

(8.9.2)

The isomorphism class of the surface is uniquely defined, via scaling the variables, by the point \([b, c] \in \mathbb{P}^1 \setminus \{0, \infty\}\).

**Proposition 8.9.8.** Any Enriques surface realizing a crystallographic basis of type II admits a birational model as an Enriques double plane given by equation (8.9.2). Moreover,

\[
\text{Aut}(S) \cong \mathfrak{S}_4.
\]

**Proof.** It remains only to describe the automorphism group.

It follows from our description of cohomologically trivial automorphisms that the group \( \text{Aut}(S) \) acts faithfully on the set of \((-2)\)-curves. The group of symmetries of the diagram is isomorphic to the symmetric group

\[
\mathfrak{S}_4 \cong (\mathbb{Z}/2\mathbb{Z})^2 \rtimes \mathfrak{S}_3.
\]

Each of the three commuting involutions switches curves in two of the three pairs \((R_1, R_{10}), (R_4, R_{11}), (R_7, R_{12})\) of curves in the diagram. They are realized by the deck transformations corresponding to the bielliptic linear system defined by a pair of elliptic fibration \(|F_i|, i = 1, 2, 3\) with reducible fibers of type \(\tilde{A}_3\) and \(\tilde{D}_5\). It can be also realized by the action of the Mordell-Weil group of the jacobian fibration of the elliptic fibration on \(S\) with a reducible fiber of type \(\tilde{D}_8\).
The Mordell-Weil group of the elliptic fibration with a reducible fiber of type $\tilde{A}_8$ is isomorphic to the group $\mathbb{Z}/3\mathbb{Z}$. It realizes elements of order 3 in $\mathfrak{S}_4$. The Mordell-Weil group of the elliptic fibration $|F_i|$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$. It realizes elements of order 4 in $\mathfrak{S}_4$. Together these elements generate $\mathfrak{S}_4$.

**Proposition 8.9.9.** A surface of type II has genus one fibrations with reducible fibers of the following types:

$$2\tilde{A}_8(4), \tilde{A}_8(4), \tilde{D}_8(6), \tilde{D}_5 + 2\tilde{A}_3(3).$$

The group $\text{Aut}(S)$ acts transitively on each group. Let $|2F_i|, i = 1, \ldots, 4$ (resp. $i = 5, 6, 7, 8$, resp. $i = 9, \ldots, 14$, resp. $i = 15, 16, 17$) be the genus one pencils of the first (resp. the second, resp. the third, resp. the fourth) type. One reorder the fibrations in the first group such that intersection matrix of the corresponding primitive isotropic vectors $f_i$ is equal to

$$\begin{pmatrix}
0 & 4 & 4 & 4 & 1 & 1 & 1 & 3 & 2 & 2 & 2 & 4 & 4 & 4 & 4 & 2 & 2 & 2 & 2 \\
4 & 0 & 4 & 4 & 1 & 1 & 3 & 1 & 2 & 2 & 2 & 4 & 4 & 4 & 4 & 2 & 2 & 2 & 2 \\
4 & 4 & 0 & 4 & 1 & 3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
4 & 4 & 4 & 0 & 3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 3 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 3 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 3 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & 0 & 3 & 3 & 3 & 3 & 3 & 3 & 1 & 2 & 1 \\
2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 3 & 0 & 3 & 3 & 3 & 3 & 3 & 1 & 2 & 1 \\
2 & 2 & 2 & 2 & 2 & 1 & 1 & 2 & 3 & 3 & 0 & 3 & 3 & 3 & 3 & 2 & 1 & 1 \\
4 & 2 & 4 & 2 & 2 & 1 & 2 & 1 & 3 & 3 & 3 & 0 & 3 & 3 & 3 & 1 & 1 & 2 \\
4 & 4 & 2 & 2 & 2 & 2 & 1 & 1 & 3 & 3 & 3 & 0 & 3 & 3 & 3 & 0 & 3 & 1 & 2 \\
4 & 2 & 2 & 4 & 2 & 2 & 2 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 0 & 2 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 2 & 0 & 1 & 1 \\
2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}$$

In particular, $\text{nd}(S) = 7$.

**Proposition 8.9.10.** The Fano nodal lattice $N_{S^{10}}$ of a surface of Type II coincides with $N_S$. The Nikulin R-invariant is $(D_9, \{0\})$. The quadratic space $\langle \text{Nod}(S) \rangle$ is an even quadratic space of rank 8 and dimension 9. The Reye lattice is isomorphic to $E_8(4) \oplus U$.

**Proof.** Let $h_{10}$ be the Fano polarization corresponding to the following canonical isotropic sequence

$$(f_1, f_1 + R_3, f_1 + R_3 + R_4, f_2, f_2 + R_6, f_2 + R_6 + R_7, f_3, f_3 + R_9, f_3 + R_9 + R_{10}, f_4)$$

where $f_1, f_2, f_3$ are the classes of half-fibers of type $\tilde{A}_3$ that contain $R_1, R_3, R_6$ as its components, respectively, and $f_4$ is the class of a half-fiber of a genus one fibration with simple fiber of type $\tilde{A}_8$ with components $R_1, \ldots, R_9$. We find that $h_{10} \cdot R_i = 0$ for $i = 1, 3, 4, 6, 7, 9, 10$ and $h_{10} \cdot R_i = 1$ otherwise. Thus $N_{S^{10}}$ is freely generated by all $R_i$. By Enriques Reducibility Lemma, $\text{Num}(S)$ is generated by $N_S$ and the classes of half-fibers of elliptic fibrations. The only half-fibers that is not composed of $(-2)$-curves are the half-fibers of elliptic fibrations of type $\tilde{D}_8$.

Thus we see that $2\text{Num}(S) \subset N_S$ and $\text{Num}(S)$ is generated by $N_S$ and $\frac{1}{2}(R_1 + R_{10} + R_7 + R_{12}), \frac{1}{2}(R_1 + R_{10} + R_4 + R_{11}), \frac{1}{2}(R_7 + R_{12} + R_4 + R_{11})$. This shows that $\text{Num}(S)/N_S \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ is a maximal isotropic subgroup of $D(N_S)$. We also know that $A_8$ and $D_8$ are realized as sublattices
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CHAPTER 8. AUTOMORPHISMS OF ENRIQUES SURFACES

of \( N_S \). Looking at possible sublattices with the previous properties we find that the only one that passes all the tests is \( A_1(-2) \oplus D_9 \cong U(4) \oplus E_8 \).

Let us compute the Nikulin \( R \)-invariant. Applying Lemma 6.4.9 (iii) to the set \( R_1, \ldots, R_9 \) we obtain that \( K \) contains a sublattice \( M \) of \( K \) isomorphic to \( D_9 \). Now note that \( R_{10} + R_{12} + R_1 + R_7 \in 2 \text{Num}(S) \) because they are components of a simple fiber of type \( D_5 \). We have also similar relations involving \( R_{10} + R_{11} \) and \( R_{11} + R_{12} \). This shows that \( K \) is generated by \( M \) and \( \delta_{R_{10}} \). Replacing the set \( R_1, \ldots, R_8, R_9 \) with \( R_2, R_3, \ldots, R_8, R_{10} \), we obtain that \( K \) contains another sublattice isomorphic to \( D_9 \). No root lattice of rank 10 has this property. So, the two sublattice coincide and are equal to \( K \).

Applying Lemma 6.4.5, we obtain that \( \langle \text{Nod}(S) \rangle \) is an even quadratic space of rank 8 and dimension 9. Thus the image of the Reye lattice in \( \text{Num}(S) \) is one-dimension space with trivial quadratic form. The Reye lattice must be isomorphic to \( 2E_8 \oplus U(2) \).

\[ \square \]

Type III

The crystallographic basis consists of 20 curves with the intersection graph of \((-2)\)-curves given in the following picture:

![Figure 8.10: Crystallographic basis of type III](image)

The vertices \( E_1, \ldots, E_{12} \) form a diagram of type (b) in Theorem 8.2.21. By Corollary 8.2.23, the surfaces containing such a diagram form a 2-dimensional family. They are obtained as double covers \( S \to \mathbb{D}_1 \) branched along the union of four conics, two from each of two pencils. The equation (8.2.6) of the double plane

\[ x_3^2 + x_1 x_2 (x_1 - x_2)(x_1 + ax_2)(x_0^2 - x_1 x_2)(x_0^2 + bx_1 x_2) = 0 \]

must be special in order to obtain the remaining 8 \((-2)\)-curves \( E_{13}, \ldots, E_{20} \). Let \( \ell_i \cap C_j = \{ q_{ij}, q'_{ij} \} \), \( i = 1, 2 \). The choice of this equation is uniquely determined by the property that there are 4 lines each passing through \( p_2 \) and a pair of points \( \{ q_{11}, q'_{22} \}, \{ q_{11}, q'_{22} \}, \{ q_{12}, q_{21} \}, \{ q_{12}, q_{21} \} \) and
4 lines each passing through \( p_4 \) and 4 pairs of points \( \{q_{11}, q_{22}\}, \{q_{11}, q_{22}\}, \{q'_{12}, q'_{21}\}, \{q_{12}, q_{21}\} \) each passes through two intersection points in \( \ell, \cap K_2 \). The pre-image of the pencil of lines through \( p_1 \) defines an elliptic fibration on \( S \) with two fibers of type \( \tilde{D}_4 \), the pre-images of the lines \( \ell_1, \ell_2 \). The pre-image of the pencil of conics through \( p_2, p_3, p_4, p_5 \) defines an elliptic fibration on \( S \) with two fibers of type \( \tilde{D}_4 \), the pre-images of the conics \( K_1, K_2 \). The components of these four fibers correspond to curves \( E_1, \ldots, E_{12} \) in the diagram. The remaining curves \( E_{13}, \ldots, E_{20} \) correspond to the proper transforms of the eight lines.

Thus we have the following equation of the double plane model of the surface:

\[
x_3^2 + x_1 x_2 (x_0^4 - x_1 x_2^2)(x_1^2 - x_2^2) = 0.
\]

**Proposition 8.9.11.** Let \( S \) be an Enriques surface realizing a crystallographic basis of type III. Then \( S \) is unique up to isomorphism and coincides with the surface birationally isomorphic to the double plane given by equation (8.9.3). Moreover, \( \text{Aut}_\text{int}(S) \cong \mathbb{Z}/2\mathbb{Z} \), and

\[
\text{Aut}(S) \cong (\mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}) \rtimes D_8, \quad \text{Aut}(S)/\text{Aut}_\text{int}(S) \cong 2^4 \rtimes D_8.
\]

**Proof.** The uniqueness follows from the equation (8.9.3). One can also argue as follows. The elliptic fibration \( |D_1| = |2(E_1 + E_2 + E_3 + E_9)| \) has two double fibers of types \( \tilde{A}_3 \). This shows that it is obtained from its jacobian extremal fibration by choosing local invariants at two reducible fibers of types \( A_3 \). According to our classification of such fibrations, it is unique, up to isomorphism. It is easy to see that the Mordell-Weil group of this fibration isomorphic to \( \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \) acts transitively on these local invariants. Thus the corresponding torsor is defined uniquely up to isomorphism.

Let us describe the group of automorphisms of \( S \). It is easy to see that the group of symmetries \( \text{Sym}(\Gamma) \) of the diagram \( \Gamma \) from 8.10 is isomorphic to the group of symmetries of the subdiagram with vertices corresponding to the curves \( E_1, \ldots, E_{12} \). It is isomorphic to the group \( 2^4 \rtimes D_8 \). The 2-elementary subgroup \( P \) consists of transformations that switch the curves in some of the pairs \( (E_2, E_9), (E_4, E_{11}), (E_6, E_{12}), (E_8, E_{10}) \). We used to determine a double plane model of \( S \) by taking the numerically trivial bielliptic involution \( g_0 \) corresponding to the \( U \)-pair of two fibrations, each with two reducible fibers of type \( \tilde{D}_4 \). Any automorphism that acts non-trivially on \( \Gamma \) by an element from \( P \) preserves the pair, and hence arises as an automorphism of the 4-nodal quartic \( D_1 \) and as an automorphism of the double plane. Suppose \( g \) acts on \( \Gamma \) as a switch of one pair, say \( (E_2, E_9) \). Then its defines an automorphism of the double plane that preserves one of the pencil, say the pencil of lines through \( p_1 \) and fixes one of the lines that defines the reducible fiber with the components \( E_2, E_9 \). We may assume that the line is \( V(x_1 - 2) \) with 2 pairs of singular points of the branch sextic \( q_1 = [1, 1, 1], q_2 = [1, -1, -1] \) and \( q'_{12} = [1, i, i], q'_{21} = [1, -i, i] \) lying on the intersection of the line with two branch conics. The transformation group \( g \) must switch only one pair of points. It is obvious that this is impossible. On the other hand, the group of projective automorphisms generated by the transformations \( [x_0, x_1, x_2] \mapsto [x_0, ix_1, ix_2], [x_0, -x_1, x_2], [x_0, x_2, x_1] \) generates the subgroup \( \tilde{H} \cong \mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} \) whose image in \( \text{Sym}(\Gamma) \) is equal to \( (\mathbb{Z}/2\mathbb{Z})^{\oplus 3} \). The square of a generator of the first summand defines an automorphism of the double plane \( [x_0, x_1, x_2, x_3] \mapsto [x_0, ix_1, ix_2, ix_3] \) whose square is the numerically trivial automorphism \( g_0 \).

So, it remains to realize the symmetries of \( \Gamma \) that belong to the subgroup \( G \) isomorphic to the dihedral group \( D_8 \) of order 8.

To realize a generator of \( G \) of order 4 we use an elliptic fibration \( |E_1 + \cdots + E_8| \) with two
simple reducible fibers of type $\tilde{A}_7$ and $\tilde{A}_1$. The Mordell-Weil group of its jacobian fibration is isomorphic to $\mathbb{Z}/4\mathbb{Z}$. Finally, we realize a generator of order 2 of $E_8$ by taking an elliptic fibration $|E_1 + E_2 + E_3 + E_8|$ with two reducible fibers of type $\tilde{A}_3$ and two reducible fibers of type $\tilde{A}_1$. The Mordell-Weil group of the jacobian fibration is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. It contains an element that switches $E_1, E_3$ and $E_5, E_7$.

**Proposition 8.9.12.** The Fano nodal of Enriques surface of type III is $(N_{h_{10}}, G_{h_{10}}) = (D_8 \oplus A_1 \oplus A_1(-1), (\mathbb{Z}/2\mathbb{Z})^{\oplus 2})$. The Nikulin $R$-invariant of an Enriques surface of Type III is equal to $(D_8 \oplus A_1^{\oplus 2}, (\mathbb{Z}/2\mathbb{Z})^{\oplus 2})$. The quadratic space $\langle \text{Nod}(S) \rangle$ is a defective quadratic space of rank 6 and dimension 8. The Reye lattice is isomorphic to $E_8(4) \oplus U(2)$.

**Proof.** First we claim that the curves $E_i$ with $13 \leq i \leq 20$ form two cosets modulo $2\text{Num}(S)$. Two curves are in the same cosets if and only if they intersect. Observe that the divisors $F_1 = E_1 + E_2 + E_3 + E_8$ and $F'_1 = E_5 + E_6 + E_7 + E_{12}$ are two half-fibers of type $\tilde{A}_3$ of a genus one fibration. Similarly, we have a genus one fibration with half-fibers $F_2 = E_3 + E_4 + E_5 + E_{11}$ and $F'_2 = E_7 + E_8 + E_1 + E_{10}$. The divisors $E_{14} + E_{18}$ and $E_{13} + E_{17}$ are simple fibers of $|2F_1|$ of type $\tilde{A}_1$. Similarly, $E_{15} + E_{19}$ and $E_{16} + E_{20}$ are simple fibers of $|2F_2|$. This shows that $E_{14} - E_{18}$, $E_{13} - E_{17}$, $E_{15} - E_{19}$, $E_{16} - E_{20} \in 2\text{Num}(S)$. Observe also that, if two different $(-2)$-curves are in the same coset modulo $2\text{Num}(S)$ then they must intersect. For example, $E_{13}$ and $E_{14}$ are in different cosets. The divisor $E_{13} + E_{15}$ is disjoint from the divisor $F_3 = E_1 + E_9 + E_2 + E_4 + E_5 + E_6 + E_7 + E_8$ that forms a half-fiber of type $\tilde{A}_7$. The divisor $F_4 = E_2 + 2E_3 + E_4 + E_9 + E_{11}$ is a simple fiber of type $\tilde{D}_4$ such that $F_1 \cdot F_3 = 2$. Since $(E_{13} + E_{15}) \cdot F_4 = 4$, we obtain that $E_{13} + E_{15}$ is a simple fiber of type $A_1$ of the genus one-fibration $|2F_1|$. Thus $E_{13} - E_{15} \in 2\text{Num}(S)$. Similarly, we prove that $E_{13} - E_{19} \in 2\text{Num}(S)$. Thus the coset of $E_{13}$ modulo $2\text{Num}(S)$ consists exactly of the curves $E_i$, $i \geq 14$ that intersect it, so the claim has been proved.

Now, we see that $N_S$ is generated by the curves $E_1, \ldots, E_{12}, E_{13}, E_{14}$. Let $\pi : X \to S$ be the K3-cover. Consider the sublattice $M_1$ of $\text{Pic}(X)$ generated by $\delta_{E_i}, i = 2, \ldots, 9$ and $\delta_{E_{13}}, \delta_{E_{14}}$. It is isomorphic to $D_8 \oplus A_1^{\oplus 2}$. Consider the sublattice $M_2$ of $\text{Pic}(S)$ spanned by $\delta_{E'_i}, i = 1, \ldots, 8$. By Lemma 6.4.9, it is isomorphic to $D_8$. Both lattices are of rank 20, hence must be of finite index in the root lattice $K$ from the Nikulin $R$-invariant of $S$. Suppose $M_1 \neq K$. Since the discriminant groups of $M_1$ and $M_2$ are isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$, the discriminant group of $K$ must be isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ or $\{0\}$. Since there are no unimodular root lattice of rank 10, only the first case is possible. However, in this case, it follows from the extended Dynkin diagram of the Nod curve that there is only one of lattice $E_8$ contains only one sublattice isomorphic to $D_8$. It is obtained by deleting the vertex $\alpha_1$ of the extended Dynkin diagram of type $\tilde{E}_8$. Thus, in any case, we see that $M_1 = M_2$ and $\delta_{E_1} \in M_1$. Similarly, we show that $\delta_{E_{10}}, \delta_{E_{12}}$ and $\delta_{E_{13}} \in M_1$. Thus $M_1 = K$. Since the cosets of $E_{13}$ and $E_{14}$ modulo $2\text{Num}(S)$ belong to the radical of the quadratic space $\langle \text{Nod}(S) \rangle$ we see that $\text{rank}(\text{Nod}(S)) \leq 8$ and hence $\dim(\text{Nod}(S)) \leq 8$. Thus the finite part $H$ of the Nikulin invariant is $(\mathbb{Z}/2\mathbb{Z})^a$, where $a \geq 2$. On the other hand, the discriminant quadratic form of $K$ is $u_1 \oplus w_1^{-1} \oplus w_2^{-1}$, hence its maximal isotropic subgroup is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. This shows that $a \leq 2$, and we conclude that $a = 2$. The assertion about the Nikulin $R$-invariant has been proved. We also proved that $\langle \text{Nod}(S) \rangle$ is a defective quadratic space of rank 6 and dimension 8.

Let $h_{10}$ be a Fano polarization defined by a canonical isotropic sequence

$$(f_1, f_1 + E_{10}, f_1, f_1 + E_{10} + E_1, f_1, f_1 + E_{10} + E_1 + E_2, f_2, f_2 + E_7, f_2 + E_7 + E_6, f_2 + E_7 + \cdots + E_4),$$
where $f_1$ is the numerical class of a half-fiber of type $\tilde A_7$, and $f_2 = [2E_1 + E_2 + E_9 + E_8 + E_{10}]$ is the class of a simple fiber of type $\tilde D_4$. Computing the intersection numbers $h_{10} \cdot E_i$, we find that $h_{10} \cdot E_i \leq 4$ for all $i = 1, \ldots, 14$. This implies that $N^S_{10} = N_S$. Using our information about $\langle \text{Nod}(S) \rangle$, the only possibility is that $N_S$ is isomorphic to $\mathbb{D}_9 \oplus A_1 \oplus A_1(-1)$.

It remains to prove the assertion about the Reye lattice. We have $\text{Rey}(S)$ is equal to the pre-image of a 2-dimensional defective quadratic subspace in $\overline{\text{Num}}(S)$. It must be isomorphic to $E_8(4) \oplus U(2)$. \hfill\qed

**Proposition 8.9.13.** A surface of type III has the following elliptic fibrations

$2\tilde A_7 + 2\tilde A_1(8), \tilde A_7 + \tilde A_1(8), \tilde D_8(16), \tilde D_4 + \tilde D_4(2), \tilde D_6 + 2\tilde A_1 + 2\tilde A_1(8), 2\tilde A_3 + 2\tilde A_3 + \tilde A_1 + \tilde A_1(2)$.

The group $\text{Aut}(S)$ acts transitively on each set of fibrations. Two half-fibers from the first or the second group intersect each other with multiplicity 2, 4, 6 or 8. Two half-fibers from the third group (resp. the fourth, resp. the fifth, resp. the sixth) group intersect each other with multiplicity 2 (resp. 1, resp. 2, resp. 1). We have $\text{nd}(S) = 6$ and a canonical isotropic 6-sequence realizing $\text{nd}(S)$ can be chosen to be formed by one fibration from the first group, one fibration from the sixth group, 2 fibrations from the fourth group and 2 fibrations from the sixth group.

**Remark 8.9.14.** One can realize an Enriques surface of type III as the quotient of the Kummer surface of the self-product $A$ of the elliptic curves $E_{\sqrt{-1}}$ with complex multiplication of order 4. To this we consider two involutions of $A$

$$\iota : (x_1, x_2) \mapsto (-x_1, -x_2),$$

$$\sigma : (x_1, x_2) \mapsto (-x_1 + \alpha, x_2 + \alpha),$$

where $\alpha$ is a non-zero 2-torsion point on $E_{\sqrt{-1}}$. The K3-cover of the surface $S$ is a minimal resolution of $\text{Kum}(A) = A/\langle \iota \rangle$ and the surface $\tilde S$ is its quotient by the fixed-point involution descended from $\sigma$.

**Type IV**

The crystallographic basis consists of 20 curves with the dual graph given in the following picture:

Let us consider a non-degenerate $U$-pair formed by $F_1 = E_1 + E_{11} \equiv E_3 + E_{12}$ and $F_2 = E_4 + E_9 \equiv E_2 + E_{10}$. The two elliptic fibrations $|2F_1|$ and $|2F_2|$ have two simple fibers of types $\tilde A_3$ and two double fibers of type $\tilde A_1$ or $A_1^1$. The other reducible fibers are $D_1 = E_6 + E_8 + E_9 + E_{10}, D_1' = E_{15} + E_{16} + E_{19} + E_{20} \in |2F_1|$ and $D_2 = E_5 + E_{11} + E_{12} + E_{17}, D_2' = E_{13} + E_{14} + E_9 + E_{20} \in |2F_2|$. They share six common irreducible fiber components $E_9, E_{10}, E_{11}, E_{12}, E_{19}, E_{20}$. This implies that the curves $E_1, E_2, E_3, E_4$ are mapped to the lines on $D_1$ and the curves $E_{11}, E_{12}, E_9, E_{10}$ are mapped to points $q_1, q_2, q_3, q_4$, respectively, lying on the lines, with $q_1, q_2$ and $q_3, q_4$ lying on the opposite lines of the quadrangle of lines.

This also means that the components $C_1, C_2$ of the branch curve on $D_1$, together with the union of 4 lines belong to the same pencil cut out by hyperplanes passing through the points $q_1, \ldots, q_4$. Recall that the equations of the lines on $D_1$ in $\mathbb{P}^4$ are $x_i = x_j = 0, i \in \{1, 2\}, j \in \{3, 4\}$ so $q_1 = [0, 1, 0, 0, a_1], q_2 = [0, 1, 0, a_2, 0], q_3 = [1, 0, 0, a_3, 0], q_4 = [1, 0, 0, 0, a_4]$, where $a_i \neq 0$. The pairs $q_1, q_3$ and $q_2, q_4$ lie on opposite sides of the quadrangle of lines. The condition that these
Figure 8.11: Crystallographic basis of type IV

points are linearly dependent is $a_2a_3 - a_1a_4 = 0$. We can use projective automorphisms of $D_1$ to assume that

$$q_1 = [0, 1, 0, 1], \quad q_2 = [0, 1, 0, 1], \quad q_3 = [1, 0, 0, 1], \quad q_4 = [1, 0, 0, 1].$$

We also see that the curves $E_{19}, E_{20}$ are mapped to the singular points of $C_1, C_2$, thus the images of $C_1, C_2$ are the curves $E_{17}, E_{18}$. The fibers $D'_1$ and $D'_2$ share the components $E_{19}, E_{20}$, hence their images are conics $K_1, K_2$ from two different pencils that pass through the singular points of $C_1, C_2$. The other fibers $D_1, D_2$ are mapped to conics $K_3, K_4$ that pass through the points $q_1, q_3$ and $q_2, q_4$.

In the plane model, we may assume that two cubics are tangent at $p_1$ with tangent direction $x_1 - x_2 = 0$ and intersect the line $x_0 = 0$ at the point $[0, 1, 1]$ so that the conics $K_3, K_4$ are mapped to the line $V(x_1 - x_2)$. The two cubics belong to the pencil that contains the cubic $x_0x_1x_2 = 0$. Thus their equations are

$$ax_1x_2(x_1 - x_2) + bx_0^2(x_1 - x_2) + cx_0x_1x_2 = 0,$$

$$ax_1x_2(x_1 - x_2) + bx_0^2(x_1 - x_2) + dx_0x_1x_2 = 0,$$

A conic $x_1x_2 - \lambda x_0^2$ intersects these cubics at $p_2, p_4$ with multiplicity 3 if and only if $a = -b$ and $\lambda = 1$. Substituting $x_2 = tx_1$, we find that the line $x_2 - tx_1 = 0$ intersects both of these curves with multiplicity 2 at some point only if and only if $d = -c$ (here we see why do we need the assumption $p \neq 2$). We check that both curves are invariant under the standard Cremona transformation corresponding to an automorphism of $D_1$ that switches the skew lines. Thus the singular point must be one of the fixed points of this transformation, and we check that indeed the point $[-1, 1, \pm 1]$ is the singular point if and only if $c = \pm 4a$. Thus we may assume that the equation of the curves correspond to $(a, c) = (1, -4), (1, 4)$.
Now we know the equation of the double plane
\[ x_3^2 + x_1x_2((x_1x_2 - x_0^2)(x_1 - x_2)^2 - 16x_0^2x_1^2x_2^2) = 0 \] (8.9.4)

**Remark 8.9.15.** The K3-cover of \( S \) is the double cover of \( \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3 \) branched over the pre-image of the branch curve on \( D_1 \). The pre-image of each of the two nodal conics are a pair of conics intersecting at 2 points. The pre-images the four conics are four lines, two in each ruling.

Figure 8.12: Branch curve for the canonical cover of surfaces of type IV

Our other remark is that, after we blow-up \( D_1 \) at the points \( q_1, q_2, q_3, q_4 \), we obtain a rational extremal elliptic surface \( V \) with two reducible fibers of type \( \tilde{A}_7 \) and \( \tilde{A}_1 \). The first fiber is the pre-image of the quadrangle of lines, and the second fiber is the pre-image of the union of the conics \( K_3, K_4 \). The surface has also two irreducible nodal fibers, the pre-images of the hyperplane sections \( C_1, C_2 \). There is only one, up to isomorphism, such surface. The surface \( S \) is birationally isomorphic to the double cover of \( V \) branched along the union of two irreducible singular fibers and four disjoint components of the fiber of type \( \tilde{A}_7 \).

**Proposition 8.9.16.** Let \( S \) be an Enriques surface realizing the crystallographic basis of type IV. It exists only if \( p \neq 2 \), and in this case \( S \) is unique up to isomorphism and its double plane model is given in (8.9.4). Moreover,

\[ \text{Aut}(S) \cong 2^4 \rtimes N, \]

where \( N = \mathbb{Z}/4\mathbb{Z} \rtimes \mathbb{Z}/5\mathbb{Z} \) is isomorphic to the normalizer of a cyclic subgroup of order 5 in \( S_5 \).

**Proof.** Let \( \Gamma \) be the diagram of type IV. Observe first that the natural homomorphism from \( \text{Aut}(S) \) to the group \( \text{Sym}(\Gamma) \) of symmetries of \( \Gamma \) is injective. This follows from Corollary 8.2.23. Observe that \( S \) has 5 elliptic pencils \( D_i \) with fibers of type \( \tilde{A}_3, \tilde{A}_3, 2\tilde{A}_1, 2\tilde{A}_1 \). So the group \( \text{Aut}(S) \) acts on these 5 fibrations \( |D_i| \) and defines a homomorphism \( r : \text{Aut}(S) \to S_5 \). It is easy to see that the kernel \( K \) of the homomorphism \( r : \text{Sym}(\Gamma) \to S_5 \) is isomorphic to \( (\mathbb{Z}/2\mathbb{Z})^4 \). It is generated by even number of permutations of two singular fibers of type \( \tilde{A}_1 \) in each of the 5 fibrations. The elements of the group \( K \) are realized by the elements of the Mordell-Weil group of the jacobian fibrations with two fibers of types \( \tilde{D}_4 \), for example \( |2E_4 + E_1 + E_{15} + E_{16} + E_3| \). The image of
\[ \text{Aut}(S) \] contains an element of order 4 defined by the Mordell-Weil group of the jacobian fibration of any of the five elliptic fibrations \(|D_i|\). It also contains an element of order 5 that is defined by the Mordell-Weil group of the jacobian fibration of any of the elliptic fibrations with 2 reducible fibers of types \(\tilde{A}_4\), for example, \([E_3 + E_4 + E_{13} + E_{19} + E_{16}]\). Since the group \(r(\text{Sym}(\Gamma))\) does not have an element of order 3 and does not have any transposition, the assertion follows from the classification of subgroups of \(S_5\).

Remark 8.9.17. One can show that the K3-cover of Enriques surfaces of IV is birationally isomorphic to the same Kummer surface as the K3-cover of the surface of type III [397, Proposition (3.4.2)]. However the action the action of the Enriques involution is different. This is an example of the different fixed-point free involutions which may occur on a K3 surface birationally isomorphic to a Kummer surface studied in [553].

Proposition 8.9.18. The Nikulin R-invariant of an Enriques surface of Type IV is equal to \((D_5 \oplus 2, \mathbb{Z}/2\mathbb{Z})\). The quadratic space \(\langle \text{Nod}(S) \rangle\) is an even quadratic space of dimension 9 and rank 8. The Reye lattice is isomorphic to \(E_8(4) \oplus U\).

Proof. Consider the genus one fibrations of type \(\tilde{D}_5 + 2\tilde{A}_3\) formed by \(E_1, E_2, E_4, E_{14}, E_{19}, E_{20}\) and \(E_5, E_6, E_7, E_{12}\). The curve \(E_9\) is its bisection. Applying Lemma 6.4.9, we find that \(K\) contains a sublattice isomorphic to \(D_5 \oplus D_5\). Its discriminant group is \(w_{2,4}^{-5} \oplus w_{2,4}^{-5}\). It maximal isotropic subgroup \(A\) is of order 2. Suppose \(K \neq M\), then \(K\) must be an overlattice with discriminant quadratic form \(w_{2,1}^{-1}\). There are no such root lattices. By Lemma 6.4.5, \(K/2K\) is an even quadratic space of dimension 10 and rank 8. Since its image cannot be the whole space, the kernel \(H\) of the reduction map \(K/2K \rightarrow \overline{\text{Num}(S)}\) is not trivial. Since the 2-torsion of the discriminant group of \(K\) is of order 2, we obtain that \(H \cong \mathbb{Z}/2\mathbb{Z}\). The quadratic space must be an even, of dimension 9 and rank 8.

Proposition 8.9.19. A surface of type IV has genus one fibrations of the following types

\[ \tilde{D}_4 + \tilde{D}_4 \ (10), \ \tilde{D}_5 + 2\tilde{A}_3 \ (40), \ \tilde{A}_4 + \tilde{A}_4 \ (16), \ \tilde{A}_3 + \tilde{A}_3 + 2\tilde{A}_1 + 2\tilde{A}_1 \ (5). \]

The intersection number of half-fibers of two fibrations from the first (the second, the third, the fourth) group is equal to 2 (resp. 2, resp. 2, resp. 1). The non-degeneracy invariant \(\text{nd}(S)\) is equal to 5.

Type V

This surface has the following crystallographic basis formed by \((-2)\)-curves

First we locate a genus one fibration

\[ |D| = 2(E_1 + E_2 + E_3 + E_8 + E_9 + E_{10})| = 2(E_5 + E_{20})| = |E_{17} + E_{18} + E_{19}| \]

with two double fibers of type \(\tilde{A}_5\) and \(\tilde{A}_1\) and one fiber of type \(\tilde{A}_2\). It shows that such diagram cannot be realized if \(p = 2\). In fact, the fibration contains two double fibers, hence \(K_S \neq 0\). Since the half-fiber is of multiplicative type, it is impossible.

So we assume, as everywhere in this section, that \(p \neq 2\).
We consider a non-degenerate $U$-pair formed by the elliptic fibrations

$|E_{10} + E_2 + 2E_1 + 2E_5 + 2E_6 + E_4 + E_7| = |2(E_9 + E_{14})| = |E_{11} + E_{13}|,$

$|E_2 + E_9 + 2E_3 + 2E_4 + 2E_5 + E_6 + E_7| = |2(E_{10} + E_{15})| = |E_{11} + E_{12}|,$

with reducible simple fibers of type $\tilde{D}_6, \tilde{A}_1$ and a double fiber of type $\tilde{A}_1$.

We easily locate a subdiagram of type (c) from Theorem 8.2.21. Thus our surface is of type (C) from Theorem 8.2.23. We use a construction of this surface as a double plane indicated in the proof of the theorem.

Recall that the branch curve on $D_1$ is the union of two conics from different pencils and a nodal hyperplane section passing through one of the intersection points of the two conics. In the plane, the hyperplane section is a cubic $C$, and the conics are a line $\ell$ through $p_1$ and a conic $K$ through $p_2, \ldots, p_5$.

Let

$\ell \cap K = \{q_1, q_2\}, \ell \cap C = \{q_1, q_3\}, C \cap K = \{q_1, q_4\}.$

The point $q_1$ is a singular point of type $d_4$ of the branch curve $W$. The remaining singular points are $q_2, q_3, q_4$ and a node $q$ of $C$.

The pre-image of the line $\ell$ (resp. the conic $K$) is the fiber of the type $\tilde{D}_6$ of the first (resp. the second) fibration. The pre-image of the line $\ell' = \langle p_1, q_4 \rangle$ (resp. the conic $K'$ through $q_3$) is a simple fiber of type $\tilde{A}_1$ of the first (resp. the second) fibration.

To realize the double fibers we have to assume additionally the following
(1) the point \(q_3\) lies on the line \(\langle p_2, p_4 \rangle\);

(2) the point \(q_4\) lies on the line on \(D_1\) equal to the exceptional curve over the point \(p_3\).

So far we see all curves \(E_i\) except \(E_8, E_{16}, E_{17}, E_{18}, E_{19}, E_{20}\). The curve \(E_8\) is the proper transform of the cubic \(C\), the exceptional curve over its node is the curve \(E_{11}\). The curve \(E_{20}\) is a hyperplane section with a node at \(q_1\) and passing through \(q\). Counting constants, we find as soon as we find our cubic \(C\). So, its existence does not impose any new conditions. We also see that \(E_{16}\) is represented by a line \(\langle q_2, q \rangle\), the same condition we used for a general member of the 4-dimensional family (C) from Theorem 8.2.23.

We observe that \(E_{17}, E_{18}, E_{19}\) are connected to \(E_6, E_4, E_7\) by double edges. This means that the corresponding curves are tangent to \(l, K, C\), respectively at the point \(q_1\). We also observe that their images on \(D_1\) are curves of degree 3. The curves \(E_{18}, E_{19}\) also are joined to \(E_{11}\) that represents the exceptional curve over the node \(q\) of \(C\). This prompts us to put the following conditions.

(3) there is a conic from \(|2e_0 - e_2 - e_3 - e_4|\) that is tangent to \(l\) at \(q_1\) and passes through \(q\) (this will give us \(E_{17}\));

(4) the tangent line to \(K\) at \(q_1\) passes through \(q\) (this will give us \(E_{18}\));

(5) the tangent line to \(C\) at \(q_1\) contains \(x_2 = [0, 1, 0]\) (this will give us the curve \(E_{19}\)).

Now, after straightforward computations, we find the equation of the double plane.

\[
x_3^2 + x_1x_2(x_1 - x_2)(x_1x_2 - x_0^2)(x_1x_2(x_1 - x_2) + x_0^2(-9x_1 + x_2) - 8x_0x_1x_2) = 0.
\] (8.9.5)

The point \(q_1\) has coordinates \([1, 1, -1]\), the singular point \(q\) of \(C\) has coordinates \([1, -3, 1]\).

We see that the construction works only if \(p \neq 2, 3\).

**Remark 8.9.20.** The canonical cover of our Enriques surface is the minimal resolution of the double cover \(S'\) of \(Q\) branched along the curve \(\tilde{W} = C_+ + C_- + L_{1,+} + L_{1,-} + L_{2,+} + L_{2,-}\). The curve \(C_+ + C_-\) is the pre-image of the nodal cubic \(C\), it splits in the cover. The remaining curves are the pre-images of \(l\) and \(K\). In the following picture borrowed from [397], we see also the curves \(L_3\) and \(F_{1,\pm}, F_{2,\pm}\). They are the pre-images of \(l'\) and \(K'\).

Figure 8.14: Branch curve for the canonical cover of surfaces of type V
8.9. ENRIQUES SURFACES WITH FINITE AUTOMORPHISM GROUP: \( P \neq 2 \)

**Proposition 8.9.21.** Let \( S \) be an Enriques surface realizing the crystallographic basis with diagram \( V \). It exists only if \( P \neq 3 \) and its double plane model is given in equation (8.9.5). Moreover,

\[
\text{Aut}(S) \cong \mathbb{Z}/2\mathbb{Z} \times S_4.
\]

**Proof.** We have only to explain the structure of the group of automorphisms. First of all, we know that our surface admits a numerically trivial involution of type (c) from Theorem 8.2.21. This is the bielliptic involution \( \sigma \) corresponding to our double plane model.

Next, as in the previous examples, we consider a homomorphism \( \text{Aut}(S) \to \text{Sym}(\Gamma) \) defined by the action of \( \text{Aut}(S) \) on the diagram \( \Gamma \) of \((-2)\)-curves. We immediately see the group \( \text{Sym}(\Gamma) \) is isomorphic to \( S_4 \). Consider the subset \( \Sigma \) of vertices \( E_1, E_3, E_5, E_8 \). Any symmetry of \( \Gamma \) that pointwise fixes this set is the identity. Thus \( \text{Sym}(\Gamma) \) is isomorphic to a subgroup of \( S_4 \). It is easy to see that it coincides with this group.

Let us prove that each symmetry of \( \Gamma \) is realized by an automorphism of \( S \). We use that the Mordell-Weil group of the jacobian fibration of any elliptic fibration on \( S \) acts by translations on \( S \). Thus, if we take an elliptic fibration with reducible fibers of type \( \tilde{A}_5, \tilde{A}_2 \) and \( \tilde{A}_1 \), we obtain the group \( \mathbb{Z}/6\mathbb{Z} \) acting on \( S \). For example, take the fiber of type \( \tilde{A}_5 \) to be \( E_1, E_2, E_3, E_9, E_8, E_{10} \). Then cube of its generator fixes the vertices \( E_1, E_2, E_3 \), hence fixes the vertices \( E_8, E_9, E_{10} \). Hence it fixes all the vertices of the hexagon defining the fiber. This also implies that it fixes all vertices of the graph and hence defines the numerically trivial automorphism \( \sigma \) of \( S \). Thus we located an element of order 3 in \( \text{Aut}(S)/(\sigma) \). Now we look at an elliptic fibration with reducible fibers of type \( \tilde{D}_6, \tilde{A}_1, \tilde{A}_1 \). The Mordell-Weil of the jacobian fibration is a cyclic group of order 4. The square of the generator interchanges \( E_6 \) with \( E_7 \) and \( E_2 \) with \( E_9 \). It defines an element of the Mordell-Weil group of the elliptic fibration \( E_9 + E_7 + E_2 + E_9 + 2E_3 + 2E_4 + 2E_5 \) of type \( \tilde{D}_6 \). Thus we obtain a generator of order 2 and 3 in \( S_4/A \cong S_3 \) and we have proved the claim. \( \square \)

**Proposition 8.9.22.** The Nikulin R-invariant of an Enriques surface of type \( V \) is equal to \( (E_7 \oplus A_2 \oplus A_1, \mathbb{Z}/2\mathbb{Z}) \). The quadratic space \( \langle \text{Nod}(S) \rangle \) is a defective quadratic space of dimension 9 and rank 8. The Reye lattice is isomorphic to \( E_8(4) \oplus A_1 \oplus A_1(-1) \).

**Proof.** The divisor \( E_1 + E_2 + E_3 + E_8 + E_9 + E_{10} \) is a half-fiber of type \( \tilde{A}_5 \). Applying Lemma 6.4.9, we find that the divisor classes of \( \delta_{E_i}, i = 1, 2, 3, 8, 9, 10 \) together with the class \( \delta_{E_7} \) generate a sublattice of \( K \) isomorphic to \( E_7 \). Together with the classes \( \delta_{E_7}, \delta_{E_{17}} \) and \( \delta_{E_{20}} \) they generate a sublattice \( L \) isomorphic to \( E_7 \oplus A_2 \oplus A_1 \). Its discriminant group is isomorphic to \( (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} \oplus \mathbb{Z}/3\mathbb{Z} \).

If \( K \neq L \), then its discriminant group must be isomorphic to \( \mathbb{Z}/3\mathbb{Z} \). Thus \( H = \{0\} \) and the image of \( K \) or \( N_{\mathbb{R}} \) in \( \text{Num}(S) \) is the whole space. However, the diagram shows that the image \( [E_{20}] \cdot x \in \mathbb{Z}/2\mathbb{Z} \) for any \( x \in N_S \), hence its image belongs to the radical of the image of \( N_S \). This contradiction proves that \( K = L \). Since \( \dim \langle \text{Nod}(S) \rangle < 10 \), we see that \( H \neq \{0\} \), hence it must coincide with the only isotropic subgroup of \( K \) isomorphic to \( \mathbb{Z}/2\mathbb{Z} \). It follows from Lemma 6.4.5 that the \( r \)-invariant \( \langle \text{Nod}(S) \rangle \) is a 9-dimensional defective quadratic space of rank 8. The Reye lattice is the pre-image in \( \text{Num}(S) \) of \( \langle \text{Nod}(S) \rangle^\perp \) and must be isomorphic to \( 2E_8 \oplus A_1 \oplus A_1(-1) \). \( \square \)
Proposition 8.9.23. A surface of type V has genus one fibrations of the following types:

\[2\bar{A}_5 + \bar{A}_2 + 2\bar{A}_1(4), \quad \bar{E}_7 + 2\bar{A}_1(12), \quad D_6 + 2\bar{A}_1 + \bar{A}_1(6), \quad \bar{A}_7 + \bar{A}_1(3), \quad \bar{E}_6 + \bar{A}_2(4).\]

The group \(\text{Aut}(S)\) has 2 orbits on each group except the last one. One of the fibrations in the first group is invariant. In particular, \(\text{nd}(S) = 7\).

Type VI

Figure 8.15: Crystallographic basis of type VI

Observe that the diagram contains a subdiagram isomorphic to the Petersen graph (6.1). Each its vertex is connected by a double edge to one of the remaining 10 vertices. They form a symmetric incidence configuration \((10_6)\). The presence of the Petersen graph prompts us to search for a construction of this surface as an Enriques surface of Hessian type which we studied in Example 6.4.18.

Recall that the most special Hessian surface is the Hessian surface of the Clebsch diagonal cubic surface. The involutions defined by the projections from the nodes descend to the involutions on the Enriques surface that generate a group of automorphisms isomorphic to the symmetric group \(S_5\). If we index the nodes as in the Petersen graph, the projection involutions \(\tau_{ab}\) act on the \((-2)\)-curves \(U_{ab}\) as transpositions [189, Corollary 4.2]. We also can interpret 10 elliptic pencils \(|F_{ab}|, 1 \leq a < b \leq 5\) defined by the double edges in the diagram. They correspond to pencil of cubic curves cut out by a pencil of planes through each of the 10 edges \(L_{ab}\) of the Sylvester pentahedron \(\Pi\) whose vertices are the ten nodes of the Hesse surface.

With all this in mind let us construct an Enriques surface of Type VI as the quotient of the Hessian
surface of the Clebsch diagonal cubic surface. It is given by equation
\[ \sum_{i=1}^{5} t_i = t_1 t_2 t_3 t_4 t_5 \sum_{i=1}^{5} \frac{1}{t_i} = 0, \] (8.9.6)
in \(?:^4\). The fixed-point-free involution \(\tau\) is given by the standard Cremona transformation in \(?:^4\) given by the formula
\[ \sigma : [t_1,\ldots,t_5] \mapsto \left[ \frac{1}{t_1},\ldots,\frac{1}{t_5} \right]. \]
The involution has no fixed points on \(X\) if \(p \neq 3, 5\). It has 5 fixed points if \(p = 3\) and one fixed point if \(p = 5\).

When \(p \neq 2\), the surface given by these equations is isomorphic to the Hessian surface of the \textit{Clebsch diagonal cubic surface}
\[ C_3 : \sum_{i=1}^{5} t_i = \sum_{i=1}^{5} t_i^3 = 0 \]
(see \[180\]). If \(p = 2\), the same is true only the equation of the Clebsch diagonal surface is the following
\[ C_3 : \sum_{i=1}^{5} t_i = \sum_{1 \leq i < j < k \leq 5} t_i t_j t_k = 0. \] (8.9.7)
If \(p \neq 3\), the surface \(X'\) has 10 ordinary double points \(P_{ab}\) with coordinates \(t_i = t_j = t_k = 0\) for \(i, j, k \notin \{a, b\}\). They are the vertices of the \textit{Sylvester pentahedron} II formed by the planes \(t_i = 0\).
We also have 30 lines
\[ L_{ab}' : t_a = t_b = t_c + t_d + t_e = 0, \{a, b, c, d, e\} = \{1, 2, 3, 4, 5\}, \]
\[ L_{cde}(\omega)' : t_a + t_b = 0, \omega t_c - t_d = \omega^2 t_e - t_e = 0, \{a, b, c, d, e\} = \{0, 1, 2, 3, 4\}, \]
where \(\omega^3 = 1, \omega \neq 1\). The first 10 lines appear on any Hessian surface, they are the edges of the Sylvester pentahedron. The additional lines \(L_{cde}(\omega)'\) are absent in the case of the Hessian surface of a general cubic surface. They correspond to splitting of the conics cut out by planes tangent to the surface along a line \(L_{ab}'\) and passing through opposite node \(P_{ab}\) with coordinates \(t_c = t_d = t_e = 0\) into two lines.

Let \(R_{ab}\) be the exceptional curve of the minimal resolution \(X \rightarrow X'\) over the singular point \(P_{ab}\) and let \(L_{ab}, L_{cde}(\omega)\) be the proper transforms of the lines in \(X'\). Together we have 40 \((-2)\)-curves on \(X\).

The Cremona involution \(\sigma\) lifts to a biregular automorphism of the minimal resolution of singularities \(X\) of \(X'\) and switches the curves \(L_{ab}\) with the exceptional curve \(R_{cde}\) over the point \(p_{cde}\) such that \(\{a, b\} \cap \{c, d, e\} = \emptyset\).

The involution \(\sigma\) makes 20 orbits on the set of 40 curves \(R_{ab}, L_{ab}, L_{cde}(\omega)\). If \(p \neq 2\), the quotient map \(X \rightarrow S = X/\langle \sigma \rangle\) is the canonical cover of an Enriques surface. The images of the forty \((-2)\) curves on \(S\) is the set of twenty \((-2)\)-curves which form the crystallographic basis of type VI. To see this we first observe that the incidence graph of the images of the orbits \(L_{ab}, R_{ab}\) is the Petersen graph 6.1. The curves \(E_1, \ldots, E_{10}\) correspond to these curves in the diagram of type VI. The remaining 10 curves \(E_{11}, \ldots, E_{20}\) correspond to 10 lines \(L_{abc}(\omega)\).
Next we will find an explicit equation of the surface as a double model corresponding to a choice of a bielliptic involution. This will prove the uniqueness of the surface up to an isomorphism and also will show that the surface exists only under assumption that \( p \neq 3, 5 \) (recall that we always assume in this section that \( p \neq 2 \)).

We have already noted in Remark 10.5.1 that the surface \( V(P) \) is obtained from the Fermat cubic by blowing up 9 points. We may assume that \( C \) is obtained by blowing up the set \( Q(a, b, c) \) and the set of 9 points is the set of base points of a Hessian pencil defined by the choice of the triad \( (abc) \).

We have already mentioned that the surface has 10 elliptic pencils \( |2F_{ab}| \) indexed by the vertices of the Petersen graph corresponding to the nodes of the Hessian surface, and also to the orbits \( U_{ab} \) of an edge and the vertex of the Sylvester pentahedrons. There are two kinds of \( U \)-pairs \( (|2F_{ab}|, |2F_{cd}|) \) formed by these fibrations corresponding to whether the two subsets \( \{a, b\}, \{c, d\} \) are disjoint or not. In the latter case the bielliptic involution defined by the pair coincides with the projection involution defined by the complementary subset of \( \{a, b, c, d\} \) in \( \{1, 2, 3, 4, 5\} \) [189, Lemma 4.5].

Let us, for convenience, reindex the curves \( E_1, \ldots, E_{10} \) by subsets \( \{a, b\} \) as in the Petersen graph:

\[
\begin{pmatrix}
E_1 & E_2 & E_3 & E_4 & E_5 & E_6 & E_7 & E_8 & E_9 & E_{10} \\
U_{15} & U_{12} & U_{34} & U_{15} & U_{24} & U_{13} & U_{35} & U_{14} & U_{25} & U_{23}
\end{pmatrix}
\]

Let us choose the \( U \)-pair

\[
|2F_{34}| = |E_3 + E_4 + E_5 + E_7 + E_8 + E_9| = |E_{12} + E_{16} + E_{19}| = |2(E_1 + E_{20})|,
\]

\[
|2F_{45}| = |E_1 + E_6 + E_5 + E_7 + E_8 + E_{10}| = |E_{12} + E_{15} + E_{18}| = |2(E_3 + E_{17})|
\]

The bielliptic involution acts on the diagram as the transposition (12). We find that the curves \( E_1, E_3, E_5, E_7, E_8, E_{12} \) are common irreducible components of both elliptic pencils. They span a sublattice of \( \text{Num}(S) \) isomorphic to \( A_3 + A_1^{33} \) and they are blown down to singular points of the branch curve that consist of one tacnode and 3 ordinary nodes. The branch curve can be found in Row 23 in Table 8.7. It consists of two hyperplane sections tangent at one point and also intersecting at two points lying on two non-skew lines on the 4-nodal anti-canonical quartic del Pezzo surface \( D_1 \), one of them has a node. In the double plane model the branch sextic curve is the union of two cubics \( C_1 \) and \( C_2 \). They intersect at a point \( q_1 \) on the line \( \langle p_2, p_4 \rangle = V(x_0) \) and at some point \( p_3' \) infinitely near to \( p_3 \). One of them, say \( C_1 \), has a node \( q_2 \) and \( C_1, C_2 \) are tangent at some point \( q_3 \).

It is easy to see from the diagram that the node \( q_2 \) is the image of \( E_{12} \), the tacnode \( q_3 \) is the image of \( E_5 + E_7 + E_8 \) where \( E_7 \) is invariant with respect to the involution, and the other components are switched. The curve \( E_1 \) (resp. \( E_3 \)) is mapped to the point \( p_3' \) (resp. \( q_1 \)). The curves \( E_{20} \) (resp. \( E_{17} \)) is mapped to the line \( V(x_0) \) (resp. is blown down to the point \( q_2 \)). The curve \( E_2 \) is mapped to the cubic \( C_1 \). The reducible fiber of type \( A_5 \) of the pencil \( |2F_{34}| \) is mapped to the line \( \ell_1 \) which passes through \( p_1, q_1, q_3 \) and its proper transform on \( S \) splits under the cover into the curves \( E_9 + E_4 \). Similarly, the fiber of the same type of \( |2F_{45}| \) is mapped to a conic \( K_1 \) that passes through \( p_2, p_5, p_4, p_5, p_3', q_3 \). Its proper transform splits into \( E_6 + E_{10} \). The curves \( E_1 \) and \( E_3 \) are the pre-images of the exceptional curves over \( q_1, p_3' \). The pairs \( (E_{15}, E_{18}) \) and \( (E_{16}, E_{19}) \) are mapped to lines \( \ell_2 \in |e_0 - e_1| \) and the conic \( K_2 \in |2e_0 - e_2 - e_3 - e_4 - e_5| \) that pass through the node \( q_2 \).

They are tangent to the cubic \( C_2 \) and split under the cover.

So far, all curves \( E_i \) have been accounted for, except the curves \( E_{11}, E_{13}, E_{14} \). We observe that the curve \( E_{14} \) intersects \( 2(F_{34} + F_{45}) \) with multiplicity 4. Since it is invariant under the involution,
it is mapped to a conic on $D_1$ passing through two opposite nodes of $D_1$. In the plane model it is the line through $p_4$ (not $p_2$ because $E_{14}$ intersects both $E_{20}$ and $E_{17}$) that tangent to $C_1, C_2$ at the point $q_3$. The curves $E_{11}$ and $E_{13}$ are switched by the involution and mapped to the cubic curve $C_3$ with a double point at $q_3$ and containing $q_2$. Since $E_{11}, E_{13}$ are connected in the diagram, the cubic $C_3$ must be tangent to $C_2$ at some point.

So, now all 20 curves are accounted for, and we may proceed to write down the equation of the double plane.

First we observe that the involution of the surface defined by the transposition (35) leaves invariant the $U$-pair and hence descends to a Cremona involution $T$ of the plane that leaves invariant the branch curve. Since it switches the two pencils it corresponds to an automorphism of the surface $D_1$ that fixes two opposite vertices of the quadrangle of lines. Using (8.7.2), it is not difficult to see that we may assume that $T$ is given by formula

$$T : [x_0, x_1, x_2] \mapsto [x_0^2, x_1 x_2, x_0 x_1].$$

Our cubics $C_1, C_2$ are invariant cubics from the linear system $| - K_{D_1}|$ that intersect at the same point on $\langle p_2, p_4 \rangle = V(z)$. They must be given by equations

$$x_1 x_2 (ax_2 - x_1) + x_0^2 (b_i x_1 - x_2) + c_i x_0 x_1 x_2 = 0, \quad i = 1, 2. \quad (8.9.8)$$

Because of the invariance, they automatically pass through the same point $q_2$ infinitely near to $x_3$.

The line $\ell = V(ax_2 - x_1)$ intersects $C_1, C_2$ at their tangency point $q_3$. Since the fixed locus of $T$ is equal to $V(x_1^2 - x_0^2)$, we may assume that $q_3 = [a, 1, a]$.

Computing the partial derivatives, we obtain the condition that the tangent line is of the form $x_0 + kx_2 = 0$ is $b_i = -(c_i - 1)/a$. They will be automatically tangent at this point.

Next we use the condition that $C_1$ has a singular point at some point $(-1, A, 1)$ on the other fixed line. This gives $A = 3/a, c_1 = -8$. Then, we use that the line $x_2$ passing through $p_1$ and this point is tangent to $C_2$. This gives $c_2 = 8/3$. Thus the equations of $C_1, C_2$ and $C_1$ become

$$C_1 : x_1 x_2 (ax_2 - x_1) + (9a/x_1 - x_2)x_0^2 - 8x_0 x_1 x_2 = 0,$$

$$C_2 : x_1 x_2 (ax_2 - x_1) + (-5/3a)x_1 - x_2)x_0^2 + 8/3)x_0 x_1 x_2 = 0.$$ 

Now we observe that, the variable change $y' = ay$ allows one to assume that $a = 1$. We verify that the condition for the existence of the cubic $C_3$ representing $E_{13}$ is automatic. So, finally we find the equation of the double plane.

$$x_3^2 + x_1 x_2 (x_1 x_2 (x_2 - x_1) + (9x_1 - x_2)x_0^2 - 8x_0 x_1 x_2)(x_1 x_2 (x_2 - x_1) + (5/3)x_1 - x_2)x_0^2 + 8/3)x_0 x_1 x_2 = 0. \quad (8.9.9)$$

We see that the surface exists if and only if $p \neq 3, 5$.

**Proposition 8.9.24.** Let $S$ be an Enriques surface with a crystallographic basis with the diagram of type VI. Then it exists only if $p \neq 3, 5$, it is unique, up to isomorphism, and

$$\text{Aut}(S) \cong S_5.$$
Proof. We have already explained that the Hessian surface admits $\mathfrak{S}_5$ group of symmetries that is inherited from the group of symmetries of the Clebsch diagonal surface and descends to the group of symmetries of the Enriques surface. The group of symmetries of the diagram is isomorphic to $\mathfrak{S}_5$ and is induced by the symmetries of the Petersen subgraph. Since the diagram has no subdiagrams of type (i),(ii),(iii) from Theorem 8.2.21, $S$ has no non-trivial numerically trivial automorphisms. So, we conclude that $\text{Aut}(S) \cong \mathfrak{S}_5$. \qed

Remark 8.9.25. If $k = \mathbb{C}$, it is known that the Kummer surface associated with the Jacobian of a curve of genus 2 is isomorphic to the irreducible Hessian surface of a nonsingular cubic surface [309]. The coefficients $(a_1, \ldots, a_5)$ in the standard equation of the Hessian surface satisfy a cubic equation

\[
\sum_{i=1}^{5} a_i^3 - 2 \sum_{1 \leq i < j < k \leq 5} a_i a_j a_k + \sum_{1 \leq i < j \leq 5} a_i a_j a_k = 0.
\]

(see [598, Theorem 7.1]). In particular, taking $a_1 = a_2 = a_3 = a_4 = a_5 = 1$, we see that the Hessian surface of the Clebsch diagonal cubic surface is not a Kummer surface. It is known that the lattice of transcendental lattice of our Hessian surface is of rank 2 defined by the matrix

\[
\begin{pmatrix}
4 & 1 \\
1 & 4
\end{pmatrix}
\]

(see [397, p. 277]).

Proposition 8.9.26. The Nikulin R-invariant of an Enriques surface of type VI is equal to $(E_6 \oplus A_4, \{0\})$. The quadratic space $\langle \text{Nod}(S) \rangle$ coincides with $\overline{\text{Num}}(S)$. The Reye lattice coincides with $2\overline{\text{Num}}(S)$.

Proof. The curves $E_3, E_4, E_5, E_6, E_7, E_8$ span a sublattice of $\mathbb{N}_E$ isomorphic to $E_6$. Thus $\delta_{E_i}, 3 \leq i \leq 8$ span a sublattice of $K$ isomorphic to $E_6$. The divisor $E_{12} + E_{15} + E_{16}$ is a half-fiber of type $\tilde{A}_2$ of the genus one fibration with a simple fiber of type $\tilde{E}_6$. It has a special bisection $E_{20}$ which is disjoint from $E_3, E_4, E_5, E_6, E_7, E_8$. Applying Lemma 6.4.9, we find that $K$ contains a sublattice isomorphic to $E_6 \oplus A_4$. Its discriminant quadratic group has no isotropic vectors. Thus $K = E_6 \oplus A_4$. Note that the projection of $E_6 \oplus A_4$ to the space $\langle \text{Nod}(S) \rangle$ is the whole space. One confirms again following Lemma 6.4.5 that it is a regular even space of dimension 10. \qed

Proposition 8.9.27. A surface of type VI has genus one fibrations of the following types:

\[
2\tilde{A}_4 + \tilde{A}_4 (12), \tilde{A}_5 + \tilde{A}_2 + 2\tilde{A}_1 (10), \tilde{D}_5 + \tilde{A}_3 (15), \tilde{E}_6 + 2\tilde{A}_2 (10).
\]

The group $\text{Aut}(S)$ acts transitively on each group. The non-degeneracy invariant $\text{nd}(S)$ is equal to 10.

The last assertion follows from the fact that ten half-fibers corresponding to the double edges in the diagram form a non-degenerate isotropic 10-sequence. Note that the surface is a special case of an Enriques surface of Hessian type, and s follows from (6.4.17), any such surface has the non-degeneracy invariant equal to 10.

Type VII

The crystallographic basis of $(-2)$-curves is given in the following diagram.
8.9. ENRIQUES SURFACES WITH FINITE AUTOMORPHISM GROUP: $P \neq 2$

We can redraw this graph differently exhibiting more explicitly its $S_5$-symmetry.

Here 5 outside vertices represent vertices $K_1, \ldots, K_5$, they are joined by thick edges.

Assume that such surfaces exists. We first determine the automorphism group of such surface. First we observe that the group of symmetries of the diagram $\Gamma$ is isomorphic to $S_5$ via its action on the complete subgraph $K(5)$ with 10 double edges. Next, looking at parabolic subdiagrams we observe the following.
Proposition 8.9.28. A surface of type VII has genus one fibrations of the following types:
\[ \tilde{A}_8 (20), \tilde{A}_7 + 2\tilde{A}_1 (15), \tilde{A}_4 + \tilde{A}_4 (12), \tilde{A}_5 + 2\tilde{A}_2 + \tilde{A}_1 (10). \]
The group \( \text{Aut}(\Gamma) \) acts transitively on the corresponding maximal parabolic subdiagrams.

Observe that the numbers 20, 15, 12, 10 are equal to the number of conjugacy classes in \( \mathfrak{S}_5 \) of cyclic subgroups of order 3, 4, 5, 6, respectively. Looking at the Mordell-Weil group of each fibration we find that they are cyclic group of order 3, 4, 5, 6, respectively. So, we can realize these symmetries by translation transformations. Since the diagram \( \Gamma \) has no subdiagram described in Theorem 8.2.21, we obtain that \( \text{Aut}_{\text{int}}(S) \) is trivial. This proves the following.

Theorem 8.9.29. Assume \( p \neq 2, 5 \). Let \( S \) be an Enriques surface with crystallographic basis of \((-2)\)-curves of Type VII. Then
\[ \text{Aut}(S) \cong \mathfrak{S}_5. \]

Proof. It is easy to see that the graph has no subgraphs described in Theorem 8.2.21. Thus \( \text{Aut}(S) \) embeds in the group of symmetries of the diagram 8.16. We know that it contains \( \mathfrak{S}_5 \). On the other hand, the group of symmetries of the diagram coincides with the group of symmetries of the subgraph \( \Gamma_1 \). It is obviously isomorphic to \( \mathfrak{S}_5 \).

Example 8.9.30. To realize this basis we consider, following H. Ohashi, the surface \( X' \) of degree 6 in \( \mathbb{P}^4 \) given by the equations
\[ \sum_{1 \leq i < j \leq 5} x_i x_j = \sum_{1 \leq i < j < k \leq 5} x_i x_j x_k = 0. \] (8.9.10)
We have already encountered this surface in Example 8.8.4, where we have shown that the group \( \mathfrak{S}_5 \) acts on \( S \) in a Mathieu fashion.

The surface has the obvious \( \mathfrak{S}_5 \)-symmetry. It also has an involution \( \sigma \) defined by the standard Cremona transformation \([x_1, \ldots, x_5] \rightarrow [1/x_1, \ldots, 1/x_5] \). The surface has 5 nodes that form an orbit of \( \mathfrak{S}_5 \) of the point \([1, 0, 0, 0, 0] \). The hyperplane sections \( A'_1 : x_i = 0 \) are curves of degree 6 and of arithmetic genus 4. Each of these curves contains 4 nodes. The proper transforms \( A_i \) of these curves on a minimal resolution \( X \), which is a K3 surface, are smooth rational curves. The lift \( \tau \) of the Cremona involution to \( X \) interchanges the exceptional curves of the resolution with the curves \( A_i \).

Assume that \( p \neq 2, 5 \). In this case the involution \( \sigma \) has no fixed points on \( X \) and the quotient \( S = X/\langle \sigma \rangle \) is an Enriques surface. The images of the curves \( A_i \) on \( S \) is a set of 5 curves \( K_1, \ldots, K_5 \) whose dual graph is a complete graph \( \Gamma_1 \) with 5 vertices and double edges.

For each even involution \( \sigma = (ij)(kl) \) in \( \mathfrak{S}_5 \) we have two smooth rational curves \( \ell^\pm_\sigma \). If \( \sigma = (12)(34) \), the line \( \ell^+_\sigma \) is the span of the points \([1, -1, \pm \sqrt{1}, \mp \sqrt{1}, 0]\) and \([0, 0, 0, 0, 1]\). The remaining lines are obtained by applying permutations of the coordinates.

Their proper inverse transforms to \( X \) is the set of 30 smooth rational curves. The involution \( \tau \) acts on this set via switching \( \ell^+_\sigma \) with \( \ell^-_\sigma \). The images of these curves on \( S \) is the set of 15 curves \( E_1, \ldots, E_{15} \) whose incidence graph is a 4-regular graph \( \Gamma_2 \). The exceptional curve over the singular point \([1, 0, 0, 0, 0]\) is mapped under the Cremona involution to the proper transform
on $X$ of the hyperplane section $H_1 = X' \cap V(x_1)$. It is a curve of degree 6 in $\mathbb{P}^3$ with 4 nodes $[1,0,0,0], \ldots, [0,0,0,1]$. It intersects with multiplicity 2 the exceptional curves over the other four nodes of $X'$. The orbit of this curve is a curve $K_1$ in the graph. In this way we realize the curves $K_1, \ldots, K_5$ in the diagram.

**Remark 8.9.31.** In characteristic 2, there exist classical and $\alpha_2$-Enriques surfaces with crystallographic basis of $(-2)$-curves of Type VII (see §8.9, Example 8.10.8).

**Remark 8.9.32.** A different construction of an Enriques surface with a crystallographic basis of type VII was given by G. Fano [222], and reproduced in [397, (3.7)].

The surface is obtained by a quadratic twist construction from an extremal rational elliptic surface $j : J \to \mathbb{P}^1$ with reducible fibers of types $\tilde{A}_7$ and $\tilde{A}_1$. It can be realizes as the blow-up the base points of a pencil of cubic curves given in (4.9.14) but we will use different way to blow down. In notation of Figure 4.5, we blow down the exceptional configuration $E_0 + R_0 + \cdots + R_8$ to one base point $x_1$ of the pencil. The image of $R_8$ is a plane cubic $C$ with a node at $x_1$. We choose coordinates in a such a way that the equation of $C$ is $x_0x_1x_2 + x_1^3 + x_2^3 = 0$. All other cubics from the pencil are tangent to it. We choose the parametrization $x_0 = u^3 - v^3, x_1 = -vu^2, y = uv^2$ we find the conditions for a cubic intersect $C$ as above. This gives us the equation of the pencil

$$
C_{\lambda, \mu} = \lambda(x_1^3 - x_2^3 - x_0x_1x_2) + \mu(x_0^3x_1 + x_1^2x_2 + x_2^2x_0) = 0.
$$

The double cover of the rational elliptic surface defined by this pencil branched along two nonsingular fibers is a nonsingular model of a Cayley quartic symmetroid. If we choose the fibers $F_{\pm}$ corresponding to the parameters $[\lambda, \mu] = [1 \pm \sqrt{-1}, 1]$, then the lines $(1 \pm \sqrt{-1})x_1 + x_2 = 0$ define two bisections tangent to $F_{\pm}$. The corresponding quadratic twist defines an Enriques surface of type VII.

Assume $p = 5$. Then the points $[1,1,1,1,1]$ is a fixed points of the Cremona transformation, and it is also an ordinary double point of $X'$. When we resolve this point and take the quotient, we obtain a rational Coble surface. We will discuss such surfaces in the next section.

**Proposition 8.9.33.** The Nikulin $R$-invariant of an Enriques surface of type VII is equal to $(A_0 \oplus A_1, \mathbb{Z}/2\mathbb{Z})$. The quadratic space $\langle \text{Nod}(S) \rangle$ is a hyperplane of rank 8 in $\text{Num}(S)$. Since $\langle \text{Nod}(S) \rangle$ is the same as those of type I and type V, the Reye lattice is $E_8(4) \oplus A_1 \oplus A_1(-1)$.

**Proof.** The nodal cycle $E_1 + \cdots + E_8 + E_{12}$ is of type $A_9$ and the sublattice of $K$ generated by $\delta_{E_i}$ is isomorphic to $A_9$. Together with $\delta_{K_{i}}$, it generates a sublattice $M$ of $K$ isomorphic to $A_9 + A_1$.

**Proposition 8.9.34.** Recall that a surface of type VII has genus one fibrations of the following types (Proposition 8.9.28):

$$
\tilde{A}_8(20), \tilde{A}_7 + 2\tilde{A}_1(15), \tilde{A}_4 + \tilde{A}_4(6), \tilde{A}_5 + 2\tilde{A}_2 + \tilde{A}_1(10).
$$

The group $\text{Aut}(S)$ acts transitively on each group. The non-degeneracy invariant $\text{nd}(S)$ is equal to 10 and can be realized by taking 6 elliptic fibrations of type $A_4 + \tilde{A}_4$ and 4 elliptic fibrations $|K_1 + K_i|, i = 2, 3, 4, 5$. 

□
Unfortunately we were not able to find an explicit formula for a double plane model of a surface of type VII. Nevertheless, it is true that all such surfaces are isomorphic. We use an argument from [469]

**Proposition 8.9.35.** An Enriques surface with finite automorphism group of type VII is unique up to an isomorphism.

**Proof.** This was certainly true in the case \( \kappa = \mathbb{C} \). The proof follows from the knowledge of the Nikulin \( R \)-invariant of the surface. However, if \( p > 0 \), we need other argument to prove the assertion. Choose an elliptic fibration on \( S \) with a double fiber of type \( \tilde{A}_1 \). Then \( S \) is isomorphic to a torsor over the corresponding Jacobian fibration \( j : J \to \mathbb{P}^1 \). It is defined by a data \( (J_1, \epsilon, J_2, \epsilon) \), where \( J_1 \) is a smooth fiber, \( J_2 \) is fiber of type \( \tilde{A}_1 \), and \( \epsilon \) are non-trivial 2-torsion points components in \( \text{Pic}(J_i) \). The pair \( (F_2, \epsilon_2) \) is unique but there are three choices for \( \epsilon_1 \). However, the Mordell-Weil group acts transitively on non-trivial 2-torsion points on \( F_1 \). This shows that the data is unique up to an isomorphism, hence the torsor is unique up to an isomorphism. \( \square \)

Note that the argument from the proof can be used to prove the uniqueness of all surfaces of types III-VII and the dependence on one parameter for surfaces of types I and II.

Since we have not been able to find an explicit equation for a double plane model as in the previous cases, we have to prove the following.

**Proposition 8.9.36.** A surface of type VII does not exist if \( p = 5 \).

**Proof.** We take an elliptic fibration with two reducible fibers of type \( \tilde{A}_4 \). For example, we take the fibers \( E_1 + E_2 + E_{10} + E_7 + E_9 \) and \( E_4 + E_5 + E_6 + E_{13} + E_7 \). Then the permutation \( (13)(24) \) defines an automorphism \( g \) of the surface that permutes these two fibers. If \( p = 5 \), the Jacobian fibration has one irreducible singular cuspidal fiber. It must be invariant and define an isolated fixed point. The trace of the action of \( g^* \) on the \( l \)-adic cohomology \( H^r_{\text{et}}(S, \mathbb{Q}_l) \) is equal to 4, so, applying the Lefschetz fixed-point-formula we obtain \( e(S^g) = 4 \). On the other hand, there is another invariant fiber which must be smooth. The involution \( g \) has 4 fixed points on it or identical on it, and we get a contradiction. \( \square \)

### 8.10 Enriques surfaces with a finite automorphism group: \( p = 2 \)

In the section 8.9 we discussed Enriques surfaces with finite automorphism group in characteristic \( p \neq 2 \). Table 8.9 summarizes all the information about the existence and the number of parameters for these surfaces. In this section we deal with the case \( p = 2 \). Throughout this section we assume that \( p = 2 \).

First we check whether a surface with a crystallographic basis of type I-VII exist in characteristic 2, and then we discuss whether there are new types of crystallographic bases in characteristic 2.

We will use the following easy observation that follows from Theorem 4.10.3.
Lemma 8.10.1. Suppose $S$ has an elliptic fibration with a singular half-fiber of multiplicative type. Then $S$ is a $\mu_2$-surface. Moreover if there are two half-fibers of this type, then $S$ does not exist.

The crystallographic basis of $(-2)$-curves on the surface allows one to check whether a fiber is multiple or not and determine its type, however in some cases we cannot distinguish between the types $\tilde{A}_1$ (resp. $\tilde{A}_2$) or $\tilde{A}_1^*$ (resp. type $\tilde{A}_2^*$). The following corollary gives an easy conclusion.

Corollary 8.10.2. (1) A surface with a crystallographic basis of type I, II or VI may exist only as a $\mu_2$-surface.

(ii) A surface with a crystallographic basis of type III, IV or V does not exist.

(iii) A surface with a crystallographic basis of type VII cannot be a $\mu_2$-surface.

Proof. (i) By Proposition 8.9.6, a surface with a crystallographic basis of $(-2)$-curves of type I contains a genus one fibration with a double fiber of type $\tilde{A}_7$. So, it could exist only as a $\mu_2$-surface. By Proposition 8.9.9, a surface with a crystallographic basis of $(-2)$-curves of type II contains a genus one fibration with a double fiber of type $\tilde{A}_3$, so we have same conclusion. By Proposition 8.9.27, a surface with a crystallographic basis of $(-2)$-curves of type VI contains a genus one fibration with a double fiber of type $\tilde{A}_4$, so it could exist only as a $\mu_2$-surface.

(ii) By Propositions 8.9.13, 8.9.19, 8.9.23, a surface with a crystallographic basis of $(-2)$-curves of type III, IV, or V contains a genus one fibration with two double fibers, and hence it is classical. Moreover they have a genus one fibration with a half-fiber of multiplicative type. So they do not exist in characteristic 2.

(iii) By Proposition 8.9.34, a surface with a crystallographic basis of $(-2)$-curves of type VII contains a genus one fibration with reducible fibers of types $\tilde{A}_5$, $\tilde{A}_2(\tilde{A}_2^*)$ and $\tilde{A}_1(\tilde{A}_1^*)$ and the fiber of type $\tilde{A}_2(\tilde{A}_2^*)$ is a double fiber. However, the classification of extremal genus one fibrations on a rational surface shows that this fiber is of additive type $\tilde{A}_5^*$. So, the surface cannot be a $\mu_2$-surface. \qed

The next task is to realize surfaces with a crystallographic basis of type I, II, or VI as a $\mu_2$-surfaces.

Example 8.10.3. Assume that a surface contains a crystallographic basis of type I. Recall that, if $p \neq 2$, we have constructed a one-dimensional family of such surfaces as a double cover of the 4-nodal quartic del Pezzo surface $D_1$ branched along the union of two conics and a hyperplane section $C$ passing through the two intersection points. We also have two conics $Q_1 \in |e_0 - e_2|$ and $Q_2 \in |e_0 - e_4|$, each passing through one of the intersection points of $C_1$ with $C_2$.

To extend this construction to the case $p = 2$, we consider the anti-canonical quartic Del Pezzo surface $D_2$ instead of $D_1$. Let $C_1 \in |e_0 - e_1|$ and $C_2 \in |2e_0 - e_2 - e_3 - e_4 - e_5|$ be two conics, and $C$ be a hyperplane section passing through their two intersection points. We also consider two conics $Q_1, Q_2 \in |e_0 - e_2|$, each passing through one of the points $q_1, q_2$. Now, we consider the Artin-Schreier cover of $D_2$ locally given by equation $z^2_U + a_U z + b_U = 0$, where $(a_U)$ defines a section of $L \cong O_{D_2}(e_0 - e_3 - e_5)$ vanishing on $C_1 + C_2$ and $(b_U)$ define a section of $L^{\otimes 2}$ vanishing on $C + C_1 + C_2$. At each point $q_1, q_2$, the local equation of the cover is $z^2 + uvz + uv(u + v) = 0$. So, the cover has two singular points of
type $D_4^{(1)}$ over $q_1, q_2$. When we resolve then, the 8 exceptional curves and the pre-images of $C_1, C_2$ has the intersection diagram given in Theorem 8.2.21 (a). The additional two $(-2)$-curves are the pre-images of $Q_1$ and $Q_2$. Note that the elliptic fibration of type $2\tilde{A}_7 + \tilde{A}_1$ now becomes of type $2\tilde{A}_7 + A_1^\ast$.

Note that we can obtain the K3-cover of $S$ as an Artin-Schreier cover of the quadric $F_0$ as in [360, 4.1]. The surface $D_2$ is the quotient of $F_0$ by the involution $\sigma$ that acts as $(x, y) \mapsto (x^{-1}, y^{-1})$ in affine toric coordinates on $F_0$. The pre-image of the pencil $|e_0 - e_2|$ on $F_0$ is the pencil of conics that are tangent at the unique fixed point $(1, 1)$ of $\sigma$. Each passes through two opposite vertices of the triangle $T$ of lines equal to the pre-image of $C_1 + C_2$. The branch curve is the union of $T$ and the pre-image of $C$. We get the picture as in Figure 8.6. The only difference is that two conics $Q_1, Q_2$ are tangent at one point.

**Example 8.10.4.** Recall that in characteristic $p \neq 2$ a surface of Type II can be obtained as a double cover of $D_1$ branched along the union of two conics $C_1, C_2$ as in the previous example and a hyperplane section $C \in |-K_{D_1}|$ that is tangent to $C_1$ and $C_2$ at points $q_3$ and $q_4$ lying on two non-skew lines $L_1, L_2$ on $D_1$.

To extend this construction to characteristic 2 we consider, again as in the previous example, the surface $D_2$. We take the split Artin-Schreier cover of $D_2$ locally given by $z^2 + az + b = 0$, where $(a, b)$ is a section of $L$ vanishing on $C$ and $(b, a)$ is a section of $L \otimes 2$ vanishing on $C_1$ and $C_2$. Locally at the tangency points $q_1, q_2$ of $C$ with $C_1$ and $C_2$, the local equation is $z^2 + z + (u + v^2)u = 0$. In the first case, by the change of coordinates

$$t = z + \omega u + v^2, \quad s = z + \omega^2 u + v^2, \quad v = v$$

($\omega^3 = 1, \omega \neq 1$), then we have $v^4 + ts = 0$ which gives a rational double point of type $A_3$. This gives two singular points of type $A_3$ as in the case of the cover in characteristic $p \neq 2$. On the other hand, at a point of intersection of $C_1$ with $C_2$, we have a local equation $z^2 + uv = 0$. This gives two singular points of type $A_1$, again as in the case $p \neq 2$. Comparing with Figure 8.7, we find that the proper transforms of $L_1$ (resp. $L_2$) on $S$ is $R_9$ (resp. $R_5$). The proper transform of $C_1$ (resp. $C_2$) is $R_3$ (resp. $R_5$). The pre-image of the exceptional curve at $q_1$ (resp. $q_2$) is $R_1 + R_2 + R_{10}$ (resp. $R_6 + R_7 + R_{12}$). The pre-images of the exceptional curves at $q_3, q_4$ are $R_4$ and $R_{11}$. So all $(-2)$-curves are accounted for. As in the case $p \neq 2$, a choice of $C_1, C_2, C$ depends on 1-parameter.

We can also find the lift of the Arin-Schrier cover to the Artin-Schrei cover of $X \to F_0$, where $X \to S$ is the K3-cover of $S$ as it is constructed in [?]. The pre-images of curves $C_1, C_2, L_1, L_2, C$ are exhibited in Picture 8.9.

**Example 8.10.5.** To construct a $\mu_2$-surface with a crystallographic basis of type VI we use the same construction of this surface in characteristic $p \neq 2, 3, 5$ as a quotient of a Hessian surface. Although the Hessian surface of the Clebsch cubic surface degenerates, we can still consider the corresponding quartic surface given by equations

$$t_0 + \cdots + t_4 = \sum_{i=0}^{4} t_i^{-1} = 0.$$ 

The standard Cremona involution inversing the coordinates acts freely on this surface and the quotient is an Enriques surface. We discover the same configuration of $(-2)$ curves on it forming a crystallographic basis of type VI.
8.10. Enriques Surfaces with a Finite Automorphism Group: $P = 2$

In the following Table we summarize the existence or non-existence of Enriques surfaces of type I, II, ..., VII in characteristic 2.

<table>
<thead>
<tr>
<th>Type</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_2$-surface</td>
<td>✓</td>
<td>✓</td>
<td>●</td>
<td>●</td>
<td>✓</td>
<td>●</td>
<td>●</td>
</tr>
<tr>
<td>classical</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
</tr>
<tr>
<td>$\alpha_2$-surface</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
</tr>
</tbody>
</table>

Table 8.10: Surfaces of types I-VII in characteristic 2

In Table 8.10, $\bigcirc$ means the existence and $\times$ means the non-existence of an Enriques surface with the dual graph of type I, ..., VII.

The only lacking justification of this Table is the existence of classical and $\alpha_2$-surfaces with a crystallographic basis of type VII.

In any characteristic except classical and $\alpha_2$-Enriques surfaces in characteristic 2, Enriques surfaces of type I and II form a 1-dimensional family and each of other types is unique. On the other hand, classical and $\alpha_2$-Enriques surfaces of type VII form a 1-dimensional family in which a special fiber is $\alpha_2$-surface and other fibers are classical.

In the following, we introduce the classification of classical and $\alpha_2$-Enriques surfaces. Since any double fiber of a genus one fibration on classical or $\alpha_2$-Enriques surfaces are additive type contrary to the case of characteristic $p \neq 2$ or $\mu_2$-Enriques, possible crystallographic basis of $(-2)$-curves are completely different. Only the dual graph of type VII is common between classical/$\alpha_2$-surfaces and the other cases. On the other hand, in case of classical and $\alpha_2$-surfaces, the canonical covers have singularities which are invariants. We use the classification of possible conductrices due to Ekedahl and Shepherd-Barron [208]. We take a possible conductrix and fix it, and then determine possible crystallographic basis of $(-2)$-curves by using Proposition 8.9.1. Thus we have the following.

**Theorem 8.10.6.** Let $X$ be an $\alpha_2$-Enriques surface in characteristic 2.

1. $X$ has a finite group of automorphisms if and only if the dual graph of all $(-2)$-curves on $X$ is one of the graphs in Table 8.11 (A).

2. There exist families of these surfaces with the automorphism groups and dimensions given in Table 8.11.

**Theorem 8.10.7.** Let $X$ be a classical Enriques surface in characteristic 2.

1. $X$ has a finite group of automorphisms if and only if the dual graph of all $(-2)$-curves on $X$ is one of the graphs in Table 8.12 (A).

2. There exist families of these surfaces with the automorphism groups and dimensions given in Table 8.12 (B).
### Table 8.11: \(\alpha_2\)-Enriques surfaces with finite automorphism group

<table>
<thead>
<tr>
<th>Type</th>
<th>Dual Graph of ((-2))-curves</th>
<th>(\text{Aut}(X))</th>
<th>(\text{Aut}_{\text{ct}}(X))</th>
<th>(\text{dim})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\tilde{E}_8)</td>
<td><img src="image" alt="Dual Graph" /></td>
<td>(\mathbb{Z}/11\mathbb{Z})</td>
<td>(\mathbb{Z}/11\mathbb{Z})</td>
<td>0</td>
</tr>
<tr>
<td>(\tilde{E}_7^2)</td>
<td><img src="image" alt="Dual Graph" /></td>
<td>(\mathbb{Z}/2\mathbb{Z}) or (\mathbb{Z}/14\mathbb{Z})</td>
<td>({1}) or (\mathbb{Z}/7\mathbb{Z})</td>
<td>1 or 0</td>
</tr>
<tr>
<td>(\tilde{E}_6 + \tilde{A}_2)</td>
<td><img src="image" alt="Dual Graph" /></td>
<td>(\mathbb{Z}/5\mathbb{Z} \times \mathfrak{S}_3)</td>
<td>(\mathbb{Z}/5\mathbb{Z})</td>
<td>0</td>
</tr>
<tr>
<td>(\tilde{D}_8)</td>
<td><img src="image" alt="Dual Graph" /></td>
<td>(\mathbb{Q}_8)</td>
<td>(\mathbb{Q}_8)</td>
<td>1</td>
</tr>
<tr>
<td>VII</td>
<td><img src="image" alt="Dual Graph" /></td>
<td>(\mathfrak{S}_5)</td>
<td>({1})</td>
<td>0</td>
</tr>
</tbody>
</table>
### 8.10. Enriques Surfaces with a Finite Automorphism Group: $P = 2$

<table>
<thead>
<tr>
<th>Type</th>
<th>Dual Graph of $(-2)$-curves</th>
<th>$\text{Aut}(\tilde{X})$</th>
<th>$\text{Aut}_{\text{int}}$</th>
<th>Dim</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{E}_8$</td>
<td><img src="image" alt="Graph" /></td>
<td>${1}$</td>
<td>${1}$</td>
<td>1</td>
</tr>
<tr>
<td>$\tilde{E}_7^{2}$</td>
<td><img src="image" alt="Graph" /></td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
<td>${1}$</td>
<td>2</td>
</tr>
<tr>
<td>$\tilde{E}_7^{1}$</td>
<td><img src="image" alt="Graph" /></td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
<td>1</td>
</tr>
<tr>
<td>$\tilde{E}_6 + \tilde{A}_2$</td>
<td><img src="image" alt="Graph" /></td>
<td>$S_3$</td>
<td>${1}$</td>
<td>0</td>
</tr>
<tr>
<td>$\tilde{D}_8$</td>
<td><img src="image" alt="Graph" /></td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
<td>1</td>
</tr>
<tr>
<td>$\tilde{D}_4 + \tilde{D}_4$</td>
<td><img src="image" alt="Graph" /></td>
<td>$(\mathbb{Z}/2\mathbb{Z})^3$</td>
<td>$(\mathbb{Z}/2\mathbb{Z})^2$</td>
<td>2</td>
</tr>
<tr>
<td>VII</td>
<td><img src="image" alt="Graph" /></td>
<td>$S_5$</td>
<td>${1}$</td>
<td>1</td>
</tr>
<tr>
<td>VIII</td>
<td><img src="image" alt="Graph" /></td>
<td>$S_4$</td>
<td>${1}$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 8.12: Classical Enriques surfaces with finite automorphism group
The last three column only give examples, no classification is known. Also we remark that the examples of \( \alpha_2 \)-Enriques surfaces of type \( \tilde{E}_7^2 \) form a 1-dimensional family, but some of their automorphism groups jump up.

In the following we will give examples of classical and \( \alpha_2 \)-surfaces given in Theorems 8.10.6, 8.10.7. The method is as follows. First we take a special genus one fibration on the dual graphs and consider a genus one fibration \( \pi \) on a rational surface with the same singular fibers. Then by taking the Frobenius base change of \( \pi \) we get a new genus one fibration \( \tilde{\pi} \). We define a suitable rational derivation on the total space of \( \tilde{\pi} \) whose quotient is birational to the desired Enriques surface.

**Example 8.10.8.**: Type VII.

First we consider surfaces of type VII. Recall that an Enriques surface with a crystallographic basis of \( (−2) \)-curves of type VII has an elliptic fibration with singular fibers of type \( \tilde{A}_4 + \tilde{A}_4 \) and with ten bi-sections. For example, the linear system

\[
|2(E_1 + E_2 + E_{10} + E_{12} + E_9)| = |2(E_4 + E_5 + E_6 + E_7 + E_{13})|
\]

gives such a fibration with bi-sections \( E_3, E_8, E_{11}, E_{14}, E_{15}, K_1, \ldots, K_5 \) in Figure 8.16.

Now assume that a classical or \( \alpha_2 \)-Enriques surface \( S \) in characteristic 2 contains fifteen \( (−2) \)-curves as in the Figure 8.18. Here lines in the figure mean \( (−2) \)-curves on \( S \). Each \( (−2) \)-curve passes through two dotted points (we will discuss this fact in Lemma 10.2.9 later). Note that there exist two disjoint pentagons of \( (−2) \)-curves which form two singular fibers of type \( \tilde{A}_4 \) of an elliptic fibration \( f : S \to \mathbb{P}^1 \), and the remaining five \( (−2) \)-curves are bi-sections of this fibration. The dual graph of these fifteen \( (−2) \)-curves on \( S \) coincides with the one of \( E_1, \ldots, E_{15} \) in Figure 8.16.

![Figure 8.18: Fifteen (−2)-curves](image)

The fibration \( f \) has singular fibers of type \( \tilde{A}_4 + \tilde{A}_4 + \tilde{A}_0^5 + \tilde{A}_0^* \) (Part I, Table 4.7) and the twelve singular points of singular fibers are the images of the twelve singular points of type \( \tilde{A}_1 \) of the canonical cover \( \pi : X \to S \) (see Corollary 10.2.7). Assume that there exist additionally five \( (−2) \)-curves which are bi-sections of the fibration. Moreover we assume that they pass through two singular points of singular fibers of type \( \tilde{A}_0^* \) and meet together at only these two points transversely (see Lemma 10.2.9). Then their dual graph coincides with the complete graph with double edges in Figure 8.16 defined by \( (−2) \)-curves \( K_1, \ldots, K_5 \). Thus if the additional five \( (−2) \)-curves meet with fifteen \( (−2) \)-curves in Figure 8.18 correctly, then we have an Enriques surface with the crystallographic basis of type VII.

Now let \( Y \) be the minimal resolution of singularities of the canonical cover \( X \) of \( S \) and \( R_i (1 \leq \ldots) \).
the exceptional curves. Then the fibration $f$ induces an elliptic fibration $p : Y \to \mathbb{P}^1$ which has two singular fibers of type $\tilde{A}_9$ and two of type $\tilde{A}_1$, and $R_i$ are components of these singular fibers. Moreover $p$ has ten sections which are pullback of ten bi-sections of $f$. Then Shioda-Tate formula implies that $Y$ is the supersingular K3 surface with Artin invariant 1. Instead of the canonical cover $X$ of $S$, we consider a rational divisorial derivation $\partial$ on $Y$ such that the quotient $Y^\partial$ of $Y$ by $\partial$ is the blowing-ups twelve points on $S$ which are the images of singular points of type $A_1$ on $X$. We denote by $\varphi : Y^\partial \to S$ the contraction morphism. Since $K_Y$ and $K_S$ are numerically trivial, Proposition 0.3.14 implies that

$$0 = K_Y = \pi^*(K_{Y^\partial}) + D = \pi^*(\varphi^*(K_S) + \sum_{i=1}^{12} R_i) + D = \pi^*\varphi^*(\sum R_i) + D, \quad (8.10.1)$$

where $D$ is the divisor of the derivation $\partial$. Since $\pi(R_i)^2 = -1$, by Proposition 0.3.19, $R_i$ is integral with respect to $D$. Therefore $\pi^*(\pi(R_i)) = R_i$ and hence $\partial$ has poles along each $R_i$ with multiplicity 1. Thus we should first find an elliptic fibration $p : Y \to \mathbb{P}^1$ on the supersingular K3 surface $Y$ with Artin invariant 1 which has singular fibers of type $\tilde{A}_9 + \tilde{A}_9 + \tilde{A}_1 + \tilde{A}_1$ and ten sections. Next find a rational divisorial vector field $\partial$ on $Y$ such that twelve disjoint $(-2)$-curves $R_i$ in fibers are integral and $D = -\sum_{i=1}^{12} R_i$. In Chapter 10, we will discuss Enriques surfaces in characteristic 2 whose canonical covers are birational to supersingular K3 surfaces. For the divisor $D$, see Lemma 10.3.13.

To get such an Enriques surface, we start with a rational elliptic surface with Weierstrass equation

$$y^2 + sx'y + y + x^3 + x^2 + s = 0.$$

The change of variables $(x, y, s) \mapsto (x + t + 1, y, s = t + 1)$ transforms this equation to the Weierstrass equation from Table 4.7 for an elliptic fibration with two fibers of type $\tilde{A}_4$ and two irreducible singular fibers $\tilde{A}_9$. We see that the discriminant is equal to $(s + 1)^5(s^2 + s + 1)$ and the $j$-invariant is equal to $\frac{s^{12}}{(s+1)^5(s^2+s+1)}$. It has two reducible fibers of type $\tilde{A}_4$ over $s = 1, \infty$ and two singular fibers of type $\tilde{A}_9$ over $s = \omega, \omega^2$, where $\omega^3 = 1, \omega \neq 1$. By taking the Frobenius base change $t^2 = s$, we have an elliptic fibration with Weierstrass equation

$$y^2 + t^2xy + y + x^3 + x^2 + t^2 = 0. \quad (8.10.2)$$

Let $p : Y \to \mathbb{P}^1$ be the relatively minimal non-singular model of the fibration (8.10.2). The discriminant of the elliptic surface $Y$ is equal to $h = (t + 1)^{10}(t^2 + t + 1)^2$ and the $j$-invariant is given by $j = t^{24}/(t + 1)^{10}(t^2 + t + 1)^2$.

Hence the elliptic fibration $p$ has two reducible fibers of type $\tilde{A}_9$ over $t = 1, \infty$ and two singular fibers of type $\tilde{A}_1$ over $t = \omega, \omega^2$. The relatively minimal elliptic surface $p : Y \to \mathbb{P}^1$ defined by

$$i \leq 12$$
(8.10.2) has ten sections \( s_i, m_i \) \((i = 0, 1, 2, 3, 4)\) given as follows:

\[
\begin{align*}
\text{s}_0 & : \text{the zero section} \\
\text{s}_1 & : x = 1, y = t^2 \\
\text{s}_2 & : x = t^2, y = t^2 \\
\text{s}_3 & : x = t^2, y = t^4 + t^2 + 1 \\
\text{s}_4 & : x = 1, y = 1 \\
\text{m}_0 & : x = \frac{1}{t^2}, y = \frac{1}{t^2} + \frac{1}{t^2} + t \\
\text{m}_1 & : x = t^3 + t + 1, y = t^4 + t^3 + t \\
\text{m}_2 & : x = t, y = t^3 \\
\text{m}_3 & : x = t, y = 1 \\
\text{m}_4 & : x = t^3 + t + 1, y = t^5 + t^4 + t^2 + t + 1
\end{align*}
\]

Then Shioda-Tate formula implies that \( Y \) is the supersingular K3 surface with Artin invariant 1.

The Weierstrass model \( W \) is singular at the point \( P = (x, y, t) = (1, 1, 1) \) and the fiber \( F_1 \) over \( t = 1 \) is an irreducible nodal curve. The singular points is a rational double points of type \( A_8 \) and the fiber of the resolution of singularities map \( Y \to W \) over this point consists of nine \((-2)\) curves \( E_{1,i} \) \((i = 1, 2, \ldots, 9)\). We index the components of the decagon \( F_1 E_{1,1} E_{1,2} \ldots E_{1,9} \) in a clockwise manner. Here we denote by the same symbol \( F_1 \) the proper transform of \( F_1 \) on \( Y \). The blowing-up at the singular point \( P \) gives two exceptional curves \( E_{1,1} \) and \( E_{1,9} \), and they intersect each other at a singular point of the obtained surface. The blowing-up at the singular point again gives two exceptional curves \( E_{1,2} \) and \( E_{1,8} \). The exceptional curve \( E_{1,2} \) (resp. \( E_{1,8} \)) intersects \( E_{1,1} \) (resp. \( E_{1,9} \)) transversely. Exceptional curves \( E_{1,2} \) and \( E_{1,8} \) intersect each other at a singular point, and so on. By successive blowing-ups, the exceptional curve \( E_{1,5} \) finally appears to complete the resolution of singularity at the point \( P \), and it intersects \( E_{1,4} \) and \( E_{1,6} \) transversely. Summarizing these results, we see that \( F_1 \) intersects \( E_{1,1} \) and \( E_{1,9} \) transversely, and that \( E_{1,i} \) intersects \( E_{1,i+1} \) \((i = 1, 2, \ldots, 8)\) transversely. We choose \( E_{1,1} \) to be the irreducible component which intersects the section \( m_2 \).

Let \( F_\infty \) be the fiber over the point defined by \( t = \infty \). The open subset of \( W \) over this point is given by equation

\[
y^2 + xy + t^6y + x^3 + t^4x^2 + t^{10} = 0,
\]

where \( t' = 1/t \). Similarly we have nine exceptional curves \( E_{\infty,i} \) \((i = 1, 2, \ldots, 9)\) over the singular point \( P_\infty = [x, y, t'] = [0, 0, 0] \) of the surface (8.10.2), and \( F_\infty \) and these 9 exceptional curves make a decagon \( F_\infty E_{\infty,1} E_{\infty,2} \ldots E_{\infty,9} \) clockwise. \( F_\infty \) intersects \( E_{\infty,1} \) and \( E_{\infty,9} \) transversely, and that \( E_{\infty,i} \) intersects \( E_{\infty,i+1} \) \((i = 1, 2, \ldots, 8)\) transversely.

The singular fiber of \( p : Y \to \mathbb{P}^1 \) over the point defined by \( t = \omega \) (resp. \( t = \omega^2 \)) consists of two irreducible components \( F_\omega \) and \( E_\omega \) (resp. \( F_{\omega^2} \) and \( E_{\omega^2} \)), where \( F_\omega \) (resp. \( F_{\omega^2} \)) is the proper transform of the fiber over the point \( t = \omega \) (resp. \( t = \omega^2 \)).

Then, the 10 sections above intersect singular fibers of elliptic surface \( p : Y \to \mathbb{P}^1 \) as follows:

Now consider a rational derivation \( \partial = \partial_{\alpha,\beta} \) defined by

\[
\partial = \frac{1}{(t + 1)} \left( (t + 1)(t + \alpha)(t + \beta) \frac{\partial}{\partial t} + (1 + t^2x) \frac{\partial}{\partial x} \right)
\]
where \( \alpha, \beta \in \mathbb{k}, \alpha + \beta = \alpha \beta, \alpha \neq 1 \). Note that the invariant differential form \( \omega \) on an elliptic curve given in the Weierstrass form
\[
y^2 + a_1y + a_3y + x^3 + a_2x^2 + a_4x + a_6 = 0
\]
is given by
\[
\omega = \frac{dx}{2y - a_1x + a_3} = \frac{dy}{a_1y + 3x^2 + 2a_2x + a_4}
\]
This shows that the restriction of the second summand \( \delta_2 \) of \( \partial \) to the general fiber coincides with the invariant differential form on it. So, we modify this form by adding the first summand \( \delta_1 \).

**Lemma 8.10.9.** (i) \( \partial^2 = \alpha \beta \partial \), namely, \( \partial \) is 2-closed and \( \partial \) is of additive type if \( \alpha = \beta = 0 \) and of multiplicative type otherwise.

(ii) On the surface \( Y \), the divisorial part \( D \) of \( \partial \) is given by
\[
D = -\left( F_1 + F_\infty + \sum_{i=1}^{4} (E_{1,2i} + E_{\infty,2i}) + E_\omega + E_{\omega^2} \right)
\]
and \( D^2 = -24 \).

(iii) The integral curves with respect to \( \partial \) in the fibers of \( p : Y \to \mathbb{P}^1 \) are the following:
the smooth fibers over \( t = \alpha, \beta \) (in case \( \alpha = \beta = 0 \), the smooth fiber over \( t = 0 \)) and
\[
F_1, F_\infty, E_{1,2i}, E_{\infty,2i} (1 \leq i \leq 4), E_\omega, E_{\omega^2}.
\]

**Proof.** We check this statement only on the complement of the fiber over \( \infty \). First change the parameters \( u = t + 1, x = X + 1, y = Y + 1 \), so the equation of the Weierstrass surface \( W \) becomes
\[
F = Y^2 + XY(u^2 + 1) + u^2Y + X^3 + u^2X = 0.
\]
The singular point now is \((0, 0, 0)\). Blowing up this point, we introduce new coordinates \( X = uX', Y = uY' \) and get a new equation
\[
F' = X'Y' + Y'^2 + u^2X'Y' + u(X'^3 + X' + Y') = 0.
\]
We see that the exceptional divisor \( u = 0 \) is given by \( Y'(X' + Y') = 0 \) and consists, as was observed before, of two components intersecting at a singular point. In new coordinates,
\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial u}, \quad \frac{\partial}{\partial X} = u^{-1} \frac{\partial}{\partial X'}
\]
and \( \frac{1}{u} (1 + t^2 x^2) \frac{\partial}{\partial X} \) transforms to \( eu^{-1} \frac{\partial}{\partial X'} \), where \( e \) is a unit in a neighborhood of the new singular point. Differentiating \( F' \) in \( X' \), we find that it has pole of order 1 at the component given by \( Y' = 0 \). We choose this component for \( E_{1,2} \). Multiplying \( \partial \) by \( u \) we see that \( \partial \) is equivalent to a vector

<table>
<thead>
<tr>
<th>sections</th>
<th>( s_0 )</th>
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<th>( s_2 )</th>
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<th>( s_4 )</th>
<th>( m_0 )</th>
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</tr>
</thead>
<tbody>
<tr>
<td>( t = 1 )</td>
<td>( F_1 )</td>
<td>( E_{1,8} )</td>
<td>( E_{1,6} )</td>
<td>( E_{1,4} )</td>
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<td>( E_{1,3} )</td>
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<td>( E_{1,9} )</td>
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<tr>
<td>( t = \infty )</td>
<td>( F_\infty )</td>
<td>( E_{\infty,6} )</td>
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Table 8.13: Intersection between sections and fibers
field whose normal component along the component $F_1, E_{1,2i}, E_\omega, E_{\omega^2}$ is equal to zero. Thus these components are integral curves of $\partial$.

Similarly, we deal with singular points $(\omega, 1, \omega)$ and find that $D$ has pole of order 1 on the exceptional curve of the minimal resolution.

By Proposition 0.3.18, we have
\[ c_2(Y) = \deg(Z) - K_Y \cdot D - D^2 \]
where $\langle Z \rangle$ is the scheme of non-divisorial zeros of $\partial$. Hence $Z = 0$ and the quotient surface $Y^{\partial_{\alpha, \beta}}$ is non-singular (Theorem 0.3.9). It has twelve exceptional curves of the first kind which are the images of the above integral $(-2)$-curves. By contracting these curves we get a non-singular surface $S_{\alpha, \beta}$. It follows from the formula (8.10.1) that $K_S$ is numerically trivial. Since the quotient morphism $Y \to Y^{\partial}$ is finite and purely inseparable, $b_2(Y^{\partial}) = b_2(Y) = 22$ and hence $b_2(S_{\alpha, \beta}) + 12 = 22$. Thus $S_{\alpha, \beta}$ is an Enriques surface.

The elliptic fibration $p : Y \to \mathbb{P}^1$ induces an elliptic fibration $f : S_{\alpha, \beta} \to \mathbb{P}^1$ which has two singular fibers of type $\tilde{A}_4$ and two singular fiber of type $\tilde{A}_0^*$. If $a \neq 0$, then the images of two smooth integral curves are double fibers of the elliptic fibration, and hence $S_{\alpha, \beta}$ is classical. If $a = b = 0$, then the fibration has one double fiber, and hence $S_{\alpha, \beta}$ is an $\alpha_2$-surface (it can not be a $\mu_2$-surface because the canonical cover is supersingular). The surface $S_{\alpha, \beta}$ contains 20 $(-2)$-curves which are the images of ten components of the fibers of $p$ over $t = 0, \infty$ and ten sections. The incidence relation between components of fibers of $p$ and sections given in Table 8.13 shows that the dual graph of twenty $(-2)$-curves on $S_{\alpha, \beta}$ is the crystallographic basis of type VII. There exist twelve points on $S_{\alpha, \beta}$ which are the images of twelve integral $(-2)$-curves. Each $(-2)$-curve on $S_{\alpha, \beta}$ passes through two points from the twelve points (see Figure 8.18).

Thus we have proved the following theorem.

**Theorem 8.10.10.** There exists a 1-dimensional family $\{S_{\alpha, \beta}\}$ of classical and $\alpha_2$-Enriques surfaces $S_{\alpha, \beta}$ where $\alpha, \beta \in k$, $\alpha + \beta = \alpha \beta$, $\alpha^3 \neq 1 \neq \beta^3$. The K3-cover of $S_{\alpha, \beta}$ has twelve nodes and the resolution of singularities is the supersingular K3 surface $Y$ with Artin invariant 1. If $\alpha = \beta = 0$, then $S_{\alpha, \beta}$ is an $\alpha_2$-surface, and otherwise classical. Each $S_{\alpha, \beta}$ contains 20 $(-2)$-curves whose dual graph coincides with the crystallographic basis of type VII. In particular the automorphism group $\text{Aut}(S_{\alpha, \beta})$ is finite and isomorphic to $\tilde{S}_5$.

**Proposition 8.10.11.** There exist exactly four types of elliptic fibrations on $S$ as follows:

\[ \tilde{A}_4 + \tilde{A}_4 + \tilde{A}_0^* + \tilde{A}_0^*, \quad \tilde{A}_5 + \tilde{A}_1 + 2\tilde{A}_2^*, \quad \tilde{A}_7 + 2\tilde{A}_1^*, \quad \tilde{A}_8 + \tilde{A}_0^* + \tilde{A}_0^* + \tilde{A}_0^*. \]

Compare it with Proposition 8.9.28.

Next we discuss the problem of the existence of crystallographic bases of $(-2)$-surfaces of different types. Of course, we already know some of them: they contain less than 12 $(-2)$-curves and are classified in Theorem 6.2.3. Three of them are realized for extra-special surfaces of type $\tilde{E}_8, \tilde{D}_8$ and $\tilde{E}_7$ with $\text{nd}(S) \leq 2$.

**Example 8.10.12.** Type $\tilde{E}_6 + \tilde{A}_2$: $\alpha_2$-surfaces.
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We consider a rational elliptic surface \(\pi : R \to \mathbb{P}^1\) associated with the Weierstrass model
\[
y^2 + sy = x^3
\]
which has a singular fiber of type \(\tilde{A}^*_2\) over \(s = 0\) and of type \(\tilde{E}_6\) over \(s = \infty\). By taking the Frobenius base change \(t^2 = s\), we have a rational elliptic surface \(\tilde{\pi} : \tilde{R} \to \mathbb{P}^1\) associated with
\[
y^2 + t^2y = x^3
\]
which has a singular fiber of type \(\tilde{E}_6\) over \(t = 0\) and of type \(\tilde{A}^*_2\) over \(t = \infty\). Note that \(\pi\) and \(\tilde{\pi}\) are isomorphic by changing the coordinate \(x' = x/s^2, y' = y/s^3, s = 1/t\). Let \(E_{0,1} + E_{0,2} + E_{0,3} + 2(E_{0,4} + E_{0,5} + E_{0,6}) + 3E_{0,7}\) be the singular fiber of \(\tilde{\pi}\) of type \(\tilde{E}_6\) and let \(E_{\infty,1} + E_{\infty,2} + E_{\infty,3}\) be the singular fiber of type \(\tilde{A}^*_2\). We assume that \(E_{0,i}\) meets with \(E_{0,i+3}\) \((i = 1, 2, 3)\). There exist three sections denoted by \(S_1, S_2, S_3\) which are not meeting together. We assume that \(S_i\) intersects \(E_{0,i}\) and \(E_{\infty,i}\).

Now we discuss what kind of a derivation we need. Assume that \(\tilde{R}\) is birational to a resolution of the normalization of the canonical cover of the desired Enriques surface \(S\). Then \(\tilde{\pi}\) should induce a special elliptic fibration on \(S\) with singular fibers of type \(\tilde{A}^*_2\) over the point corresponding to \(s = 0\) and of type \(\tilde{E}_6\) over the point corresponding to \(s = \infty\). This means that four components of the fiber of type \(\tilde{E}_6\) over \(t = 0\) map to a point on \(S\) to get a singular fiber of type \(\tilde{A}^*_2\) and the fiber of \(\tilde{A}^*_2\) over \(t = \infty\) should be blowing up to get a fiber of type \(\tilde{E}_6\) on \(S\). So we first blow up \(\tilde{R}\) at the singular point of the singular fiber of type \(\tilde{A}^*_2\) and denote by \(E_{\infty,4}\) the exceptional curve. Then blow up the three points which are the intersection of \(E_{\infty,4}\) and the proper transforms of three components of the original fiber. Denote by \(E_{\infty,5}, E_{\infty,6}, E_{\infty,7}\) the three exceptional curves and by the same symbols \(E_{\infty,1}, E_{\infty,2}, E_{\infty,3}\) their proper transforms under these blow-ups. We assume that \(E_{\infty,i}\) meets \(E_{\infty,i+4}\) \((i = 1, 2, 3)\). Then
\[
E_{\infty,1}^2 = E_{\infty,2}^2 = E_{\infty,3}^2 = E_{\infty,4}^2 = -4, \quad E_{\infty,5}^2 = E_{\infty,6}^2 = E_{\infty,7}^2 = -1.
\]
Also
\[
E_{0,i}^2 = -2 \quad (i = 1, 2, \ldots, 7), \quad S_1^2 = S_2^2 = S_3^2 = -1.
\]
We denote by \(Y\) the obtained surface by these blowing ups.

Now assume that there exists a derivation \(\partial\) on \(Y\) without isolated zeros such that the integral curves with respect to \(\partial\) are given by
\[
E_{0,4}, E_{0,5}, E_{0,6}, E_{\infty,1}, E_{\infty,2}, E_{\infty,3}, E_{\infty,4}. \tag{8.10.3}
\]
Then by taking the quotient by \(\partial\) we get a smooth surface \(Y^\partial\) and a purely inseparable double covering \(p : Y^\partial \to Y^\partial\). Since \(2(p(C))^2 = C^2\) if \(C\) is integral and \((p(C))^2 = 2C^2\) otherwise, we have
\[
E_{\infty,i}^2 = S_j^2 = -2 \quad (i = 1, 2, \ldots, 7; j = 1, 2, 3),
\]
\[
E_{0,k}^2 = -4 \quad (k = 1, 2, 3, 7), \quad E_{0,l}^2 = -1 \quad (l = 4, 5, 6).
\]
Now contract three \((-1)\)-curves \(E_{0,4}, E_{0,5}, E_{0,6}\) and then contract the image of \(E_{0,7}\). Then we have a smooth surface \(S\) and the contraction map \(q : Y^\partial \to S\). By construction, the corresponding elliptic fibration has a singular fiber of type \(\tilde{A}^*_2\) consisting of the images of \(E_{0,1}, E_{0,2}, E_{0,3}\) and a singular fiber of type \(\tilde{E}_6\). Since \(S_i, E_{0,i}, i = 1, 2, 3\) are non-integral and \(E_{\infty,i}, i = 1, 2, 3\) are integral, the images of \(S_i\) are tangent to \(E_{0,i}\) at one point and \(S_i\) meets \(E_{\infty,i}\) at one point transversally. Therefore
Assume that $S$ is an Enriques surface and hence $\deg(\partial)$ is double. Then $\partial$ is 2-closed. We denote by the same symbol $\partial$ the induced derivation on $Y$. It follows from local calculations that

$$K_Y = q^*(K_S) + 2(E_{0,4} + E_{0,5} + E_{0,6}) + E_{0,7}.$$  

Since $E_{0,4}, E_{0,5}, E_{0,6}$ are integral and $E_{0,7}$ is not, it follows that

$$K_Y = p^*K_Y + D = p^*q^*(K_S) + 2(E_{0,4} + E_{0,5} + E_{0,6} + E_{0,7}) + D$$

where $D$ is the divisor of $\partial$. On the other hand, $Y$ is obtained by blowing ups 4 points of the rational elliptic surface and hence

$$K_Y = -(E_{\infty,1} + E_{\infty,2} + E_{\infty,3}) - 2(E_{\infty,4} + E_{\infty,5} + E_{\infty,6} + E_{\infty,7}).$$

Assume that $S$ is an Enriques surface. Then $p^*q^*(K_S) = 0$. Thus we have an another condition

$$D = -2(E_{0,4} + E_{0,5} + E_{0,6} + E_{0,7} + E_{\infty,4} + E_{\infty,5} + E_{\infty,6} + E_{\infty,7}) - (E_{\infty,1} + E_{\infty,2} + E_{\infty,3}).$$  

(8.10.4)

We conclude that $\partial$ should satisfy the two conditions (8.10.3) and (8.10.4) to get a desired Enriques surface $S$.

Now we consider the following derivation

$$\partial = \frac{\partial}{\partial t} + t^2 \frac{\partial}{\partial x}.$$  

Then $\partial^2 = 0$, that is, $\partial$ is 2-closed. We denote by the same symbol $\partial$ the induced derivation on $Y$. It follows from local calculations that

**Lemma 8.10.13.** (1) The derivation $\partial$ on $Y$ satisfies the two conditions (8.10.3) and (8.10.4).

(2) $D^2 = -12$, $(D, K_Y) = -4$.

**Remark 8.10.14.** We blow up $\tilde{R}$ at four points. These points are nothing but the isolated singular points of the derivation on $\tilde{R}$ induced by $\partial$.

**Lemma 8.10.15.** The derivation $\partial$ is divisorial.

**Proof.** Let $Z$ be the scheme of non-divisorial zeros of $\partial$. By using the facts $c_2(Y) = 16$ and $D^2 = -12$, $(D, K_Y) = -4$, we have

$$16 = c_2(Y) = \deg(Z) - (K_Y, D) - D^2 = \deg(Z) + 16.$$  

Hence $\deg(Z) = 0$ which means $\partial$ is divisorial.  

It follows that $Y^{\partial}$ is smooth. Then applying the previous argument we can see $S$ is the desired Enriques surface. Since $\partial^2 = 0$, $S$ is an $\alpha_2$-surface.

**Theorem 8.10.16.** The surface $S$ is an $\alpha_2$-Enriques surface with a crystallographic basis of type $\tilde{E}_6 + A_2$.

**Proposition 8.10.17.** There are exactly one elliptic fibration with singular fibers of type $2\tilde{E}_6 + \tilde{A}_2^*$ and three quasi-elliptic fibrations with singular fibers of type $\tilde{E}_7 + 2\tilde{A}_1^*$.  

$S_i$ are bisections of the fibration, the fiber of type $\tilde{E}_6$ is double and the one of type $\tilde{A}_2^*$ is simple. Thus we have 13 $(-2)$-curves whose dual graph is a crystallographic basis of type $\tilde{E}_6 + \tilde{A}_2$ in Theorem 8.10.6. Since the Euler number of $\tilde{R}$ is 12 and $p$ is purely inseparable double cover, the Euler number of $Y$ and $Y^{\partial}$ is 16 and hence that of $S$ is 12. Obviously

$$K_{Y^{\partial}} = q^*(K_S) + 2(E_{0,4} + E_{0,5} + E_{0,6}) + E_{0,7}.$$
**Proof.** For the elliptic fibration the assertion is clear. Note that both $S_1$ and $E_{0,1}$ are non-integral and meet at one point transversally on $Y$, their images on $S$ are tangent together. The twice of the sum of these two curves defines a genus one fibration with a double fiber of type $\tilde{A}_1^\ast$. By the classification of extremal genus one fibrations, this fibration is quasi-elliptic.

**Remark 8.10.18.** To construct $S$ we started from the elliptic surface $y^2 + t^2y = x^3$ and the derivation $\partial = \frac{\partial}{\partial t} + t^2 \frac{\partial}{\partial x}$. Then we can determine the function field $k(x, y, t)^{\partial}$ of the quotient surface and have the following equation of the surface birationally isomorphic to $S$:

$$Y^2 + TY + TX^4 + X^3 + T^3X + T^7 = 0.$$ (8.10.5)

For more details we refer the reader to [363].

**Theorem 8.10.19.** The automorphism group $\text{Aut}(S)$ is isomorphic to $\mathbb{S}_3 \times \mathbb{Z}/5\mathbb{Z}$ in which $\mathbb{Z}/5\mathbb{Z}$ is cohomologically trivial.

**Proof.** Note that the symmetry group of the dual graph of $(-2)$-curves is isomorphic to the symmetric group $\mathbb{S}_3$ of degree 3. It is easy to prove the existence of automorphisms generating $\mathbb{S}_3$ by using the Jacobian fibrations of genus one fibrations on $S$. On the other hand we can determine the automorphism group of the surface defined by 8.10.5 preserving the fibration defined by the projection $(X, Y, T) \to T$ as follows:

$$\varphi: (X, Y, T) \to (\zeta^4X, \zeta Y, \zeta T), \quad \psi: (X, Y, T) \to (X, Y + T, T)$$

where $\zeta$ is a primitive 5-th root of unity. Then $\varphi, \psi$ and an automorphism of order 3 in $\mathbb{S}_3$ generate $\mathbb{S}_3 \times \mathbb{Z}/5\mathbb{Z}$. There are no symmetries of order 5 of the dual graph of $(-2)$-curves, $\varphi$ is cohomologically trivial.

**Remark 8.10.20.** By construction, the canonical cover of $S$ has a rational double point of type $D_4$. On the other hand, any bisection of a quasi-elliptic fibration on an Enriques surface is contained in the conductrix. Therefore the conductrix of $S$ is non-empty. In fact the support of the conductrix is the union of 7 components of the singular fiber of type $\tilde{E}_6$. In our construction we take a non-singular model of the normalization of the canonical cover of $S$, and hence the canonical cover itself does not appear here.

**Example 8.10.21.** **Type $\tilde{E}_6 + \tilde{A}_2$: Classical Case.**

In this case we start with a rational elliptic surface associated with the Weierstrass model

$$y^2 + xy + sy + x^3 = 0$$

which has a singular fiber of type $\tilde{A}_2$ over $s = 0$, a singular fiber of type $\tilde{A}_0^\ast$ over $s = 1$ and a singular fiber of type $\tilde{E}_6$ over $s = \infty$. Then by taking the Frobenius base change $t^2 = s$ we have a rational elliptic surface $\tilde{\pi}: \tilde{R} \to \mathbb{P}^1$ associated with the Weierstrass equation which has a singular fiber of type $\tilde{A}_5$ over $t = 0$, a singular fiber of type $\tilde{A}_1$ over $t = 1$ and a singular fiber of type $\tilde{A}_2^\ast$ over $t = \infty$.

Blowing ups of the fiber over $t = \infty$ four times, we have a rational surface $Y$ which has the same configuration of curves on the fiber over $l = \infty$ as that of the $\alpha_2$-surface. We use the same symbols:

$$E_{\infty,1}, E_{\infty,2}, E_{\infty,3}, E_{\infty,4}, E_{\infty,5}, E_{\infty,6}, E_{\infty,7}.$$
We also denote by $E_{0,1},\ldots,E_{0,6}$ the component of the fiber over $t = 0$ such that $E_{0,i} \cdot E_{0,i+1} = 1$ $(i \in \mathbb{Z}/6\mathbb{Z})$, and by $E_{1,1},E_{1,2}$ the components of the fiber over $t = 1$. We may assume that the three sections $S_i$ meets $E_{0,2i-1},E_{1,1},E_{\infty,i}$ $(i = 1,2,3)$.

Now we consider the following derivation.

$$
\partial_a = (t + a) \frac{\partial}{\partial t} + (x + t^2) \frac{\partial}{\partial x},
$$

where $a \in k, a \neq 0, 1$. Then $\partial_a^2 = \partial_a$, that is, $\partial_a$ is 2-closed. By calculations we have the following two lemmas.

**Lemma 8.10.22.** (1) The divisor $D_a$ of $\partial_a$ is given by

$$
-(E_{0,2} + E_{0,4} + E_{0,6} + E_{1,2} + E_{\infty,1} + E_{\infty,2} + E_{\infty,3}) - 2(E_{\infty,4} + E_{\infty,5} + E_{\infty,6} + E_{\infty,7}).
$$

(2) $D_a^2 = -12$, $(D_a,K_Y) = -4$.

(3) The derivation $\partial_a$ has no isolated zeros.

**Lemma 8.10.23.** The integral curves with respect to $\partial_a$ are the smooth fiber over $t = a$ and

$$
E_{0,2}, E_{0,4}, E_{0,6}, E_{1,2}, E_{\infty,1}, E_{\infty,2}, E_{\infty,3}, E_{\infty,4}
$$

Now we contract four $(-1)$-curves on $Y^{\partial_a}$, the images of $E_{0,2}$, $E_{0,4}$, $E_{0,6}$, $E_{1,2}$ and denote the obtained surface by $S_a$. By combining these two lemmas and by the same argument as in the case of the $\alpha_2$-Enriques surface, we have the following theorem.

**Theorem 8.10.24.** The surfaces $S_a$ form a 1-dimensional family of classical Enriques surfaces with a crystallographic basis of type $\tilde{E}_6 + \tilde{A}_2$.

**Proof.** Since the fiber of $\tilde{\pi}$ over $t = a$ is integral, its image on $S_a$ is a double fiber. Thus the fibration on $S_a$ has two double fibers (another one is the fiber of type $\tilde{E}_6$), $S_a$ is classical.

It follows from the dual graph of $(-2)$-curves that the following Proposition holds.

**Proposition 8.10.25.** There are exactly one elliptic fibration with singular fibers of type $2\tilde{E}_6 + \tilde{A}_2$ and three quasi-elliptic fibrations with singular fibers of type $\tilde{E}_7 + 2\tilde{A}_1$.

**Theorem 8.10.26.** The automorphism group $\text{Aut}(S_a)$ is isomorphic to $\mathfrak{S}_3$.

**Proof.** It suffices to see that there are no numerically trivial automorphisms. If $g \in \text{Aut}(S_a)$ is numerically trivial, it preserves each of 13 $(-2)$-curves. In particular $g$ fixes pointwisely each component of the singular fiber of type $\tilde{A}_2$ because $g$ fixes three points on each component. Now consider a quasi-elliptic fibration $p$. Since a component of the fiber of type $\tilde{A}_2$ is a bisection of $p$, $g$ acts trivially on the base of $p$. Since $p$ has two bisections, $g$ fixes three points on a general fiber $F$ of $p$ and hence $g$ acts trivially on $F$.

**Remark 8.10.27.** By construction, the canonical cover of $S_a$ has four rational double point of type $A_1$. On the other hand, the conductrix of $S_a$ is the same as in the case of the $\alpha_2$-Enriques surface.
Example 8.10.28. Type VIII.

We take a parabolic subdiagram of type $\tilde{D}_5 + \tilde{A}_3$ in the dual graph of type VIII. We consider an extremal rational elliptic fibration $\pi : \tilde{R} \to \mathbb{P}^1$ associated with the Weierstrass model

$$y^2 + sxy = x^3 + s^2x$$

which has a singular fiber of type $\tilde{D}_5$ over $s = 0$ and a singular fiber of type $\tilde{A}_3$ over $s = \infty$. By the Frobenius base change $t^2 = s$ we have a rational elliptic fibration $\tilde{\pi} : \tilde{R} \to \mathbb{P}^1$ associated with

$$y^2 + t^2xy = x^3 + t^4x$$

which has a singular fiber of type $\tilde{A}_1^1$ over $t = 0$ and a singular fiber of type $\tilde{A}_7$ over $t = \infty$. The elliptic fibration $\tilde{\pi}$ has four sections:

$$S_1 : \text{the zero section, } S_2 : x = y = 0, \ S_3 : x = t, \ y = 0, \ S_4 : x = 0, \ y = t.$$  

Moreover there exist two bisections defined by

$$B_1 : x + y = x^2 + tx + t = 0, \ B_2 : x + y + tx + t = x^2 + tx + t = 0$$

both of which pass the singular point of the fiber of type $\tilde{A}_1^1$. We refer the reader to Figure 10 in [363] for the incidence relation between components of fibers and sections, bisections.

To get the dual graph of type VIII, we blow up on four points on the fiber over $t = 0$ and contract four disjoint components of the fiber over $t = \infty$. Let $E_{0,1}, E_{0,2}$ be components of the fiber of type $\tilde{A}_1^1$ and let $E_{\infty,1}, \ldots, E_{\infty,8}$ the components of the fiber of type $\tilde{A}_7$ such that $E_{\infty,i} \cdot E_{\infty,i+1} = 1$ $(i \in \mathbb{Z}/8\mathbb{Z})$. We first blow up the singular point of the fiber of type $\tilde{A}_1^1$, and then blow up at the point of the intersection of the proper transforms of $E_{0,1}$ and $E_{0,2}$. Denote by $E_{0,3}$ the exceptional curve of the first blowing-up and by $E_{0,4}$ the exceptional curve of the second blowing-up. We also use the same symbols $E_{0,1}, S_1, B_k$ for their proper transforms. Now three curves $E_{0,1}, E_{0,2}, E_{0,3}$ are disjoint. Note that $B_1$ and $B_2$ meet $E_{0,3}$. Blow up the two points $B_1 \cap E_{0,3}, B_2 \cap E_{0,3}$. The obtained surface is denoted by $Y$.

Now we consider the following derivation.

$$\partial_a = t(at + 1) \frac{\partial}{\partial t} + (x + 1) \frac{\partial}{\partial x},$$

where $a \in k, a \neq 0$. Then $\partial_a^2 = \partial_a$, that is, $\partial_a$ is 2-closed. By calculations we have the following two lemmas.

Lemma 8.10.29. (1) The divisor $D_a$ of $\partial_a$ is given by

$$-(E_{0,1} + E_{0,2} + E_{0,3} + 2E_{0,4} + E_{\infty,2} + E_{\infty,4} + E_{\infty,6} + E_{\infty,8}).$$

(2) $D_a^2 = -12, \ (D_a, K_Y) = -4$.  

(3) The derivation $\partial_a$ has no isolated zeros.

Lemma 8.10.30. The integral curves with respect to $\partial_a$ are the smooth fiber over $t = 1/a$ and

$$E_{0,1}, E_{0,2}, E_{0,3}, E_{\infty,2}, E_{\infty,4}, E_{\infty,6}, E_{\infty,8}.$$  

Let $S_a$ be the surface obtained by contracting four $(-1)$-curves on $Y^\partial_a$ which are the images of $E_{\infty,2}, E_{\infty,4}, E_{\infty,6}, E_{\infty,8}$. By combining these two lemmas and by the same argument as in the
case of the $\alpha_2$-Enriques surface of type $\tilde{E}_8$, we have

**Theorem 8.10.31.** The surfaces $S_\alpha$ form a 1-dimensional family of classical Enriques surfaces with a crystallographic basis of type VIII.

**Proof.** Since the fiber of $\tilde{\pi}$ over $t = 1/a$ is integral, its image on $S$ is a double fiber. Thus the fibration on $X$ has two double fibers (another one is the fiber of type $\tilde{D}_5$), so $S$ is classical. □

It follows from the dual graph of $(-2)$-curves that

**Proposition 8.10.32.** There are exactly three elliptic fibrations with singular fibers of type $2\tilde{D}_5 + \tilde{A}_3$, three quasi-elliptic fibrations with singular fibers of type $\tilde{D}_6 + 2\tilde{A}_1^* + 2\tilde{A}_1^*$ and eight elliptic fibrations with singular fibers of type $\tilde{E}_6 + \tilde{A}_2 + \tilde{A}_1^*$.

Using an argument similar to one from the proof of Theorem 8.10.26, we have the following.

**Theorem 8.10.33.** The automorphism group $\text{Aut}(S)$ is isomorphic to $S_4$.

**Remark 8.10.34.** By construction, the canonical cover of $S$ has four rational double point of type $A_1$. The support of the conductrix of $S$ is the union of four $(-2)$-curves each of which appears as a component with multiplicity 2 of a singular fiber of type $\tilde{D}_6$.

**Example 8.10.35.** Type $\tilde{E}_8$: $\alpha_2$-surface.

Recall that the crystallographic root basis of type $\tilde{E}_8$ is given as follows:

- $\tilde{E}_8$ has a unique quasi-elliptic fibration with a bisection $R_{10}$.

First we discuss how to find a rational surface and a derivation on it whose quotient is birational to $S$. Assume that the canonical cover has a rational double point of type $D_4$ over the point $\tilde{P} = R_9 \cap R_{10}$. First blow up $P$ and denote by the exceptional curve by $E_1$. Then blow up the intersection points of $E_1$ and the proper transforms of $R_9$ and $R_{10}$ and denote the exceptional curves by $E_2$ and $E_3$ respectively. And blow up one point on $E_1$ not lying on $R_9$ and $R_{10}$ and denote the exceptional curve by $E_4$. We denote by $S'$ the obtained surface. Assume that a resolution $Y$ of the normalization of the canonical cover of $S$ is an inseparable double cover $\pi : Y \to S'$, that is, $S' = Y^0$ for a derivation $\partial$ on $Y$. We denote by $\tilde{R}_i$, $\tilde{E}_i$ the preimages on $Y$ of $R_i$, $E_i$. Note that the cycle $\tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3 + \tilde{E}_4$ on $Y$ is the exceptional curve corresponding to the rational double point of type $D_4$. Since $\tilde{E}_i^2 = -2$ ($i = 1, 2, 3, 4$) and $E_1^2 = -4$, $E_2^2 = E_3^2 = E_4^2 = -1$, $\tilde{E}_1$, $\tilde{E}_3$, $\tilde{E}_4$ should be integral and $\tilde{E}_1$ not with respect to the derivation $\partial$. We assume that $\tilde{R}_2$, $\tilde{R}_4$, $\tilde{R}_6$ and $\tilde{R}_8$ are integral with respect to $\partial$, and other $\tilde{R}_i$ not. Then

$$\tilde{R}_2^2 = \tilde{R}_4^2 = \tilde{R}_6^2 = \tilde{R}_8^2 = -4, \quad \tilde{R}_1^2 = \tilde{R}_3^2 = \tilde{R}_5^2 = \tilde{R}_7^2 = -1, \quad \tilde{R}_9^2 = \tilde{R}_{10}^2 = -2.$$  

Now we contract $(-1)$-curves and then contract new $(-1)$-curves except $\tilde{E}_4$ successively. Finally we get a projective plane $\mathbb{P}^2$. Thus $Y$ should be obtained from $\mathbb{P}^2$ by 13 times blowing ups.
On the other hand, as in the case of the $\alpha_2$-Enriques surface of type $\tilde{E}_6 + \tilde{A}_2$, the triviality of the canonical bundle $K_S$ implies some conditions on the divisor $D$ of $\partial$ and $K_Y$. Assume $K_S = 0$. Then $K_{S'} = E_1 + 2(E_2 + E_3 + E_4)$. Since $E_1$ is not integral and other $E_i$ are integral, we have

$$K_Y = \pi^* K_{S'} + D = 2(\tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3 + \tilde{E}_4) + D.$$  

This holds if

$$D = -(5\tilde{E}_4 + 2\tilde{R}_{10} + 6\tilde{E}_3 + 8\tilde{E}_1 + 7\tilde{E}_2 + 4\tilde{R}_9 + 3\tilde{R}_8 + 2\tilde{R}_2 + 4\tilde{R}_6 + 5\tilde{R}_4 + 6\tilde{R}_7 + 8\tilde{R}_5 + 4\tilde{R}_1 + 6\tilde{R}_3)$$

and

$$K_Y = -(3\tilde{E}_4 + 2\tilde{R}_{10} + 4\tilde{E}_3 + 6\tilde{E}_1 + 5\tilde{E}_2 + 4\tilde{R}_9 + 3\tilde{R}_8 + 2\tilde{R}_2 + 4\tilde{R}_6 + 5\tilde{R}_4 + 6\tilde{R}_7 + 8\tilde{R}_5 + 4\tilde{R}_1 + 6\tilde{R}_3).$$

Moreover, we can see that $D^2 = -12$, $K_Y \cdot D = -4$, and hence $\partial$ has no isolated zeros. Thus we need to find a rational surface $Y$ and a derivation $\partial$ on $Y$ satisfying these conditions.

Now we consider the affine plane $\mathbb{A}^2 \subset \mathbb{P}^2$ with a coordinate $(x, y)$ and the following derivation

$$\partial = \frac{1}{x^5} \left( (xy^6 + x^3) \frac{\partial}{\partial x} + (x^6 + y^7 + x^2y) \frac{\partial}{\partial y} \right).$$

Then $\partial^2 = 0$, that is, $\partial$ is 2-closed. Note that $\partial$ has a pole with order 5 along the line $\ell$ defined by $x = 0$ and $\ell$ is integral with respect to $\partial$. Moreover $\partial$ has a unique isolated singularity at $P = (0, 0)$. We blow up at the point $P$. Then the exceptional curve is not integral with respect to the induced derivation. The induced derivation has a pole of order 2 along the exceptional curve and has a unique isolated singularity at the intersection of the exceptional curve and the proper transform of $\ell$.

Then continue this process until the induced derivation has no isolated singularities. The resulting surface is denoted by $Y'$ which, together with the induced derivation, satisfies the desired conditions. Here the proper transform of $\ell$ corresponds to $\tilde{E}_4$ and the exceptional curve of the first blowing up corresponds to $\tilde{R}_{10}$. Thus we have proved the following theorem.

**Theorem 8.10.36.** The surface $S$ is an $\alpha_2$-Enriques surface with a crystallographic basis of type $\tilde{E}_8$. There exists a unique quasi-elliptic fibration which has a singular fiber of type $2\tilde{E}_8$.

**Remark 8.10.37.** As in the case of the $\alpha_2$-Enriques surface of type $\tilde{E}_6 + \tilde{A}_2$, we can determine the function field $k(x, y)^\partial$ of the quotient surface and have the following equation of the surface birationally isomorphic to $S$:

$$Y^2 + TX^4 + X + T^7 = 0. \quad (8.10.6)$$

**Theorem 8.10.38.** The automorphism group $\text{Aut}(S)$ is isomorphic to $\mathbb{Z}/11\mathbb{Z}$ which is cohomologically trivial.

**Proof.** Note that the symmetry group of the dual graph of $(-2)$-curves is trivial and hence any automorphism is cohomologically trivial. We can determine the automorphism group of the surface defined by 8.10.6 preserving the fibration defined by the projection $(X, Y, T) \to T$ as follows:

$$\varphi : (X, Y, T) \to (\zeta^7 X, \zeta^9 Y, \zeta T)$$

where $\zeta$ is a primitive 11-th root of unity. ☐

**Remark 8.10.39.** By construction, the canonical cover of $S$ has a rational double point of type $D_4$. The support of the conductrix is the union of ten $(-2)$-curves on $S$. 
Example 8.10.40. **Type \( \tilde{E}_8 \): Classical Case.** Classical Enriques surface of type \( \tilde{E}_8 \) has the same dual graph as in Figure 8.19. In this case the quasi-elliptic fibration has two double fibers. We denote by \( F_0 \) the irreducible one. Difference from the \( \alpha_2 \)-surface is that the canonical cover has 4 rational double points of type \( A_1 \) instead of a rational double point of type \( D_4 \). So we blow up at \( F_0 \cap R_{10}, R_{10} \cap R_9, \) a nonsingular point on \( F_0 \) and a point on \( R_9 \). We denote by \( E_1, E_2, E_3, E_4 \) the exceptional curves, respectively. We use the same notation as in the case of the \( \alpha_2 \)-Enriques surface of type \( \tilde{E}_8 \). We also assume that \( \tilde{E}_i, (1 \leq i \leq 4), \tilde{R}_2, \tilde{R}_4, \tilde{R}_6 \) and \( \tilde{R}_8 \) are integral with respect to a derivation \( \partial \). The pullback of \( F_0 \) to \( Y \), denoted by \( \tilde{F}_0 \), is a nonsingular rational curve. Obviously

\[
\begin{align*}
\tilde{E}_1^2 &= \tilde{E}_3^2 = \tilde{E}_4^2 = -2, & \tilde{R}_2^2 &= \tilde{R}_4^2 = \tilde{R}_6^2 = \tilde{R}_8^2 = -4, \\
\tilde{F}_0^2 &= \tilde{R}_1^2 = \tilde{R}_2^2 = \tilde{R}_7^2 = -1, & \tilde{R}_3^2 &= \tilde{R}_{10}^2 = -2.
\end{align*}
\]

Now we contract \((-1)\)-curves and then contract new \((-1)\)-curves except \( \tilde{E}_3, \tilde{E}_4, \tilde{R}_{10} \) successively. Finally we get a nonsingular quadric surface \( \mathbb{P}^1 \times \mathbb{P}^1 \). Thus \( Y \) should be obtained from \( \mathbb{P}^1 \times \mathbb{P}^1 \) by 12 times blowing up.

Now we consider the affine open set \( \mathbb{A}^1 \times \mathbb{A}^1 \) of \( \mathbb{P}^1 \times \mathbb{P}^1 \) with coordinates \((x, y)\) and the following derivation

\[
\partial_a = \frac{1}{x^3y^2} \left( x^4y^2 \frac{\partial}{\partial x} + (x^2 + ax^4y^4 + y^4) \frac{\partial}{\partial y} \right), \quad (a \neq 0 \in k)
\]

Then \( \partial_a^2 = \partial_a \), that is, \( \partial_a \) is 2-closed. Note that \( \partial_a \) has a pole with order 3 along the divisor defined by \( x = 0 \), a pole of order 1 along the divisor defined by \( x = \infty \) and a pole of order 2 along the divisor defined by \( y = 0 \). Moreover \( \partial_a \) has two isolated singularities at \((x, y) = (0, 0), (\infty, 0)\). As in the case of the \( \alpha_2 \)-Enriques surface of type \( \tilde{E}_8 \), we blow up at the singular points of \( \partial_a \) and those of the induced derivations successively. The resulting surface is denoted by \( Y \) and the configuration of curves is as desired as above. Here the proper transform of curves defined by \( x = 0, x = \infty \) or \( y = 0 \) is \( \tilde{E}_4, \tilde{E}_5 \) or \( \tilde{R}_{10} \), respectively.

**Lemma 8.10.41.** Let \( D_a \) be the divisor of \( \partial_a \). Then

\[
D_a = -(2\tilde{R}_{10} + 3\tilde{E}_4 + \tilde{E}_3 + 2\tilde{E}_1 + 4\tilde{E}_2 + 4\tilde{R}_9 + 3\tilde{R}_8 + 6\tilde{R}_7 + 4\tilde{R}_6 + 8\tilde{R}_5 + 5\tilde{R}_4 + 6\tilde{R}_3 + 2\tilde{R}_2 + 4\tilde{R}_1)
\]

and

\[
K_Y = -(2\tilde{R}_{10} + 2\tilde{E}_4 + \tilde{E}_1 + 3\tilde{E}_2 + 4\tilde{R}_9 + 3\tilde{R}_8 + 6\tilde{R}_7 + 4\tilde{R}_6 + 8\tilde{R}_5 + 5\tilde{R}_4 + 6\tilde{R}_3 + 2\tilde{R}_2 + 4\tilde{R}_1).
\]

It follows from the formula \( K_Y = \pi^*K_{Y/\alpha_a} + D_a \) that \( K_{Y/\alpha_a} = E_1 + E_2 + E_3 + E_4 \) because all \( \tilde{E}_i \) are integral. Moreover we can see that \( D_a^2 = -12 \), \( K_Y \cdot D_a = -4 \), and hence \( D_a \) has no isolated zeros. By contracting \((-1)\)-curves \( E_1, ..., E_4 \) on \( Y/\partial_a \) we get a smooth surface \( S_a \) as desired. Thus we have proved the following theorem.

**Theorem 8.10.42.** The surface \( S_a \) is a classical Enriques surface with a crystallographic basis of type \( \tilde{E}_8 \). There exists a unique quasi-elliptic fibration which has a singular fiber of type \( 2\tilde{E}_8 \).

**Remark 8.10.43.** As in the case of the \( \alpha_2 \)-Enriques surface of type \( \tilde{E}_6 + \tilde{A}_2 \), we can determine the function field \( k(x, y)_{\partial_a} \) of the quotient surface and have the following equation of the surface birationally isomorphic to \( S_a \):

\[
Y^2 + TX^4 + bT^3X + T^3 + T^7 = 0, \quad b = a^{\frac{1}{2}}.
\]

(8.10.7)
Theorem 8.10.44. The automorphism group \( \text{Aut}(S_a) \) is trivial.

Proof. Note that the symmetry group of the dual graph of \((-2)\)-curves is trivial and hence any automorphism is numerically trivial. We can see that the automorphism group of the surface defined by 8.10.7 preserving the fibration defined by the projection \((X, Y, T) \to T\) is trivial. \(\square\)

Remark 8.10.45. By construction, the canonical cover of \(S\) has four rational double points of type \(A_1\). The support of the conduitrix is the union of 10 \((-2)\)-curves on \(S\).

Remark 8.10.46. We have already given a construction of classical extra-special surfaces of \(E_8\) type in Example 6.2.10. The equation (8.10.7) is transformed to equation (6.2.11) by the variable change

\[
T = \frac{t_1}{t_2}, \quad X = \frac{t_0}{t_2}, \quad Y = \frac{t_3}{t_2}.
\]

It uses the unique (degenerate) bielliptic linear system on \(S\) and constructs a birational model of \(S\) as an inseparable double plane. It follows from this construction that all surfaces of this type form an irreducible one-dimensional family and hence coincide with the surfaces constructed in the previous example.

Example 8.10.47. Type \(E_7^1\).

Recall that the crystallographic root basis of type \(E_7^1\) is given as follows:

![Figure 8.20](image)

On such an Enriques surface \(S\), there exists a quasi-elliptic fibration with singular fibers of type \(2\bar{E}_7 + 2\bar{A}_1^2\) and a bisection \(R_0\). Assume that the canonical cover has rational double points of type \(A_1\) over two points of \(R_2 \setminus R_3\) and two points of \(R_{11} \setminus R_{10}\). First blow up these four points and denote by the exceptional curves over the points of \(R_2 \setminus R_3\) by \(E_1, E_2\) and those over the points of \(R_{11} \setminus R_{10}\) by \(E_3, E_4\), respectively. We denote by \(S'\) the obtained surface. Assume that a resolution \(Y\) of the normalization of the canonical cover of \(S\) is an inseparable double cover \(\pi : Y \to S'\), that is, \(S' = Y^\partial\) for a derivation \(\partial\) on \(Y\). We denote by \(\bar{R}_i, \bar{E}_i\) the preimages on \(Y\) of \(R_i, E_i\). Note that \(\bar{E}_1, \bar{E}_2, \bar{E}_3, \bar{E}_4\) on \(Y\) is the exceptional curves corresponding to the four rational double points of type \(A_1\) on \(S\). Assume that \(\bar{E}_1, \bar{E}_2, \bar{E}_3, \bar{E}_4, \bar{R}_3, \bar{R}_5, \bar{R}_7, \bar{R}_9\) are integral with respect to a derivation \(\partial\) on \(Y\). Then

\[
\bar{E}_i^2 = -2 \quad (i = 1, 2, 3, 4), \quad \bar{R}_2^2 = \bar{R}_{11}^2 = -2, \quad \bar{R}_3^2 = \bar{R}_5^2 = \bar{R}_7^2 = \bar{R}_9^2 = -4,
\]

\[
\bar{R}_1^2 = \bar{R}_4^2 = \bar{R}_6^2 = \bar{R}_8^2 = \bar{R}_{10}^2 = -1.
\]

Now we contract \((-1)\)-curves and then contract new \((-1)\)-curves successively as in the following order:

\[
\bar{R}_1, \bar{R}_4, \bar{R}_6, \bar{R}_8, \bar{R}_{10}, \bar{R}_5, \bar{R}_7, \bar{R}_3, \bar{R}_{11}.
\]

On the other hand, we contract two \((-2)\)-curves \(\bar{E}_1, \bar{E}_2\). Finally we get a surface \(R\) with two rational double points of type \(A_1\). The quasi-elliptic fibration on \(S\) induces a conic bundle structure.
on $R$ which has singular fibers $\tilde{E}_3 + \tilde{E}_4$, $2\tilde{R}_2$ and a bisection $\tilde{R}_9$. Two rational double points of $R$ lie on $\tilde{R}_2$.

To get an Enriques surface we need the following conditions:

\[ D = -(2\tilde{E}_1 + 2\tilde{E}_2 + \tilde{E}_3 + \tilde{E}_4 + 2\tilde{R}_1 + 2\tilde{R}_2 + 2\tilde{R}_3 + 4\tilde{R}_4 + 3\tilde{R}_5 + 4\tilde{R}_6 + 2\tilde{R}_7 + 2\tilde{R}_8 + \tilde{R}_9) \]

where $D$ is the divisor of $\partial$ and

\[ K_Y = -(\tilde{E}_1 + \tilde{E}_2 + 2\tilde{R}_1 + 2\tilde{R}_2 + 2\tilde{R}_3 + 4\tilde{R}_4 + 3\tilde{R}_5 + 4\tilde{R}_6 + 2\tilde{R}_7 + 2\tilde{R}_8 + \tilde{R}_9). \]

Now we consider a surface $R$ defined by

\[ aSX_0^2 + T(X_1^2 + aX_1X_2) = 0 \quad (a \in k, a \neq 0). \]

The surface has a structure of a conic bundle $R \rightarrow \mathbb{P}^1$ with $(S, T) \in \mathbb{P}^1$. The fiber over $(S, T) = (0, 1)$ is a union of two lines (corresponding to $\tilde{E}_3$, $\tilde{E}_4$) defined by $X_1(X_1 + aX_2) = 0$ and the fiber over $(S, T) = (1, 0)$ is a double line (corresponding to $\tilde{R}_2$) defined by $X_2^2 = 0$. The line defined by $X_2 = 0$ is a bisection (corresponding to $\tilde{R}_9$) of the fiber space. The surface has two rational double points $((0, 0), (1, 0), (0, a, 1), (1, 0))$ of type $A_1$ (corresponding to $\tilde{E}_1, \tilde{E}_2$). Let $(x = X_0/X_2, y = X_1/X_2, s = S/T)$ be affine coordinates. Define

\[ \partial = \frac{1}{s} \left( (s^2 + 1) \frac{\partial}{\partial x} + s^2a^2 \frac{\partial}{\partial y} \right). \]

Then $\partial^2 = \partial$, that is, $\partial$ is 2-closed. A calculation shows that $\partial$ has two isolated singularities at the intersection of the bisection and the two fibers over $(S, T) = (1, 0), (0, 1)$. As in the previous cases, we blow up the two rational double points and isolated singularities of $\partial$ successively, and finally get the surface $Y$ and a desired derivation. Since $R$ has a parameter $a$, we denote by $S_a$ the obtained Enriques surface.

**Theorem 8.10.48.** The surface $S_a$ is a classical Enriques surface with a crystallographic basis of type $\tilde{E}_1^4$.

By the dual graph of $(-2)$-curves, we can prove the following Proposition.

**Proposition 8.10.49.** There are exactly one quasi-elliptic fibration with a singular fiber of type $\tilde{E}_8$ and one quasi-elliptic fibration with singular fibers of type $2\tilde{E}_7 + 2A_1^*$. 

**Theorem 8.10.50.** The automorphism group $\text{Aut}(S_a)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ which is numerically trivial.

**Proof.** Since there are no non-trivial symmetries of the dual graph of $(-2)$-curves, all automorphisms are numerically trivial. Let $\pi_1 : S_a \rightarrow \mathbb{P}^1$ be the quasi-elliptic fibration with a singular fiber of type $\tilde{E}_8$ and let $\pi_2 : S_a \rightarrow \mathbb{P}^1$ be the quasi-elliptic fibration with singular fibers of type $2\tilde{E}_7 + 2A_1^*$. Let $F_1, F_2$ be the double fibers of $\pi_1$. Note that $F_1, F_2$ are bi-sections of $\pi_2$. If $g \in \text{Aut}(S_a)$ preserves each $F_i$, then $g$ fixes three points on $F_i$ (the cusp and the two intersection points with the two double fibers of $\pi_2$), and hence acts trivially on $F_i$. Therefore $g$ fixes three points on a general fiber of $\pi_2$, that is, the cusp and two intersection points with $F_1, F_2$. Thus $g$ fixes a general fiber of $\pi_2$ pointwise and hence $g$ is trivial. Thus we have $|\text{Aut}(S_a)| \leq 2$. On the other hand, the Mordell-Weil group $\mathbb{Z}/2\mathbb{Z}$ of the Jacobian fibration of $\pi_2$ acts on $S_a$. Thus we have the assertion. \qed
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Remark 8.10.51. By construction, the canonical cover of $S$ has four rational double points of type $A_1$. The support of the conductrix is the union of $9$ $(-2)$-curves $R_1, \ldots, R_9$.

Remark 8.10.52. We have already given a construction of classical extra-special surfaces of type $\tilde{E}_7^1$ in Example 6.2.11. It uses a unique non-degenerate bielliptic map of $S$ to the 4-nodal anti-canonical quartic del Pezzo surface $D_1$. It follows from this construction that all surfaces of this type form an irreducible one-dimensional family and hence coincide with the family of surfaces constructed in the previous example.

Example 8.10.53. **Type $\tilde{E}_7^2$: $\alpha_2$-surface.**

Recall that the crystallographic root basis of type $\tilde{E}_7^2$ is given as follows:

\[ R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8, R_9, R_{10}, R_{11} \]

Figure 8.21:

We use the same idea to find a suitable rational surface and a derivation as in the case of classical Enriques surfaces of type $\tilde{E}_7^2$ (Example 8.10.47). Consider the quasi-elliptic fibration with a singular fiber of type $2\tilde{E}_7$ and with a bisection $R_9$. Assume that the canonical cover of $S$ has a rational double point of type $D_4$ over a point $P$ of $R_2 \setminus R_3$. First blow up $P$ and denote the exceptional curve by $E_1$. Then blow up the intersection point of $E_1$ and the proper transform of $R_2$, and denote the exceptional curve by $E_2$. And blow up two points on $E_1$ and denote the exceptional curves by $E_3$, $E_4$. We denote by $S'$ the obtained surface. Assume that a resolution $Y$ of the normalization of the canonical cover of $S$ is an inseparable double cover $\pi : Y \to S'$, that is, $S' = Y^\partial$ for a derivation $\partial$ on $Y$. We denote by $\tilde{R}_i, \tilde{E}_i$ the preimages on $Y$ of $R_i, E_i$, respectively. Note that $\tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_4$ on $Y$ is the exceptional curves over the rational double point of type $D_4$. Assume that $\tilde{E}_2, \tilde{E}_3, \tilde{E}_4, \tilde{R}_3, \tilde{R}_5, \tilde{R}_7$ and $\tilde{R}_9$ are integral with respect to a derivation $\partial$ on $Y$. Then

\[
\begin{align*}
\tilde{E}_1^2 &= -2 \quad (i = 1, 2, 3, 4), \quad \tilde{R}_2^2 = -2, \\
\tilde{R}_3^2 &= \tilde{R}_5^2 = \tilde{R}_9^2 = -4, \\
\tilde{R}_1^2 &= \tilde{R}_4^2 = \tilde{R}_6^2 = \tilde{R}_8^2 = \tilde{R}_{10}^2 = \tilde{R}_{11}^2 = -1.
\end{align*}
\]

Note that the preimage of the fiber $R_{10} + R_{11}$ of type $A_1^*$ is a union of two $(-1)$-curves $\tilde{R}_{10}, \tilde{R}_{11}$ meeting at one point transversally.

Now we contract $(-1)$-curves and then contract new $(-1)$-curves successively as in the following order:

$\tilde{R}_1, \tilde{R}_4, \tilde{R}_6, \tilde{R}_8, \tilde{R}_5, \tilde{R}_7, \tilde{R}_3, \tilde{R}_2, \tilde{E}_2$.

On the other hand, we contract two $(-2)$-curves $\tilde{E}_3, \tilde{E}_4$. Thus we get a surface $\tilde{R}$ with two rational double points of type $A_1$. The quasi-elliptic fibration on $R$ induces a conic bundle structure on $R$ which has two singular fibers $\tilde{R}_{10} + \tilde{R}_{11}$, $2\tilde{E}_1$ and a bisection $\tilde{R}_9$. Two rational double points of $R$ lie on $\tilde{E}_1$.

On the other hand, to get an Enriques surface we need the following conditions:

\[
D = -(4\tilde{E}_1 + 4\tilde{E}_2 + 3\tilde{E}_3 + 3\tilde{E}_4 + 2\tilde{R}_1 + 2\tilde{R}_2 + 2\tilde{R}_3 + 4\tilde{R}_4 + 3\tilde{R}_5 + 4\tilde{R}_6 + 2\tilde{R}_7 + 2\tilde{R}_8 + \tilde{R}_9)
\]

where $D$ is the divisor of $\partial$ and

\[
K_Y = -(2\tilde{E}_1 + 2\tilde{E}_2 + \tilde{E}_3 + \tilde{E}_4 + 2\tilde{R}_1 + 2\tilde{R}_2 + 2\tilde{R}_3 + 4\tilde{R}_4 + 3\tilde{R}_5 + 4\tilde{R}_6 + 2\tilde{R}_7 + 2\tilde{R}_8 + \tilde{R}_9).
\]
Now we consider a surface $R$ defined by

$$S(X_0^2 + a^3X_2^2) + T(X_1^2 + X_1X_2 + a^2X_0X_2) = 0 \quad (a \in k, a \neq 0).$$

The surface has a structure of a conic bundle $R \to \mathbb{P}^1$ with $(S, T) \in \mathbb{P}^1$. The fiber over $(S, T) = (1, 0)$ is a double line and the fiber over $(S, T) = (a^3, 1)$ is a union of two lines. The line defined by $X_2 = 0$ is a bisection of the fiber space. The surface has two rational double points $((X_0, X_1, X_2), (S, T)) = ((\alpha, \beta_i, 1), (1, 0))$ of type $A_1$, where $\alpha = \sqrt{a^3}$ and $\beta_i$ is a root of the equation $z^2 + z + a^3\sqrt{a} = 0$.

Let $(x = X_0/X_2, y = X_1/X_2, s = S/T)$ be affine coordinates. Define

$$\partial = (s^2 + a) \frac{\partial}{\partial x} + (x^2 + a^2s^2) \frac{\partial}{\partial y}.$$  

Then $\partial^2 = 0$, that is, $\partial$ is $2$-closed. A calculation shows that $\partial$ has an isolated singularity at the intersection of the bisection and the fiber over the point $(S, T) = (1, 0)$. As in the previous cases, we blow up the two rational double points and the isolated singularity of $\partial$ successively, and finally get the surface $Y$ and a desired derivation. Since $R$ has a parameter $a$, we denote by $S_a$ the obtained Enriques surface.

**Theorem 8.10.54.** The surface $S_a$ is an $\alpha_2$-Enriques surface with a crystallographic basis of type $\tilde{E}_7^1$.

By the dual graph of $(-2)$-curves, we can prove the following Proposition.

**Proposition 8.10.55.** There are exactly one quasi-elliptic fibration with singular fibers of type $2\tilde{E}_7 + \tilde{A}_1^1$ and two quasi-elliptic fibrations with a singular fiber of type $\tilde{E}_8$.

**Remark 8.10.56.** As in the case of the $\alpha_2$-Enriques surface of type $\tilde{E}_6 + \tilde{A}_2$, we can determine the function field $k(x, y, s)^0$ of the quotient surface and have the following equation of the surface birationally isomorphic to $S_a$:

$$Y^2 + Y + TX^4 + (T + a^4)^7 = 0. \quad (8.10.8)$$

By using this equation we can determine the automorphism group.

**Theorem 8.10.57.** If $a^7 \neq 1$, then the automorphism group $\text{Aut}(S_a)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ which is not numerically trivial. If $a^7 = 1$, then $\text{Aut}(S_a)$ is isomorphic to $\mathbb{Z}/14\mathbb{Z}$ and $\text{Aut}_{\text{rl}}(S_a)$ is isomorphic to $\mathbb{Z}/7\mathbb{Z}$.

**Proof.** Here we give only a generator $\sigma$ of $\text{Aut}(S_a)$ in terms of the equation (8.10.8): In case $a^7 \neq 1$,

$$\sigma(T, X, Y) = (T, X, Y + 1).$$

In case $a^7 = 1$,

$$\sigma(T, X, Y) = (\zeta T, \zeta^{-2}X + (1 + \zeta^{-2})a^6 + (\zeta + \zeta^{-2})a^2T, Y + 1 + (1 + \zeta^2)a^6T^2 + (1 + \zeta^3)a^2T^3)$$

where $\zeta$ is a primitive 7th root of unity.

**Remark 8.10.58.** By construction, the canonical cover of $S$ has a rational double point of type $D_4$. The support of the conductrix is the union of 9 $(-2)$-curves $R_1, \ldots, R_9$. 

8.10. ENRIQUES SURFACES WITH A FINITE AUTOMORPHISM GROUP: \( P = 2 \)

Example 8.10.59. Type \( \tilde{E}^2_7 \): Classical Case.

Classical Enriques surface \( S \) of type \( \tilde{E}^2_7 \) has the same dual graph as in Figure 8.21. We use the same idea to find a suitable rational surface and a derivation as in the case of classical Enriques surfaces of type \( \tilde{E}^2_7 \) (Example 8.10.47). In fact, the type \( \tilde{E}^2_7 \) is a degeneration of this case. In this case the quasi-elliptic fibration with a singular fiber of type \( 2\tilde{E}_7 \) and with a bisection \( R_9 \) has two double fibers. We denote by \( F_0 \) the irreducible double fiber. Assume that the canonical cover of \( S \) has four rational double points of type \( A_1 \) over two points of \( R_2 \setminus R_3 \) and two points of \( F_0 \setminus R_0 \). First blow up these four points and denote by the exceptional curves over the points of \( R_2 \setminus R_3 \) by \( E_1, E_2 \) and those over the points of \( F_0 \setminus R_0 \) by \( E_3, E_4 \), respectively. We denote by \( S' \) the obtained surface. Assume that a resolution \( Y \) of the normalization of the canonical cover of \( S \) is an inseparable double cover \( \pi : Y \to S' \), that is, \( S' = Y^{\partial} \) for a derivation \( \partial \) on \( Y \). We denote by \( R_{i1}, E_i, F_0 \) the preimages on \( Y \) of \( R_i, E_i, F_0 \), respectively. Note that \( \tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_4 \) on \( Y \) is the exceptional curves corresponding to four rational double points of type \( A_1 \). Assume that \( \tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_4, \tilde{R}_5, \tilde{R}_7 \) and \( \tilde{R}_9 \) are integral with respect to the derivation \( \partial \) on \( Y \). Then

\[
\tilde{E}_i = -2 \quad (i = 1, 2, 3, 4), \quad \tilde{R}_2 = -2, \quad \tilde{R}_3 = \tilde{R}_5 = \tilde{R}_7 = \tilde{R}_9 = -4, \\
\tilde{R}_1 = \tilde{R}_4 = \tilde{R}_6 = \tilde{R}_8 = \tilde{R}_{10} = \tilde{R}_{11} = -1.
\]

Note that the preimage of the fiber \( R_{10} + R_{11} \) of type \( \tilde{A}^+_1 \) is a union of two \((-1)\)-curves \( \tilde{R}_{10}, \tilde{R}_{11} \) meeting at one point transversally. Also the preimage of the cuspidal curve \( F_0 \) is a nonsingular rational curve \( \tilde{F}_0 \) with \( \tilde{F}_0^2 = -1 \).

Now we contract \((-1)\)-curves and then contract new \((-1)\)-curves successively as in the following order:

\[
\tilde{R}_1, \tilde{R}_4, \tilde{R}_6, \tilde{R}_8, \tilde{R}_5, \tilde{R}_7, \tilde{R}_3, \tilde{F}_0, \tilde{E}_3.
\]

On the other hand, we contract two \((-2)\)-curves \( \tilde{E}_1, \tilde{E}_2 \). Finally we get a surface \( R \) with two rational double points of type \( A_1 \). The quasi-elliptic fibration on \( S \) induces a conic bundle structure on \( R \) which has two singular fibers \( \tilde{R}_{10} + \tilde{R}_{11}, 2\tilde{R}_2 \) and a bisection \( \tilde{R}_9 \). Two rational double points of \( R \) lie on \( \tilde{R}_2 \).

On the other hand, to get an Enriques surface we need the following conditions:

\[
D = -2\tilde{E}_1 + 2\tilde{E}_2 + \tilde{E}_3 + \tilde{E}_4 + 2\tilde{R}_1 + 2\tilde{R}_2 + 2\tilde{R}_3 + 4\tilde{R}_4 + 3\tilde{R}_5 + 4\tilde{R}_6 + 2\tilde{R}_7 + 2\tilde{R}_8 + \tilde{R}_9
\]

where \( D \) is the divisor of \( \partial \) and

\[
K_Y = -\tilde{E}_1 + \tilde{E}_2 + 2\tilde{R}_1 + 2\tilde{R}_2 + 2\tilde{R}_3 + 4\tilde{R}_4 + 3\tilde{R}_5 + 4\tilde{R}_6 + 2\tilde{R}_7 + 2\tilde{R}_8 + \tilde{R}_9.
\]

Now we consider a surface \( R \) defined by

\[
S(aX_0^2 + bX_2^2 + T(X_1^2 + aX_1X_2 + bX_0X_2)) = 0 \quad (a, b \in k, a \neq 0, b \neq 0).
\]

The surface has a structure of a conic bundle \( R \to \mathbb{P}^1 \) with \( (S, T) \in \mathbb{P}^1 \). The fiber over \( (S, T) = (0, 1) \) is nonsingular (corresponding to the above \( \tilde{E}_4 \)), the fiber over \( (S, T) = (1, 0) \) is a double line and the fiber over \( (S, T) = (b^2, a^3) \) is a union of two lines. The line defined by \( X_2 = 0 \) is a bisection of the fiber space. The surface has two rational double points \((X_0, X_1, X_2, (S, T)) = ((\alpha, \beta, 1), (1, 0))\) of type \( A_1 \), where \( \alpha = \sqrt{b/a} \) and \( \beta_i \) is a root of the equation \( z^2 + a z + \sqrt{b^2/a} = 0 \).
Let \((x = X_0/X_2, \ y = X_1/X_2, \ s = S/T)\) be affine coordinates. Define
\[
\partial = \frac{1}{s} \left( a(s^2 + c) \frac{\partial}{\partial x} + (as^2x^2 + bc) \frac{\partial}{\partial y} \right), \quad (b \neq a^2c)
\]
where \(c\) is a root of the equation of \(z^2 + (b/a)z + 1 = 0\). Then \(\partial^2 = a\partial\), that is, \(\partial\) is 2-closed. A calculation shows that \(\partial\) has two isolated singularities at the intersection of the bisection and the two fibers over \((S, T) = (1, 0), (0, 1)\). As in the previous cases, we blow up at the two rational double points and isolated singularities of \(\partial\) successively, and finally get the surface \(Y\) and a desired derivation. Since \(R\) has parameters \(a, b\), we denote by \(S_{\alpha, \beta}\) the obtained Enriques surface.

**Theorem 8.10.60.** The surface \(S_{\alpha, \beta}\) is a classical Enriques surface with a crystallographic basis of type \(\tilde{E}_7^2\).

By the dual graph of \((-2)\)-curves we have the following Proposition.

**Proposition 8.10.61.** There are exactly one quasi-elliptic fibration with singular fibers of type \(2\tilde{E}_7 + \tilde{A}_1^*\) and two quasi-elliptic fibrations with a singular fiber of type \(\tilde{E}_8\).

**Theorem 8.10.62.** The automorphism group \(\text{Aut}(S_{\alpha, \beta})\) is isomorphic to \(\mathbb{Z}/2\mathbb{Z}\) which is not numerically trivial.

**Proof.** First we show that there are no numerically trivial involutions. Let \(\pi_1 : S_{\alpha, \beta} \to \mathbb{P}^1\) be a quasi-elliptic fibration with a singular fiber of type \(\tilde{E}_8\) and let \(\pi_2 : S_{\alpha, \beta} \to \mathbb{P}^1\) be the quasi-elliptic fibration with singular fibers of type \(2\tilde{E}_7 + \tilde{A}_1^*\) and with a bisection \(R_0\). Let \(F_1, F_2\) be the double fibers of \(\pi_1\) and let \(F_0\) be the remaining double fiber of \(\pi_2\). Note that \(F_1, F_2\) (resp. \(F_0\)) are bisections of \(\pi_2\) (resp. \(\pi_1\)). If \(g \in \text{Aut}(S_{\alpha, \beta})\) is numerically trivial, then \(g\) fixes four points on \(F_0\) (the cusp and the three intersection points with \(F_1, F_2\) and \(R_0\)). Hence \(g\) fixes \(F_0\) pointwisely. Then \(g\) fixes three points on \(F_i \ (i = 1, 2)\) (the cusp and the two intersection points with \(F_0\) and \(R_2\)) and hence fixes \(F_i\) pointwisely. Therefore \(g\) fixes three points on a general fiber of \(\pi_2\), that is, the cusp and two intersection points with \(F_1, F_2\). Thus \(g\) fixes a general fiber of \(\pi_2\) pointwise and hence \(g\) is trivial. Thus we have \(|\text{Aut}(S_{\alpha, \beta})| \leq 2\). On the other hand, the Mordell-Weil group \(\mathbb{Z}/2\mathbb{Z}\) of the Jacobian fibration of \(\pi_2\) acts on \(S_{\alpha, \beta}\). Thus we have the assertion.

**Remark 8.10.63.** By construction, the canonical cover of \(S_{\alpha, \beta}\) has four rational double points of type \(A_1\). The support of the conductrix is the union of 9 \((-2)\)-curves \(R_1, \ldots, R_9\).

**Example 8.10.64.** Type \(\tilde{D}_8\): \(\alpha_2\)-surface.

Recall that the crystallographic root basis of type \(\tilde{D}_8\) is as follows:

\[\begin{array}{ccccc}
R_3 & R_5 & R_6 & R_7 & R_9 \\
R_1 & R_2 & R_4 & R_8 & R_{10}
\end{array}\]

Figure 8.22:
We use the same idea to find a suitable rational surface and a derivation as in the case of the $\alpha_2$-Enriques surface of type $\tilde{E}_8$ (Example 8.10.35). On such an Enriques surface $S$, there exists a quasi-elliptic fibration with a singular fiber of type $2\tilde{D}_8$ and with a bisection $R_1$. Assume that the canonical cover has a rational double point of type $D_4$ over the point $P = R_1 \cap R_2$. First blow up $P$ and denote the exceptional curve by $E_1$. Then blow up the intersection points of $E_1$ and the proper transforms of $R_1$ and $R_2$ and denote the exceptional curves by $E_2$ and $E_3$ respectively. And blow up one point on $E_1$ not lying on $R_1$ and $R_2$ and denote the exceptional curve by $E_4$. We denote by $S'$ the obtained surface. Assume that a resolution $Y$ of the normalization of the canonical cover of $S$ is an inseparable double cover $\pi : Y \to S'$, that is, $S' = Y^\partial$ for a derivation $\partial$ on $Y$. We denote by $\tilde{R}_i$, $\tilde{E}_i$ the preimages on $Y$ of $R_i$, $E_i$. Note that the cycle $\tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3 + \tilde{E}_4$ on $Y$ is the exceptional divisor corresponding to the rational double point of type $D_4$. Since $E_i^2 = -2$ $(i = 1, 2, 3, 4)$ and $E_i^2 = E_i^2 = E_i^4 = -1$, $\tilde{E}_2, \tilde{E}_3, \tilde{E}_4$ should be integral and $\tilde{E}_1$ not with respect to the derivation $\partial$. We assume that $\tilde{R}_4$, $\tilde{R}_6$ and $\tilde{R}_8$ are integral with respect to $\partial$. Then

$$\tilde{R}_1 = \tilde{R}_2 = \tilde{R}_3 = \tilde{R}_7 = \tilde{R}_8 = 4, \quad \tilde{R}_4 = \tilde{R}_5 = \tilde{R}_6 = \tilde{R}_{10} = -1, \quad \tilde{R}_2 = \tilde{R}_2 = -2.$$  

Now we contract $(-1)$-curves and then contract new $(-1)$-curves except $\tilde{E}_4$ successively. Finally we get a projective plane $\mathbb{P}^2$. Thus $Y$ should be obtained from $\mathbb{P}^2$ by 13 times blowing ups.

To get an Enriques surface $S$, we need the following conditions:

$$D = -(2\tilde{R}_1 + 6\tilde{E}_2 + 8\tilde{E}_1 + 7\tilde{E}_3 + 5\tilde{E}_4 + 4\tilde{R}_2 + 3\tilde{R}_4 + 2\tilde{R}_3 + 4\tilde{R}_5 + 2\tilde{R}_6 + 2\tilde{R}_7 + \tilde{R}_8)$$  

where $D$ is the divisor of $\partial$ and

$$K_Y = -(2\tilde{R}_1 + 4\tilde{E}_2 + 6\tilde{E}_1 + 5\tilde{E}_3 + 3\tilde{E}_4 + 4\tilde{R}_2 + 3\tilde{R}_4 + 2\tilde{R}_3 + 4\tilde{R}_5 + 2\tilde{R}_6 + 2\tilde{R}_7 + \tilde{R}_8).$$  

Now we consider an affine plane $\mathbb{A}^2 \subset \mathbb{P}^2$ with affine coordinates $(x, y)$ and the following derivation

$$\partial_a = \frac{1}{x^a} \left( x(x^4 + x^2 + y^6) \frac{\partial}{\partial x} + (ax^6 + y(x^4 + x^2 + y^6)) \frac{\partial}{\partial y} \right),$$  

where $a \in k$, $a \neq 0$. Then $\partial_a^2 = 0$, that is, $\partial_a$ is 2-closed. Note that $\partial_a$ has a pole with order 5 along the line $\ell$ defined by $x = 0$ (corresponding to $\tilde{E}_4$) and $\ell$ is integral with respect to $\partial_a$. Moreover $\partial_a$ has a unique isolated singularity at $P = (0, 0)$. We blow up the point $P$. Then the exceptional curve is not integral with respect to the induced derivation. The induced derivation has a pole of order 2 along the exceptional curve and has a unique isolated singularity at the intersection of the exceptional curve and the proper transform of $\ell$. Then continue this process until the induced derivation has no isolated singularities. The resulting surface is denoted by $Y$ and the configuration of curves is as desired. Then contracting exceptional curves on $S' = Y^\partial$ described above, we obtain an Enriques surface $S_a$.

**Theorem 8.10.65.** The surface $S_a$ is an $\alpha_2$-Enriques surface with a crystallographic basis of type $\tilde{D}_8$. There exist exactly one quasi-elliptic fibration with a singular fiber of type $2\tilde{D}_8$ and two elliptic fibrations with a singular fiber of type $\tilde{E}_8$.

**Proof.** The non-trivial assertion is that any genus one fibration with singular fiber of type $\tilde{E}_8$ is an elliptic fibration. This follows from the fact that the conductrix is contained in the fiber of type $\tilde{E}_8$ (see Lemma 3.3 in [363]).
Finally we get a nonsingular quadric surface $P^{12}$ times blowing ups. To get an Enriques surface, we need the following conditions:

\[ Y^2 + TX^4 + TX^2 + aX + T^7 = 0. \] (8.10.9)

By using this equation we can determine the automorphism group.

**Theorem 8.10.67.** The automorphism group $\text{Aut}(S_a)$ is isomorphic to the quaternion group $Q_8$ of order 8 which is cohomologically trivial.

**Proof.** Here we give only a generator $\{\sigma, \alpha\}$ of $\text{Aut}(S_a)$ in terms of the equation (8.10.9):

\[ \sigma_{\omega, \alpha}(T, X, Y) = (T + \omega, X + \alpha + \omega^2 T, Y + \omega^2 X^2 + \omega X + \omega^2 T^3 + \sqrt{\omega} \alpha + \sqrt{\omega}) \]

where $\omega$ is a primitive cube root of unity and $\alpha$ is a root of the equation $z^2 + z + \omega \sqrt{\omega} + 1 = 0$. \qed

**Remark 8.10.68.** By construction, the canonical cover of $S_a$ has a rational double point of type $D_4$. The support of the conductrix is the union of eight $(-2)$-curves $R_1, \ldots, R_8$ on $S_a$.

**Remark 8.10.69.** We have already given a construction of classical extra-special surfaces of type $\tilde{D}_8$ in Example 8.10.40. It uses a non-degenerate bielliptic map of $S$ to the 4-nodal anti-canonical quartic del Pezzo surface $D_1$. It follows from this construction that all surfaces of this type form an irreducible one-dimensional family and hence coincide with the family of surfaces constructed in the previous example.

**Example 8.10.70.** Type $\tilde{D}_8$: Classical Case.

Classical Enriques surface of type $\tilde{D}_8$ has the same dual graph as in Figure 8.22. We use the same idea to find a suitable rational surface and a derivation as in the case of classical Enriques surfaces of type $\tilde{E}_8$ (Example 8.10.40). In this case a quasi-elliptic fibration with a singular fiber of type $2\tilde{D}_8$ has two double fibers. We denote by $F_0$ the irreducible one. Difference from the $\alpha_2$-surface is that the canonical cover has 4 rational double points of type $A_1$ instead of a rational double point of type $D_4$. So we blow up $F_0 \cap R_1, R_1 \cap R_2$, a nonsingular point on $F_0$ and a point on $R_2$. We denote by $E_1, E_2, E_3, E_4$ the exceptional curves, respectively. We use the same notation as in the case of the $\alpha_2$-Enriques surface of type $\tilde{D}_8$. We also assume that $\tilde{E}_i, (1 \leq i \leq 4), \tilde{R}_4, \tilde{R}_6$ and $\tilde{R}_8$ are integral with respect to a derivation $\partial$. The pullback of $F_0$ to $Y$, denoted by $\tilde{F}_0$, is a nonsingular rational curve. Obviously

\[
\tilde{E}_1^2 = \tilde{E}_2^2 = \tilde{E}_3^2 = \tilde{E}_4^2 = \tilde{R}_1^2 = \tilde{R}_2^2 = -2, \quad \tilde{R}_4^2 = \tilde{R}_5^2 = \tilde{R}_6 = \tilde{R}_8 = -4,
\]

\[
\tilde{F}_0^2 = \tilde{R}_3^2 = \tilde{R}_7^2 = \tilde{R}_9 = \tilde{R}_{10}^2 = -1
\]

Now we contract $(-1)$-curves and then contract new $(-1)$-curves except $\tilde{R}_1, \tilde{E}_3, \tilde{E}_4$ successively. Finally we get a nonsingular quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$. Thus $Y$ should be obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by 12 times blowing ups. To get an Enriques surface, we need the following conditions:

\[ D = -(2\tilde{E}_1 + 4\tilde{E}_2 + \tilde{E}_3 + 3\tilde{E}_4 + 2\tilde{R}_1 + 4\tilde{R}_2 + 2\tilde{R}_3 + 3\tilde{R}_4 + 4\tilde{R}_5 + 2\tilde{R}_6 + 2\tilde{R}_7 + \tilde{R}_8) \]

where $D$ is the divisor of $\partial$ and

\[ K_X = -(\tilde{E}_1 + 3\tilde{E}_2 + 2\tilde{E}_4 + 2\tilde{R}_1 + 4\tilde{R}_2 + 2\tilde{R}_3 + 3\tilde{R}_4 + 4\tilde{R}_5 + 2\tilde{R}_6 + 2\tilde{R}_7 + \tilde{R}_8). \]
Now we consider an affine open set \( \mathbb{A}^1 \times \mathbb{A}^1 \) of \( \mathbb{P}^1 \times \mathbb{P}^1 \) with affine coordinates \((x, y)\) and the following derivation

\[
\partial_{\alpha, \beta} = \frac{1}{xy^2} \left( ax^2y^2 \frac{\partial}{\partial x} + (x^4y^4 + by^4 + x^2y^2 + x^2) \frac{\partial}{\partial y} \right), \quad (a, b \neq 0 \in k)
\]

Then \( \partial_{\alpha, \beta}^2 = a \partial_{\alpha, \beta} \), that is, \( \partial_{\alpha, \beta} \) is 2-closed. Note that \( \partial_{\alpha, \beta} \) has a pole with order 1 along the divisor defined by \( x = 0 \) (corresponding to \( \tilde{E}_3 \)), a pole of order 3 along the divisor defined by \( x = \infty \) (corresponding to \( \tilde{E}_4 \)) and a pole of order 2 along the divisor defined by \( y = 0 \) (corresponding to \( \tilde{R}_1 \)). Moreover \( \partial_{\alpha, \beta} \) has two isolated singularities at \((x, y) = (0, 0), (\infty, 0)\).

Theorem 8.10.71. The surface \( S_{\alpha, \beta} \) is a classical Enriques surface with a crystallographic basis of type \( \tilde{D}_8 \).

Theorem 8.10.72. The surface \( S_{\alpha, \beta} \) has exactly one quasi-elliptic fibration with a singular fiber of type \( 2\tilde{D}_8 \) and two elliptic fibrations with a singular fiber of type \( \tilde{E}_8 \).

Remark 8.10.73. As in the case of the \( \alpha_2 \)-Enriques surface of type \( \tilde{E}_8 \), we can determine the function field \( k(x, y)^{\partial_{\alpha, \beta}} \) of the quotient surface and have the following equation of the surface birationally isomorphic to \( S_{\alpha, \beta} \):

\[
Y^2 + TX^4 + aT^3X^2 + bT^3X + T^3 + T^7 = 0. \tag{8.10.10}
\]

Theorem 8.10.74. The automorphism group \( \text{Aut}(S_{\alpha, \beta}) \) is \( \mathbb{Z}/2\mathbb{Z} \) which is numerically trivial.

Proof. Here we give only a generator \( \sigma \) of \( \text{Aut}(S_{\alpha, \beta}) \) in terms of the equation (8.10.10):

\[
\sigma(T, X, Y) = (T, X + \sqrt{a}T, Y + \sqrt{a}\sqrt{b}T^2)
\]

Remark 8.10.75. By construction, the canonical cover of \( S_{\alpha, \beta} \) has four rational double points of type \( A_1 \). The support of the conductrix is the union of eight \((-2)\)-curves \( R_1, \ldots, R_8 \) on \( S_{\alpha, \beta} \).

Example 8.10.76. Type \( \tilde{D}_4 + \tilde{D}_4 \).

Recall that the crystallographic root basis of type \( \tilde{D}_4 + \tilde{D}_4 \) is as follows:

\[
\begin{array}{c}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
R_2 & R_1 & R_4 & R_5 & R_6 & R_7 & R_8 & R_9 & R_{10} & R_{11}
\end{array}
\]

Figure 8.23:

We use the same idea to find a suitable rational surface and a derivation as in the case of classical Enriques surfaces of type \( \tilde{E}_8 \) (Example 8.10.40). In this case, the canonical cover of the desired
Enriques surface $S$ has four rational double points of type $A_1$. So we blow up $R_5 \cap R_6$, $R_6 \cap R_7$, a nonsingular point on $R_5$ and a point on $R_7$. We denote by $E_1$, $E_2$, $E_3$, $E_4$ the exceptional curves, respectively. We denote by $S'$ the obtained surface. Assume that a resolution $Y$ of the normalization of the canonical cover of $S$ is an inseparable double cover $\pi : Y \to S'$, that is, $S' = Y^{\varnothing}$ for a derivation $\partial$ on $Y$. We denote by $\tilde{E}_i$, $\tilde{E}_i$ the preimages on $Y$ of $R_i$, $R_i$. Note that the cycle $\tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_4$ on $Y$ is the exceptional curves corresponding to rational double points of type $A_1$. We assume that $\tilde{E}_i$, $(1 \leq i \leq 4)$, $\tilde{R}_4$ and $\tilde{R}_8$ are integral with respect to the derivation $\partial$. Then we have
\[ E_1^2 = E_2^2 = E_3^2 = E_4^2 = -2, \quad R_4^2 = R_8^2 = -4, \]
\[ \tilde{E}_1^2 = \tilde{E}_2^2 = \tilde{E}_3^2 = \tilde{E}_4^2 = -2, \quad \tilde{R}_4^2 = \tilde{R}_8^2 = -2. \]

Now we contract $(-1)$-curves and then contract new $(-1)$-curves except $\tilde{E}_3, \tilde{E}_4, \tilde{R}_6$ successively. Finally we get a nonsingular quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$. Thus $Y$ should be obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by $12$ times blowing ups.

To get an Enriques surface, we need the following conditions:
\[ D = -(3\tilde{E}_1 + 3\tilde{E}_2 + 2\tilde{E}_3 + 2\tilde{E}_4 + \tilde{R}_4 + 2\tilde{R}_5 + 2\tilde{R}_6 + 2\tilde{R}_7 + \tilde{R}_8) \]
where $D$ is the divisor of $\partial$ and
\[ K_Y = -(2\tilde{E}_1 + 2\tilde{E}_2 + \tilde{E}_3 + \tilde{E}_4 + \tilde{R}_4 + 2\tilde{R}_5 + 2\tilde{R}_6 + 2\tilde{R}_7 + \tilde{R}_8). \]

We consider an affine open set $\mathbb{A}^1 \times \mathbb{A}^1$ of $\mathbb{P}^1 \times \mathbb{P}^1$ with affine coordinates $(x, y)$ and the following derivation
\[ \partial_{\alpha, \beta} = \frac{1}{x^2 y^2} \left( b x^3 y^3 \frac{\partial}{\partial x} + (a x^2 y^2 + x^2 + x^4 y^4 + 4 x^2 y^3) \frac{\partial}{\partial y} \right), \quad (a, b \in k, \ b \neq 0 \in k) \]
Then $\partial_{\alpha, \beta}^{\alpha, \beta} = b^2 \partial_{\alpha, \beta}$, that is, $\partial_{\alpha, \beta}$ is $2$-closed. Note that $\partial_{\alpha, \beta}$ has poles of order $2$ along the divisors defined by $x = 0$ (corresponding to $\tilde{E}_3$), $x = \infty$ (corresponding to $\tilde{E}_4$) and $y = 0$ (corresponding to $\tilde{R}_6$). Moreover $\partial_{\alpha, \beta}$ has two isolated singularities at $(x, y) = (0, 0), (\infty, 0)$. The divisors defined by $x = 0$ and $x = \infty$ are integral with respect to $\partial_{\alpha, \beta}$. We first blow up $(x, y) = (0, 0), (\infty, 0)$. The induced derivation has poles of order $3$ along the two exceptional curves and has isolated zeros at the intersection points of the exceptional curves and the proper transforms of the divisors defined by $x = 0$ and $x = \infty$. The both two exceptional curves are integral. As in the case of the $\alpha_2$-Enriques surface of type $\tilde{E}_8$, we blow up the singular points of the induced derivations successively. The resulting surface is $Y$ and the configuration of curves is as desired. After contracting exceptional curves on $Y^{\varnothing}$ we obtain an Enriques surface $S_{\alpha, \beta}$.

**Theorem 8.10.77.** The surface $S_{\alpha, \beta}$ is a classical Enriques surface with a crystallographic basis of type $\tilde{D}_4 + \tilde{D}_4$.

**Remark 8.10.78.** As in the case of the $\alpha_2$-Enriques surface of type $\tilde{E}_6 + \tilde{A}_2$, we can determine the function field $k(x, y)^{\partial_{\alpha, \beta}}$ of the quotient surface and have the following equation of the surface birationally isomorphic to $S_{\alpha, \beta}$:
\[ Y^2 + TX^4 + a T^3 X^2 + b T^4 X + T^3 + T^7 = 0. \] \hspace{1cm} (8.10.11)

**Theorem 8.10.79.** The automorphism group $\text{Aut}(S_{\alpha, \beta})$ is $(\mathbb{Z}/2\mathbb{Z})^3$ and $\text{Aut}_n(S_{\alpha, \beta})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$.
Proof. Here we give only a generator \( \{ \sigma, \tau \} \) of \( \text{Aut}(S_{\alpha, \beta}) \) in terms of the equation (8.10.11):

\[
\sigma(T, X, Y) = (T, X + \alpha T, Y), \quad \tau(T, X, Y) = (1/T, X/T^2, Y/T^5),
\]

where \( \alpha \) is a root of the equation \( z^3 + az + b = 0 \).

Remark 8.10.80. There is a unique quasi-elliptic fibration \( p \) with singular fibers of type \( 2 \tilde{D}_4 + 2 \tilde{D}_4 \) and nine elliptic fibrations with a singular fiber of type \( \tilde{D}_8 \). By construction, the canonical cover of \( S_{\alpha, \beta} \) has four rational double point of type \( A_1 \). The support of the conductrix is the union of 5 \((-2\)-curves \( R_4, \ldots, R_8 \) on \( S_{\alpha, \beta} \).

Remark 8.10.81. Salomonsson [607] studied Enriques surfaces of type \( \tilde{E}_8, \tilde{E}_7, \tilde{E}_6 + \tilde{A}_2 \) in a different point of view and gave equations of such Enriques surfaces.

Bibliographical notes

Enriques himself knew that a general Enriques surface has infinite discrete group of birational automorphisms because it contains elliptic pencils [214].

The discovery of the precise structure of the automorphism group of a general Enriques surface had to wait until arriving of the transcendental methods based on the theory of periods of K3 surfaces developed in the work of I. Pyatetsky-Shapiro and I. Shafarevich [573]. Using these methods W. Barth and C. Peters [46], and independently, V. Nikulin [533] proved Theorem 8.3.4 that asserts that this group is isomorphic to the 2-level congruence subgroup of the Weyl group \( W_{237} \). The first geometric proof of this result, based on ideas of A. Coble, was given by the first author in [169]. The assumption that \( p \neq 2 \) was eliminated later in [103].

The structure of the automorphism group given in Theorem 8.4.7 of a general nodal Enriques surface over complex numbers can be also deduced from Nikulin’s paper. The first geometric proof of Theorem was given by F. Cossec and the first author [136], the details of the proof are given first time in sections 8.4 and 8.5 of the book. Both cases rely on Coble’s lattice theoretical description of natural involutions of Enriques surface [121] given by him, essentially, without proof. The computational proofs of these results given in [134] are now replaced by conceptual proofs provided by D. Allcock [6], [8]. based on the known structure of the automorphism group, Cossec computed the degrees of the forgetting map from the moduli space of polarized general Enriques surfaces to the moduli space of Enriques surfaces which we reproduced in Table 8.4.

The first example of a numerically trivial automorphism of an Enriques surface was constructed by D. Lieberman in 1976 [438]. Then second one was found by the first author [168] and Barth, Peters [46]. It was erroneously claimed by Mukai and Namikawa in that that there are no more example [503]. However, the second author has found a third example in [397]. It was later proven by Mukai that there are no more examples over the field of complex numbers [?]. One of the main tools of the Mukai-Namikawa classification is the Global Torelli Theorem for K3 covers of Enriques surfaces. The absence of these tools in the case of characteristic \( p > 0 \) requires different methods. A paper [183] of the first author was the first attempt to extend the work of Mukai and Namikawa to this case. Although the main result of the paper is correct when \( p \neq 2 \), some arguments were not complete and the analysis of possible groups in characteristic 2 was erroneous and far from giving a classification of possible groups. In section 8.2 we follow a paper of the first author and G. Martin [190] which gives possible structures of groups of numerically or cohomologically trivial automorphisms of Enriques surfaces over fields of arbitrary characteristic. The extension of Mukai-Namikawa classification to the case of positive odd characteristic given in Theorem 8.2.23 is new.

A systematic study of possible groups of automorphisms of complex Enriques surfaces was undertaken by S. Mukai and Ohashi [508],[509], [510]. The goal here is to find them in terms of their Nikulin \( R \)-invariant. In his fundamental work Mukai classified finite automorphisms groups of K3 surfaces which act trivially
on the space of regular 2-forms [504]. They turned out to be subgroups of the Mathieu group $M_{23}$. By analogy, Mukai introduces finite automorphisms groups of Enriques surfaces of Mathieu type and together with Ohashi they classify Enriques surfaces admitting such a group of automorphisms [509]. We give an exposition of his results in Section 8.7. The classification of cyclic groups of automorphisms of complex Enriques surfaces was given by H. Ito and H. Ohashi [330]. In Section 8.6 we reproduce this classification in any characteristic except 2 by using different methods that do not rely on the theory of periods of K3 surfaces and give the classification.

In his fundamental paper [214] Enriques asked whether the sextic model of an Enriques surface could degenerate in such a way that the group of its birational automorphisms becomes finite. An example of a family of such surfaces was given by the first author in [168]. It is now the family of type I in the Kondo-Nikulin classification. It was later discovered by the first author that much earlier Gino Fano gave another example, although his construction lacks of rigor and his claim about the automorphism group was wrong [222]. Fano’s example is a surface of type VII with the group of automorphisms isomorphic to $S_5$ (instead of $S_3$ claimed by Fano).

V. Nikulin was the first to give a classification of complex Enriques surfaces with finite automorphisms in terms of their periods expressed by his $R$-invariant [535]. One of the cases was missing and no geometric construction was given. In [397] the second author gave a geometric classification of such surfaces. Both results rely on the theory of periods of Enriques surfaces and some of the surfaces do not exist over fields of positive characteristic. The classification of Enriques surfaces with finite automorphism group in positive characteristic $p \neq 2$ was give by G. Martin [469]. Together with T. Katsura and the second author, they were able to finish the classification in characteristic 2 [361], [363].
Chapter 9

Rational Coble surfaces

9.1 Rational Coble surfaces of K3 type

Recall from Volume 1 Chapter 5 that a Coble surface is a rational surface $V$ with $|−K_V| = ∅$ and $|−2K_V| ≠ ∅$. We will be concerned only with Coble surfaces of K3 type for which we assume additionally that $|−2K_V| = \{C\}$, where $C$ is a reduced divisor. It follows from [174, Lemma 1.4] that $C$ is the union of smooth rational curves forming a simple normal crossing divisor. Blowing up its singular points, we obtain a Coble surface $V'$ with smooth anti-bicanonical divisor $C'$. Such a Coble surface of K3 type is called terminal. Its irreducible components $C_1, \ldots, C_n$ are smooth rational curves with self-intersection $−4$ (−4-curves for short). They are called the boundary components of $V$. We have

$$K_V^2 = −n,$$

(9.1.1)

where $s$ is the number of connected components of $C$.

If $p ≠ 2$, the $\mu_2$ cover $X' \to V'$ branched over $C$ is a K3 surface. The ramification divisor of the cover is the disjoint union of $n$ smooth rational curves. Conversely, if $g$ is an involution of a K3 surface $X$ with $X^g$ equal to the union of smooth rational curves, then the quotient $X/(g)$ is a terminal Coble surface of K3 type. If $p = 2$, the cover is an inseparable $\mu_2$-cover.

Recall that a rational surface $Z$ is called a basic rational surface if it admits a birational morphism to $\mathbb{P}^2$. An example of a non-basic rational surface is a minimal ruled surface $F_n, n ≠ 1$.

Lemma 9.1.1. A non-minimal rational surface $Z$ is not basic if and only if $Z$ satisfies the following property:

\( (*) \) There exists a birational morphism $\sigma : X \to F_n, n ≥ 2$, such that the full pre-image $\sigma^*(\epsilon)$ of the divisor class $\epsilon$ of the exceptional section $\epsilon$ on $F_n$ is equal to $[E] + e_1 + ⋯ + e_k$, where $[E]$ is the class of the proper transform of $\epsilon$ and $(e_1, \ldots, e_k)$ are the classes of the exceptional curves on $X$ with $e_i^2 = 0, e_i \cdot e_j = 0$ and $K_X = \sigma^*(K_{F_n}) + e_1 + ⋯ + e_k$.

Proof. Suppose $X$ is not a basic rational surface. A rational surface $Z$ always admits a birational morphism to a minimal rational surface, $\mathbb{P}^2$ or $F_n$. Suppose $\pi : Z \to F_0$ is a birational morphism.
Since \( Z \neq \mathbb{F}_0 \), it factors through the blow-up \( Z' \to \mathbb{F}_0 \) of a point \( x \in \mathbb{F}_0 \) and a birational morphism \( \pi' : Z \to Z' \). Composing \( \pi' \) with the birational morphism \( Z' \to \mathbb{P}^2 \) that blows down the proper transforms of two lines passing through \( x \), we obtain that \( Z \) is a basic rational surface. So, we may assume that \( Z \) admits a birational morphism \( \pi : Z \to \mathbb{F}_n \), where \( n \geq 2 \) chosen to be minimal possible.

Suppose \((*)\) is not satisfied. Without loss of generality, we may assume that \( \sigma^* (\epsilon) = [E] + e_1 + \cdots + e_s \), where \( s < k \). Then the image of the exceptional configuration representing \( e_k \) in \( X \) is a point \( p \) not lying on \( e \). This shows that \( \sigma \) is a composition of a birational morphism \( X \to Z \to \mathbb{F}_n \), where \( Z \) is the blow-up of \( p \). Let \( Z \to \mathbb{F}_{n-1} \) be the blowing down of the fiber of \( \mathbb{F}_n \) containing \( p \). Then the composition \( X \to Z \to \mathbb{F}_{n-1} \) gives a contradiction with the minimality of \( n \).

To prove the converse, suppose \((*)\) is satisfied. Then \( k = \rho(X) - \rho(\mathbb{F}_n) = \rho(X) - 2 \) and \( E^2 = e^2 - k = -n - \rho(X) \). If \( X \) is basic, then it is obtained from \( \mathbb{P}^2 \) by blowing up \( k+1 = \rho(X - \rho(\mathbb{P}^2)) \) times. The image of \( E \) in \( \mathbb{P}^2 \) has self-intersection \( \leq E^2 + k + 1 = -n - k + k + 1 = -n_1 \leq -1 \), a contradiction.

\[ \square \]

**Proposition 9.1.2.** A Coble surface \( V \) admits a birational morphism \( f : V \to \mathbb{P}^2 \) such that the image of the anti-bicanonical curve \( C \) is a curve \( W \) of degree 6. The fundamental points of \( f^{-1} : \mathbb{P}^2 \to V \) are among singular points of \( W \) (maybe infinitely near).

**Proof.** Assume that \( V \) is not a basic type and let \( \pi : V \to \mathbb{F}_n \) be a minimal model with minimal possible \( n \). Obviously \( n = 0 \) is excluded and hence \( n \geq 2 \). Let \( s'_n \) be the proper transform of the negative section \( s_n \) of the projection \( \mathbb{F}_n \to \mathbb{P}^1 \). Then \( s'^2_n \leq -n \). On the other hand, by the adjunction formula, \( s'^2_n = -4, \ -2 \) or \(-1\), and hence \( n \) is at most 4. If \( n = 3, 4 \), by using the facts that \( \rho(V) \geq 11 \) and any smooth rational curve on \( V \) has self-intersection number \(-4, 2 \) or \(-1\), one can easily see that the map \( \pi \) factorizes \( V \to V' \to \mathbb{F}_n \) where \( \sigma \) is the blow-up of a point \( x \notin s_n \).

Then we have a birational morphism \( V \to \mathbb{F}_{n-1} \) which contradicts the minimality of \( n \).

Now consider the case \( n = 2 \). We freely use the facts that the minimality of \( n \), \( \rho(V) \geq 11 \) and any smooth rational curve on \( V \) has self-intersection number \(-4, 2 \) or \(-1\). To get the map \( \pi \), we should first blow-up a point \( x \in s_n \) and then blow-up the intersection of the exceptional curve \( E \) and the proper transform of \( s_n \). Next we should blow-up a general point of the \((-2)\)-curve which is the proper transform of \( E \). Continuing this, we have a sequence of \((-4)\)-, \((-1)\)- and \((-2)\)-curves, otherwise the map \( \pi \) factorizes \( V \to V' \to \mathbb{F}_n \) as above. We denote the sequence of these curves by

\[ B_1, E_1, B_2, E_2, \ldots, B_{k-1}, E_{k-1}, B_k, E_k, R \]

where \( B^2_1 = -4, E^2_1 = -1, R^2 = -2 \), \( B_1 \) is the proper transform of \( s_n \) and adjacent curves meet at one point transversely. There are two possibilities:

(A) \( V \) has been obtained;

(B) \( V \) is obtained after blowing-ups two general points on \( R \).

We consider the case (A) (the case (B) is similar). Let \( N \) be the number of boundary components. Since \( V \) is obtained from \( \mathbb{F}_2 \) by blowing-ups \( 2k \) points, \( \rho(V) = 2 + 2k \) and \( \rho(\mathbb{F}_2) = 11 \).
$-K_{p_2}^2 + 2k = -8 + 2k$. Since $E_k$ meets only $B_k$ with multiplicity 1 among boundary components $B_1, \ldots, B_k$, there exists a $(-4)$-curve meeting with $E_k$. Therefore $N = 2k - 8 \geq k + 1$ and hence $k \geq 9$. Let $B_{k+1}, \ldots, B_N$ be the remaining boundary components with $E_k \cdot B_{k+1} = 1$. Note that $B_1, E_1, B_2, \ldots, B_{k-1}, E_{k-1}, B_k, E_k, R$ generate a negative definite sublattice of $\text{Pic}(V)$ because these curves are perpendicular to the total transform of a positive section of $\mathbb{F}_2 \to \mathbb{P}^1$. Moreover $B_{k+2}, \ldots, B_N$ are perpendicular to these curves, all of them generate a negative definite sublattice of rank $3k - 8$. By Hodge index theorem, we have $2 + 2k > 3k - 8$, that is, $k < 10$. Thus we conclude $k = 9, n = 10$.

Now we show that there exists a birational morphism from $V$ to $\mathbb{P}^2$. Let $F$ be the proper transform of the fiber of $\mathbb{F}_2 \to \mathbb{P}^1$ containing the point $x$. We can easily check that $F^2 = -1$, otherwise we can not blow-ups $\mathbb{F}_2$ 18 times. By blowing down the nineteen curves $F, E_1, E_2, \ldots, E_9, B_2, B_3, B_1, R, B_9, \ldots, B_5$ successively, we have a smooth surface with the Picard number 1, that is, $\mathbb{P}^2$. This is a contradiction. Thus we have proved that $V$ is a basic type.

Finally we show the second claim. Let $C \in | - 2K_V|$. Then $\pi_*(C) = -2K_{p_2}$, hence the image of $C$ on $\mathbb{P}^2$ is a curve $W$ of degree 6. We have $\pi^*(-2K_{p_2}) \sim \phi^*(W) \sim C + 2E$, where $E = \sum E_i$ is the exceptional divisor. This shows that $\pi_*(E)$ is contained in the set of points of $W$ of multiplicity $\geq 2$.

**Remark 9.1.3.** We used in the proof that each boundary component of a terminal surface is $(-4)$-curve. For a Coble surface of not K3 type, the minimal model of $V$ could be different from $\mathbb{P}^2$ (see [174]). Also, the assumption that $V$ is a rational surface is essential too. There are ruled non-rational surfaces satisfying the conditions on the $| - mK_V|$ as above.

**Proposition 9.1.4.** A Coble surface $V$ of K3 type can be obtained from an Halphen surface $Z$ of index 2 by blowing-up some singular points of one singular reduced fiber of a genus one fibration or from a jacobian rational elliptic surface by blowing up some singular points on two reduced fibers. The second case does not happen if $n = 1$. The surface is terminal if and only if one blows up all singular points of one fiber (resp. two fibers).

**Proof.** Obviously, we may assume that $V$ is terminal. Let $f : V \to \mathbb{P}^2$ be a birational morphism. Since $K_V^2 = -n < 0$, the map $f$ is a composition of a birational morphism $\sigma : V \to Z$ and $\phi' : Z \to \mathbb{P}^2$, where $K_Z^2 = 0$. By Riemann-Roch, $h^0(-K_Z) \geq 1$. We use induction on $n = -K_V^2$.

Suppose $n = 1$, then $\sigma$ is the blowing-down of a $(-1)$-curve $E$. We have $E \cdot C = -2K_V \cdot E = 2$. The image of $C$ under the blowing down is an irreducible curve $\bar{C} \in | - 2K_Z|$ of arithmetic genus one with a double point. Suppose $h^0(-2K_Z) = 1$, then $| - 2K_Z| = \{2A\}$, where $A \in | - K_Z|$. Since $\sigma_*(-2K_Z) - 2E = 2\sigma^*(A) - 2E = C \geq 0$, we must have $-K_V = \sigma^*(A) - E \geq 0$, contradicting the definition of a Coble surface. Thus, $a = \dim | - 2K_Z| > 0$ and the exact sequence

$$0 \to \mathcal{O}_Z \to \mathcal{O}_Z(\bar{C}) \to \mathcal{O}_{\bar{C}}(\bar{C}) \to 0$$

shows that $\mathcal{O}_{\bar{C}}(\bar{C}) \cong \mathcal{O}_C$ and $| - 2K_Z|$ is a genus one pencil. By definition, $Z$ is an Halphen surface of index 2.

Now suppose $n > 1$. Let $\sigma : V \to V'$ $\to Z$, where $\sigma' : V \to V'$ is the blowing down of a $(1)$-curve $E$. Then, again $C \cdot E = 2$, and $E$ intersects one boundary component, say $C_1$, with multiplicity 2 or two boundary components, say $C_1, C_2$, each with multiplicity 1. In the first case,
the image of \( C_1 \) is an irreducible curve \( \bar{C} \) of arithmetic genus one with a double point and the images of \( C_2, \ldots, C_s \) are \((-4)\)-curves \( \bar{C}_i \). If \( h^0(-K_{V'}) = 0 \), \( V' \) is a Coble surface of K3 type, applying induction, we are done. Suppose \( h^0(-K_{V'}) > 0 \), as above we show that \( \dim | -2K_{V'} | = 1 \) and \( | -C_1 | \) is a genus one pencil, the movable part of the linear system. Composing with the birational morphism to a relatively minimal genus one surface \( Z' \) with \( | -2K_{Z'} | \) containing an irreducible reduced divisor or containing a disjoint sum of \( s \) genus one curves. This must be an Halphen surface of index 1, i.e. a jacobian rational genus one surface. Thus we arrive at an Halphen surface of index 2.

In the second case, the image of \( C_1 + C_2 \) is a divisor \( D_1 \) that consists of two \((-3)\)-curves intersecting transversally at one point. We have \( | -2Z' | = \{ D_1 + \bar{C}_3 + \cdots + \bar{C}_s \} \), hence \( h^0(-2K_{V'}) = 1 \). By above this forces \( h^0(-K_{V'}) \) to be equal to 0. Thus \( V' \) is a Coble surface of K3 type with \( K_{V'}^2 = s - 1 \). By induction we are done.

**Corollary 9.1.5.** Let \( V \) be a Coble surface of K3 type. Then

\[
K_V^2 \geq -10.
\]

In particular, the number of boundary components of a terminal Coble surface of K3 type is at most 10.

**Proof.** We know that \( V \) is obtained by blowing up singular point of one reduced singular fiber (resp. two such fibers) of an Halphen surface \( H \) of index two (resp. one). We may assume that \( V \) is terminal. In the first case the fiber must be of type \( A_n \) and in the second case the two fibers are of types \( A_k, A_m \) (or their degenerations \( A^*_n \)). We know that \( n \leq 8 \) and \( k + m \leq 8 \). This gives that the maximal number of singular points is equal to 9 in the first case and 10 in the second case.

**Remark 9.1.6.** If an Halphen surface has a non-reduced singular fiber, we can blow up points on a non-reduced component as many times as we want to obtain a Coble surface which is not of K3 type. So, for an arbitrary Coble surface \( V \), there is no lower bound for \( K_V^2 \).

**Example 9.1.7.** The case of a terminal Coble surface of K3 type with \( n = 1 \) was the case originally considered by A. Coble. In this case the image of the anti-bicanonical curve \( C \) is a rational nodal sextic i.e. an irreducible plane sextic \( W = V(F_6) \) with 10 ordinary nodes or cusps (maybe infinitely near). If we choose nine of them, then we will be able to pass a cubic curve \( V(F_3) \) through them. The pencil \( V(\lambda F_6 + \mu F_3^2) \) is an Halphen pencil of index 2. The minimal resolution of its base points is an Halphen surface of index 2. The Coble surface \( V \) is obtained by blowing up a singular point of an irreducible simple fiber of the pencil. If we blow-up a singular point of a reducible simple fiber \( F \) instead, we obtain a non-terminal Coble surface of K3 type with \( n = 1 \). Blowing up the singular points of its connected anti-bicanonical curve equal to the proper transform of the fiber, we obtain a terminal Coble surface of K3 type with \( s \) equal to the number of components of \( F \).

Let \( V \) be a Coble surface of K3 type with boundary components \( C_1, \ldots, C_s \). Let \( H \) be nef and big divisor such that \( H \cdot C_i = 0 \). We can choose \( H \) such that the map \( V \to X \) defined by the linear system \( |H| \) is an isomorphism outside \( C \). It blows down each \( C_i \) to a singular point \( x_i \in X \). It follows from Proposition 9.1.4 that the exceptional curve over each point \( x_i \) is either a \((-4)\)-curve of a chain of rational curves \( R_1 + R_2 + \cdots + R_k \), where \( R_i^2 = R_{k+1}^2 = -3, R_i^2 = -2, i \neq i, k \) and \( R_i \cdot R_{i+1} = 0 \).
9.1. RATIONAL COBLE SURFACES OF K3 TYPE

The singular points of these types are special cases of a toric singularity of class $\mathcal{T}$. It is denoted by $\frac{1}{dn}(1, dna - 1)$. If $(p, dn) = 1$, they are special quotient singularities and the notation agrees with those. We will be interested only in singularities $\frac{1}{dn}(1,2k-1)$ which have exceptional curve with $k$ irreducible components as above. In arbitrary characteristic they are isomorphic to the quotient of a double rational point of type $A_k$ by $\mu_2$. Their Gorenstein index is equal to 2. The speciality of toric singularities of class $\mathcal{T}$ is that they admit $\mathbb{Q}$-Gorenstein smoothing.

We refer for a proof of the following theorem to [440, Theorem 4.7].

**Theorem 9.1.8.** Let $X$ be a normal projective surface over $\mathbb{k}$ with only toric singularities of type $\mathcal{T}$ besides double rational points. Assume that $X$ satisfies the following conditions

(a) $H^2(X, \Theta_{X/\mathbb{k}}) = 0$,
(b) $H^2(X, \mathcal{O}_X) = 0$.

Then there is a deformation $f : \mathcal{X} \to T$ of $X$ over nonsingular algebraic curve $T$ defined over $\mathbb{k}$ such that

(i) The morphism $f$ is projective and it is smooth over $T \setminus \{t_0\}$;
(ii) The fiber $\mathcal{X}_{t_0}$ over $t_0$ is isomorphic to $X$;
(iii) The scheme $\mathcal{X}$ is normal and $rK_{\mathcal{X}}$ is Cartier;
(iv) If $r$ is the Gorenstein index of $X$, then $\mathcal{O}_{\mathcal{X}}(rK_{\mathcal{X}}) \otimes \mathcal{O}_{\mathcal{X}_{t_0}} \cong \mathcal{O}_X(rK_X)$.
(v) For any $t \neq t_0$, $K^2_{\mathcal{X}_t} = K^2_X$, $H^i(\mathcal{X}_t, \mathcal{O}_{\mathcal{X}_t}) \cong H^i(X, \mathcal{O}_X) = 0$;
(vi) For any $t \neq t_0$, $H^2(\mathcal{X}_t, \Theta_{\mathcal{X}_t/\mathbb{k}}) = 0$.

We apply this theorem to our situation where $2K_X = 0$.

**Corollary 9.1.9.** Let $\mathcal{V}$ be a Coble surface of K3 type. Then there exists a projective flat morphism $f : \mathcal{X} \to T$ over nonsingular algebraic curve $T$ defined over $\mathbb{k}$ such that $\mathcal{X}_{t_0}$ is isomorphic to the blow-down of the anti-bicanonical curve $C$ of $\mathcal{V}$ and $\mathcal{X}_t$ is a classical Enriques surface.

The only observation which is not obvious is the fact that $\tilde{H}^2(S, \Theta_S) \cong H^0(S, \Theta_S) \neq 0$ for an exceptional classical Enriques surface (see Theorem 1.4.10 and Table after the proof of this theorem).

One can say more.
Let $\phi : V \to \mathbb{P}^1$ be a genus one fibration on $V$ (not relatively minimal). Let $\sigma : V \to H$ be the birational morphism to an Halphen surface. It coincides with the canonical morphism to a relatively minimal genus one (elliptic or quasi-elliptic) surface. Thus any genus one fibration on $V$ originates from a genus one fibration on some Halphen surface. Since a general fiber $F$ of $\phi$ is nef, and $F \cdot K_V = 0$, we see that any boundary components $C_i$ of $V$ is contained in fibers of $\phi$. Thus, after we blow-down the boundary components, we obtain a genus one fibration on the normal $\mathbb{Q}$-Gorenstein surface $X$.

**Proposition 9.1.10.** Let $\sigma : V \to H$ be a morphism of a Coble surface of K3 type to an Halphen surface $H$. Let $X_i$ be a smooth fiber of the deformation from Corollary 9.1.9. Then there exists a deformation such that the genus one fibration on $H$ deforms to a genus one fibration on $X_i$. If $V$ is obtained from blowing up singular points of one simple fiber of type $\tilde{A}_n$ (resp. two simple fibers of types $\tilde{A}_a$ and $\tilde{A}_b$), then the deformed fibration has a singular fiber of type $\tilde{A}_n$ (resp. two singular fibers of types $\tilde{A}_a$ and $\tilde{A}_b$).

Let $\sigma : V \to H$ be a morphism of a Coble surface of K3 type to an Halphen surface $H$. Let $E_1, \ldots, E_l$ be the $(-1)$-curves that are blown down to singular point $y_1, \ldots, y_l$ of a fiber $F$ of type $\tilde{A}_n$ on $X$. Then the proper transform of $F$ on $V$ is the union of $l$ disjoint $(-4)$-curves.

We know from Section 5.4 that the moduli space of complex Coble surfaces with $n$ boundaries is irreducible of dimension $10 - n$ if $n \neq 8$ and consists of two irreducible components when $n = 8$. This follows from the computation of the Picard lattice of their canonical K3-covers. Let us give a construction of a general member $V$ of an irreducible family of dimension $10 - n$ for each $n = 1, \ldots, 10$.

In the first examples we exhibit $V$ as the blow up of the plane at double points of a plane sextic curve $W$ with only nodes as singularities.

$n = 1$: $W$ is an irreducible rational sextic with 10 nodes.

$n = 2$: $W$ is the union of two nodal plane cubics intersecting at 9 distinct points.

$n = 3$: $W$ is the union of three conics intersecting pairwise at 4 distinct points.

$n = 4$: $W$ is as in the previous case only one conic is reducible.

For $5 \leq n \leq 10$ we exhibit $V$ as the blow-up of singular points of the union of five conics on an anti-canonical del Pezzo surface $D_5$ of degree 5 taken from different five pencils of conics on $V$. The proper transforms of the irreducible components of the conics is the boundary of $V$.

Recall that $D_5$ is isomorphic to the blow-up of 4 points $p_1, \ldots, p_4$ in the plane no three of which are collinear. We use the standard geometric basis $(e_0, e_1, \ldots, e_4)$ of $\text{Pic}(D_5)$ defined by the blow-up. The five pencils $C_i$ of conics are the linear systems $|e_0 - e_1|, \ldots, |e_0 - e_4|, |2e_0 - e_1 - e_2 - e_3 - e_4|$. The surface has 10 lines which we index by duads $(ab)$ from $[1, 5]$. They are the exceptional curves $E_{a5}$ over the points $p_a$ and the proper transforms of lines $\ell_{ab} \in |e_0 - e_c - e_d|$, where $\{a, b, c, d\} = \{1, 2, 3, 4\}$.

$n = 5$: $C_i \in C_i, i = 1, \ldots, 5$, are irreducible. Their images in the plane are 4 general lines passing through $p_1, \ldots, p_4$ and a conic through these points.
9.1. RATIONAL COBLE SURFACES OF K3 TYPE

\( n = 6 \): One of the conics \( C_i \) is reducible. There are three such conics in each pencil. If we choose it from \( C_5 \), then it is equal to the proper transform of two lines \( \ell_{a,b}, \ell_{cd} \). In this way we see \( V \) as the blow-up of intersection points of six lines in general linear position. If we choose \( C_i \) from \( C_i, i \neq 5 \), then it is equal to the exceptional curve \( E_i \) and a line whose image in the plane is a line \( \ell_{ij} \). There are 5 projectively non-equivalent, but Cremona equivalent, ways to exhibit \( D_5 \) as the blow-up of 4 points in the plane and hence \( 5! = # \text{Aut}(D_5) \) ways to define a geometric basis in \( \text{Pic}(V) \). All different choices of one reducible conic are equivalent under a change of a geometric basis. In another words, if we consider a family of marked del Pezzo surfaces, there will be \( 15 = 3 \times 5 \) different choices of a reducible conic, but if we forget about the marking there will be only one choice.

\( n = 7 \): Two conics are reducible. We can choose them from two pencils \( C_i, C_j, i, j \neq 5 \), say from \( C_1 \) and \( C_2 \). The image will be two lines \( \ell_{1,i}, \ell_{2,j} \), two general lines through \( p_i, p_j \) and a smooth conic through \( p_1, p_2, p_3, p_4 \). Applying a quadratic Cremona transformation with fundamental points at \( p_1, p_i, p_j \) we may assume that one of the reducible conics is taken from \( C_5 \). We can take one of them from \( C_5 \) and two from \( C_1, C_2 \). The image of the five conics in the plane could be the union of four lines \( \ell_{13}, \ell_{24}, \ell_{34} \), two general lines through \( p_3, p_4 \) and the union of two lines \( \ell_{a,b}, \ell_{cd} \). There will be two different choices: (a) the two lines are \( \ell_{14}, \ell_{23} \) or (b) one of these lines passes through \( p_3, p_4 \). We can draw the following two pictures which display the two different choices:

![Figure 9.1: Two families of Coble surfaces with 8 boundary components](#)

Here the parallel lines meet at one of the points \( p_3, p_4 \). In the second picture one of the lines is the line at infinity passing through \( p_3, p_4 \).

By using the rational map \( \mathbb{P}^2 \to Q = \mathbb{P}^1 \times \mathbb{P}^1 \) defined by the linear system of conics through \( p_3, p_4 \), we may consider a surface of type (a) as the blow-up of the intersection points of 8 lines on \( Q \), four from each family of lines and a surface of type (b) as the blow-up of the intersection points of 6 lines, three from each family, and a conic passing through one of the intersection points. We also blow-up the infinitely near points to \( q \) corresponding to the tangent directions of the two lines and the conic at this point.

We know that the moduli space of Coble surfaces with 8 boundary components consists of two
irreducible components. It is a natural guess that the two different choices (a) and (b) lead to different components. To show this we compute the Picard lattice of the canonical K3-covers $X$ of these surfaces. Using the model of $V$ as the blow-up of points on a quadric, we locate two elliptic pencils on $X$ coming from the two rulings on $Q$. In case (a), it has 4 fibers of type $D_4$ and four disjoint sections. The Picard lattice is isomorphic to $U \oplus D_8 \oplus D_8$. In case (b), it has two fibers of type $D_6$, two fibers of type $A_1$ and 4 sections. The Picard lattice is isomorphic to $U \oplus E_7 \oplus D_7 \oplus A_1 \oplus A_1$. The discriminant group in both cases is $(\mathbb{Z}/2\mathbb{Z})^{\oplus 4}$ but the discriminant form is even in case (a) and odd in case (b).

$n = 9$: four conics are reducible. We may take for the images of the 5 conics the union of four lines $\ell_{12}, \ell_{23}, \ell_{34}, \ell_{14}$ and a smooth conic through the 4 points.

$n = 10$: all conics are reducible. The surface is the blow-up of 15 intersection points of ten lines on $D_5$. The image of the union of the five conics is the complete quadrangle of lines and its two diagonals. We will discuss this surface later in Example 9.2.7.

### 9.2 Coble-Mukai lattice

From now on we will consider only terminal Coble surfaces of K3 type. Let $V$ be such a surface, $C = C_1 + \cdots + C_n$ its anti-bicanonical curve with boundary components $C_1, \ldots, C_n$. We denote by $\beta_i$ the divisor class of $C_i$. We have $\beta_i^2 = -4, K_V \cdot \beta_i = 2$. Let $\text{Pic}(V)$ be the $\mathbb{Z}$-submodule of the quadratic vector $\mathbb{Q}$-space $\text{Pic}(V)_\mathbb{Q}$ generated by $\text{Pic}(V)$ and $\frac{1}{2} \beta_1, \ldots, \frac{1}{2} \beta_n$. Following S. Mukai, we introduce the following quadratic lattice.

$$\text{CM}(V) := \{ x \in \overline{\text{Pic}}(V) : x \cdot \beta_i = 0, i = 1, \ldots, n \}. \quad (9.2.1)$$

Let us see that it is indeed a quadratic lattice. Any $x \in \text{CM}(V)$ may be expressed as $y + \sum_{i=1}^{k} r_i \beta_i/2$ where $y \in \text{Num}(S)$ and $r_1, \ldots, r_k \in \mathbb{Z}$. Rewriting this as $y = x - \sum r_i \beta_i/2$, expressing $x' \in \text{CM}(V)$ similarly, and using $x, x' \in \beta_i^\perp$ gives

$$y \cdot y' = x \cdot x' + \sum r_i r_i' \frac{\beta_i \cdot \beta_i}{4} = x \cdot x' - \sum r_i r_i'$$

This proves $x \cdot x' \in \mathbb{Z}$.

We call $\text{CM}(V)$ the Coble-Mukai lattice of $V$. If $n = 1$, the lattice coincides with $K_V^\perp$ and hence it is isomorphic to the Enriques lattice $E_{10}$. We will show later that the Coble-Mukai lattice is always isomorphic to $E_{10}$ under the assumption $k = \mathbb{C}$ (see Theorem 9.2.15).

Obviously, the automorphism group $\text{Aut}(V)$ leaves the set of curves $\{ C_1, \ldots, C_n \}$ invariant and hence acts on the lattice $\text{CM}(V)$.

By Proposition 9.1.4 there exists a birational morphism $\phi : V \to H$, where $H$ is an Halphen surface of index 1 or 2. Let $\pi : H \to \mathbb{P}^2$ be a birational morphism and let $(e_0, e_1, \ldots, e_9)$ be a geometric basis of $\text{Pic}(H)$ (see Section 0.3 in Part I). Let $(e_0, e_1, \ldots, e_9, r_1, \ldots, r_n)$ be a geometric basis of $\text{Pic}(V)$.

Let $W_{\text{nod}}^\alpha$ be the reflection group of $V$, the subgroup of $O(\text{CM}(V))$ generated by reflections $s_\alpha$, where $\alpha \in \text{CM}(V)$ is an effective divisor class with $\alpha^2 = -2$. We call such a divisor class an
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effective root. We say that an effective root \( \alpha \) is irreducible if \(|\alpha - \beta| = \emptyset \) for any other effective root \( \beta \).

Since \(|-2K_V| = \{C_1 + \cdots + C_n\}\), the only curves with negative self-intersection on \( V \) are the curves \( C_1, \ldots, C_n \), \((-2)\)-curves or \((-1)\)-curves.

Lemma 9.2.1. Let \( \alpha \) be an effective irreducible root. Then \( \alpha \) is either the divisor class of a \((-2)\)-
curve or the \( \mathbb{Q} \)-divisor class of an effective root of the form \( 2e + \frac{1}{2}\beta_j + \frac{1}{2}\beta_k \), where \( e \) is the class of a \((-1)\)-curve \( E \) that intersects two different components \( C_j, C_k \) of \( C \).

Proof. Writing an effective class \( R \) from \(|\alpha|\) as a sum of irreducible curves and the halves of the boundary components, we may assume that

\[
R = \frac{1}{2} \sum_{i \in I} C_i + D,
\]

where \( D \) has no \((-2)\)-curves among its irreducible components. Suppose \(#I = 1\). We have \( R \cdot K_V = 1 + D \cdot K_V = 0 \) and \(-2 = R^2 = -1 + C_1 \cdot D + D^2 = -1 - 2D \cdot K_V + D^2 = 1 + D^2\), hence \( D^2 = -3 \), \( D \cdot K_V = -1 \). Writing \( D \) as a sum of \((-1)\)-curves and curves \( N \) with \( N \cdot K_V \leq 0 \) and \( N^2 \geq 0 \), we get a contradiction.

Assume now that \( a = #I \geq 2 \). Arguing as above, we obtain \( D \cdot K_V = -a \) and \( D^2 = -2 - a \).

Write

\[
[D] = [m_1 E_1 + \cdots + m_t E_t + D'] = \alpha - \frac{1}{2} \sum_{i \in I} \beta_i,
\]

(9.2.2)

where \( E_i \) are all \((-1)\)-components of \( D \) and \( D' \) is a sum of curves with non-negative self-intersections. Intersecting with \( K_V \), we get \( a = \sum m_i - D' \cdot K_V \geq \sum m_i \). We also have \( D^2 = -2 - a \geq -\sum m_i^2 \).

Thus, we obtain the following inequalities

\[
\sum_{i=1}^t m_i \leq a, \quad \sum_{i=1}^t m_i^2 \geq a + 2.
\]

If all \( m_i \)’s are equal to 1, these inequalities are not satisfied. Thus we may assume that \( m_1 \geq 2 \).

Intersecting with \( C_i \), we obtain \( 0 = R \cdot C_i \geq -2 + 2E_1 \cdot C_i \). This implies that we can find \( C_i \) and \( C_j \) such that \( E_1 \cdot C_i = E_1 \cdot C_j = 1 \). Thus we find an irreducible effective root \( \frac{1}{2}(C_i + C_j) + 2E_1 \) as a part of \( R \). By assumption of irreducibility, \( R \) must be equal to this effective root.

Let

\[
\text{CM}(V)^+ = \{ x \in \text{CM}(V) : x^2 \geq 0, x \cdot h > 0 \text{ for some ample divisor } h \}\).
\]

By Riemann-Roch on \( V \), \( \text{CM}(V)^+ \) consists of effective divisor classes with non-negative self-
intersection.

Proposition 9.2.2. A divisor class \( x \in \text{CM}(V)^+ \) is nef if and only if \( x \cdot r \geq 0 \) for any effective root. In other words, the intersection of the nef cone in \( \text{Pic}(V)_\mathbb{Q} \) with \( \text{CM}(V)^+ \) is a fundamental domain for the group \( W^\text{nod}_V \) in \( \text{CM}(V)^+ \).
Proof. Suppose \( x \in \text{CM}(V)^+ \). Then \( x \) is nef if and only \( x \cdot r \geq 0 \) for every effective divisor class \( r \in \text{Pic}(V) \) with \( r^2 = -1 \) or \(-2\). We may assume that \( r \) is the class of a \((-1)\)-curve or of a \((-2)\)-curve. Let \( x \) be a nef class in \( \text{CM}(V)^+ \). Applying Lemma 9.2.1, it suffices to show that \( x \cdot r \geq 0 \) for any effective root of the form \( r = 2e + \frac{1}{2} (\beta_i + \beta_j) \). Since \( x \cdot \beta_i = 0 \), and \( x \cdot e \geq 0 \), this is obvious. Thus \( x \) belongs to the fundamental domain of \( W^\text{nod}_V \). Conversely, suppose \( x \) belongs to the fundamental domain. By definition, \( x \cdot r \geq 0 \) for any divisor class of a \((-1)\)-curve and \( x \cdot r \geq 0 \) for any positive root of the form \( 2e + \frac{1}{2} (\beta_i + \beta_j) \). Since \( x \cdot \beta_i = 0 \), we obtain \( x \cdot e \geq 0 \). Thus, for any \((-1)\)-curve \( E \) that intersects two different \( C_i \), we have \( x \cdot [E] \geq 0 \). Suppose \( E \) is a \((-1)\)-curve that intersects only one \( C_i \) with multiplicity 2. Let \( f = [E] + \frac{1}{2} \beta_i \). Then \( f^2 = 0 \) and \( f \in \text{CM}(V)^+ \). Let \( \pi : V \to X \) be the blowing-down of \( E \) to a point \( q \in X \). Then the divisor class \( 2f = 2[E] + C_i \in \text{Pic}(V) \) is equal to \( \pi^*(C_i) \), where \( C_i = \pi(C_i) \) is an irreducible curve with \( \bar{C}_i^2 = 0 \) and a node at \( q \). Suppose \( x \cdot E < 0 \), then \( x \cdot 2f = 2x \cdot E < 0 \) and we obtain \( x \cdot 2f = \pi_*(x) \cdot \bar{C}_i < 0 \). Since \( \bar{C}_i \) is obviously nef, we get a contradiction. 

We say that a Coble surface of is unnodal if it does not contain effective roots.

Proposition 9.2.3. A Coble surface \( V \) of K3-type is unnodal if and only if \( n = 1 \) and it has no \((-2)\)-curves.

Proof. Using Lemma 9.2.1, it suffices to show that \( n = 1 \) if \( V \) is unnodal. Suppose \( n > 1 \). By the same lemma, any \((-1)\)-curve \( E \) intersects one component of \( C \) with multiplicity 2. Thus no component \( C_i \) of \( C \) is blown down to a point under the morphism \( \pi : V \to \mathbb{P}^2 \) and the image \( B_i \) of \( C_i \) is a smooth rational curve of degree \( \leq 6 \). Since \( n > 1 \), the curve \( B = B_1 + \cdots + B_n \) is reducible and any two components intersect transversally (because \( V \) does not contain \((-2)\)-curves). The exceptional curve \( R \) over an intersection point will intersect two components of \( C \) contradicting the assumption.

Remark 9.2.4. One can show (see [103]) that a Coble surface is unnodal if and only if it is obtained by blowing up a set \( \{p_1, \ldots, p_{10}\} \) of ten points in the plane satisfying the following 496 conditions:

(i) no points among the ten points are infinitely near;

(ii) no three points are collinear;

(iii) no six points lie on a conic;

(iv) no plane cubic passes through 8 points with one of them being a singular point of the cubic;

(v) no plane quartic curve passes through the 10 points with one of them being a triple point.

Assume \( n = 1 \). As we remarked before, \( \text{CM}(V) \cong E_{10} \). Let

\[
\begin{align*}
f_i & = 3e_0 - e_1 - \cdots - e_{10} + e_i, \quad i = 1, \ldots, 10, \\
h & = \frac{1}{3} (f_1 + \cdots + f_{10}) = 10e_0 - 3(e_1 + \cdots + e_{10}).
\end{align*}
\]

We have \( f_i^2 = 0, f_i \cdot f_j = 1, h^2 = 10, h \cdot f_i = 3 \). The classes \( f_i \) represent the proper transforms of cubic curves passing through the points \( p_j, j \neq i \). Since \( B \) is irreducible, all \( f_i \) are nef divisors,
and \((f_1, \ldots, f_{10})\) is a maximal non-degenerate isotropic sequence of vectors in \(\text{CM}(V)\). The linear system \(|h|\) defines a birational map from \(V\) onto a surface of degree 10 in \(\mathbb{P}^5\). We call it the Fano model of a Coble surface. The images of effective divisors \(F_i\) representing \(f_i\) are plane cubics. The anti-bicanonical curve \(C\) is blown down to a singular point of type \(\frac{1}{3}(1,1)\) of the Fano model.

We can also consider the adjoint linear system \(|h + K_V|\) representing the divisor class \(7e_0 - 2(e_1 + \cdots + e_{10})\). This time the linear system is very ample and its image is a surface \(F\) of degree 9 in \(\mathbb{P}^5\). The image of \(F_i\) are still cubic curves, but the image of \(C\) is a curve of degree 2. The union of \(F\) and the plane spanned by the image of \(C\) is a reducible surface of degree 10 in \(\mathbb{P}^5\). We call it the adjoint Fano model.

**Example 9.2.5.** Assume \(V\) is a general Coble surface with \(n = 2\) and let \(|-2K_V| = \{C_1 + C_2\}\). Let \(\pi : V \to \mathbb{P}^2\) be a birational morphism. Let \(W_i = \pi(C_i)\) and \(d_i\) the degree of \(W_i\). We have the following possibilities \(\{d_1, d_2\} = \{1,5\}, \{2,4\}, \{3,3\}\). We assume \(d_1 \leq d_2\), then there are three possibilities: \(W_1\) is a line intersecting the 6-nodal quintic \(W_2\) at 5 distinct points, or \(W_1\) is a smooth conic intersecting the 3-nodal quartic at 8 points, and \(W_1, W_2\) are two nodal cubics intersecting at 9 distinct points.

Suppose \((d_1, d_2) = (1,5)\). Let \(T\) be a quadratic Cremona transformation with fundamental points at three nodes of \(W_2\). Then the image under \(T\) of \(W_1\) is a conic and the image of \(W_2\) is a 3-nodal quartic. Thus the cases \((1,5)\) and \((2,4)\) can be reduced to each other by a Cremona transformation. This means that we get the same Coble surface only the birational morphisms to \(\mathbb{P}^2\) are different. We can also reduce the case \((2,4)\) to the case \((3,3)\) by taking a quadratic Cremona transformation with two fundamental points at two nodes of the quartic and the third fundamental point taken from \(W_1 \cap W_2\).

Thus we may assume that \(W_1, W_2\) are nodal cubics and \(V\) is obtained by blowing up their intersection points \(p_1, \ldots, p_9\) and the nodes \(p_{10}, p_{11}\) of \(W_1\) and \(W_2\). Let \(H\) be the rational elliptic surface obtained by blowing up \(p_1, \ldots, p_9\). Then the proper transforms of \(W_1, W_2\) are two nodal fibers of the elliptic fibration and \(V\) is obtained by blowing the singular points of these two fibers.

Following Mukai, we consider a birational map \(\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3\) given by the linear system of conics through the points \(p_{10}\) and \(p_{11}\). The composition map \(V \to \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1\) is a birational morphism that maps \(F_1\) (resp. \(F_2\)) onto a smooth rational curve \(F_i\) of bidegree \((1,3)\) (resp. \((3,1)\)) intersecting at 9 points. The surface \(V\) is obtained by blowing up the set \(\{q_1, \ldots, q_9\} = C_1 \cap C_2\) and the additional point \(q_{10}\) equal to the image of the line \(\langle p_{10}, p_{11} \rangle\).

Let \((h, h', e_1, \ldots, e_{10})\) be a basis of \(\text{Pic}(V)\) formed by the pre-images of a canonical basis of \(\text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)\) and the divisor classes of the exceptional curves. In this basis, the boundary components are

\[
\beta_1 = [C_1] = 3h + h' - (e_1 + \cdots + e_{10}), \quad \beta_2 = [C_2] = h + 3h' - (e_1 + \cdots + e_{10}).
\]

We have \(\beta_1 + \beta_2 = -2K_V\) and \(\beta_1 - \beta_2 = 2(h - h')\). Thus we can write \(\beta_1 = -K_V + h - h'\) and \(\beta_2 = -K_V + h' - h\). This shows that \(\text{CM}(V)\) contains a sublattice of index 2 equal to the orthogonal complement of \(\langle K_V, h - h' \rangle\) in \(\text{Pic}(V)\). We have \(K_V^\perp \cong \mathbb{E}_{11}\) with canonical root basis \((\alpha_0 = e_1 - e_2, \alpha_1 = h' - h, \alpha_2 = h - e_1 - e_2, \alpha_3 = e_2 - e_3, \ldots, \alpha_{10} = e_9 - e_{10})\). The orthogonal complement of \(h - h'\) in this lattice has a basis \((h + h' - e_1 - e_2 - e_3 - e_4, e_1 - e_2, \ldots, e_9 - e_{10})\). It is easy to see this a canonical root basis of the lattice \(\mathbb{E}_{2,4,6}\) of discriminant 4. Since \(\text{CM}(V)\) contains
this lattice as a sublattice of index 2, it must be a unimodular lattice. Let

\[ v = \frac{1}{2}(h + h' - K_V) - e_1 = \frac{1}{2}\beta_1 + h' - e_1. \]

Then \( v \cdot (\beta_1 - \beta_2) = v \cdot (h - h') = 0 \) and \( v \cdot (\beta_1 + \beta_2) = -2v \cdot K_V = 0 \), hence \( v \in \text{CM}(V) \). We have \( v^2 = 0 \). This shows that \( \text{CM}(V) \) is an even lattice, and hence

\[ \text{CM}(V) \cong E_{10}. \]

Let

\[ f_i = 3(h + h') - (e_1 + \cdots + e_{10}) - 2e_i, \quad i = 1, \ldots, 10, \]

\[ h_{10} = 5(h + h') - 2(e_1 + \cdots + e_{10}). \]

We have \( f_i^2 = 0, f_i \cdot f_j = 4, h_{10}^2 = 10, h_{10} \cdot f_i = 6 \). Moreover, we have

\[ 6h_{10} = f_1 + \cdots + f_{10} \]

with a complete analogy with the Fano polarization on an Enriques surface. The classes \( f_i \) represent the proper transforms of plane quintics curves passing through \( p_1, \ldots, p_{9}, p, q \) with double points at \( p, q \) and a triple point at \( p_i \). The linear system \( |h_{10}| \) defines a birational morphism onto a surface of degree 10 in \( \mathbb{P}^5 \) which blows down each boundary component to a quotient singularity of type \( 1/4(1, 1) \). We will call the image surface a Fano-Mukai model of \( V \). We may also consider the adjoint linear system \( |H_{10} + K_V| \) representing by \( 3(h + h') - (e_1 + \cdots + e_{10}) \). It maps \( V \) onto a surface of degree 8 in \( \mathbb{P}^5 \). The images of the boundary components this time are curves of degree 2, each spanning a plane. The union of the planes and the octic surface is called a adjoint Fano-Mukai model of \( V \).

One can follow the definition of a canonical isotropic sequence in the Enriques lattice, to define a canonical isotropic sequence in the lattice \( \text{CM}(V) \). Thus, we can speak about non-degenerate (degenerate) \( U \)-pair \((f_1, f_2)\) of such vectors. Similarly to the case of an Enriques surface it defines a regular degree 2 map onto one of the anti-canonical quartic del Pezzo surfaces \( D = D'_1, D'_2, D'_3 \) (resp. \( D_1, D_2, D_3 \)). We still call it a bielliptic map. The difference in the case of Coble surfaces is that this map is never finite. It blows down each boundary component to a singular point of \( D \). If \( p \neq 2 \), the branch curve of the bielliptic map is cut out by a quadric passing through \( k \) singular points of \( D \).

The equation of the double plane model of a Coble surface in \( \mathbb{P}(1, 1, 1, 3) \) obtained from a bielliptic map (if \( p \neq 2 \)) is

\[ x_3^2 + x_1(x_0^4A_1 + x_0^3x_1x_2A_2 + x_0^2x_1x_2A_3 + x_0x_1x_2A_4 + x_1^2x_2A_5) = 0. \quad (9.2.3) \]

where \( A_1, \ldots, A_5 \) are binary forms in \( x_1, x_2 \). The branch curve is equal to the union of the line \( V(x_1) \) and a quintic curve from the linear system \( |5e_0 - e_1 - 2e_2 - 2e_3 - e_4 - e_5| \).

The following pictures the branch locus of a bielliptic map for a Coble surface with one boundary component on the weak del Pezzo surface \( D_1 \). It consists of a curve cut out by a quadric and three \(( -2 \rangle \)-curves. They are colored in blue.

**Example 9.2.6.** Let us consider the previous example and let \( I_1 \) and \( I_2 \) be two subsets of cardinality 7 among \( p_1, \ldots, p_9 \) with \( |I_1 \cap I_2| = 6 \). Let \( f_k = 3e_0 - e_{10} - e_{11} - \sum_{i \in I_k} e_i, k = 1, 2 \). Then \( f_k^2 = 0, f_1 \cdot f_2 = 1, f_k \cdot C_1 = f_k \cdot C_2 = 0 \). The linear system \( |2f_1 + 2F_j| \) defines a bielliptic map.
9.2. COBLE-MUKAI LATTICE

We have a homomorphism 

\[ \phi : V \to D_1. \]

It blows down \( C_1, C_2 \) to two different nodes of \( D_1 \).

On the other hand, let us consider another model of \( V \) where \( C_1 = |e_0 - e_7 - e_8 - e_9 - e_{10} - e_{11}| \) and \( C_2 = |5e_0 - 2(e_1 + \cdots + e_6) - (e_7 + \cdots + e_{11})| \). Let \( F_I \) be a cubic curve passing through the points \( p_1, \ldots, p_6 \) and a subset \( I \) of 3 points among \( p_7, \ldots, p_{11} \). Let \( f_I \) be the divisor class of its proper inverse transform on \( V \). Then \( f_I \) belongs to the Coble-Mukai lattice \( CM(V) \). We have 

\[ f_I \cdot f_J = 3 - \#I \cap J \in \{1, 2\}. \]

Let us now extend the definition 6.4.2 of Nikulin \( R \)-invariant for Enriques surfaces to the case of Coble surfaces.

Let \( V \) be a Coble surface with \( n \) boundary components \( C_1, \ldots, C_n \). For the convenience of notation we denote \( CM(V) \) by \( \Lambda \). Let \( \pi : X \to V \) be the double covering branched along \( C_1, \ldots, C_n \) and \( \iota \) the covering transformation. For any curve \( C \) on \( V \) we denote by \( \tilde{C} \) the proper transform of \( C \). Define \( L^+ = \pi^+(\Lambda) \). Denote by \( L^- \) the orthogonal complement of \( L^+ \) in \( \text{Pic}(X) \). Note that \( \tilde{C}_1, \ldots, \tilde{C}_n \) are contained in \( L^- \). Define

\[ h^\pm = \{ \delta^\pm \in L^\pm : \exists \delta^\mp \in L^\mp, (\delta^\pm)^2 = -4, \frac{\delta^+ + \delta^-}{2} \in \text{Pic}(X) \}. \]

(9.2.4)

Let \( \langle h^- \rangle \) be the sublattice of \( L^- \) generated by \( h^- \). Then \( \langle h^- \rangle = K(2) \) where \( K \) is a root lattice. We have a homomorphism

\[ \gamma : K/2K \cong K(2)/2K(2) \to \Lambda/2\Lambda \cong L^+/2L^+ \]

(9.2.5)

defined by

\[ \gamma(\delta^- \mod 2) = \delta^+ \mod 2. \]

We define a subgroup \( H \) of \( K/2K \) by the kernel of \( \gamma \) which is an isotropic subgroup with respect
Recall that, by Lemma 9.2.1, there are two types of effective irreducible roots. One is a \((-2)\)-

curve and the other is \(2E + \frac{1}{2}C_i + \frac{1}{2}C_j\), where \(E\) is a \((-1)\)-curve that intersects \(C_i\) and \(C_j\). If \(C\) is a \((-2)\)-curve, then \(\pi^*(C) = \tilde{C}^+ + \tilde{C}^-\) where \(\tilde{C}^+, \tilde{C}^-\) are disjoint \((-2)\)-curves on \(X\) with 
\(\iota(\tilde{C}^+ = \tilde{C}^-).\) If \(C\) is a \((-1)\)-curve, it intersects the boundary at two points and hence \(\pi^*(C) = \tilde{C}\). If \(C = C_i\), then \(\pi^*(C_i) = 2\tilde{C}_i\). Therefore \(\pi^*(2E + \frac{1}{2}C_i + \frac{1}{2}C_j) = 2\tilde{E} + \tilde{C}_i + \tilde{C}_j\). We associate 
\(\delta^+ = \tilde{C}^+ + \tilde{C}^\prime\) and \(\delta^- = \tilde{C}^- - \tilde{C}^-\) to a \((-2)\)-curve \(C\), and we associate \(\delta^+ = 2E + \tilde{C}_i + \tilde{C}_j\) and 
\(\delta^- = C_i + C_j\) to an effective irreducible root \(2E + \frac{1}{2}C_i + \frac{1}{2}C_j\). As mentioned above, \(K_H\) contains 
\((-1)\)-vectors corresponding to \(\tilde{C}_1, \ldots, \tilde{C}_n\) when \(n \geq 2\).

Note that this definition is very similar to the one for Enriques surfaces in characteristic 2 whose 
canonical covers are supersingular \(K3\) surfaces with rational double points (see Example 10.6.11).

**Example 9.2.7.** Here we consider an example of a Coble surface with 10 boundary components 
obtained as the quotient of a \(K3\) surface studied by E. Vinberg in [689]. This surface has infinite 
group of automorphisms, however, one can find a structure of the automorphism group explicitly.

We start with a quintic del Pezzo surface \(D_5\) which is unique up to isomorphisms. The surface \(D_5\) is obtained from the plane \(P^2\) blowing up four points \(p_1, p_2, p_3, p_4\) in general linear position. The surface \(D_5\) contains ten lines whose dual graph is the Petersen graph given in Figure 6.1. Note that each line on \(D_5\) meets exactly three lines. Let \(V\) be the surface obtained from \(D_5\) by blowing up 15 intersection points of the ten lines. They are represented by the edges in the Petersen graph. Thus \(V\) contains 15 \((-1)\)-curves which are exceptional curves of the blowing-up \(V \rightarrow D_5\) and ten \((-4)\)-
curves which are the proper transforms of ten lines. The surface \(V\) has the Picard number 20. We denote by \(C_{ij}\) \((1 \leq i < j \leq 5)\) the ten \((-4)\)-curves. Here we use the same subscripts as we used it for the vertices \(U_{ab}\) given in Figure 6.1. Then \(|-2K_V| = \{\sum_{1 \leq i < j \leq 5} C_{ij}\}\). Let \(\pi : X \rightarrow V\) be the double cover branched along \(\sum_{1 \leq i < j \leq 5} C_{ij}\) which is a \(K3\) surface with Picard number 20. Let \(\ell_{ij}\) be the line in \(P^2\) passing through \(p_i, p_j\). The surface \(X\) is birationally isomorphic to the double cover of the plane branched over the union of lines \(\ell_{ij}\). The blow-up of seven singular points of the branch curve is a weak del Pezzo surface of degree 2. The anti-canonical linear system defines a double cover of the plane branched along a complete quadrilateral of lines. This shows that the surface \(X\) is birationally isomorphic to the quartic surface

\[w^4 + xyz(x + y + z) = 0\]

as asserted in [689, Theorem 2.5]. It has 6 rational double points of type \(A_3\).

Let \(C_{ab}, 1 \leq a, b \leq 4\) be the proper transform of the line \(\ell_{cd} = \langle p_c, p_d \rangle\) with \(\{ab\} \cap \{cd\} = \emptyset\) 
and let \(C_{15}\) be the exceptional curve over \(p_i\). The curves \(C_{12} + C_{13} + C_{14} + C_{23} + C_{24} + C_{25}\) and 
\(C_{13} + C_{14} + C_{24} + C_{35} + C_{25}\) span a pencil on \(D_5\) with 5 base points defined by the edges 
\((U_{12}, U_{35}), (U_{34}, U_{25}), (U_{15}, U_{24}), (U_{45}, U_{13})\). The pencil originates from the pencil of plane cubics generated by \(\ell_{34} + \ell_{12} + \ell_{14} + \ell_{24} + \ell_{13} + \ell_{23}\) on \(P^2\). After we blow up the base points we 
obtain an elliptic fibration on \(X\) with singular fibers of type \(A_9\) and five sections. The Shioda-
Tate formula implies that the discriminant of \(\text{Pic}(X)\) is equal to \(-10^2/5^2 = -2\). Since \(\text{Pic}(X)\) is the invariant part of \(H^2(X, Z)\) under the action of the covering transformation \(\sigma\) of \(X \rightarrow V\),
Pic($X$) is a 2-elementary lattice. Therefore Pic($X$) is isomorphic to $U \oplus E_8^{\oplus 2} \oplus A_1^{\oplus 2}$, and hence the transcendental lattice isomorphic to $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. This K3 surface is one of the two K3 surfaces called two most algebraic K3 surfaces. The automorphism group $\text{Aut}(D_5)$ is isomorphic to $S_5$ of degree 5. It induces an action on $V$. Then this action can be lifted to the one on $X$. The involution $\sigma$ acts on Pic($X$) trivially and acts on the transcendental lattice as $-1$, and hence it is contained in the center of $\text{Aut}(X)$. Thus $\text{Aut}(V) \cong \text{Aut}(X)/\langle \sigma \rangle$.

The K3 surface $X$ has 25 $(-2)$-curves which are the pre-images of fifteen $(-1)$-curves and ten $(-4)$-curves on $V$. We picture them on Figure 9.3. Here the red vertices are the pre-images of the $(-4)$-curves.

![Figure 9.3: Twenty five $(-2)$-curves on the Vinberg’s most algebraic K3 surface](image)

Vinberg found additionally five $(-4)$-classes $c_1, ..., c_5$ in Pic($X$) each of which defines a reflection of Pic($X$), and showed that this reflection is represented by an involution of $X$. Let us explain how do these classes arise.

We may assume that four points $p_1, ..., p_4$ are the reference points $p_1 = [1, 0, 0], \quad p_2 = [0, 1, 0], \quad p_3 = [0, 0, 1], \quad p_4 = [1, 1, 1].$

Then six lines joining them by pairs form a complete quadrangle with the diagonal points $q_1 = [1, 1, 0], \quad q_2 = [1, 0, 1], \quad q_3 = [0, 1, 1].$ The quadratic Cremona transformation with fundamental points at $q_1, q_2, q_3$ is given by

$$T : [x, y, z] \mapsto [(x - y + z)(x - y - z), (x + y - z)(x - y - z), (z - x - y)(x - y + z)]. \quad (9.2.6)$$

As is well-known it acts on the blow-up of the fundamental points as a reflection in the vector $e_0 - e_1 - e_2 - e_3$, where $e_1, e_2, e_3$ are the classes of the exceptional curves over $q_1, q_2, q_3$. Since $V$ is the blow-up of a set of points in $\mathbb{P}^2$ including the points $q_1, q_2, q_3$, the class $e_0 - e_1 - e_2 - e_3$ is lifted to a divisor class $\delta$ in Pic($V$) of square-norm $-2$. The involution $T$ also lifts to an involution of $V$. There are exactly 5 different linear systems on $D_5$ that define a birational morphism $D_5 \rightarrow \mathbb{P}^2$ [180, 8.5]. So, in this way, we obtain 5 classes $\delta_1, ..., \delta_5$ in Pic($V$). Their lifts to the K3-cover $X$ are the five classes $c_1, ..., c_5$ of Vinberg. It follows from the definition that each $\delta_i$ intersects three $(-1)$-curves on $D_5$ corresponding to the diagonal points. So, each $c_i$ intersects three $(-2)$-curves on $X$. For example, we marked one of these sets of 3 points in Figure 9.3.

**Theorem 9.2.8.** Assume that $k = \mathbb{C}$. Then $\text{Aut}(X)$ is isomorphic to a central extension of $UC(5) \rtimes S_5$. 


by \( \mathbb{Z}/2\mathbb{Z} = \langle \sigma \rangle \). The subgroup \( \text{UC}(5) \) is generated by the lifts of five Cremona involutions \( T_i \) that act on \( \text{Pic}(X) \) as reflections in \( c_i \). The subgroup \( \mathcal{G}_5 \) is the lift of the group of automorphisms of the del Pezzo surface \( \mathcal{D}_5 \) of degree 5.

Recall that we use the notation \( \text{UC}(k) \) for the free product of \( k \) cyclic groups of order 2 (the universal Coxeter group with \( k \) generators).

As a Corollary we now have

**Corollary 9.2.9.** Assume that \( k = \mathbb{C} \). Then \( \text{Aut}(V) \) is isomorphic to \( \text{UC}(5) \rtimes \mathcal{G}_5 \).

The five Cremona involutions \( T_1, \ldots, T_5 \) define five involutions of \( V \) (and \( X \)), and generate the subgroup \( \text{UC}(5) \) of \( \text{Aut}(V) \). It is known that \( \text{Aut}(\mathcal{D}_5) \) is isomorphic to \( \mathcal{G}_5 \). It lifts to a group of automorphisms of \( V \) and permutes the classes \( \delta_1, \ldots, \delta_5 \). This defines a subgroup \( \text{UC}(5) \rtimes \mathcal{G}_5 \) of \( \text{Aut}(V) \).

We extend the result for \( \text{char}(k) \neq 2 \). To do this, we show that the Coble-Mukai lattice of \( V \) contains 20 \((-2)\)-classes forming a crystallographic basis of type VII. Recall that the diagram in Figure 8.16 is the union of the complete graph \( K(5) \) and the dual of the Petersen graph whose vertices are the edges in the Petersen graph and the edges are its vertices. Each vertex of the first subgraph is joined by a double edge to 3 vertices of the second subgraph. This is very visible in the Figure 8.17. Let

\[
\alpha_{ab,cd} = 2E + \frac{1}{2}(C_{ab} + C_{cd})
\]

be an effective root where \( E \) is the exceptional curve over the intersection point of two lines \( \ell_{ab} \) and \( \ell_{cd} \) (1 \( \leq \) \( a, b, c, d \leq 5 \), \( \{a, b\} \cap \{c, d\} = \emptyset \)). We immediately check that the intersection graph of \( \alpha_{ab,cd} \)’s is the dual Petersen graph, and the intersection graph of \( \delta_1, \ldots, \delta_5 \) is the complete graph \( K(5) \) and each \( K_i \) intersects with multiplicity 2 exactly three roots \( \alpha_{ab,cd} \) (note that in the graph 9.3, Vinberg normalized the norms of vectors corresponding to vertices as \((-1)\)-vectors). Thus the twenty roots \( \alpha_{ab,cd}, \delta_i \) form the crystallographic basis of type VII. The five involutions of \( V \) work as the reflections associated with five roots \( \delta_1, \ldots, \delta_5 \).

Thus we now conclude that

**Theorem 9.2.10.** Assume that \( \text{char}(k) \neq 2 \). Then \( \text{Aut}(V) \) is isomorphic to \( \text{UC}(5) \rtimes \mathcal{G}_5 \).

In \( \text{char}(k) = 2 \), we will show that five classes \( \delta_1, \ldots, \delta_5 \) are represented by \((-2)\)-curves and thus \( \text{Aut}(V) \) is isomorphic to \( \mathcal{G}_5 \) (see Example 9.8.16).

**Remark 9.2.11.** The Coble-Mukai lattice \( \text{CM}(V) \) is isomorphic to the Enriques lattice \( E_{10} \). The crystallographic basis of type VII is realized on an Enriques surface \( S \) with finite automorphism group of type VII. The twenty classes are all \((-2)\)-curves on \( S \) which also determine all genus one fibrations on \( S \). It follows from Enriques reducibility lemma that 20 classes of the crystallographic basis generate \( E_{10} \). This implies that \( \text{CM}(V) \) contains \( E_{10} \) and hence they coincide.

Next we calculate the \( R \)-invariant of this example.
Proposition 9.2.12. Assume that $k = \mathbb{C}$. Then the $R$-invariant $(K, H)$ of $V$ is $(D_{10}, \mathbb{Z}/2\mathbb{Z})$.

Proof. Since ten $\tilde{C}_{ij}$ generate a root lattice $A_1^{\oplus 10}$, $K$ is a sublattice of $A_1(\frac{1}{2})^{\oplus 10}$. Since $K$ is generated by $(-2)$-vectors $\tilde{C}_{ij} \pm \tilde{C}_{kl}$ ($\{i, j\} \cap \{k, l\} = \emptyset$), $K$ is a root sublattice of $A_1(\frac{1}{2})^{\oplus 10}$ of index 2, and hence isomorphic to $D_{10}$. Since $\tilde{C}_{ij}$ gives a non-zero element in $H$, we conclude that $H = \mathbb{Z}/2\mathbb{Z}$.

Remarks 9.2.13. The most algebraic K3 surface $X$ from the previous example is a special case of the following family of K3 surfaces. It is known that the smooth quintic anti-canonical del Pezzo surface $D_5$ contains five pencils of conics. They correspond to the pencils of lines through $p_1, \ldots, p_4$. Choose one conic from each pencil and consider the double cover of $D_5$ branched over the union of the five conics. The blow-up of 10 intersection points of the five conics is a Coble surface with 5 boundary components. Our surface $X$ is obtained when each conic is the union of two lines. The canonical covers of the Coble surfaces forms a 5-dimensional family of K3 surfaces that contains a codimension one family of surfaces birationally isomorphic to a 15-nodal quartic surface [191].

Let $V$ be a Coble surface with $n$ boundary components. It is obvious that the Coble-Mukai lattice of $V$ is isomorphic to the Enriques lattice $E_{10}$ if $n = 1$. We also saw in Example 9.2.5 that it is true if $n = 2$. In Example 6.4.18 we considered Coble surfaces with $n \leq 4$ boundary components constructed as the quotients of the Hessian quartic surfaces of cubic surfaces with $n$ nodes by the Cremona involution. It was proven in [9, Lemma 2.4] that the Coble-Mukai lattice is isomorphic to the Enriques lattice in these cases too. The isomorphism also holds in the case $n = 10$. Indeed, as we saw in Example 9.2.7, the twenty roots $\alpha_{ab,cd}, \delta_i$ form a crystallographic basis of type VII. Since on the corresponding Enriques surface with finite automorphism group the roots from this basis represent all $(-2)$-curves on $S$, and, by Theorem 2.3.5, they generate $\text{Num}(S)$, we see that $CM(V)$ contains a sublattice isomorphic to $E_{10}$ and hence coincides with it.

All these examples lead to the following conjecture.


In the following we prove that this conjecture is true under the assumption that $k = \mathbb{C}$.

Theorem 9.2.15. Assume $k = \mathbb{C}$. Then the Coble-Mukai lattice of a Coble surface is isomorphic to the Enriques lattice $E_{10}$.

Proof. Assume $V$ is general in the irreducible component of the moduli space of such surfaces. Let $X$ be its canonical K3 cover. Then Table ?? from section 5.4 in Volume I shows that the Picard lattice of $X$ is uniquely determined by the number $n$ unless $n = 8$.

As we remarked earlier the assertion is true if $n \leq 4$ and $n = 10$. We use the construction of the irreducible families with $n = 5, \ldots, 10$ from the end of the previous section. Let $V_n$ be a general, in sense of moduli, surface with $n$ boundaries $C_1, \ldots, C_n$. We know that it is obtained as the blow-up of the singular points of the union of 5 conics $K_1, \ldots, K_5$ on a quintic del Pezzo surface $D_5$. We
may assume that $K_1, \ldots, K_{10-n}$ of them are irreducible and the rest $n-5$ are reducible. We can specialize $V_n$ to $V_{n+1}$ by replacing $K_{10-n}$ with a reducible conic $L_1 + L_2$ from the same pencil.

The Picard lattice of $V_n$ is generated by the classes of $5 + n$ exceptional curves $E_i$ of the blow-up $\pi_n : V_n \to D_5$ and $\pi^*(\Pic(D_5))$. The $\mathbb{Q}$-rational lattice $\widetilde{\Pic}(V_n)$ is generated by $\Pic(V_n)$ and $\frac{1}{2}[C_1], \ldots, \frac{1}{2}[C_{10}].$ Without loss of generality, we may assume that $C_1$ is the proper transform of a conic from the pencil $K_{10-n}$. Consider the isometry embedding

$$\iota : \widetilde{\Pic}(V_n) \hookrightarrow \widetilde{\Pic}(V_{n+1})$$

which is the identity on the set of classes of exceptional curves $E_i$, on $\pi^*(\Pic(D_5))$, and on the classes $\frac{1}{2}[C_i], i \neq 1$. It sends the class $\frac{1}{2}[C_1]$ to the class $\frac{1}{2}[L_1]$. The image of this embedding is equal to the orthogonal complement of $\frac{1}{2}[L_2]$. It follows from the definition of the Coble-Mukai lattice that the embedding $\iota$ maps $\CM(V_n)$ isomorphically onto $\CM(V_n)$. Since $\CM(V_{10}) \cong E_{10}$, we are done.

Remark 9.2.16. In case $n = 8$ (a), the Coble-Mukai lattice contains 40 roots that span a remarkable polytope with denoted by $\Gamma_{MII}$ (see Section 10.6). Recall that a general member $V_8$ in this family can be obtained by the blow-up of the intersection points of 8 lines $\ell_i, \ell'_i$ ($i = 1, 2, 3, 4$) on a smooth quadric $Q$ with $\ell_i \cdot \ell'_i = 1$. Denote by $p_{ij}$ the intersection point of $\ell_i$ and $\ell'_j$ and let $E_{ij}$ be the exceptional curve over $p_{ij}$. Let $C_i, C'_j$ be the proper transforms of $\ell_i, \ell'_j$ which are the boundary curves of $V$, and let $h, h'$ be the total transforms of $\ell_i, \ell'_j$. Then $\CM(V_8)$ contains sixteen effective roots $\alpha_{ij} = 2E_{ij} + \frac{1}{2}(C_i + C'_j)$ and 24 non-effective roots $\delta_{\sigma} = h + h' - (E_{1\sigma(1)} + E_{2\sigma(2)} + E_{3\sigma(3)} + E_{4\sigma(4)})$ indexed by $\mathcal{G}_4$. The 40 roots $(e_{ij}, \delta_{\sigma})$ are the vertices of the polytope $\Gamma_{MII}$. The subgraph with 16 vertices $e_{ij}$ is dual to the complete bipartite graph $BK(4)$ on two sets of cardinality 4 (see the left hand side of Figure 10.8). Let $H_0$ be the 2-elementary subgroup of $\mathcal{G}_4$ generated by $(12)(34), (13)(24)$ and consider the coset decomposition $\mathcal{A}_4 = H_0 + H_1 + H_2$ and $\mathcal{A}_4 \setminus \mathcal{A}_4 = H_3 + H_4 + H_5$. Then $\delta_{\sigma} \cdot \delta_{\sigma'} = 2$ for any $\sigma, \sigma' \in H_i$ ($0 \leq i \leq 4$), and $\delta_{\sigma} \cdot \delta_{\sigma'} = 1$ for any $\delta, \delta' \in \mathcal{A}_4$ or $\delta, \delta' \notin \mathcal{A}_4$ which belong to different cosets. Thus the subgraph with vertices $\{\delta_{\sigma}\}$ is the union of two 3 complete graphs $K(4)$ with double edges, each vertex of one subgraph is joined to all other vertices from two remaining subgraphs (see right hand side of Figure 10.8).

### 9.3 Rational quadratic twist construction

Suppose $V$ is a Coble surface (as always terminal of K3 type). First we extend construction of a rational quadratic twist for Enriques surfaces from Section 4.8 to Halphen surfaces of index 2. Let $f : H \to \mathbb{P}^1$ be the elliptic fibration on such a surface $H$ and let $R$ be its smooth rational bisection. The double cover $f : R \to \mathbb{P}^1$ ramifies at two points $p_0 \in F_0, p \in F$, where $F_0$ is the unique half-fiber, and $F$ is another fiber. For our application to Coble surfaces, it would be enough to assume that $F$ is nonsingular or of type $\tilde{A}_m$. We also assume that $F_0$ is of type $\tilde{A}_m$ (smooth if $m = 0$).

Two possibilities may arise: $p$ is a nonsingular or a double point of $F$. If $V \to H$ is the blow-up of singular points of $F$, then the proper inverse transform of $R$ on $V$ is a bisection of the elliptic fibration $f' : V \to \mathbb{P}^1$ that lifts $f$. It is a $(-1)$-curve if $p$ is nonsingular, and a $(-2)$-curve otherwise. Let us say that a bisection on $H$ (resp. on $V$) is of the first kind if $p$ is nonsingular and of the second kind otherwise.
9.3. RATIONAL QUADRATIC TWIST CONSTRUCTION

Since $F \sim 2F_0$, we can consider the double cover $\pi' : X' \to H$ branched over $F$. If $F$ is singular, then $X'$ has ordinary double points over the singular points of $F$. Let $\pi : X \to X' \to H$ be the composition of $\pi'$ with the minimal resolution of singularities of $X'$. The pre-image $\tilde{F}$ of $F$ on $X$ is a singular fiber of type $A_{2n+1}$. The Hurwitz type formula for the canonical class implies that $X$ is a K3 surface.

Assume $R$ is a bisection of the first kind. Then $R$ is tangent to the branch curve at its nonsingular point $p \in F$, hence its pre-image on $X$ splits into two components $R_+$ and $R_-$ intersecting at the point $\pi^{-1}(p)$. Each curve $R_\pm$ is a $(-2)$-curve and a section of the elliptic fibration $\tilde{f} : X \to \mathbb{P}^1$ lifted from $f$. As in the case of Enriques surfaces, we define an involution $\sigma_+ = t_{R_-} \circ \tau$, where $\tau$ is the covering involution, and $t_{R_-}$ is the translation involution by $R_-$ with respect to the group law with zero section $R_+$ on the set of sections of $\tilde{f}$. The involution $\tau$ acts identically on $n + 1$ components of $\tilde{F}$ and acts as an involution on each other component with two fixed points. The involution $t_{R_-}$ fixes the point $\pi^{-1}(p) = R_+ \cap R_-$ on one of the components of the ramification locus. This implies that $\sigma_+$ acts identically on this component and hence identically on $n + 1$ components of $\tilde{F}$ including the pre-image of the component on $H$ that intersects $R$. As in the Enriques case, we see that $\sigma_+$ acts identically on $m + 1$ components of the half-fiber $\tilde{F}_0$ of type $A_{2m+1}$. The quotient surface $X/(\sigma_+)$ is smooth, after we blow down $m + 1$ components on the image of $\tilde{F}_0$ and $n + 1$ components of the image of $\tilde{F}$, we obtain a rational jacobian elliptic surface $j : J \to \mathbb{P}^1$ with fibers $\tilde{F}_0$ and $F$ of types $\tilde{A}_m$ and $\tilde{A}_n$. As in the Enriques case, $\sigma_+(R_-) = R_-$ and $\sigma_+(R_+) = t_{R_-}(R_-) = R'_-$. The curve $R_-$ descends to a section of $j$ which we can choose as the zero section $C_0$ in the Mordell-Weil group. Since $R_+ \cap R_- = \{\pi^{-1}(p)\}$, the curve $R_+ + R'_-$ descends to a rational bisection $C$ of $j$ that intersects the zero section at one point and at this point it is tangent to the fiber passing through this point.

Suppose $R$ is a bisection of $f : H \to \mathbb{P}^1$ of the second kind. In the previous notation, $R_+$ and $R_-$ are now disjoint on $X$ and intersect the same component of $F$ which does not belong to the locus of fixed points of $\sigma_+$. Everything else remains unchanged and we arrive at a rational jacobian elliptic surface $j : J \to \mathbb{P}^1$ such that $X$ is a minimal resolution of singularities of two fibers $\tilde{F}_0$ and $\tilde{F}$ of multiplicative types. The curve $R_-$ descends to a section $C_0$ of $j$ and the curves $R_+ + R'_-$ descends to a rational bisection $C$ of $j$ that is tangent to $\tilde{F}_0$ and $\tilde{F}$ at their nonsingular points. In both cases, $f : V \to \mathbb{P}^1$ is a torsor of $j : J \to \mathbb{P}^1$ defined by the cocycle $\text{Gal}(C/\mathbb{P}^1) \to J_\eta(C_\eta)$ with the image equal to $C_\eta$. The Halphen fibration $f$ is reconstructed from $j$ by the Weil descent.

The bisection $C$ that gives the inverse construction is defined similar to the case of Enriques surfaces. We start with a jacobian rational surface $j : J \to \mathbb{P}^1$, then choose two fibres $F_{t_1}, F_{t_2}$, smooth or of multiplicative type. Then we fix a section $C_0$ to define a group law on the set of sections with the zero section $C_0$. Next we look for a rational curve $C$ satisfying conditions 1)-5) from Section 4.9 where we defined an Enriques bisection. If $R$ is a bisection of the first kind, we replace property 3 with the following property

(3') $C$ intersects $C_0$ at one point over a point $t_2$ with multiplicity 2.

We will call such a bisection $C$ an Halphen bisection. Let $X \to J$ be the double cover branched at the fibers $F_{t_1}, F_{t_2}$ which in the following we assume to be smooth for brevity of notation. The pre-image of $C$ in $X$ splits into $R_1$ and $R_2$. We consider the translation automorphisms $t_{R_1}$ with
respect to the group law on the pre-image of the elliptic fibration on $X$ with zero section $R_0$ equal to the pre-image of $C_0$. As in the case of an Enriques surface, we have $R_1 \boxplus R_2 = 0$. The involution $\sigma_1 = t_{R_1} \circ \tau$ acts identically on the pre-image $\tilde{F}_{t_1}$ of the fiber $F_{t_1}$ and acts as a translation by a 2-torsion point on the pre-image $\tilde{F}_{t_2}$ of the fiber $F_{t_2}$. The quotient $H = X/(\sigma_1)$ is an Halphen surface of index 2 with the double fiber equal to the image $\tilde{F}_{t_2}$ on $H$. We have $\sigma_1(R_0) = R_1$ and the image of $R_0 + R_1$ is a smooth rational bisection $R$ on $H$. Since $C$ is tangent to the fiber $F_{t_1}$ at its nonsingular point where it intersects $C_0$, we obtain that $R_0$ intersects $R_1$ at one point on the fiber $\tilde{F}_{t_1}$. Thus $R$ is tangent to the image of $\tilde{F}_{t_1}$ on $H$ at a nonsingular point.

To find $C$ we look at the quotient $J \to J/\langle \iota \rangle \cong F_2$ by the negation involution defined by the zero section $C_0$. Then we look for a section $E$ of $F_2 \to \mathbb{P}^1$ from the linear system $|3f + \epsilon|$ that intersects transversally the branch curve $B$ at one nonsingular point and intersects the exceptional section transversally at another point. We have $E \cdot B = (3f + \epsilon) \cdot (6f + 3\epsilon) = 9$, so in order the pre-image of $E$ on $J$ be a rational curve we must require that $E$ intersects $B$ with multiplicity 2 at four additional points.

Now suppose $V$ is a Coble surface obtained from an Halphen surface $H$ of index 2 by blowing up singular points of a singular simple fiber $F$ of type $\tilde{A}_n$. Let $R'$ be a smooth bisection of the elliptic fibration on $V$ lifted from a bisection $R$ of the elliptic fibration on $H$. If $R$ is of the first kind, then $R'$ is $(-1)$-curve, otherwise $R'$ is a $(-2)$-curve. We assume that $R$ is tangent to $F$ at one nonsingular point. This means that $R'$ is tangent to one of the boundary components at a nonsingular point. In particular, $R'$ is a $(-1)$-curve. Using the image of $R$, we can apply the rational quadratic twist construction to $H$.

Example 9.3.1. Here we consider an analogue of Example 4.10.12 from Section 4.8 that gives a construction of a Coble surface. In this example $j : J \to \mathbb{P}^1$ is defined by the Hesse pencil of cubic curves and has 4 singular fibers of type $\tilde{A}_2$. We fix the section defined by the base point $p_0 = [0, 1, -1]$ and consider the Weierstrass model with the curve $B$ coming from the curve

$$V(u_2^3 + 12u_1(u_0^3 - u_1^3)u_2 + 2(u_0^6 - 20u_0^3u_1^3 - 8u_1^6)) \subset \mathbb{Q} \cong \mathbb{P}(1, 1, 2).$$

It has 4 cusps $c_1, c_2, c_3, c_4$ with coordinates

$$[u_0, u_1, u_2] = [0, 1, -2], \ [1, -1, -\frac{3}{2}, 2], \ [1, -1, -\frac{1}{2}, \frac{3}{2} \omega], \ [1, -\frac{1}{2}, \omega^2, \frac{3}{2} \omega].$$

Suppose we want to construct an Halphen torsor with a nonsingular half-fiber $F_1$ and a bisection of the first kind. To do this need to consider a rational curve $K$ of degree 3 on $Q$ that passes through the vertex of $Q$, passes through all cusps and intersects $B$ transversally at one point $b_1$ that does not lie over a branch point of $B \to \mathbb{P}^1$. Since $\dim \langle 3f + \epsilon \rangle = 5$, we can find a unique $K$ satisfying these conditions. We find $K$ as the residual curve in the intersection of $Q$ with a quadric in $\mathbb{P}^3$ passing through the vertex of $Q$ and containing a fiber of $Q \dashrightarrow \mathbb{P}^1$ such that the residual curve is a curve from $3f + \epsilon$ satisfying our conditions. The linear system of such curves is a pencil generated by the curves

$$6u_0^2u_1 - 6u_0u_1^2 + 6u_1^3 + (3u_0 - u_1)u_2 = 0$$

and

$$3u_0^3 - 6u_0^2u_1 - 12u_0u_1^2 + 4u_1u_2 = 0.$$
hence the second fiber $F_2$.

Next, let us construct $H$ with a singular half-fiber $F_1$. There are two possible bisections on $H$ which we need to do this job. Assume first that it is of the first kind and $F_2$ is nonsingular. Then we look for $K$ as above but require that it intersects $B$ at one of the cusps with multiplicity 3. If it intersects the exceptional section at the point lying on the ruling containing another cusp, we get $F_2$ singular too.

Applying this to Coble surfaces, we have to construct an Halphen surface with a singular non double fiber. So, the previous construction gives us a bisection on the Coble surface which intersects one of the singular fibers at two points in different irreducible components. We choose one of this fibers to blow up its singular points and get a Coble surface $V$. The proper transform of the bisection is a $(-1)$-curve with class $e$ that intersects two boundary components $\beta_1, \beta_2$. It gives rise to an effective root $\alpha = 2e + \frac{1}{2} \beta_1 + \frac{1}{2} \beta_2$ on $V$.

### 9.4 Self-projective rational nodal plane sextics

Here, nodal means that we allow only ordinary nodes or cusps, maybe infinitely near. This is equivalent to that the blow up of $\mathbb{P}^2$ at the singular points is a Coble surface with one boundary component.

In this section, following [704], we classify such plane sextics that admit a non-trivial group of projective automorphisms. The interest to this classification is partially explained by the following.

**Lemma 9.4.1.** Let $V$ be a Coble surface with one boundary component $C$. Then a finite group $G$ of automorphisms of $V$ preserves a Fano polarization if and only if $V$ is isomorphic to the blow-up of $\mathbb{P}^2$ at ten double points of an irreducible nodal plane sextic that admits a group of projective automorphisms isomorphic to $G$.

**Proof.** Suppose $G \subset \text{Aut}(V)$ preserves a Fano polarization $h$. Let $3h = F_1 + \cdots + F_{10}$, where $f_i = [F_i]$ form a canonical isotropic sequence in $K_V^1 \cong E_{10}$. Let $e_i = f_i + K_V$. Then $e_i^2 = -1, e_i \cdot K_V = -1$. By Riemann-Roch, $e_i$ is effective and linearly equivalent to a sum of irreducible curves $Z_1, \ldots, Z_s$. Assume $F_i$ is nef. Since $F_i \cdot e_i = 0$ each $Z_i$ is either $(-1)$ or $(-2)$-curve. Since $-2K_V \cdot E = 2$ for any $(-1)$-curve, there is only one $(-1)$-curve and the rest is a chain of $(-2)$-curves. In other words, $e_i$ is the class of an exceptional configuration $E_i$. If $F_i$ is not nef, then we know that $F_i \sim F_j + R_{ij}$ for some nef $F_j$ and a chain of $(-2)$-curves $R_{ij}$. Thus $F_i + K_V \sim E_i = E_j + R_{ij}$ is an exceptional configuration extending the exceptional configuration $E_j$. This shows that we can blow down $E_1, \ldots, E_{10}$ to points $p_1, \ldots, p_{10}$ (maybe infinitely near) in $\mathbb{P}^2$. The image of $C$ is a plane sextic with double points at $p_1, \ldots, p_{10}$. One checks that

$$e_0 = h + 3K_V = \frac{1}{3}(f_1 + \cdots + f_{10}) + 3K_V$$

satisfies $e_0^2 = 1, e_0 \cdot e_i = 0$, hence it is equal to the pre-image of the class of a line in the plane. Since $G$ leaves $e_0$ invariant, it descends to a group of projective transformations of $\mathbb{P}^2$ that leaves $W$ invariant.
Conversely, assume that a rational nodal plane sextic $C$ admits a non-trivial group $G$ of projective symmetries. Then $G$ lifts to an automorphism of $V$ that leaves invariant the classes $e_0, e_1, \ldots, e_{10}$. Thus it leaves invariant the divisor class $h$.

**Definition 9.4.2.** Let $S$ be an Enriques or a Coble surface. We say that $S$ is Fano-symmetric with respect to a finite group $G$ and a Fano polarization $h$ if $G$ is isomorphic to a subgroup of $\text{Aut}(S)$ that leaves invariant $h$.

Let $C$ be a rational plane sextic admitting a group $G$ as its group of projective automorphisms. This means that $G$ admits a faithful projective representation $\rho_1 : G \to \text{Aut}(\mathbb{P}^2) = \text{PGL}_3(\mathbb{k})$ such that $W$ is invariant (since $\deg C > 1$, the restriction of the action to $C$ is also faithful). The action of $G$ on $C$ induces a faithful action on its normalization $\overline{C} \cong \mathbb{P}^1$ and defines a projective representation $\rho_2 : G \to \text{Aut}(\mathbb{P}^1) \cong \text{PGL}_2(\mathbb{k})$. The map $\mathbb{P}^1 \cong \overline{C} \to C \subset \mathbb{P}^2$ is a rational parametrization

$$s : [t_0, t_1] \mapsto [P_1(t_0, t_1), P_2(t_0, t_1), P_3(t_0, t_1)].$$

(9.4.1)

Here $P_i(t_0, t_1)$ are mutually coprime binary forms of degree 6. By construction, the map $s$ is $G$-equivariant. Let $\mathbb{P}^1 = [U] = \mathbb{P}(U^\vee)$ and $\mathbb{P}^2 = [V] = \mathbb{P}(V^\vee)$. A $G$-equivariant map $s$ originates from a morphism of projective representations $|V| \to |S^6 U^\vee|$.

We assume that $G$ is a tame group, i.e.

$$(p, \#G) = 1.$$ 

As is well-known, a finite tame subgroup $G$ of $\text{PGL}_2(\mathbb{k})$ is isomorphic to one of the *polyhedral groups* given by the following proposition due to Felix Klein.

**Proposition 9.4.3.** Each finite tame subgroup of $\text{PGL}_2(\mathbb{k})$ is isomorphic to one of the following groups:

- a cyclic group $C_n$ of order $n$;
- a dihedral group $\mathbb{D}_{2n}$ of order $2n$;
- a tetrahedral group $\mathbb{T}$ isomorphic to $\mathfrak{A}_4$;
- an octahedral group $\mathbb{O}$ isomorphic to $\mathfrak{S}_4$;
- an icosahedral group $\mathbb{I}$ isomorphic to $\mathfrak{A}_5$.

Two isomorphic subgroups are conjugate in $\text{PGL}_2(\mathbb{k})$. A cyclic group $G$ of odd order lifts isomorphically to a subgroup of $\text{SL}_2(\mathbb{k})$. All other groups lift to a central extension $\tilde{G} = 2.G$ subgroup of $\text{SL}_2(\mathbb{k})$.

The group $\tilde{G} = 2.G$ is called a *binary polyhedral group*. Recall that any projective representation $G \to \text{PGL}_n(\mathbb{k})$ of a tame group $G$ can be obtained from a linear representation $\rho' : G' \to \text{GL}_{n+1}(\mathbb{k})$, where $G' = K.G$ is a central extension of $G$ with a group $K$, called the group of Schur multipliers. It is isomorphic to $H^2(G, \mathbb{k}^*)$. It is known that any prime divisor $p$ of $\#K$ is equal to the order of a non-cyclic $p$-Sylow subgroup of $G$ [325], 11.2. Applying this to a polyhedral group $G$ we find that $K$ is a cyclic group of order 2 or trivial if $G$ is cyclic. Thus, every projective
representation of $G$ originates from a linear representation of $\tilde{G}$. Thus our rational parametrization $\mathbb{P}^1 = |U| \to |V| = \mathbb{P}^2$ is given by a projective representation $|S^6U| \to |V|$ of $G$, and hence by a linear representation $S^6U \to V$ of $\tilde{G}$. Passing to the duals, we get a linear 3-dimensional summand $V^\vee$ of the natural representation of $\tilde{G}$ in the space $S^6U^\vee$ of binary forms of degree 6. Note that the center of $\tilde{G}$ acts trivially on $S^6U^\vee$, hence $V^\vee$ is a 3-dimensional linear sub-representation of $G$ in $S^6U^\vee$. Note that two linear representations define isomorphic projective representations if and only if they differ by a one-dimensional character of $G$.

Our task is to find all possible such representations that define a rational parametrization whose image is a rational nodal plane sextic.

Let us recall first some known facts about irreducible linear representations of a binary polyhedral (non-cyclic) group. The group $\tilde{G}$ can be generated by elements $g_1, g_2, g_3$ of orders $2p, 2q, 2r$, where

\[
(p, q, r) = \begin{cases} 
(2, 2, n) & G = D_{2n}, \\
(2, 3, 3) & G = T, \\
(2, 3, 4) & G = O, \\
(2, 3, 5) & G = I.
\end{cases}
\]

The generators satisfy the following basic relations

\[
g_1^p = g_2^q = g_3^r = g_1g_2g_3 = -1. \tag{9.4.2}
\]

The conjugacy classes are represented by the elements $1$, $-1$, $g_1^a$ ($0 < a < p$), $g_2^b$ ($0 < b < q$), $g_3^c$ ($0 < c < r$).

Let $\Gamma(p, q, r)$ be the Dynkin diagram of an affine root system of one of the types $\tilde{D}_{n+2}, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$, where

\[
(p, q, r) = (2, 2, n), (2, 3, 3), (2, 3, 4), (2, 3, 5), \tag{9.4.3}
\]

respectively. In the last three cases, we set

\[
(c, b, a) = (3, 3, 3), (2, 4, 4), (2, 4, 6)
\]

for the lengths of the arms of these diagrams.

The group $\tilde{G}$ has $a + b + c - 2$ irreducible characters indexed by vertices of the diagram. Their degrees coincide with the multiplicities of the irreducible components entering in a fiber of an elliptic fibrations with the dual graph of components equal to $\Gamma(p, q, r)$ (see Figure 4.1 from Volume I).

All of this is well-known and a part of the McKay correspondence (see [383] and other references therein).

The character of the standard 2-dimensional representation of $\tilde{G}$ in $U$ corresponds to the neighbor of the vertex used to extend a finite type diagram to the affine one. We denote this character by $x$. We have the following McKay rule. For any character $\chi_v$ corresponding to a vertex $v$ of the Dynkin diagram, a vertex $v'$ incident to $v$ enters into the tensor product $x \otimes \chi_v$ with multiplicity equal to the degree of $\chi_{v'}$.

The following Table 9.4 summarizes the known facts about irreducible representations of binary polyhedral groups $\tilde{G}$. The squares correspond to representations that factor through a representation
The characters $\rho_k$ of 2-dimensional representations of the binary dihedral groups are defined on generators

$$r = \begin{pmatrix} \epsilon_{2n} & 0 \\ 0 & \epsilon_{2n}^{-1} \end{pmatrix}, \quad s = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

(in the standard 2-dimensional representation of these groups) by the matrices

$$\begin{pmatrix} \epsilon_{2n}^{k+1} & 0 \\ 0 & \epsilon_{2n}^{-k-1} \end{pmatrix}, \quad \begin{pmatrix} 0 & i^{k+1} \\ i^{k+1} & 0 \end{pmatrix}.$$ 

Here, $\epsilon_k$ denotes a generator of a cyclic subgroup of order $k$ of $k^*$. The one-dimensional characters $\chi_k$ are defined by their values on the generators as follows

$$\chi_0 = (1,1), \quad \chi_1 = (1,-1), \quad \chi_2 = (-1,1), \quad \chi_3 = (-1,-1).$$

The characters $y$ and $z$ are uniquely determined by the McKay rule for decomposition of the character $x\chi_.$.

We start with a cyclic group $C_n = \langle g \rangle$ of order $n$.

We may assume that it acts faithfully on $|U|$ and lifts to a linear representation of a cyclic group $\tilde{G} = \langle \tilde{g} \rangle$ of order $2n$ on $U$ such that $\tilde{g}$ acts by $(t_0, t_1) \mapsto (\epsilon_{2n} t_0, \epsilon_{2n}^{-1} t_1)$.

The monomials $t_0^{k} t_1^{6-k}$ are eigenvectors with eigenvalues $\epsilon_{2n}^{2k-6} = \epsilon_n^{k-3}$. For brevity of notation, we give a rational parametrization $(x_0, x_1, x_2) = (P_0(t_0, t_1), P_1(t_0, t_1), P_2(t_0, t_1))$ in terms of the affine coordinate $t = t_1/t_0$.

If $n > 6$, the eigenvalues are different, and, the curve is a bimonomial curve. It is a not a nodal
a sextic. It is easy to see, that up to a projective transformation of \( \mathbb{P}^2 \), the parametrization is given by

\[
(x_0, x_1, x_2) = (1, t, t^6)
\]

and the curve is \( V(x_2^6 - x_1 x_0^5) \).

So, we may assume that \( n \leq 6 \). To choose a parametrization, we may use a projective transformation of the plane and a Moebius transformation of \( t \). We also use that the polynomials \( t_0 \) and \( t_1 \) do not divide all \( P_i(t_0, t_1) \), hence one of the dehomogenized polynomials must be equal to \( t^6 + \cdots \) and another (maybe the same) to \( 1 + \cdots \). We also use that the curve is not monomial.

Suppose \( n = 2 \). Then \( S^6 U^\vee \) is the direct sum of two eigensubspaces of \( \tilde{g} \). They are \( V_0 = \langle t_1^6, t_0^2 t_1^4, t_0^4 t_1^2, t_0^6 \rangle \) and \( V_1 = \langle t_0 t_1^5, t_0^3 t_1^3, t_0^5 t_1 \rangle \) with eigenvalues \(-1\) and \(1\), respectively. Thus we can choose one parametrization of the form

\[
(x_0, x_1, x_2) = (t^6 + a_1 t^4 + a_2 t^2 + a_3, a_4 t^4 + a_5 t^2 + a_6, a_7 t^5 + a_8 t^3 + a_9 t),
\]

where \((a_3, a_6) = (1, 0)\) or \((0, 1)\) and another one of the form

\[
(x_0, x_1, x_2) = (t^6 + a_1 t^4 + b_2 t^2 + a_3 t^5 + a_4 t^3 + a_5 t, a_6 t^3 + a_7 t).
\]

Suppose \( n = 3 \). We have three eigensubspaces \( \langle t_1^6, t_0^3 t_1^3, 1 \rangle, \langle t_0 t_1^5, t_0^2 t_1^4, t_0^4 t_1^2 \rangle, \langle t_0^2 t_1^4, t_0^5 t_1 \rangle \), with eigenvalues \(1, \epsilon_3, \epsilon_2^3\), respectively. We must take one of the polynomials \( P_i \) from the first eigensubspace and get the following three possible cases

\[
\begin{align*}
(x_0, x_1, x_2) &= (t^6 + a_1 t^3 + 1, a_2 t^5 + a_3 t, a_4 t^4 + a_4 t^2), \\
(x_0, x_1, x_2) &= (t^6 + a_1 t^3, a_2 t^5 + 1, a_3 t^4 + a_4 t), \\
(x_0, x_1, x_2) &= (t^6 + a t^3 + 1, t^4, t).
\end{align*}
\]

Using MAPLE we check that in the last two cases the curve has a triple point and hence these cases must be omitted.

Suppose \( n = 4 \). We have four eigensubspaces \( \langle t_1^6, t_0 t_1^5, t_0^5 t_1 \rangle, \langle t_0^2 t_1^4, t_0^4 t_1^2 \rangle, \langle t_0^4 t_1^2 \rangle \), with eigenvalues \(1, -1, i, -i\). We must take two of the polynomials \( P_i \) to be equal to \( t^6 + a t^2 \) and \( b t^4 + c \) with \( c \neq 0 \). This leaves us with two possibilities

\[
\begin{align*}
(x_0, x_1, x_2) &= (t^6 + a_1 t^2, a_2 t^4 + 1, t^3), \\
(x_0, x_1, x_2) &= (t^6 + a_1 t^2, a_2 t^4 + 1, a_3 t^5 + a_4 t).
\end{align*}
\]

Again, we check, using MAPLE, that the coefficient \( a_2 \) must be nonzero, so we may assume that \( a_2 = 1 \). In the second equation \( a_2 \) or \( a_3 \) (but not both) could be equal to zero.

Assume \( n = 5 \). We have five eigensubspaces \( \langle t_0^2 t_1^4 \rangle, \langle t_0^4 t_1^2 \rangle, \langle t_0^4 t_1 \rangle, \langle t_0 t_1^5 \rangle, \langle t_0^2 t_1^4 \rangle \) with eigenvalues \(1, \epsilon_5, \epsilon_3^2, \epsilon_2^3, \epsilon_4^2\), respectively. This leaves us with the following representatives of three isomorphism classes of parametrizations:

\[
\begin{align*}
(x_0, x_1, x_2) &= (t^6 + a t, b t^5 + 1, t^2), \\
(x_0, x_1, x_2) &= (t^6 + a t, b t^5 + 1, t^3), \\
(x_0, x_1, x_2) &= (t^6 + a t, b t^5 + 1, t^4).
\end{align*}
\]

Here \( b \neq 0 \) in the first two cases, otherwise the curve is not a nodal sextic. In the third case \( b \) could be equal to zero. For example, taking \( b = 0 \) we obtain a nodal sextic with 5 ordinary nodes and 5 infinitely near cusps (i.e. a simple singularity of type \( A_{10} \)).
Finally, assume that \( n = 6 \). We have 6 eigensubspaces \( \langle t_0^6t_1^3 \rangle, \langle t_0^6t_1^6 \rangle, \langle t_0^5t_1^5 \rangle, \langle t_0^5t_1^1 \rangle, \langle t_0^4t_1^4 \rangle, \langle t_0^4t_1^2 \rangle \) with eigenvalues \( 1, -1, \epsilon_3, \epsilon_3^2, \epsilon_6, \epsilon_6^{-1} \). We have to take one of the polynomials equal to \( t_0^6 + t_1^6 \). After scaling, we may assume that \( a = b = 1 \). Up to switching \( t_0 \) and \( t_1 \), we have the following parametrizations:

\[
(x_0, x_1, x_2) = (1 + t^6, t, t^a), \quad a = 2, 3, 4, 5,
\]

and

\[
(x_0, x_1, x_2) = (1 + t^6, t^2, t^3).
\]

Using MAPLE, we check that only one of them gives a nodal sextic:

\[
(x_0, x_1, x_2) = (1 + t^6, t, t^5)
\]

with equation

\[
y^6 - x^4yz + 4x^2y^2z^2 - 2y^3z^3 + z^6 = 0.
\]

It has 10 ordinary double points and has the dihedral group \( \mathbb{D}_{12} \) as its group of projective symmetries.

Before we go to non-cyclic groups, let us prove the following.

**Lemma 9.4.4.** Suppose that \( G \) is not a cyclic group. Then the action of \( G \) in \(|V|\) has only isolated fixed points.

**Proof.** Suppose we have a line \( \ell \) of fixed points. Let \( \eta \) be its generic point. Since \( G \) is tame, it is mapped injectively into the group of automorphisms of the local ring \( \mathcal{O}_{\mathbb{P}^2, \eta} \) and acts trivially on its residue field. Thus it acts by sending a local parameter \( u \) to \( \chi(g)u \), where \( \chi : G \to k^* \) is an injective homomorphism. Since \( G \) is not cyclic, this is impossible. \( \square \)

The lemma shows that all characters of \( V \) enter with multiplicity 1.

We use Table 9.4 and paragraphs before the Table to describe such representations. We have to find all homomorphisms of linear representations

\[
\rho : V \otimes \chi \to S^6U,
\]

where \( \chi : G \to \mathbb{G}_m \) is a one-dimensional representation of \( G \). Since \( U \cong U^\vee \), we find an expression of the character \( x_6 \) of \( S^6U^\vee \) as a sum of irreducible characters and take \( V^\vee \) to be the direct sum of three irreducible characters up to tensoring them simultaneously by a one-dimensional character of \( G \). We also have to take care that the rational parametrization given by \( V \) defines a nodal sextic.

We start with dihedral groups \( \mathbb{D}_{2n} \) of order \( 2n \). It follows from the classification of rational nodal sextics with a cyclic group of symmetries that \( n \in \{2, 3, 4, 5, 6\} \).

If \( n = 2 \), the character \( x_6 \) of \( S^6U^\vee \) decomposes as \( x_6 = \chi_0 + 2\chi_1 + 2\chi_2 + 2\chi_3 \) with irreducible summands \( \chi_0 = \langle t_0t_1^5 - t_0^5t_1 \rangle, 2\chi_1 = \langle t_0^6t_1 + t_0t_1^5 + t_0^5t_1^2 \rangle, 2\chi_2 = \langle t_0^5t_1^2 - t_0^6t_1, t_0^6t_1^3 - t_0^5t_1^2 \rangle, 2\chi_3 = \langle t_0^6 + t_0^5t_1^2 + t_0^4t_1^4 \rangle \). One of the summands of \( V \) must be \( \chi_2 \) or \( \chi_3 \). We have the following possibilities:

\[
(x_0, x_1, x_2) = (t^6 + 1 + a(t^4 + t^2), t^5 - t, b(t^5 + t) + ct^3), \quad (9.4.5)
\]

\[
(x_0, x_1, x_2) = (t^6 + 1 + a(t^4 + t^2), b(t^6 - 1) + c(t^2 - t^4), d(t^5 + t) + ct^3), \quad (9.4.6)
\]

\[
(x_0, x_1, x_2) = (t^6 + 1 + a(t^4 + t^2), b(t^6 - 1) + c(t^2 - t^4), t^5 - t).
\]
corresponding to the sum of characters $\chi_3 + \chi_0 + \chi_1, \chi_3 + \chi_2 + \chi_1$ and $\chi_3 + \chi_2 + \chi_0$, respectively. If $c = 0$, the last two parametrizations become isomorphic under the change $t \mapsto \epsilon_8 t$.

If $n = 3$, we have two 2-dimensional characters $x$ and $\rho_1$. The representation $x_6$ decomposes as $x_6 = 2\rho_1 + \chi_0 + 2\chi_1$, where $2\rho_1 = \langle t_0 t_1^2, t_0^2 t_1 \rangle + \langle t_0^2 t_1^2, t_0^2 t_1 \rangle$, $2\chi_1 = \langle t_0^3 t_1^2, t_0^6 + t_0^6 \rangle, \chi_0 = \langle t_0^6 - t_0^6 \rangle$.

We have two possible decompositions of $V$, namely $V = \rho_1 + \chi_0$ or $V = \rho_1 + \chi_1$. This gives us two possible isomorphism classes of parametrizations

$$\begin{align*}
(x_0, x_1, x_2) &= (at^5 + bt^2, bt^4 + at, t^6 + 1 + ct^3), \\
(x_0, x_1, x_2) &= (at^5 + bt^2, bt^4 + at, t^6 - 1).
\end{align*}$$

However, if $c = 0$, they become isomorphic under the change $t \mapsto it$.

If $n = 4$, we get $x_6 = 2\rho_1 + \chi_1 + \chi_2 + \chi_3$, where $2\rho_1 = \langle t_0 t_1^2, t_0^2 t_1^2 \rangle + \langle t_0^2 t_1^2, t_0^2 t_1 \rangle, \chi_1 = \langle t_0^3 t_1^2 \rangle, \chi_2 = \langle t_0 t_1 - t_0 t_1 \rangle, \chi_3 = \langle t_0 t_1 + t_0^3 t_1 \rangle$. We obtain three possible parametrizations $V = \rho_1 + \chi_i, i = 1, 2, 3$.

$$\begin{align*}
(x_0, x_1, x_2) &= (t^6 + at^2, at^4 + 1, t^3), \\
(x_0, x_1, x_2) &= (t^6 + at^2, at^4 + 1, t^5 + t), \\
(x_0, x_1, x_2) &= (t^6 + at^2, at^4 + 1, t^5 - t).
\end{align*}$$

The projective representations in the last two cases are isomorphic under the change $t \mapsto \epsilon_8 t$. Note that the first two curves are special members of the family of curves with $D_4$-symmetry defined in Row 9 from Table 9.4. They correspond to parameters $d$ or $e$ equal to zero. Also, the second family from Row 10 can also specialize to a member of the first family with $D_8$-symmetry. We give to parameters the values $b = 0, e = 1$ and $a = -3$ and change the generator $g_1$ to the transformation $(t_0, t_1) \mapsto (\frac{1}{\sqrt{2}}(t_0 - it_1), \frac{1}{\sqrt{2}}(t_0 + it_1))$.

Suppose that $n = 5$. Here, for the future use, we slightly change our generators of the binary dihedral group by taking

$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We find that $x_6 = \rho_1 + 2\rho_3 + \chi_1$, where $\rho_1 = \langle t_0 t_1^2, t_0^2 t_1 \rangle, 2\rho_3 = \langle t_0^2 t_1^2, t_0^2 t_1 \rangle + \langle t_0 t_1^2, t_0^2 t_1 \rangle$, and $\chi_1 = \langle t_0^3 t_1^2 \rangle$. There is only one isomorphism class of parametrizations $V = \rho_3 + \chi_1$ represented by

$$(x_0, x_1, x_2) = (t^6 + at, -at^5 + 1, t^3).$$

Finally, if $n = 6$, we know that there is only one isomorphism class of rational nodal sextics with cyclic symmetry of order 6 given by equation (9.4.25). It has additional projective symmetry given by interchanging $y$ with $z$.

Assume now that $G = T, O$, or $I$. We would like to know how the character $x_k$ of $S^kU^\vee$ decomposes into irreducible summands. Let $m_k(v) = \langle \chi_v, x_k \rangle$ be the multiplicity of the character $\chi_v$ corresponding to the vertex $v$ of the extended graph of $\Gamma(p, q, r)$ from Table 2 in the representation $S^kU$ with character $x_k$. Let

$$P_v(T) = \sum_{k=0}^{\infty} m_k(v)T^k.$$

The proof of the following result can be found in [653, 4.2].
Proposition 9.4.5. Assume $G = \mathbb{T}, \mathbb{O}$ or $\mathbb{I}$. Let $a$ be the length of the arm of the Dynkin diagram that contains the trivial representation $\chi_0$. The Poincaré series $P_v(T)$ is a rational function with denominator 

$$(1 - T^{2a})(1 - T^{4a-4}).$$

Denoting the numerator of $P_v(T)$ corresponding to the characters $\chi_v$ equal to $x_h, 0 \leq h \leq a - 1, yx_i, 0 \leq i \leq b - 2$ and $zx_j, 0 \leq j \leq c - 2$ by $X_h, Y_i, Z_j$, we have

$$X_h(T) = T^h + T^{6a-6-h} + (T^{2a-h} + T^{4a-4-h}) \frac{1 - T^{2h}}{1 - T^2}, \quad v = x_h,$$

$$Y_i(T) = T^{a+b-i-2}(1 + T^{2a-2}) \frac{1 - T^{2a}}{1 - T^2} \frac{1 - T^{2i+2}}{1 - T^2}, \quad v = yx_i,$$

$$Z_j(T) = T^{a+c-j-2}(1 + T^{2a-2}) \frac{1 - T^{2a}}{1 - T^2} \frac{1 - T^{2j+2}}{1 - T^2}, \quad v = zx_j.$$ 

A straightforward computation gives the following.

Corollary 9.4.6. The character $x_6$ is equal to

$$x_6 = \begin{cases} 
\chi_0 + 2x_2 & \text{if } G = \mathbb{T}, \\
x_2 + y + x_2y & \text{if } G = \mathbb{O}, \\
x_2y + z & \text{if } G = \mathbb{I}.
\end{cases}$$

Using these tools, we can proceed.

Without loss of generality, we may assume that $k = \mathbb{C}$. Each such group is conjugate to a finite subgroup of $SU(2)$. Let $\mathbb{H}^*$ be the multiplicative group $\mathbb{H}^*$ of quaternions. We use an isomorphism

$$\mathbb{H}^* \to SU(2), \quad (a + bi) + (c + di)j \mapsto \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}.$$ 

This allows one to realize $\widetilde{G}$ as a finite subgroup of $\mathbb{H}^*$.

Let $g_1 = i$, $g_2 = \frac{1}{2}(1 + i + j + k)$, $g_3 = \frac{1}{2}(1 + i + j - k)$. They form a set of standard generators of $\widetilde{G}$ satisfying $g_1^2 = g_2^3 = g_3^3 = g_1g_2g_3 = -1$. We will use other generators

$$g_1 = \begin{pmatrix} e_4 & 0 \\ 0 & e_4^{-1} \end{pmatrix}, \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 + i & 1 + i \\ -\frac{1}{2} + i & -\frac{1}{2} \end{pmatrix}.$$

We see that the subgroup generated by $i$ and $k$ is isomorphic to the binary dihedral group $\tilde{D}_4$.

The transformation $g_3$ acts on the space $S^*U^\vee$ with a basis $(t_{10}^6, t_{10}^5t_{11}, t_{10}^4t_{11}^2, t_{10}^3t_{11}^3, t_{10}^2t_{11}^4, t_{10}t_{11}^5, t_{11}^6)$ via the matrix

$$S = 1/8 \cdot \begin{pmatrix} 1 & -i & -1 & i & 1 & -i & -1 \\ 6 & -4i & -2 & 0 & -2 & 4i & 6 \\ 15 & -5i & 1 & -3i & -1 & -5i & -15 \\ 20 & 0 & 4 & 0 & 4 & 0 & 20 \\ 15 & 5i & 1 & 3i & -1 & 5i & -15 \\ 6 & 4i & -2 & 0 & -2 & -4i & 6 \\ 1 & i & -1 & -i & 1 & i & -1 \end{pmatrix} \quad (9.4.13)$$
Assume that $G = \mathbb{T}$. By Corollary 9.4.6, $x_6 = \chi_0 + 2x_2$. Hence $V$ is given by the character $x_2$. The group contains a subgroup isomorphic to $\mathbb{D}_4$. We have two families of rational nodal sextics with $\mathbb{D}_4$-symmetry. We check that no member of the second family given in Row 9 is invariant with respect to $g_3$. Using the matrix $S$, we check that a member of the first family with parameters $(a, b, c, d, e)$ is invariant with respect to $g_3$ if and only if the following relations are satisfied:

$$(2d + e)a = 3e - 10d, \quad (2d - e)c = (3e + 10d)b, \quad b(15 - a) = c(a + 1). \quad (9.4.14)$$

The third relation follows from the first two relations. Also, the conditions $b = 0$, $e = 2d$ and $a = -1$ are equivalent.

It follows that we have a one-parameter family of tetrahedral rational nodal sextics. Assume that $b \neq 0, d \neq 0$, hence we can take $b = d = 1$.

We check that $g_3$ changes the basis of $V$ via a scalar multiple of the matrix

$$
\begin{pmatrix}
0 & \frac{i(2+e)}{e-2} & 0 \\
0 & 0 & \frac{2-e}{8} \\
\frac{8i}{e+2} & 0 & 0
\end{pmatrix}
$$

This can be applied only if $e^2 - 4 \neq 0$. As we remarked before, under our assumption, $e \neq 2$ and also $e = -2$ does not satisfy the first of relations (9.4.14).

After scaling the variables $(x, y, z) \mapsto (-8i(e-2)x, 8(e+2)y, (e^2-4)z)$, we obtain the following equation

$$(e^2 - 4)^2(x^6 + y^6 + z^6) + (e + 2)(3e^3 - 6e^2 + 36e - 8)(x^4y^2 + x^2z^4 + y^4z^2)$$

$$+ (e - 2)(3e^3 + 6e^2 + 36e + 8)(x^4z^2 + x^2y^4 + y^2z^4) - 3(7e^4 + 40e^2 - 16)x^2y^2z^2 = 0. \quad (9.4.15)$$

Note that the equation has a full $S_3$-symmetry if and only if $e = 0$. In this case we get a rational nodal sextic

$$x^6 + y^6 + z^6 - (x^4y^2 + x^2z^4 + y^4z^2 + x^4z^2 + x^2y^4 + y^2z^4) + 3x^2y^2z^2 = 0 \quad (9.4.16)$$

with an octahedral symmetry. The plane sextic curve has 10 ordinary nodes forming 4 orbits of points $[1, 1, 1], [-1, 1, 1], [i, 1, 0]$ and $[-i, 1, 0]$.

Assume now that $b = 0$. In this case, $a = -1, e = 2d$ and we may also take $c = d = 1$. The transformation $g_3$ is given by matrix

$$
\begin{pmatrix}
0 & \frac{i}{4} & 0 \\
0 & 0 & 2 \\
-2i & 0 & 0
\end{pmatrix}.
$$

So, we take a basis $(i(t^6 + 1 - (t^2 + t^4)), 4(t^2 - t^4), 2(t^5 + 1 + 2t^3))$ in $V$ and obtain the following equation of the rational nodal sextic

$$x^4y^2 + x^2z^4 + y^4z^2 - 3x^2y^2z^2 = 0. \quad (9.4.17)$$

It has an obvious tetrahedral symmetry.

The curve has three tacnodes at the $S_3$-orbit of $[1, 0, 0]$ and 4 ordinary nodes at the $S_3$-orbit of $[1, 1, 1]$ and $[-1, -1, 1]$. 
Finally, we assume that $d = 0$. In this case $V = (t^6 + 1 + 3(t^2 + t^4), t^6 - 1 + 3(t^2 - t^4), t^3)$. The action of the transformation $g_3$ is given by the following matrix

$$
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1/8 \\
-8i & 0 & 0
\end{pmatrix}.
$$

After scaling the basis, we get the following equation

$$x^6 + y^6 + z^6 - 21x^2y^2z^2 + 3(x^4y^2 + x^4z^2 + x^2y^4 + x^2z^4 + y^2z^2 + y^2z^4) = 0. \quad (9.4.18)
$$

The curve has 10 ordinary nodes forming the $S_3$-orbits of the points $[1, 1, 1], [1, 1, -1]$ and $[0, \pm i, 1]$.

Next we consider the case $G = \mathbb{I}$. It follows from Corollary 9.4.6 that there are two isomorphism classes of octahedral rational sextics realizing the irreducible representations of $\mathbb{I} \cong S_4$ with characters $x_2$ and $x_2y$. We have already found both of them as special members of the tetrahedral family. They correspond to the parameters $d = 0$ and $e = 0$. They are also special members of the family of nodal sextics given by (9.4.9) with $a = -5$ and in (9.4.9) with $a = 3$.

Finally, let $G = \mathbb{I}$. The group $\tilde{G}$ has generators from $SU(2)$

$$
\begin{align*}
g_1 &= \frac{1}{2}(\lambda^{-1}i + j + k), \\
g_2 &= \frac{1}{2}(1 + i + j + k) \\
g_3 &= \frac{1}{2}(\lambda + \lambda^{-1}i + j)
\end{align*}
$$

satisfying $g_1^2 = g_2^2 = g_3^2 = g_1g_2g_3 = -1$.

However, we will use Klein’s generators [379, p.213])

$$
S = \left( \begin{array}{cc}
\epsilon_{10}^3 & 0 \\
0 & \epsilon_{10}^{-3}
\end{array} \right), \quad T = \frac{1}{\sqrt{5}} \left( \begin{array}{ccc}
\epsilon_5 - \epsilon_3^2 & \epsilon_2 - \epsilon_5^2 & \epsilon_2 - \epsilon_3^2 \\
\epsilon_2 - \epsilon_5^2 & \epsilon_2 - \epsilon_3^2 & \epsilon_5 - \epsilon_3^2
\end{array} \right), \quad U = \left( \begin{array}{cc}
0 & -1 \\
1 & 0
\end{array} \right)
$$

of orders 10, 4 and 4.

By Corollary 9.4.6, $x_6 = xy + z$. Thus $V \cong z$ and there is only one isomorphism class of nodal rational plane sextics with icosahedral projective symmetry. Since $D_{10}$ is a subgroup of $\mathbb{I}$, our plane sextic belongs to the one-parameter family given in Row (14) in the Table. Applying transformation $T$, we see that the parametrization $t_1^6 + at_0^5t_1 - at_0t_1^5 + t_0^6, t_0^3t_1^3$ is invariant if and only if $a = 3$. If we multiply $z$ by $5i$, we obtain equation

$$
32x^6 + 27xy^5 - 120x^4yz + 150x^2y^2z^2 + 5y^3z^3 + 27xz^5 = 0 \quad (9.4.19)
$$

which can be found in [705, §2].

We summarize our classification in Table 9.4.

The following are equations of some of the curves from the Table.

$$
\begin{align*}
4(a - 2)^4xy^5 + a(a - 2)^4y^4z^2 - 4(a - 2)^4x^3y^3 - (a^2 - 3a - 8)(a - 1)^2xyz^4 \\
+ (5a^2 - 5a - 2)(a - 2)x^2y^2z^2 - x^5y + x^4z^2 + (a - 1)^2z^6 = 0. \quad (9.4.20)
\end{align*}
$$

Here we give to the parameters the values $a_1 = a_4 = a_6 = 0, a_2 = a, a_3 = a_6 = 1, a_7 = 2$, otherwise the formula is too long.
9.4. SELF-PROJECTIVE RATIONAL NODAL PLANE SEXTICS

<table>
<thead>
<tr>
<th>G</th>
<th>V</th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>C₂</td>
<td>(x_2 \oplus x_2 \oplus 1)</td>
<td>(t^6 + a_1 t^4 + a_2 t^2)</td>
<td>(a_3 t^4 + a_4 t^2 + 1)</td>
<td>(a_5 t + a_6 t^2 + a_7 t)</td>
</tr>
<tr>
<td>(2)</td>
<td>C₂</td>
<td>(x_2 \oplus 1 \oplus 1)</td>
<td>(t^6 + a_1 t^4 + a_2 t^2 + 1)</td>
<td>(a_3 t^4 + a_4 t^2 + a_5 t)</td>
<td>(a_6 t + a_7 t^2)</td>
</tr>
<tr>
<td>(3)</td>
<td>C_1</td>
<td>(1 \oplus x_4 \oplus x_2)</td>
<td>(t^6 + a_1 t^4 + 1)</td>
<td>(a_2 t^2 + a_3 t^2)</td>
<td>(a_4 t + a_5 t)</td>
</tr>
<tr>
<td>(4)</td>
<td>C_4</td>
<td>(x_4 \oplus x_2 \oplus 1)</td>
<td>(t^6 + a_1 t^4)</td>
<td>(t^2 + 1)</td>
<td>(t^2)</td>
</tr>
<tr>
<td>(5)</td>
<td>C_4</td>
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<td>(a_2 t^2 + 1)</td>
<td>(a_3 t + a_4 t)</td>
</tr>
<tr>
<td>(6)</td>
<td>C_2</td>
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<td>(t^6 + a t)</td>
<td>(t^2 + 1)</td>
<td>(t^2)</td>
</tr>
<tr>
<td>(7)</td>
<td>C_2</td>
<td>(x_2 \oplus x_2 \oplus 1)</td>
<td>(t^6 + a t)</td>
<td>(t^2 + 1)</td>
<td>(t^4)</td>
</tr>
<tr>
<td>(8)</td>
<td>C_2</td>
<td>(x_2 \oplus x_2 \oplus x_2)</td>
<td>(t^6 + a t)</td>
<td>(t^2 + 1)</td>
<td>(t^4)</td>
</tr>
<tr>
<td>(9)</td>
<td>D_4</td>
<td>(x_3 \oplus x_2 \oplus 1)</td>
<td>(t^6 + 1 + a(t^4 + t^2))</td>
<td>(b(t^4 - 1) + e(t^4 - t^2))</td>
<td>(d(t^4 + t) + e t^2)</td>
</tr>
<tr>
<td>(10)</td>
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<td>(x_3 \oplus x_0 \oplus 1)</td>
<td>(t^6 + 1 + a(t^4 + t^2))</td>
<td>(t^2 - t)</td>
<td>(b(t^4 + t) + e t^2)</td>
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<tr>
<td>(11)</td>
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<td>(t^4)</td>
</tr>
<tr>
<td>(12)</td>
<td>D_4</td>
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<tr>
<td>(13)</td>
<td>D_4</td>
<td>(x_0 \oplus x_3)</td>
<td>(t^6 + a t^2)</td>
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<td>(t^4)</td>
</tr>
<tr>
<td>(14)</td>
<td>D_4</td>
<td>(x_0 \oplus x_3)</td>
<td>(t^6 - a t^2)</td>
<td>(t^2 + 1)</td>
<td>(t^4)</td>
</tr>
<tr>
<td>(15)</td>
<td>D_4</td>
<td>(x_0 \oplus x_2)</td>
<td>(t^6 + a t^2)</td>
<td>(t^2 + 1)</td>
<td>(t^4)</td>
</tr>
<tr>
<td>(16)</td>
<td>T</td>
<td>(x_2)</td>
<td>(t^6 + 1 + a(t^4 + t^2))</td>
<td>(b(t^4 - 1) + e(t^4 - t^2))</td>
<td>(d(t^4 + t) + e t^2)</td>
</tr>
<tr>
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<td>T</td>
<td>(x_2 y)</td>
<td>(t^6 + 2 t^4)</td>
<td>(t^6 + 1)</td>
<td>(t^4)</td>
</tr>
<tr>
<td>(18)</td>
<td>T</td>
<td>(x_2 y)</td>
<td>(t^6 - 5 t^4)</td>
<td>(-5 t^4 + 1)</td>
<td>(t^4 + t)</td>
</tr>
<tr>
<td>(19)</td>
<td>T</td>
<td>(x_2 y)</td>
<td>(t^6 + 5 t^4)</td>
<td>(3 t^4 - 1)</td>
<td>(-5 t^4)</td>
</tr>
</tbody>
</table>

Table 9.1: Symmetric rational nodal sextics

\[(a^3 - 1)^2 y^6 + a^5 x^2 y^4 - a^5 x^4 y^2 - 6a(a-1)y^5 z + a(4a^3 - 5)x^2 y^2 z^2 + (2a^3 - 1)x^2 y^3 z \]
\[-2(a^3 - 1)y^3 z^3 + 9a^2 y^4 z^2 - 6a^2 x^2 y^2 z^3 + 6ay^2 z^4 - 6az^6 = 0,\]
\[(9.4.21)\]

where \(a_1 = a_2 = a_4 = a_5 = 0, a_3 = a_6 = 1, a_7 = a.\)

\[((a + 2)^3 y^6 - (a + 2)^2(a + 4)x^2 y^2 z + 2(a^2 + 20)x^2 y^2 z^2 - 8x^3 y^3 + 8x^4 yz\]
\[+ (a - 4)(a - 2)^2 x^2 y^2 z^2 - 8x^3 y^3 z^2 + (a^2 - 4)^2 y^3 z^3 - (a - 2)^3 z^6 = 0,\]
\[(9.4.22)\]

where \(a_1 = a, a_2 = a_3 = a_4 = 1, a_5 = -1.\)

\[a^3 y^4 z - 3 x^3 y^3 (a + 3)(a - 1)^2 x^2 y^2 z^2 - (a - 3) x^2 y^2 z^2 + x^4 z^2 + (a - 1)^4 z^6 = 0.\]
\[(9.4.23)\]

\[a^5 y^3 z^2 + a^2 (5 - 3a)x^2 y^2 z^2 - a(a - 5)(a - 1)^2 x^2 y^2 z^2 - x^5 y + (a - 1)^4 z^6 = 0.\]
\[(9.4.24)\]

where \(a_1 = a, a_2 = a_3 = 1, a_4 = 0.\)

\[-a^2 y^5 z + x^2 y^4 - (a - 1)^2(3a + 2)x^2 y^2 z^2 + a^2 (a + 4)(a - 1)x^3 y^2 z^2 - x^5 z - (a - 1)^3 z^6 = 0.\]
\[(9.4.25)\]

\[x^5 z - x^3 y^3 (a + 3)x^2 y^2 z^2 + (2a + 3)(a - 1)x^3 y^2 z^2 - a^3 y^3 z^2 - (a - 1)^2 z^6 = 0.\]
\[(9.4.26)\]

\[-a^4 y^5 + a^2 (a + 4)(a - 1)^2 y^3 z^2 + x^4 y^2 - (a + 2)(a - 1)^2 x^2 y^2 z^3 - (a - 1)^5 z^6 = 0.\]
\[(9.4.27)\]
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\[ y^6 + 2(a^3 - 3a^2 + 12)yz^2 - 3x^2y^4 + (a + 1)(a - 2)(a^3 - 3a^2 + 12)y^2z^4 + 6(a^3 - 4)x^2y^2z^2 + 3x^4y^2 + 6(a^2 + 2)x^4z^2 - 3a^2(3a^2 - 8)x^2z^4 - x^6 + 4(a + 1)^2(a - 2)^2z^6 = 0, \]  

(9.4.28)

where \( b = e = 1, c = d = 0. \)

\[ 4(a + 1)^2x^2y^4 - x^4y^2 - 2(a + 1)(a - 3)yz^4 + (a + 1)^2(a - 3)^2y^2z^2 + z^6 + 4ax^2y^2z^2 = 0 \]  

(9.4.29)

where \( b = 0, c = 1. \)

\[ -(b^4 + 4b^2 - 2)x^3yz^3 + 2(b^2 + 4)x^2y^2z^2 + 4bx^4yz + x^6 + y^6 - xy^2 + 4(a + 1)x^2z^4 = 0, \]  

(9.4.30)

where \( a = 0. \)

\[ (a^2 - 1)^4z^6 - (a^2 + 3)(a^2 - 1)^2xyz^4 + a^3x^4z^2 + a^3y^4z^2 - (a^2 - 3)x^2y^2z^2 = 0. \]  

(9.4.31)

\[ (a + 1)^4z^6 - (a^2 + 8a + 9)x^2y^2 - 2x^3y^3 - 2y^5 - 2(a + 3)(a + 1)^2xyz^4 + ax^4z^2 + ay^4z^2 = 0. \]  

(9.4.32)

\[ (a^2 + 1)^5z^6 + (2a^2 - 3)(a^2 + 1)^3xyz^4 + (a^2 - 1)(a^4 + a^2 + 3)x^2y^2z^2 + a^3x^5z - a^3y^5z = 0. \]  

(9.4.33)

Remark 9.4.7. Among Coble sextics with \( \mathbb{D}_3 \)-symmetry there is a remarkable one. We specialize equation (9.4.29) by taking \( a = 3 \), change \((x, y, z)\) to \((2x, iy/4, z)\) and obtain the equation

\[ M = x^4y^4 + x^2y^4 - 3x^2y^2z^2 + z^6 = 0. \]  

(9.4.34)

The polynomial has a \( \mathbb{D}_3 \)-symmetry. The curve is isomorphic to a curve from the family given by (9.4.31) with parameter \( a \) equal to 3.

The polynomial \( M \) takes only non-negative real values in \( \mathbb{P}^2(\mathbb{R}) \) and cannot be represented as a sum of squares of ternary forms. It is called the Motzkin polynomial (see a survey on non-negative ternary forms in [591]).

Another ternary sextic with the same property is the Robertson ternary sextic. It coincides with our tetrahedral sextic given by equation (9.4.17).

Using Timofte’s test positivity for homogeneous polynomials in \( d \) variables with \( \mathcal{G}_d \)-symmetry (see [593]), we find that the polynomial \( P \) defining the octahedral sextic given by equation (??) from Row (17) of the Table is also non-negative on all real values of the variables.

Remark 9.4.8. The equation of the icosahedral sextic can be expressed as

\[ 5A^3 + 27F = 0 \]

where \( A = x_0x_1 + x_2^2 \) and \( F = x_2(x_0^3 + x_1^3 + x_2^3 + 5x_0^2x_1^2x_2 - 5x_0x_1x_2^2) \) are fundamental invariants of the icosahedron group of degree 2 and 6 (see [704], [180], 9.5.4).

9.5 Self-projective rational nodal sextics and quartic symmetroids

Let \( G \) be a polyhedral group. In the previous section we found a 3-dimensional subrepresentation \( V \) of \( G \) in \( S^6U^\vee \) that defines a \( G \)-equivariant rational parametrization of a Coble sextic. Let

\[ S^6U^\vee = V \oplus W, \]

\[ V \] and \( W \) being complementary eigenspaces of the action of \( G \).
9.5. SELF-PROJECTIVE RATIONAL NODAL SEXTICS AND QUARTIC SYMMETROIDS

where $W$ is a 4-dimensional linear representation of $G$. The linear space $S^6U^\vee$ admits a unique $SL(U)$-invariant symmetric bilinear form such that the two summands $V$ and $W$ become orthogonal summands.

Let us recall the definition of this bilinear form. The linear representations of $SL(U)$ in $U$ and its dual representation in $U^\vee$ are isomorphic via the correlation isomorphism $c : U \to U^\vee$ defined by choosing a volume form $\Omega \in \wedge^2 U \cong \mathbb{k}$. If we choose a basis $(e_0, e_1)$ of $U$ and the dual basis $(t_0, t_1)$ of $U^\vee$, correlation isomorphism is given by $(t_0, t_1) \mapsto (-e_1, e_0)$. It defines an isomorphism of the linear representations $S^6U^\vee \to S^6U$ of $SL(U)$ whose composition with the polarization isomorphism $S^6U \to (S^6U^\vee)^\vee$ is a $SL(U)$-invariant symmetric bilinear form on $S^6(U^\vee)$. In coordinates, it is given by

$$\sum_{i=0}^6 a_i t_i^6 - i, \sum_{i=0}^6 b_i t_i^6 - i = \sum_{i=0}^6 (-1)^i \binom{6}{i} a_i b_{6-i}.$$  

It corresponds to the quadratic invariant

$$I_2 = a_0a_6 - 6a_1a_5 + 15a_2a_4 - 10a_3^2.$$  

on the space of binary sextics $\sum_{i=0}^6 \binom{6}{i} a_i t_i^6 - i$. Of course, all of this applies to binary forms of any even degree (in odd degree we obtain a skew-symmetric bilinear form).

The linear space $W = V^\perp \subset S^6U^\vee$ defines a web $|W|$ of harmonic quadrics and its discriminant quartic is a quartic symmetroid $D(|W|)$ whose group of projective automorphisms contains $G$ as a subgroup. In this section we will find its equation for each possible polyhedral group $G$ and its linear 3-dimensional representation $V$ in $S^6U^\vee$ that leads to a rational nodal sextic. We will restrict ourselves with non-cyclic groups $G$.

Recall from Section 7.5 that there exists a $SL(U)$-equivariant isomorphism from the space $S^3(L^\vee)_{\text{harm}}$ of harmonic quadratic forms on $L = S^3U$ and the space $S^6U^\vee$ of binary sextics. Using the explicit definition of harmonic forms given in Section 7.5, we can choose a basis in $S^3(V^\vee)_{\text{harm}}$ formed by

$$p_1 = xw + 9yz, p_2 = 2xz + 3y^2, p_3 = 2yw + 3z^2, p_4 = x^2, p_5 = w^2, p_6 = xy, p_7 = zw.$$  

(see equation (7.5.4)). They correspond to the following basis in $S^6U^\vee$

$$10t_0^3t_1^3, 5t_0^4t_2, 5t_0^2t_1^4, t_0^6, t_0^5t_1, t_0t_1^5.$$  

Let us identify the representation $W$ with a linear subspace of harmonic quadratic form. Then the associated quartic symmetroid $D(|W|)$ admits $G$ as its group of projective symmetries. If the web of harmonic quadrics has no base points, the quotient of the minimal nonsingular model of the symmetroid by the Reye involution is an Enriques surface that admits a Fano-Reye polarization with the group of symmetry isomorphic to $G$.

We use the classification of symmetric Coble sextics from Table (9.4).

We start with the group $G = \mathbb{D}_4$. There are two families corresponding to Coble sextics in Rows 9 and 10 of the Table with $V$ given in (9.4.5) and (9.4.6). Let us consider the first case. We have

$$V = \langle t^6 + 1 + a(t^4 + t^2), b(t^6 - 1) + c(t^4 - t^2), d(t^5 + t) + et^3 \rangle.$$
Assume first that $ac \neq 0$. We can choose the following basis of $V^\perp$

$$e_1 = t^5 - t, \quad e_2 = -3e(t^5 + t) + 20dt^3, \quad e_3 = a(t^6 + 1) + 15(t^2 + t^4), \quad e_4 = c(t^6 - 1) + 15b(t^4 - t^2).$$

The action of $g_1$ and $g_2$ is given by diagonal matrices $(-1, -1, 1, 1)$ and $(-1, 1, 1, -1)$. The web of harmonic quadrics is

$$x_0(p_7 - p_6) + x_1(-3e(p_6 + p_7) + 2dp_1) + x_2(a(p_4 + p_5) - 3(p_2 + p_3)) + x_3(c(p_5 - p_4) - 3b(p_3 - p_2)) = 0.$$

Using the MAPLE, we find the following equation of the discriminant quartic

$$x_0^4 + (e^2 - 4d^2)x_1^4 + 16(a + 1)^2x_2^4 + 16b^2(b + c)^2x_3^4 - 2(e^2 - 4d^2)x_0^2x_1^2 + 8(a + 1)x_0^2x_2^2
\quad - 8b(b + c)x_0^2x_3^2 + 8(-2a^2d^2 + 4ade + ae^2 - 6d^2 - 4de - e^2)x_1^2x_2^2
\quad + 8(6b^2d^2 - 4b^2de + b^2e^2 + 4bced - bce^2 + 2c^2d^2)x_1^2x_3^2 + 16(-a^2b^2 + 6ab^2 - 4b^2
\quad + 6bc - c^2)x_2^2x_3^2 + 16(2abcd - abe + 4bd + 4cd + ce)x_0x_1x_2x_3 = 0.$$ (9.5.1)

If $a = 0, c \neq 0$, we may assume $c = 1$ and take the following basis of $V^\perp$

$$e_1 = t^5 - t, \quad e_2 = -3e(t^5 + t) + 20dt^3, \quad e_3 = t^4 - t^2, \quad e_4 = (t^6 - 1) + 15b(t^4 + t^2).$$

The equation of the web of harmonic quadrics is

$$x_0(p_7 - p_6) + x_1(-3e(p_6 + p_7) + 2dp_1) + x_2(p_3 - p_2) + x_3(a(p_5 - p_4) + 3b(p_2 + p_3)) = 0.$$

The equation of the discriminant quartic is

$$x_0^4 + 81(e^2 - 4d^2)^2x_1^4 + 16x_2^4 + 1296b^2(b^2 - 1)x_3^4 - 18(e^2 - 4d^2)x_0^2x_1^2 - 8x_0^2x_2^2
\quad + (72b^2x_0^2x_3^2 + 72(6d^2 - 4de + e^2))x_1^2x_3^2 + 648(-6b^2d^2 - 4b^2de - b^2e^2 + 2d^2)x_1^2x_3^2
\quad + 144(-2b^2 + 1)x_2^2x_3^2 - 576bdx_0x_1x_2x_3 + 2592bx_2^3x_3^3 + 96x_2^2x_3^2 - 432b(2d + e)x_0x_1x_2^2
\quad - 24x_0^2x_2x_3 + 216e(4d - e)x_1^2x_2x_3 = 0.$$ (9.5.2)

If $c = 0$, we take $d = 1$ and find the following basis of $V^\perp$

$$e_1 = t^5 - t, \quad e_2 = -3e(t^5 + t) + 20t^3, \quad e_3 = t^4 - t^2, \quad e_4 = a(t^5 + t) - 50t^2.$$

The equation of the web of harmonic quadrics is

$$x_0(p_7 - p_6) + x_1(-3e(p_6 + p_7) + 2p_1) + x_2(p_3 - p_2) + x_3(a(p_2 + p_3) - 6p_2) = 0.$$

The equation of the discriminant quartic is

$$x_0^4 + 81(e^2 - 4d^2)^2x_1^4 + 16x_2^4 + (4a^2 - 6a - 9)(4a^2 + 6a + 9)x_3^4 - 18(e^2 - 4d^2)x_0^2x_1^2
\quad + 2(4a^2 + 9)x_0^2x_3^2 + 72(e^2 - 4de + 6d^2)x_1^2x_2^2 - 4(8a^2 - 27)x_2^2x_3^2 - 192adx_0x_1x_2x_3
\quad - 8x_0^2x_2^2 + 18(-24a^2d^2 - 16a^2de - 4a^2e^2 + 108d^2 - 9e^2)x_1^2x_3^2 + 72(6a^2d + 2a^2e - 9d)x_1x_2^3
\quad + 96ax_0x_2x_3^2 + 648d(e^2 - 4d^2)x_3^2x_3 + 144(e - 3d)x_1x_2^2x_3 - 72dx_0^2x_1x_3 = 0.$$ (9.5.3)
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In particular, if \( a = c = 0 \), we take \( d = 1 \) and get the equation
\[
x_3^4 + 81(e^2 - 4d^2)x_1^4 + 16x_2^4 + 81x_3^4 - 18(e^2 - 4d^2)x_0^2x_2^2 + 18x_0x_2^3
\]
\[+ 72(e^2 - 4de + 6d^2)x_1^2x_2^2 + 108x_2^2x_3^2 - 8x_0^2x_2^2 + 162(12d^2 - e^2)x_1^2x_3^2
\]
\[+ 648dx_1^3 + 648d(e^2 - 4d^2)x_1x_3^2 + 144(e - 3d)x_1^2x_3 - 72dx_0^2x_1x_3 = 0.
\]

Next we consider the second family with \( \mathbb{D}_4 \)-symmetry.

This time \( V = \langle t^6 + 1 + a(t^4 + t^2), t^5 - t, b(t^5 + t) + ct^3 \rangle \) and we choose a basis of \( V^\perp \) formed by
\[e_1 = t^6 - 1, \ e_2 = t^2 - t^4, \ e_3 = 3c(t^5 + t) - 10bt^3, \ e_4 = a(t^6 + 1) - 15(t^2 + t^4).
\]
The group acts by diagonal matrices \((1, 1, -1, 1)\) and \((-1, -1, 1, 1)\). The web of harmonic quadrics is
\[x_0(p_5 - p_4) + x_1(p_2 - p_3) + x_2(-3c(p_6 + p_7) - bp_1) + x_3(a(p_4 + p_5) - 3(p_2 + p_3)) = 0.
\]
The equation of the discriminant quartic (after scaling of the unknowns) is
\[
(b^2 - c^2)^2x_2^4 + (a + 1)^2x_3^4 + x_1^4 + (b^2 - c^2)x_2^2x_3^2 + (3b^2 + 4bc + 2c^2)x_1^2x_2^2 - x_0^2x_3^2
\]
\[- (a^2 - 6a + 2)x_2^2x_3^2 + 2c(2b + c)x_0x_1x_2^2 + b^2x_0^2x_2^2 - 6x_0x_1x_3^2 + x_3^2x_1^2 - 2x_0x_3^3 = 0.
\]
Assume \( G = \mathbb{D}_6 \). Then \( V = \langle t^5 + at^2, at^4 + t, t^6 + 1 + bt^3 \rangle \) and \( V^\perp \) has a basis
\[e_1 = 5t^4 + 2at, \ e_2 = 2at^5 + 5t^2, \ e_3 = b(t^6 + 1) + 40t^3, \ e_4 = t^6 - 1.
\]
The web of harmonic quadrics is
\[x_0(p_3 + 2ap_6) + x_1(p_2 + 2ap_7) + x_2(b(p_4 + p_5) + 4p_1) + x_3(p_5 - p_4) = 0.
\]
To simplify the equation, we replace the generators \( Q_0, Q_1, Q_2 \) with \( Q_2 + e_3^2Q_1 + e_3Q_0, \ Q_2 + e_3Q_1 + e_3^2Q_0, \ Q_2 + Q_1 + Q_0, \) where \( e_3 \) is a primitive third root of unity. In this way the generator \( g_1 \) of the group \( \mathbb{D}_6 \) acts as a cyclic permutation of the coordinates \( x_0, x_1, x_2 \). The generator \( g_2 \) acts in the projective space by permuting \( x_0 \) and \( x_1 \). The equation of the discriminant hypersurface becomes
\[
(a^2 - 2a + 15b - 29)(a^2 + 2a - 21b - 41)(x_0^4 + x_1^4 + x_2^4) + 657(x_0x_1 + x_0x_2 + x_1x_2)x_3^3
\]
\[+ 315(x_0^2 + x_1^2 + x_2^2)x_3^2 + 3(a^4 + 6a^2b - 2a^2 - 648b^2 - 24a + 2b + 2593)(x_0^2x_1^2 + x_0^2x_2^2 + x_1^2x_2^2)
\]
\[+ (-2a^4 + 3a^2b - 68a^2 - 72ab - 128ab^2 - 12a + 3b + 5074)(x_0^3(x_1 + x_2) + x_1^3(x_0 + x_2) + x_2^3(x_0 + x_1)
\]
\[+ 9(-2a^2b + 24a^2 - 24ab - 435b^2 + 8a - 2b + 1764)(x_0^2x_1x_2 + x_0x_1^2x_2 + x_0x_1x_2^2)
\]
\[- 9\sqrt{3}(a^2 - 1)(x_0^3(x_2 - x_1) + x_1^3(x_2 - x_0) + x_2^3(x_0 - x_1)x_3 = 0.
\]

Next we consider the case of dihedral groups \( \mathbb{D}_8 \).

We have two families of those corresponding to Rows (12) and (13) in the Table. We start with the first case. It is a specialization of the first case with \( G = \mathbb{D}_4 \) by taking \( d = 0, e = 1, a = c, b = 1 \). We have \( V = \langle t^6 + at^2, at^4 + 1, t^3 \rangle \) and take the following basis of \( V^\perp \)
\[e_1 = t^6, \ e_2 = t, \ e_3 = a - 15t^4, \ e_4 = at^6 - 15t^2.
\]
The web of harmonic quadrics is given by
\[ x_0p_6 + x_3p_7 + x_0(ap_4 - 3p_3) + x_1(ap_5 - 3p_2) = 0. \]
The equation of the discriminant quartic is
\[ 1296a(x_2^4 + x_3^4) + 1296(a^2 + 1)x_2^2x_3^2 + x_0^2x_1^2 + 36a(x_0^2 + x_1^2)x_2x_3 - 72x_0x_1x_2x_3 = 0. \] (9.5.7)
The transformation \( g_1 \) is given by the diagonal matrix \((1, -1, i, -i)\) and the transformation \( g_2 \) switches \( x_0, x_1 \) and \( x_2, x_3 \). Without destroying the symmetry we can scale the variables \( x_1 \) and \( x_2 \) to obtain equation
\[ a(x_2^4 + x_3^4) + (a^2 + 1)x_2^2x_3^2 + x_0^2x_1^2 + a(x_0^2 + x_1^2)x_2x_3 - 2x_0x_1x_2x_3 = 0. \] (9.5.8)
Let us look at the second family with \( \mathbb{D}_8 \)-symmetry. It is a specialization of the first family with \( D_4 \)-symmetry obtained by taking \( a = c, b = 1, d = 1, e = 0 \). In this case \( V = \langle t^6 + 1 + a(t^4 + t^2), t^6 - 1 + a(t^2 - t^4), t^5 + 1 \rangle \) and \( V^\perp \) has a basis
\[ e_1 = t^5 - t, \ e_2 = t^3, \ e_3 = at^6 - 15t^2, \ e_4 = -15t^4 + a. \]
The transformation \( g_1 \) acts via a diagonal matrix \((i, -i, -1, 1)\) and the transformation \( g_2 \) multiplies \( x_0 \) by \(-1\) and switches the last two coordinates.

The web of harmonic quadrics is
\[ x_0(p_7 - p_6) + x_1p_1 + x_2(ap_5 - 3p_2) + x_3(ap_4 - 3p_3) = 0. \]
The equation of the discriminant quartic (after scaling the first two variables) is
\[ x_0^4 + x_1^4 + a(x_2^4 + x_3^4) + (a^2 + 1)x_2^2x_3^2 + 2x_0^2x_1^2 - (a^2 + 3)x_0^2x_2x_3 + 2(a + 1)(x_0x_1x_2^3 + x_1x_2x_3 - x_0x_1^2x_2) = 0. \] (9.5.9)
Next is the case \( G = \mathbb{D}_{10} \). We have only one family with \( V = \langle t^6 + at, at^5 - 1, t^3 \rangle \) and use a basis of \( V^\perp \)
\[ e_1 = 6t^5 + a, \ e_2 = 6t - at^6, \ e_3 = t^4, \ e_4 = t^2. \]
The web of harmonic quadrics is
\[ x_0(6p_7 + ap_4) + x_1(ap_5 + 6p_6) + x_2p_3 + x_3p_2 = 0. \]
The equation of the discriminant quartic (after scaling the last two variables) is
\[ (a^2 - 2)x_0x_1x_2x_3 + ax_1x_3^3 + ax_0x_3^3 + ax_0x_1x_2 + x_2^2x_3^2 + x_0^2x_1^2 = 0. \] (9.5.10)
Let \( G = \mathbb{D}_{12} \). Again we have only one isomorphism class of rational nodal sextics with \( \mathbb{D}_{12} \)-symmetry. The linear space \( V \) is spanned by \( t^6 + 1, t^5, t \) and \( V^\perp = \langle t^6 - 1, t^2, t^3, t^4 \rangle \). The web of harmonic quadrics is
\[ x_0(p_4 - p_5) + x_1p_1 + x_2p_2 + x_3p_3 = 0. \]
The equation of the symmetroid (after an equivariant scaling of the variables) is
\[ x_0x_3^3 + x_2^2x_3^2 - x_0^2x_2x_3 - 3x_1^2x_2x_3 + x_0^2x_1^2 - x_0x_2^3 + x_1^4 = 0. \] (9.5.11)
The next is the tetrahedral case \( G = \mathbb{T} \). Here
\[ V = \langle t^6 + 1 + a(t^4 + t^2), b(t^6 - 1) + c(t^2 - t^4), d(t^5 + t) + et^3 \rangle, \]
where the parameters satisfy the relations
\[(2d + e)a = 3e - 10d, \quad (e - 2d)c = (3e + 10d)b.\] (9.5.12)

Let us first omit the special cases \(b = 0, d = 0\). Thus we may assume that \(b = d = 1\). The relations imply that \(e \neq \pm 2\) and
\[a = \frac{3e - 10}{e + 2}, \quad c = \frac{3e + 10}{e - 2}.\]

If \(e \neq 0\), we can take the following basis of \(V^\perp\)
\[e_1 = t^5 - t, \quad e_2 = 10t^3 - 3et, \quad e_3 = 15(3e^2 + 20)t^2 + 240et^4 - (9e^2 - 100)t^6, \quad e_4 = 16e + 15(e^2 - 4)t^2 - (3e^2 + 20)t^6.\]

If \(e = 0\), we take
\[e_1 = t^5 - t, \quad e_2 = t^3, \quad e_3 = 3t^2 + t^6, \quad e_4 = 3t^4 + 1.\]

Let us postpone the later case until later.

We check that \(g_1, g_2\) and \(g_3\) act on the basis via the following matrices
\[g_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} -1 & -3e & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{3e^2 + 20}{16e} & \frac{e^2 - 4}{16e} \\ 0 & 0 & \frac{100 - 9e^2}{16e} & -\frac{3e^2 + 20}{16e} \end{pmatrix}\]

Using (9.4.13), we also check that \(g_3\) acts via the matrix
\[g_3 = \begin{pmatrix} 1 & \frac{3e}{2} & \frac{6ie(3e + 10)}{128e} & \frac{6ie(e - 2)}{128e} \\ 0 & 0 & -4i(3e + 10) & -4i(e - 2) \\ 0 & -\frac{e - 2}{32e} & \frac{-i(e + 2)(3e - 10)}{32e^2} & \frac{-i(e + 2)^2}{32e} \\ \frac{1}{128e} & \frac{3e - 10}{32e} & \frac{i(3e - 10)^2}{32e} & \frac{i(e + 2)(3e - 10)}{32e} \end{pmatrix}.\]

We would like to change a basis of \(V^\perp\) so that the group \(T \cong A_4\) would act by even permutations of coordinates. Let
\[C = \begin{pmatrix} 1 & -\frac{3e}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e + 2 & 2 - e \\ 0 & 0 & 10 - 3e & 3e + 10 \end{pmatrix}.\]

We check that \(C^{-1}g_2C\) is the diagonal matrix \((-1, 1, 1, -1)\) and
\[g'_3 = C^{-1}g_3C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -128ie & 0 \\ 0 & 0 & 0 & -i \\ 0 & \frac{1}{128e} & 0 & 0 \end{pmatrix}.\]

Replacing \(C\) with the product \(C \cdot J\), where \(J\) is the diagonal matrix \((1, 1, \frac{i}{128e}, -\frac{i}{128e})\), we obtain
\[C^{-1}g_3C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.\]
Thus we achieve our goal by taking a new basis
\[ f_1 = e_1, \quad f_2 = e_2 - \frac{3e}{2}, \quad f_3 = -\frac{i}{128e}((e+2)e_3+(10-3e)e_4), \quad f_4 = \frac{1}{128e}((e-2)e_3-(3e+10)e_4). \]

The web of harmonic quadrics is now given by
\[ x_0(p_7 - p_6) + x_1(p_1 - \frac{3e}{2}(p_6 + p_7)) + \frac{i}{8}x_2((3e - 10)(p_4 + p_5) - 3(e + 2)(p_2 + p_3)) + \frac{1}{8}x_3((3e + 10)(p_5 - p_4) + 3(e - 2)(p_3 - p_2)) = 0. \]

After equivariant scaling \((x_0, x_1, x_2, x_3) \mapsto (ix_0, x_1/3, x_2/3, x_3/3)\), we obtain the following equation of the quartic symmetroid
\begin{align*}
x_0^4 + (e^2 - 4)^2(x_1^4 + x_2^4 + x_3^4) &+ 2(e^2 - 4)x_0(x_1^2 + x_2^2 + x_3^2) \\
&- (e^4 - 40e^2 - 112)(x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_1^2) + 16(3e^2 + 4)x_0x_1x_2x_3 = 0. \tag{9.5.14}
\end{align*}

Let us now consider the case \(e = 0\). Here we may assume that \(b = d = 1\) and it follows from relations (9.5.12) that \(a = -5\). We have a basis \(V^\perp\) formed by \(t^5 - t, t^3, t^6 + 3t^2, 3t^4 + 1\).

The matrices \(g_2, g_3\) and \(C\) change to
\[
g_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad g_3 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -4i & -4i \\ 0 & \frac{1}{8} & \frac{i}{2} & \frac{-i}{2} \\ 0 & -\frac{1}{8} & \frac{3}{2} & \frac{3}{2} \end{pmatrix},
\]

In a new basis
\[
f_1 = e_1, \quad f_2 = e_2, \quad f_3 = \frac{i}{8}(e_3 + e_4), \quad f_4 = \frac{i}{8}(e_3 - e_4),
\]

the equation of the discriminant quartic is
\[
x_0^4 + x_1^4 + x_2^4 + x_3^4 - 2x_0^2(x_1^2 + x_2^2 + x_3^2) + 7(x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_1^2) + 8x_0x_1x_2x_3 = 0. \tag{9.5.15}
\]

It is obtained from equation (9.5.14) by substituting \(e = 0\) and replacing \(x_1, x_2, x_3\) with \(\frac{1}{7}x_1, \frac{1}{7}x_2, \frac{1}{7}x_3\). The equation has obvious \(G_3\)-symmetry in coordinates \(x_1, x_2, x_3\), also it admits a symmetry group \(D_4\) of transformations defined by diagonal matrices \((1, \pm 1, \pm 1, \pm 1)\).

The total symmetry is the octahedral group \(O \cong S_4\). We see that it is also a specialization of the second family of surfaces with dihedral symmetry \(D_8\). It corresponds to the parameter \(a = -5\).

The surface has 10 ordinary nodes at \([1, 2, 1, -1], [1, 2, -1, 1], [1, -2, 1, 1], [1, -2, -1, -1]\) and the \(S_4\) orbits of the points \([1, 0, 1, 0], [1, 0, -1, 0]\).

In the second omitted case we have \(b = 0\). We may assume that \(c = 1\) and relations (9.5.12) imply that \(e = 2d\), so we may also assume that \(d = 1\) and \(e = 2\). We have
\[
V = (t^6 + 1 - (t^2 + t^4), t^2 - t^4, (t^5 + t) + 2t^3).
\]

We start with the following basis of \(V^\perp\)
\[
e_1 = t^5 - t, \quad e_2 = -5t^3 + 3t, \quad e_3 = t^6 - 1, \quad e_4 = 2t^6 + 15(t^2 + t^4).\]
The equation of the discriminant quartic is
\[ \text{The equation of the quartic symmetroid (after an equivariant scaling of the variables) is} \]
\[ A \]
of type
\[ \text{the equation of the web of harmonic quadrics is} \]
\[ \text{In a new basis} \]
The surface has 10 ordinary nodes at
\[ g \]
The surface has four ordinary nodes at the
\[ S \]
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The transformation \( g_3 \) acts in this basis via the following matrix
\[ C := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2i \\ 0 & -2 & -\frac{i}{2} & -i \\ 0 & -2 & \frac{i}{4} & \frac{i}{2} \end{pmatrix}. \]
In a new basis
\[ f_1 = e_1, \ f_2 = 2i(3e_1 + 2e_2), \ f_3 = -4ie_3, \ f_4 = -e_3 + e_4 \]
the equation of the web of harmonic quadrics is
\[ x_0(p_7 - p_6) + ix_1(-2p_1 + 6p_6 + 6p_7) - 4ix_2(p_5 - p_4) + x_3(3p_2 + 3p_3 + p_4 + p_5) = 0. \]
The equation of the quartic symmetroid (after an equivariant scaling of the variables) is
\[ x_0^4 + x_1^2x_3^2 + x_2^2x_3^2 + x_1^2x_2^2 + 4x_0x_1x_2x_3 = 0 \] (9.5.16)
The surface has four ordinary nodes at the \( G_4 \) orbit of \([1, 1, 1, -1]\) and three rational double points of type \( A_2 \) at \([0, 0, 1, 0], [0, 0, 0, 1], [0, 0, 0, 1]\).
Finally assume \( d = 0 \). Then relations (9.5.12) let us also assume that \( b = e = 1, a = -5 \). We take a basis
\[ e_1 = t^5 - t, \ e_2 = t^5 + t, \ e_3 = t^6 - 5t^2, \ e_4 = -5t^2 + 1 \]
and find that \( g_3 \) acts in this basis via the following matrix
\[ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -2i & -2i \\ 0 & -\frac{1}{4} & -\frac{i}{2} & \frac{i}{2} \\ 0 & \frac{1}{4} & -\frac{i}{2} & \frac{i}{2} \end{pmatrix} \]
In a new basis
\[ v_1 = e_1, \ v_2 = -ie_2, \ v_3 = \frac{i}{4}(e_3 - e_4), \ v_4 = \frac{i}{4}(e_3 + e_4) \]
the transformation \( g_3 \) acts as cyclic permutation \((234)\) of the basis.

The web of harmonic quadrics in the new basis is
\[ x_0(p_7 - p_6) - 2ix_1(p_7 + p_6) + \left(\frac{i}{2}\right)x_2(p_3 - p_2 + p_5 - p_4) + \frac{1}{2}x_3(p_4 + p_5 - p_2 - p_3) = 0. \]
The equation of the discriminant quartic is
\[ x_0^4 + x_1^4 + x_2^4 + x_3^4 + 2x_0^2(x_1^2 + x_2^2 + x_3^2) - (y^2z^2 + y^2w^2 + z^2w^2) = 0. \] (9.5.17)
The surface has 10 ordinary nodes at \([0, 1, 1, 1]\) and \( G_3 \) orbits of point \([1, \pm i, 0, 0]\) (acting on the last tree coordinates).

Now we go to the octahedral cases. We know that there are two non-isomorphic dihedral rational nodal sextics given in Rows (17) and (18) in the Table. They are obtained as specializations of the families of dihedral sextics from Rows (12) and (13) by taking the parameters \( a = -5, 3 \), respectively. As we saw in above, they are also special members of the tetrahedral family by taking \( d = 0 \) or \( e = 0 \). Their equations are given in (9.5.17) and (9.5.15). In the first case the linear representation of the group \( \mathfrak{g} \cong G_4 \) in \( V^\perp \) is isomorphic to the standard 4-dimensional representation twisted by
the sign character. Note that in the second case \( V^\perp \) is isomorphic to the standard 4-dimensional permutation representation of \( S_4 \).

Let \( P_1 \) be the permutation matrix of the cyclic permutation \((1234)\). We have

\[
S^{-1}g_1S = iP_1,
\]

where

\[
S = \begin{pmatrix}
-1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 \\
-i & 1 & i & -1 \\
i & 1 & -i & -1
\end{pmatrix}
\]
is the inverse of a matrix of eigenvectors of \( P \). We have

\[
h_2 := S^{-1}g_2S = \frac{1}{2} \begin{pmatrix}
-1 & 1 & 1 & 1 \\
1 & 1 & 1 & -1 \\
1 & 1 & -1 & 1 \\
1 & -1 & 1 & 1
\end{pmatrix}
\]

and

\[
h_3 := S^{-1}g_3S = \frac{1}{16} \begin{pmatrix}
4 - i & 4 - 31i & 4 - i & 4 + 33i \\
-4 & 4 + 8i & 12 & 4 - 8i \\
4 + i & 4 - 33i & 4 + i & 4 + 31i \\
12 & 4 + 8i & -4 & 4 - 8i
\end{pmatrix}.
\]

A matrix \( A \) commuting with \( P_1 \) must be of the form

\[
\begin{pmatrix}
a & b & c & d \\
d & a & b & c \\
c & d & a & b \\
b & c & d & a
\end{pmatrix}.
\]

If we take its first column to be an eigenvector of \( h_3 \) with eigenvalue 1, then we obtain that

\[
P_2 := A^{-1}h_2A = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{pmatrix}, \quad P_3 := A^{-1}h_3A = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\]

The eigensubspace of \( h_3 \) with eigenvalue 1 is two-dimensional and spanned by vectors \((41+4i, 16-4i, 0, 17)\) and \((1, 1, 1, 1)\). A choice of an eigenvector such that the matrix \( A \) is invertible defines a symmetric form of the symmetroid. However, the coefficients of the equation may be complex numbers. However, if we take \( A \) such that

\[
M = SA = \begin{pmatrix}
-8i & 8i & -8i & 8i \\
1 & 1 & 1 & 1 \\
-1 - i & 1 - i & 1 + i & -1_i \\
-1 + i & 1 + i & 1 - i & -1_i
\end{pmatrix}.
\]

the equation of the discriminant quartic becomes

\[
407569(x_0^4 + x_1^4 + x_2^4 + x_3^4) - 689852(x_0^3x_1 + \cdots) + 791142(x_0^2x_2^2 + \cdots) + 368076(x_0^2x_1x_2 + \cdots) - 2355816x_0x_1x_2x_3 = 0.
\]

(9.5.18)
Finally, let us consider the icosahedral case. It is a member of the one-dimensional family (9.5.10) corresponding to the value of the parameter $a = 3$. We have $V = \langle t^6 + 3t, 3t^5 - 1, t^3 \rangle$ and $V^\perp = \langle 2t^5 + 1, 2t - t^6, t^2, t^4 \rangle$. The web of harmonic quadrics is

$$x_0(2p_7 + p_4) + x_1(2p_6 - p_5) + x_2p_2 + x_3p_3 = 0.$$  

We get the following equation of the discriminant quartic

$$F = -3x_0^2x_2 + x_0^2x_1^2 - 11x_0x_1x_2x_3 - 3x_0x_3^3 + 3x_1^3x_3 + 3x_1x_2^3 + x_2^3x_3 = 0. \quad (9.5.19)$$

**Remark 9.5.1.** It is shown in [188, Section 5] that, after a change of variables, the equation of the icosahedral symmetroid can be reduced to the form

$$30 \sum_{i=1}^{5} y_i^4 - 7(\sum_{i=1}^{5} y_i^2)^2 = y_1 + \cdots + y_5 = 0. \quad (9.5.20)$$

It becomes a member of the Hashimoto pencil of quartic surfaces with icosahedron symmetry.

**Remark 9.5.2.** The advantage of the symmetrical form (9.5.18) is that we can apply Timofte’s test of positivity to check the polynomial $F$ defining the equation takes only non-negative values on real values of the unknowns. According to the test, it is enough to check this property for vectors of the form $(s, s, s, t)$ and $(s, s, t, t)$. Substituting these vector we find

$$F(s, s, s, t) = (369s^2 - 1098st + 1129t^2)(-19t + 39s)^2 \geq 0$$

and

$$F(s, s, t, t) = 16(17s - 7t)^2(7s - 17t)^2 \geq 0.$$  

The same test applied to the equation from the previous Remark shows that the polynomial

$$F = 30 \sum_{i=0}^{3} x_i^4 + 30(x_0 + x_1 + x_2 + x_3)^4 = 7(\sum_{i=0}^{3} x_i^2 + (x_0 + x_1 + x_2 + x_3^2)^2)^2$$

is non-negative.

Another example of a non-negative polynomial is from equation (9.5.16) of a tetrahedral symmetroid with parameter $b = 0$. It is known that it is a not a sum of squares of real polynomials and, in fact, it is one of the first examples of quarternary quartic forms with this property [591].

The polynomials from equations (9.5.17) and (9.5.17) defining two symmetroids with octahedral symmetry seem to be non-negative too.

**Remark 9.5.3.** We know from Reye Theorem 7.5.3 from Section 7.5 that a general quartic symmetroid admits a unique pair $(C_1, C_2)$ of apolar rational normal curves. This means that the linear system of apolar quadrics to the web $|W|$ defining the symmetroid is spanned by the nets of quadrics $|I(C_1)\perp|$ and $|I(C_2)\perp|$ vanishing on the curves. If $W$ is a linear representation of a group $G$ then the pair $(C_1, C_2)$ must be invariant and a subgroup of index $\leq 2$ of $G$ leaves the subspaces $I(C_1)\perp$ and $I(C_2)\perp$ invariant. However, in our cases the web $|W|$ is not a general web so the Reye Theorem may not apply. In fact, this happens in some of our cases. By definition of a harmonic quadric, one of the curves $C_i$ must coincide with the dual Veronese curve $P_3^*$. The trouble could be in finding another rational cubic curve.

For example, if $G = D_{12}$. In this case

$$W = \langle x^2 - w^2, xw + 9yz, 2xz + 3y^2, 2yw + 3z^2 \rangle$$
and there is a unique decomposition of linear representations of $G$ in the space of apolar quadrics $W^\perp$
\[
W^\perp = \langle x^* w*, y^* z^*, y^* w^* \rangle \oplus I(R_3^\perp),
\]
where $x^*, y^*, z^*, w^*$ are the dual coordinates. The first summand is isomorphic to $V$ and the discriminant quartic of the corresponding net $\lambda(x^* y^* + y^* w^*) + \mu x^* z^* + \gamma y^* w^*$ of quadrics is equal to the union of two double lines $\mu^2 = 0$ and $\gamma^2 = 0$. However, the discriminant curve of a net of quadrics vanishing on a rational conic taken with multiplicity 2.

On the other hand, if we take $G = \mathbb{D}_{10}$, then everything is as expected. We find that
\[
W^\perp = \langle -az^* w^* + 3x^* z^*, -ax^* y^* + 3w^* z^*, 9x^* w^* - a^2 y^* z^* \rangle \oplus I(R^\perp_3),
\]
and
\[
I(C_2) = \langle z^* w^* + 3x^* z^*, -ax^* y^* + 3w^* z^*, 27u^2 w^*, 3au^2 v, [u, v] \in \mathbb{P}^1 \rangle.
\]
In the case $G = \mathbb{O}$, the linear system $|W|$ contains a unique $G$-invariant net of quadrics whose discriminant curve is a double conic but singular quadrics in the net are all reducible. This must be considered as a degenerate case of the Reye Theorem.

The composition of the Veronese map $v_3^* : |U| \to R_3^*$ with the map $R_3^* \to |V^\vee| \cong |V|$ is the rational parametrization of a rational nodal sextic from which we derived our $G$-symmetric symmetroid. This reconstructs the sextic from the symmetroid.

In the case when the Reye Theorem holds for $|W|$, the two nets $|I(C_1)|$ and $|I(R^\perp_3)|$ define the same self-dual rational nodal sextics.

### 9.6 Automorphisms of an unnodal Coble surface

**Theorem 9.6.1.** Let $V$ be an unnodal Coble surface. Then the homomorphism $\rho : \text{Aut}(V) \to O(\text{CM}(V))$ is injective and the image $\text{Aut}(V)^* \subseteq \text{Aut}(V)$ contains the 2-congruence subgroup $W(\mathbb{E}_{10})(2)$ of $W(\mathbb{E}_{10})$.

**Proof.** The proof is similar to the proof of Theorem 8.3.4. We skip some details referring to the full proof in [103]. Let $V$ be obtained by blowing up 10 nodes of an irreducible plane sextic. An automorphism $g$ of $V$ that acts identically on $\text{CM}(V) = K_V^+$ sends the geometric basis $(e_0, \ldots, e_{10})$ defined by the blowing down map $V \to \mathbb{P}^2$ to itself. This implies that $g$ originates from a projective symmetry of the 10 points that preserves the ordering of the set of points. Clearly this is impossible.

Let $F_i \in |3e_0 - (e_1 + \cdots + e_{10}) + e_i| = |-K_V + e_i|$. It is represented by the unique plane cubic passing through the points $p_j \neq p_i$. We have $F_i^2 = 0$ and $F_i \cdot F_j = 1$. The divisor classes $f_i = [F_i]$
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and \( f_j = [F_j] \) generate a hyperbolic plane and define a primitive embedding \( U \hookrightarrow K^\perp_V \cong E_{10} \).

Conversely, any such embedding defines two divisor classes \( f_i, f_j \) with \( f_i^2 = f_j^2 = 0, f_if_j = 1 \). By Riemann-Roch, any isotropic vector \( f \in K^\perp_V \) is effective. Let \( f = [F] \), where \( F \) is an effective divisor. Write \( F \) as a sum of irreducible curves \( C_i \neq C \) and some multiple \( mC \) of \( W \). Since all effective primitive isotropic vectors in \( E_{10} \) are in the same orbit of the reflection group \( W(E_{10}) \), there exists \( w \in W(E_{10}) \) such that \( w(f) = f_i \). One can show that the image of any effective curve on \( V \) under an element \( w \in W(E_{10}) \) is effective (see Lemma 3.4 in [103]). Since \( F_i \) is an irreducible curve, we obtain that \( f \) is represented by an irreducible curve \( F \) of arithmetic genus one.

The exact sequence

\[ 0 \to \mathcal{O}_V(F) \to \mathcal{O}_V(2F) \to \mathcal{O}_F(2F) \to 0, \]

together with the adjunction formula \( \mathcal{O}_F(2F) \cong \mathcal{O}_F(-2K_V) \cong \mathcal{O}_F \) shows that \( \dim |2F| = 1 \), thus \(|2F| \) is an irreducible pencil of genus one curves. For each \( U \hookrightarrow K^\perp_V \) we thus find two genus one pencils \(|2F| \) and \(|2F'| \) with \( F \cdot F' = 1 \).

Next, as in section 9.2, we consider a bielliptic map \( f : V \to \mathbb{P}^4 \) given by the linear system \(|2F + 2F'|\) of degree 8. One shows that the image of this map is a 4-nodal anti-canonical del Pezzo surface \( D_1 \). Since \( V \) has no \((-2)\)-curves the map \( f \) must be a separable map of degree 2. Since \( F \cdot C = F' \cdot C = 0 \), the image of the curve \( C \) is one of the nodes. The deck transformation defines an involution on \( V \) and we apply Proposition 8.3.3 to conclude the proof.

As we see the deck involution defined by a pair of genus one curves \( F, F' \) with \( F \cdot F' = 1 \) is similar to the one we used for the description of automorphism group of an unnodal Enriques surface. When \( V \) is unnodal, each such pair is defined by a primitive embedding \( j : U \hookrightarrow K^\perp_V \). If \( p \neq 2 \), the double cover \( V \to D \) factors through a degree 2 map \( V \to D' \), where \( D' \) is the blow-up of one of the nodes of \( D \), the image of the anti-canonical curve \( C \). The branch curve is a smooth curve of genus 4 equal to a proper inverse transform of a curve from \( |\mathcal{O}_D(2)| \) passing through a node. The fixed points of the deck involution is the union of the pre-image of this curve on \( V \) and three isolated fixed points over the nodes of \( D' \).

In a geometric basis of the blow-up of five points \( p_1, p_2, p_3, p_4, p_5 \) (see Section 0.4), the branch curve becomes the union of the curve from \( |e_0 - e_3 - e_4| \) and a curve from \( |5e_0 - e_1 - e_2 - e_3 - 2p_4 - 2p_5| \). So its image in the plane is the union of a line through \( p_1, p_2 \) and a quintic curve of geometric genus 4 with tacnode at \( p_4 \) which intersects the line at \( p_1 \) and is tangent to the line at \( p_2 \).

**Remark 9.6.2.** Assume \( p \neq 2 \). Let \( p_1, \ldots, p_{10} \) be the ten nodes of a rational plane sextic \( B \). Let \( g_{ij} \in \text{Aut}(V) \) be the automorphism of \( V \) defined by the deck transformation of the double cover \( f : V \to D \) defined by the linear system \(|2F_i + 2F_j|\), where \( F_i, F_j \) are the proper transforms on \( V \) of the cubic curves passing through points \( p_1, \ldots, p_{10} \) excluding \( p_i, p_j \). Let \( V_{ij} \to \mathbb{P}^2 \) be the blow-up of the points \( p_k, k \neq i, j \). The surface \( V_{ij} \) is a weak del Pezzo surface of degree 1, the linear system \(|-2K_{V_{ij}}|\) defines a degree 2 map \( \phi_{ij} : V_{ij} \to Q \subset \mathbb{P}^3 \) onto a quadratic cone \( Q \) with the branch curve \( W \) cut out by a cubic surface (see [180], Theorem 8.3.2). The proper transform \( \tilde{B} \) of the curve \( B \) on \( V_{ij} \) belongs to this linear system, and hence it is equal to the pre-image of a hyperplane section \( C \) of \( Q \). The restriction of \( \phi_{ij} \) to \( \tilde{B} \) is a degree 2 map

\[ \phi_{ij} : \tilde{B} \to C \cong \mathbb{P}^1. \]
Since each point $p_i, p_j$ (we identify it with a point on $V_{ij}$) is a double point of $\tilde{B}$, the hyperplane section $C$ is tangent to the branch curve of $\phi_{ij}$ at the images of these points in $C$. This implies that the points $p_i, p_j$ are on the ramification curve of $\phi_{ij}$. Let $\beta_{ij}$ be the deck transformation of the map $\phi_{ij}$ which defines, via the blow-up map, a birational Bertini involution of $\mathbb{P}^2$ (see [180], 8.8.2). The points $p_i, p_j$ are among its fixed points. Thus, $\beta_{ij}$ extends to an automorphism $\tau_{ij}$ of $V$. We claim that it coincides with the deck transformation $g_{ij}$ of the map $f: V \to D$ defined by the linear system $|2F_i + 2F_j|$. To see this, it is enough to check that $g_{ij}$ and $\tau_{ij}$ act identically on $Pic(V)$. In fact, it is known that $g_{ij}^* act as the minus identity on the sublattice $K_{V_{ij}}^+$ of $Pic(V_{ij})$. In the geometric basis $(e_0, e_1, \ldots, e_{10})$ of $Pic(V)$, this lattice is identified with $(3e_0 - (e_1 + \cdots + e_{10}) + e_i + e_j)^\perp$. This shows that $\tau_{ij}^* (F_k) = F_k, k \neq i, j$, and $\tau_{ij}^*$ acts identically on $\mathbb{Z}[F_i] + \mathbb{Z}[F_j]$. This implies that $\tau_{ij}^* = g_{ij}^*$.

Note that the Bertini involution $\beta_{ij}$ has one isolated fixed point $q_i \in E_i, q_j \in E_j$ on each of the exceptional curves. The quotient $D'$ of $V$ by this involution has 3 ordinary double points, the image of the base point of the pencil $| - K_{V_{ij}}^+$ and the images of the points $q_i, q_j$. One can show that the surface $D'$ is obtained from the 4-nodal anti-canonical quartic del Pezzo surface $D$ by blowing up one of its four ordinary double points. The image of the curve $\tilde{B}$ is the exceptional curve $R$ over this point. We have $\tilde{B}^2 = -4$ and $R^2 = -2$, so this agrees. It is known that the ramification map of the map $f_{ij} : V_{ij} \to Q$ is the proper transform $\tilde{Z}$ on $V_{ij}$ of a curve $Z$ of geometric genus 4, of degree 9 with triple points at $p_1, \ldots, p_8$. It passes through the points $p_i, p_j$, hence $\tilde{Z} \cdot \tilde{B} = 54 - 6 \cdot 8 - 2 \cdot 2 = 2$. The intersection points are the ramification points of the map $\tilde{B} \to R$.

**Remark 9.6.3.** Here is another description of a Bertini involution of a Coble surface defined by a pair of genus one curves $F_1, F_2$ with $F_1 \cdot F_2 = 1$.

Suppose $V$ is of Halphen type and let $\sigma : V \to H$ be a blowing-down morphism. Let $|F|$ be the elliptic fibration on $H$ and let $R$ be a special bisection of $|F|$. It is a $(-1)$-curve on $V$. The linear system $|R + F|$ on $H$ is of degree $(R + F)^2 = 3$ and has one base point $x_0 = R \cap F_0$, where $2F_0 \in |F|$. Let $\alpha : H' \to V$ be the blow-up of $x_0$ and $E$ the exceptional curve. The linear system $\alpha^{-1}(R + F)| = |\alpha^*(R + F) - E|$ is of dimension 2 and degree 2. Its restriction to a general member of $|F|$ is of degree 2. Hence it defines a degree 2 map $\phi : H' \to \mathbb{P}^2$. It blows down the proper transforms $\tilde{R}$ and $\tilde{F}_0$ of $R$ and $F_0$ on $H'$ to points $p, q \in \mathbb{P}^2$. The image of $E$ is the line $\ell = \langle p, q \rangle$. Assume that $p \neq 2$. The branch curve $W$ of $\phi$ is a curve of degree 6, the union of $\ell$ and a curve of degree 5 that passes through $p$ and has a tacnode at $q$ with tangent directional line equal to $\ell$. The pencil $|F|$ is the pre-image of the pencil of lines through $p$. The pre-image of the pencil of lines through $q$ is an elliptic pencil $|F'|$ with $F'F' = 2$. It has a double fiber defined by the pre-image of the line $\ell$, so that $|F'| = |2F_0'|$, where $F_0 : F_0' = 1$.

The deck transformation of the cover $\phi$ leaves invariant $E$ and hence descends to a biregular involution $g'$ of $H$. It is another analogue of a Bertini involution. If $W$ has no singular points except $p$ and $q$, then the locus of fixed points of $g'$ on $H$ consists of a smooth curve genus 4 (the pre-image of the quintic component of $W$) and two isolated points on $F_0$. One of them is the intersection point $z_1 = R \cap F_0$ and another $z_2$ is obtained from $z_1$ by the translation on the divisor class of the normal bundle of $F_0$ (i.e. $\mathcal{O}_{F_0}(F_0 + R) \cong \mathcal{O}_{F_0}(z_2)$).

Let $F_1$ be an irreducible nodal member of $|F|$. It is obviously fixed by $g'$, and hence lifts to a biregular involution $g$ on the Coble surface $V$. Its set of fixed points consists of a smooth curve
of genus 4 and three isolated points: the pre-images of \( z_1, z_2 \) and one additional point \( z_3 \) on the exceptional curve of \( \sigma : V \to H \).

We see that the fixed locus of \( g \) is the same as the fixed locus of the Bertini involutions used in the proof of Theorem 9.6.1. The branch curve of the cover \( V \to \mathbb{P}^2 \) is also the same as the branch cover of the composition of the map \( V \to D \dashrightarrow \mathbb{P}^2 \). Applying the Lefschetz fixed-point formula, we obtain that the trace of \( g^* \) on \( K_V^1 \equiv E_{10} \) is equal to \(-6\), the same as the trace of the Bertini involution. In fact, \( g \) coincides with such an involution. To see this we consider the divisor class \([R - K_V]\). We have \((R - K_V)^2 = (R - K_V) \cdot K_V = 0\), so, by Riemann-Roch and the adjunction formula, \([R - K_V]\) is an effective isotropic vector \( f \) in \( K_V^\perp \). Since \( f \cdot [F_0] = 1 \), it is also primitive. The Bertini involution corresponding to the pair \([F_0], f\) coincides with \( g \).

In a plane model of \( V \) as the blow-up of 10 nodes \( p_1, \ldots, p_{10} \) of a rational sextic \( B \), we may take \( R \) to be an exceptional curve over a point \( p_1 \) and the Halphen surface \( H \) be the blow-up of the first nine base points. Then the curve \( G \) representing the class \([R - K_V]\) can be written in terms of the geometric basis \((e_0, e_1, \ldots, e_{10})\) as \(3e_0 - (e_2 + \cdots + e_{10})\). This shows that the involution \( g \) coincides with the Bertini involution \( \beta_{1,10} \).

**Example 9.6.4.** Assume \( \text{char}(k) \neq 5 \). Consider the 10-nodal plane sextic \( W \) given in parametric form by the equation
\[
[x_0, x_1, x_2] = [t_1^6 + t_0^6, t_0(3t_1^5 - t_0^5), -5t_1^2t_0^2].
\]
The icosahedron group \( \mathfrak{A}_5 \) is generated by the dihedral group \( D_{10} \) that acts on \([t_0, t_1]\) by \([t_0, t_1] \mapsto [t_1, t_0], [-t_0, t_1] \) and the Klein involution \( T \) that acts by
\[
T : [t_0, t_1] \mapsto [(e^2 - e^4)t_0 + (e - e^4)t_1, (e^4 - e)t_0 + (e^2 - e^3)t_1].
\]
Here \( e \) is the 5th root of 1. According to R. M. Winger [704], the action of \( \mathfrak{A}_5 \) on the parameters defines a projective automorphism of the plane sextic \( W \).

Assume \( k = \mathbb{C} \). Let \( X \) be the canonical cover of \( V \). The group \( \mathfrak{A}_5 \) lifts isomorphically to a finite group acting symmetrically on \( X \). It acts on the cohomology \( H^*(X, \mathbb{C}) \) and acts trivially on the 6-dimensional subspace generated by the divisor class of the pre-image of an ample divisor on \( V \), the divisor class of the ramification \((-2)\)-curve, and the part \( H^{0,0} \oplus H^{2,2} \oplus H^{0,2} \oplus H^{2,0} \) of the Hodge decomposition. According to the classification of such groups due to S. Mukai [504], the subspace of invariant cohomology classes for an action of a symplectic group isomorphic to \( \mathfrak{A}_5 \) on a K3 surface is equal to 6. This is proven by analyzing the Mathieu character of the action of the group on the cohomology. Assume that \( V \) has a \((-2)\)-curve. Then \( \mathfrak{A}_5 \) has one additional invariant divisor class, namely, the sum \( D \) of the curves in the orbit of any \((-2)\) curve whose image on \( V \) is a \((-2)\)-curve. Note that this class is different from the orbit of an ample class which we considered before. In fact, the sum of the curves in the new orbit does not intersect the ramification \((-2)\)-curve, but the sum of the curves in the old orbit does intersect this curve. This contradicts Mukai’s computation.\(^1\) So, our surface \( V \) is unnodal, hence its group of automorphisms contains the group \( W(E_{10})(2) \) and the group \( \mathfrak{A}_5 \). Applying Proposition 8.6.1, we find that the group generated by these two subgroups is isomorphic to a semi-direct product \( W(E_{10})(2) \rtimes \mathfrak{A}_5 \).

\(^1\)We thank H. Ohashi for this argument.
9.7 Enriques and Coble surfaces of Hessian type

In section 6.4 we discussed Enriques surfaces of Hessian type of a nonsingular cubic surface \( C = V(F) \) defined as quotients of the Hessian quartic surface \( H(C) \) by the natural birational involution \( T \) that exchanges the 10 nodes and the ten lines. The involution extends to a biregular involution \( \tau \) of a minimal resolution \( X \) of \( H(C) \). If the cubic surface acquires a singular point, the Hessian surface acquires a fixed point too, and the involution \( T \) fixes this point. The lifted involution of \( X \) has now a pointwise fixed \((-2)-\)curve over this node, and the quotient surface \( X/\langle \tau \rangle \) becomes a Coble surface with the image of this curve as a boundary component. We call a Coble surface obtained in this way a \textit{Coble surface of Hessian type}. The moduli space of such surfaces is 3-dimensional and coincides with the moduli space of Sylvester non-degenerate nodal cubic surfaces. The number \( k \) of boundary components coincides with the number of nodes and hence takes values in \( \{1, 2, 3, 4\} \). It is shown in \cite[Proposition 5.1]{189} that the moduli space of Coble surfaces with one boundary component (together with a geometric marking defined by a blowing down \( V \to \mathbb{P}^2 \)) is naturally isomorphic to the moduli space of \textit{Desargues configurations} of 10 lines and ten points in the projective plane.

The group of automorphisms of a general Enriques surface of Hessian type was described in terms of generators and relations by I. Shimada \cite{642}. Let us describe their result.

The group of birational automorphisms of the Hessian surface of a general cubic surface was described in \cite{175}. A part of its set of generators are the deck transformations of the projections from the ten nodes \( p_\alpha \). It is proven in loc.cit. that they commute with the involution \( T \) and hence descend to biregular involutions \( j_\alpha \) of the Enriques quotient surface \( S \).

\textbf{Theorem 9.7.1.} Let \( X \) be a minimal resolution of the Hessian surface \( H(C) \) of a general cubic surface \( C \) and let \( S = X/\langle \tau \rangle \) be the Enriques surface of Hessian type. The 10 involutions \( j_\alpha \) generate the group \( \text{Aut}(S) \). The defining relations are of the following type:

- \( j_\alpha^2 = \text{id} \),
- \( (j_\alpha \circ j_\beta \circ j_\gamma)^2 = \text{id} \) for each triple of collinear nodes;
- \( (j_\alpha \circ j_\beta)^2 = \text{id} \) for each pair of nodes not lying on the same line contained in \( H(C) \).

In his proof of the theorem Shimada describes a fundamental polytope for the action of the automorphism group in the nef cone. It is equal to the crystallographic reflection polytope (8.15) which corresponds to Kondo’s Enriques surfaces of type VI. The ten vertices \( E_1, \ldots, E_{10} \) correspond to outer walls, i.e. walls lying on the boundary of the nef cone. The remaining 10 nodes correspond to non-effective primitive vectors of square norm \(-2\) such that the projection involution \( j_\alpha \) acts as the reflection in these vectors.

Similar description of the nef cone can be given for any Enriques or Coble surface of Hessian type. The following theorem is proven in \cite[Lemma 2.4 and Theorem 3.2]{9}.

\textbf{Theorem 9.7.2.} Let \( \Lambda = \text{Num}(X) \) if \( X \) is an Enriques surface and the Coble-Mukai lattice \( \text{CM}(X) \) if \( X \) is a Coble surface. Then \( \Lambda \cong E_{10} \). Let \( G_\alpha \) be the subgroup of \( W(\Lambda) \) generated by \( j_\alpha^* \), where \( p_\alpha \)
is not an Eckardt point of the cubic surface. Then \( \text{Nef}(X) \cap \Lambda_\mathbb{R} \) is the closure \( Q \) of the union of \( G_0 \)-images of the polytope \( P \) whose faces are orthogonal to \((-2)\)-vectors \( E_1, \ldots, E_{20} \) with intersection diagram \( 8.15 \). The vectors \( E_1, \ldots, E_{10} \) are the classes of \((-2)\)-curves \( U_{ab} \) and the number of vectors \( E_{11}, \ldots, E_{20} \) represented by the classes of \((-2)\)-curves is equal to the number of Eckardt points on the cubic surface.

Recall that the group of projective automorphisms of a general cubic surface is trivial. The classification of automorphism groups of nonsingular cubic surfaces in arbitrary characteristic can be found in [187]. When the cubic is Sylvester non-degenerate and acquires a non-trivial automorphism group, the corresponding Enriques surface becomes special and its automorphism group changes. Here we consider the family of cubic surfaces with automorphism group isomorphic to \( S_4 \). The cubic surfaces with this group of symmetry can be characterized by the property that they contain six Eckardt points. Each such cubic surface is isomorphic to a member of the following pencil of cubic surfaces

\[
y_1^3 + y_2^3 + y_3^3 + y_4^3 + t(y_1 + y_2 + y_3 + y_4)^3 = 0
\]  

(9.7.1)

The Hessian surfaces \( H_t \) of the members of this family are defined by equations

\[
\left( \frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} + \frac{1}{y_4} \right) - \frac{t}{y_1 + y_2 + y_3 + y_4} = 0.
\]  

(9.7.2)

There are 4 special values of the parameters \( t \). If \( t = \infty \), the cubic surface degenerates to the plane taken with multiplicity 3 and the Hessian surface contains a plane as an irreducible component. If \( t = 1 \) the cubic surface is isomorphic to the Clebsch diagonal cubic surface \( C \) with \( \text{Aut}(C) \cong S_5 \). The Hessian surface acquires the same group of projective symmetries. As we saw in Section 8.9, the corresponding Enriques surface is the Kondo surface of type VI with finite automorphism group isomorphic to \( S_5 \). If \( t = \frac{1}{16} \) or \( t = \frac{1}{4} \), the cubic surface acquires one or four nodes, respectively. The corresponding quotient \( X/\tau \) is a Coble surface with one or four boundary components.

The surface \( V_{\frac{1}{16}} \) is obtained by blowing up 10 nodes of an irreducible plane sextic that admits a group \( S_4 \cong \mathbb{O} \) of projective symmetries. We found its equation (??) in Section 9.4. It follows from Remark ?? that one can choose more symmetric equation given by the Robertson ternary sextic (??).

The surface \( V_{\frac{1}{4}} \) is obtained by blowing up the singular points of a reducible sextic with 4 irreducible components:

\[
(y + z)(y + z + 4x)(xy + yz + xz)(xy + yz + xz + 4x^2) = 0
\]  

(see [189, 5.5]).

It turns out that the group of automorphisms of any Enriques or Coble surface in the pencil (9.7.2) are isomorphic to the same group unless the automorphism group becomes finite and isomorphic to \( S_5 \). The proof of the following theorem can be found in [9].

**Theorem 9.7.3.** Let \( S_t \) be the minimal resolution of the Hessian surface \( H_t, t \neq 1, \infty \). Then \( \text{Aut}(S_t) \) is isomorphic to the semi-direct product \( UC(4) \rtimes \mathbb{S}_4 \) of the free product of four copies of the group \( \mathbb{Z}/2\mathbb{Z} \) and the group \( \mathbb{S}_4 \). The generators of \( UC(4) \) are the projection involutions...
corresponding to 4 nodes. The remaining six projection involutions correspond to transpositions \((ab)\) in \(S_4\). The nef cone of \(S_1\) is generated by 20 faces.

One can find in [9] various interesting realizations of the group \(\text{UC}(4) \rtimes S_4\). One of them is its realization as a group of isometries of the Euclidean space generated by the isometries of a regular tetrahedron and the reflections across its faces.

**Remark 9.7.4.** In [124] A. Coble discussed the following problem. Let \(V\) be a Coble surface with one boundary component and \(C \in |-2K_V|\). Then there is a natural restriction homomorphism

\[
\tau : \text{Aut}(V) \to \text{Aut}(C) \cong \text{PGL}(2).
\]

What is the kernel and the image of this homomorphism? Coble conjectured that the kernel is trivial for a general Coble surface. The answer for the question is known for the Coble surface \(V_{1,1}^\tau\): the kernel is trivial and the image is conjugate to the subgroup of isometries of a regular tetrahedron.

A slightly different family of Enriques quotients of quartic surfaces with \(S_4\) symmetry is discussed in [510]. The family is the following:

\[
ls_2^2 + ks_4 + ls_1s_3 = 0,
\]

where \(s_k\) are elementary symmetric polynomials in 4 variables \(y_1, y_2, y_3, y_4\) and \(t, k, l\) are parameters. Note that the subfamily with \(t = 0\) coincides with our family of quartic Hessian surfaces with \(S_4\)-symmetry. The points \(x_i\) with coordinates \([1, 0, 0, 0], \ldots, [0, 0, 0, 1]\) are the common nodes of the surfaces in the family. The Enriques involution of the general member of the family is defined by the standard Cremona transformation \(y_i \mapsto y_i^{-1}\). It has 8 fixed points in \(\mathbb{P}^3\) with coordinates \([1, \pm 1, \pm 1, \pm 1]\). If the surface does not contain any of these points, the involution \(\tau\) of a minimal resolutions has no fixed points and the quotient is an Enriques surface \(S(t, k, l)\). This condition is equivalent to that \(-36t + k + 16l \neq 0, 4l + k \neq 0, 4t + k \neq 0\). The group \(S_4\) acts on the set of 8 fixed points with orbits

\[
\begin{align*}
&\{[1, 1, 1, 1]\}, \quad -36t + k + 16l = 0, \\
&\{[1, -1, 1, 1], [1, 1, -1, 1], [1, -1, -1, 1], [1, 1, -1, -1], [1, -1, 1, -1]\}, \quad k + 4l = 0, \\
&\{[1, -1, -1, 1], [1, 1, -1, -1], [1, -1, 1, 1]\}, \quad 4t + k = 0.
\end{align*}
\]

Thus we obtain examples of Coble surfaces with \(n = 1, 4\) and 3 which admit the group \(S_4\) as its group of automorphisms. When \(t = 0\), the first examples are our surfaces \(V_{1,-\frac{15}{2}}\) and \(V_{1,\frac{1}{2}}\).

Note that the Coble surfaces with \(n = 1\) define a one-parameter family of 10-nodal plane sextics with \(S_4\)-Cremona isometry. Only one of them with parameter \(t = 0\) has the projective \(S_4\)-symmetry.

The four projection involutions of nodal members of the family descend to four involutions of the quotient surface \(X/\langle \tau \rangle\). They generate the free product \(\text{UC}(4)\) and this gives an embedding

\[
\iota : \text{UC}(4) \rtimes S_4 \hookrightarrow \text{Aut}(S)
\]

for all members of the family. When \(t = 0\), the 4 nodes of specialize to the four nodes of the Hessian surface that define non-projective involutions coming from the projections from the nodes. The bijectivity of \(\iota\) in the Hessian case is the assertion of Theorem 9.7.3.

**Theorem 9.7.5.** Suppose \(t = 1, l = 0\) and \(k \neq 0, 4, -36\). Then the homomorphism \(\iota\) is bijective.
9.8 Coble surfaces with finite automorphism group

We do not know the structure of the automorphism group of a general member of the family of surfaces $S(t,k,l)$.

Note that the Jacobian Kummer surface admits a projective embedding in $\mathbb{P}^3$ isomorphic to the Hessian of a cubic surface [309]. The equation of the Hessian surface birationally isomorphic to $\text{Kum}(\text{Jac}(C))$ is of the form

$$A_0x_0^{-1} + A_1x_1^{-1} + A_2x_2^{-1} + A_3x_3^{-1} + (A_0x_0 + A_1x_1 + A_2x_2 + A_3x_3)^{-1} = 0$$

(see [309], [175, Theorem 4.1]). Note that after coordinate change $y_i = -A_ix_i$, the equation becomes the equation of the Hessian surface of the cubic surface

$$A_0^2x_0^3 + A_1^2x_1^3 + A_2^2x_2^3 + A_3^2x_3^3 + (x_0 + x_1 + x_2 + x_3)^3 = 0.$$ 

It follows that our pencil (9.7.2) contains only one member birationally isomorphic to a Jacobian Kummer surface. It corresponds to the parameter $(\lambda : \mu) = (1 : 1)$. Note that the Hessian of the Clebsch diagonal cubic corresponds to the parameter $(\lambda : \mu) = (1 : -1)$ and hence it is not birationally isomorphic to the Jacobian Kummer surface. We noted this fact in Section 8.9 while discussing Kondo surfaces of type VI. We do not know the structure of the lattice of transcendental cycles of this Kummer surface.

Note that the family (9.7.3) of quartic surfaces studied by Mukai and Ohashi also contains the members birationally isomorphic to a Jacobian Kummer surface. They correspond to parameters satisfying $t = 1$, $(k-4)(l-4) = 0$ and, as we noticed before $t = 0$, $k = l$.

Finally let us see what happens if we assume that the characteristic is positive. Although the Hessians of cubic surfaces in characteristic $p = 2, 3$ can degenerate, we may still consider the quartic surfaces given by equations $\sum_{i=0}^4 \frac{1}{a_ix_i} = \sum_{i=0}^4 x_i = 0$ and continue to call them Hessian quartic surfaces. The pencil (9.7.2) and the Mukai-Ohashi family still makes sense. In characteristic $p = 2$, the Cremona involution has only one fixed points $[1, 1, 1, 1, 1]$ and it lies on the member $X(t, k, l)$ of the family if and only if $k = 0$. In particular, in the pencil (9.7.2) the two Coble surfaces degenerate to reducible surfaces $s_1s_3 = 0$. However, if $p = 3$ they just coincide and contain 5 boundary components corresponding to 5 nodes $[1, 1, 1, 1], [-1, 1, 1, 1], [1, -1, 1, 1], [1, 1, -1, 1], [1, 1, 1, -1]$. Also if $p = 5$, we see that the Clebsch diagonal cubic surface and its Hessian surface has a singular point $[1, 1, 1, 1]$ and the corresponding Enriques surface of Hessian type specializes to the Coble surface $V_{(1; \frac{1}{4})} = V_{(1; 1)}$ with one boundary component.

9.8 Coble surfaces with finite automorphism group

In this section we will extend Kondo-Nikulin classification of complex Enriques surfaces with finite automorphisms to complex Coble surfaces and also study examples of such surfaces in positive characteristic.

The following theorem extends Theorem 8.1.7 to the case of Coble surfaces. Its proof uses Proposition 9.2.2 and is left to the reader.

**Theorem 9.8.1.** Let $V$ be a Coble surface. Suppose $W^\text{mod}_V$ is of finite index in $O(CM(V))$. Then $\text{Aut}(V)$ is finite.
We believe that the converse is also true.

Another general fact about Coble surfaces with finite automorphism group is given in the following.

Let \( \sigma : V \to H \) be the blow-up of an Halphen surface \( H \). Let \( \text{MW}(\sigma) \) be the Mordell-Weil group of the jacobian elliptic (quasi-elliptic) surface of \( f : Y \to \mathbb{P}^1 \). Then it acts on \( H \) by translations leaving invariant the set of singular points of fibers of \( f \). A subgroup of finite index of \( \text{MW}(\sigma) \) will fix the fundamental points of \( \sigma^{-1} \), and hence could be lifted to \( \text{Aut}(V) \). Of course, in general, an automorphism of \( V \) does not descend to any \( H \) obtained by blowing down morphism \( \sigma \).

**Proposition 9.8.2.** Suppose \( \text{Aut}(V) \) is finite. Let \( \sigma : V \to H \) be a birational morphism onto an Halphen surface \( f : H \to \mathbb{P}^1 \). Then the jacobian genus one fibration of \( f \) has finite Mordell-Weil group.

It follows from this Proposition that a Coble surface with finite automorphism group is obtained as the blow-up of either an extremal rational elliptic (or quasi-elliptic) surface of index \( \leq 2 \).

Next, we first assume that \( \mathbb{k} = \mathbb{C} \) and give a characterization of such surfaces in terms of their Nikulin \( R \)-invariant similar to what we gave for Enriques surfaces in Section Theorem 8.9.3 in Section 8.9.

The following theorem extends to Coble surfaces the classification of \( R \)-invariants of surfaces with finite automorphism group.

**Theorem 9.8.3.** Assume that \( \mathbb{k} = \mathbb{C} \). Then \( \text{Aut}(V) \) is finite if and only if the \( R \)-invariant \( (K, H) \) is isomorphic to one of the following:

\[ (E_8 \oplus A_1, \{0\}), \ (E_8 \oplus A_1^{\oplus 2}, \ Z/2Z), \ (D_9, \{0\}). \]

**Proof.** The proof is similar to the one of Theorem 8.9.3.

The sufficiency follows from examples below where we construct Coble surfaces with finite automorphism group and compute their Nikulin \( R \)-invariants. Let us show the necessity.

First we assume \( n = 1 \). Then any effective irreducible root is a \((-2)\)-curve which is perpendicular to the boundary \( C_1 \). Thus \( \text{rank} \ K \leq 9 \) and hence this case is reduced to the case of Theorem 8.9.3. The same proof implies that \((K, H)\) is isomorphic to \((E_8 \oplus A_1, \{0\}), \ (D_9, \{0\})\).

Next we consider the case \( n \geq 2 \). The classes \( \hat{C}_1 \pm \hat{C}_2, \hat{C}_2 + \hat{C}_3, \ldots, \hat{C}_{n-1} + \hat{C}_n \) generate the lattice \( D_n(2) \) where we assume \( D_2 = A_1^{\oplus 2} \) and \( D_3 = A_3 \). Since any \((-2)\)-curve on \( V \) is perpendicular to \( C_1, \ldots, C_n \), \( K = D_n \oplus K_0 \) where \( K_0 \) is generated by classes corresponding to \((-2)\)-curves. Since \( D_n/2D_n \) contains the kernel \( Z/2Z \) generated by \( 2\hat{C}_1 \), it suffices to consider the sublattice \( K' = A_{n-1} \oplus K_0 \) where \( A_{n-1}(2) \) is generated by \( \hat{C}_1 + \hat{C}_2, \hat{C}_2 + \hat{C}_3, \ldots, \hat{C}_{n-1} + \hat{C}_n \). Since \( \text{rank} \ K' \leq 9 \), again the proof is reduced to the case of the one of Theorem 8.9.3. The same proof implies that \( K' = E_8 \oplus A_1 \) and hence we have \( (E_8 \oplus A_1^{\oplus 2}, \ Z/2Z) \) (note that \( n \geq 2 \) implies that \( K' \) contains a factor \( A_{n-1} \), and hence the case \( D_9 \) is excluded).

The following examples show that the three isomorphism classes of Coble surfaces with finite automorphism group lie in the boundary of two one-dimensional families of Enriques surfaces with
finite automorphism groups. The first two are two points in the boundary of the family of surfaces of type I and the third one is one point in the boundary of surfaces of type II.

**Example 9.8.4.** This surface lies in the boundary of the one-dimensional family of surfaces with crystallographic basis of \((-2)\)-curves of type I.

We use the equation of the double plane model of the family of Enriques surfaces type I given in (8.9.1) and specialize the parameters by taking \(a + c = b = d = 0\). The equation becomes

\[
x^3 + x_1 x_2^2 (x_1 - x_2)(x_1 x_2 - x_6^2)(x_1^2 - x_6^2) = 0.
\]

After normalization we get rid of \(x_2^2\) and get a special case of the equation of a double plane for a Coble surface given in (9.2.3). We see that the branch curve is equal to the union of a line \(\ell\) passing through \(p_1\) representing the divisor class \(e_0 - e_1\) on \(\tilde{D}_1\), a conic \(K\) through \(p_2, p_3, p_4, p_5\) representing the divisor class \(2e_0 - e_2 - e_3 - e_4 - e_5\), and two lines \(\ell_1, \ell_2\) passing through \(p_1\) and each passing through one of the intersection points \(q_1, q_2 \in \ell \cap K\). The points \(q_1, q_2\) are ordinary triple points of the quintic branch curve \(W\). The double cover \(Y \to \tilde{D}_1\) has two singular points of type \(D_4\) over \(q_1, q_2\). Let \(2E + E_1 + E_2 + E_3\) and \(2E' + E'_1 + E'_2 + E'_3\) be the fundamental cycles of the minimal resolutions \(X' \to Y\) of these singularities. The picture of curves on the minimal resolution \(\tilde{D}_1\) of the branch curve is as follows.

Here we kept the notations for the proper transforms of the curves \(\ell, \ell_1, \ell_2, K\) which are drawn in blue color.

Since the branch curve has ordinary triple points at \(q_1, q_2\), the double plane has singular points of type \(D_4\) over these points. Let \(2R + R_1 + R_2 + R_3\) and \(2R' + R'_1 + R'_2 + R'_3\) the fundamental cycles of the minimal resolution \(\tilde{V}\) of the double cover over these points. The pre-images \(C_1, C_2\) of \(A_1, A_3\) are \((-4)\)-curves. The pre-images of the \((-1)\)-curves are \((-2)\)-curves, and the pre-images of \(A_2, A_4\) are \((-1)\)-curves (taken with multiplicity 2). Under the blowing down \(V \to V\) of these curves to points \(x_1, x_2\), we obtain a Coble surface \(V\). The images of \(C_1, C_2\) is the two boundary components, the images \(E_1, E_2, E_3, E_4\) of \((-2)\)-curves are \((-1)\)-curves on \(V\), each intersecting \(C_1, C_2\).

We denote the images in \(V\) of the pre-images of \(\ell_1, \ell_2, \ell, K\) on \(\tilde{V}\) by \(L_1, L_2, L_3, L_4\), respectively. We have the following picture on \(V\). We draw \((-1)\)-curves in green, the pre-images of the line \(\ell\) and the conic \(K\) are drawn in red. The curves drawn in black are \((-2)\)-curves, and, of course, the curves \(C_1, C_2\) are the boundary \((-4)\)-curves.
After we blow down the $(-1)$-curves $L_1, L_2$, we obtain an Halphen surface of index 2 with an elliptic fibration with a reducible double fiber $F_1$ of type $\tilde{A}_7$ and a reducible simple fiber $F_2$ of type $\tilde{A}_1$. The images of the curves $R_1$ and $R'_1$ are two bisections which pass through the singular points of $F_2$. The images of the curves $R_1, R_2, R_3, R_4$ are also bisections of the fibration.

Let $\alpha_1 = \frac{1}{2}(C_1 + C_2) + 2L_1, \alpha_2 = \frac{1}{2}(C_1 + C_2) + 2L_2$ be two irreducible effective roots in $\text{CM}(V)$. We have $\alpha_1 \cdot \alpha_2 = 2$. The effective roots $\alpha_1, \alpha_2, R, R_1, R_2, R_3, L_3, R', R'_1, R'_2, R'_3, L_4$ form a crystallographic basis of type I.

Applying Theorem 9.8.1, we obtain that $\text{Aut}(V)$ is finite. Let us find its automorphism group. The covering involution $g_0$ fixes all 12 effective roots, and, as in the case of Enriques surfaces of type I, define a cohomologically trivial involution of $\tilde{V}$. The projective transformation $[x, y, z] \mapsto [-x, -y, z]$ defines another involution of $\tilde{V}$. It switches the points $q, q'$ and the lines $\ell_1, \ell_2$. It acts identically on the root basis of the diagram (8.5), however it does not act identically on $\text{Pic}(\tilde{V})$. This is the peculiarity of our Coble surface because two of its effective roots are not $(-2)$-curves. As in the case of Enriques surfaces (Proposition 8.9.5), there exists an automorphism of order 4 whose square is $g_0$. The group of automorphisms of $\tilde{V}$ is isomorphic to $D_8$.

Let us calculate the $R$-invariant of $\tilde{V}$. The classes $\tilde{C}_1 \pm \tilde{C}_2$ generate the lattice $A_1(2) \oplus A_1(2)$. There are $(-2)$-curves orthogonal to $\tilde{C}_1, \tilde{C}_2$ which generate $E_8(2)$. Thus $K$ contains $E_8 \oplus A_1 \oplus A_1$. Since there are no root lattices containing $E_8 \oplus A_1 \oplus A_1$ properly, we have $K = E_8 \oplus A_1 \oplus A_1$. The sum of $\tilde{C}_1 + \tilde{C}_2$ and $\tilde{C}_1 - \tilde{C}_2$ gives a generator of $H = \mathbb{Z}/2\mathbb{Z}$. Thus the $R$-invariant of this surface is $(E_8 \oplus A_1^{\oplus 2}, \mathbb{Z}/2\mathbb{Z})$.

Since all effective irreducible roots are the classes of $(-2)$-curves and we have the same crystallographic basis, the $R$-invariant is the same as for the Enriques surfaces of type I.

Example 9.8.5. We can also degenerate an Enriques surface $S$ with a crystallographic basis of type I to a Coble surface with one boundary component. This type we specialize equation (8.9.1) of the double plane model of $S$ by taking $c = 0, a = -b - d, a^2 - 4bd = 0$. Thus we can take $a = -2, b = d$, and, after normalization, we obtain the following equation of the double plane

$$x_3^2 + x_2(x_1 - x_2)(x_0^2 - x_1x_2)(-2x_1x_2 + x_0^2 + x_2^2) = 0.$$ 

We see that now the branch quintic becomes the union of a line $\ell$, a conic $K$ (as in the previous example) and an irreducible conic $K' = V(-2x_1x_2 + x_0^2 + x_2^2)$ passing through $\{q_1, q_2\} = \ell \cap K$ which replaces the lines $\ell_1, \ell_2$ from the previous example. In the picture of the branch curve on $\tilde{D}_1$, we have to replace $\ell_1, \ell_2$ with $K'$ that intersects $A_1$ at two points and intersects two $(-1)$-curves.
Its pre-image on $\tilde{V}$ is a $(-1)$-curve. The lines $V(x_0 \pm x_2)$ are tangent to the conic $K'$ at one of the points $q_1, q_2$. Their pre-images on $\tilde{V}$ are $(-4)$-curves, and their pre-images on $V$ are two $(-2)$-curves intersecting at two points. They intersect the curves $R, R'$ with multiplicity 2. Now we see the same diagram as in the case of Enriques surfaces of Type I. The crystallographic basis consists of the divisor classes of $(-2)$-curves. After we blow down any $(-1)$-curve $E$, we obtain an Halphen surface of index 2 with an elliptic fibration whose type depends on a choice of $E$. For example, if we take $E$ equal to the proper transform of $K'$, we obtain an elliptic fibration with fibers of types $\tilde{A}_7$ and $A_1$.

The group of automorphism does not change and is isomorphic to $D_8$ as in the case of Enriques surfaces. The deck transformation defined by the double plane is a cohomologically trivial involution.

The $R$-invariant of $V$ is the same as for the Enriques surfaces of type I, i.e. $(E_8 \oplus A_1, \{0\})$.

**Example 9.8.6.** Let us consider a degeneration of an Enriques surface with crystallographic basis of $(-2)$-curves of type II to a Coble surface.

In equation (8.9.2) of the double plane model of an Enriques surface of type II,

$$x_3^2 + x_1 x_2 (x_1 - x_2)(x_1 x_2 - x_0^2)(x_1 x_2 (x_1 - x_2) + x_0^2 (bx_1 + cx_2)) = 0,$$

we take $b = 0$ (or $c = 0$ it does not change anything). After normalization, we obtain the following equation

$$x_3^2 + x_1 (x_1 - x_2)(x_1 x_2 - x_0^2)(x_1 (x_1 - x_2) + x_0^2) = 0$$

The picture of the branch curve on the weak del Pezzo surface $\tilde{D}_1$ is the following.

![Figure 9.5: The branch curve of the bielliptic map for type II Coble surfaces](image)

The curve $C$ is cut out by a hyperplane passing through one of the node of $D_1$. The slanting line exhibits the exceptional curve over this point on $D_1$. The proper transform of $C$ on $V$ is a $(-1)$-curve. The surface has one boundary component, the pre-image of the exceptional curve over the node. We still have twelve $(-2)$-curves on $V$, the proper transforms of $C_1, C_2$ and the exceptional curves over the singular points of the branch curve. They form a crystallographic basis of type II.

The group of automorphisms of the Coble surface is finite. It contains a subgroup $(\mathbb{Z}/2\mathbb{Z})^2$ generated as in the previous example by the deck transformation $g_1$ and the transformation $g_2 : [x_0, x_1, x_2] \mapsto [-x_0, x_1, x_2]$. This time the deck transformation does not act identically on the Picard lattice. It switches the exceptional curves $R_1, R_{10}$ and $R_4, R_{11}$, where we use the notation...
from Figure 8.7. The second transformation switches the points \( q, q' \) and hence the exceptional \((-2)\)-curves \( R_7, R_{12} \) over them. The exceptional curve over \( p_1 = [1, 0, 0] \) is invariant with respect to \( g_2 \), its pre-image on \( V \) is a \((-1)\)-curve \( \bar{C} \). When we blow down it, we obtain an Halphen surface of index 2 with an elliptic fibration with reducible fibers of types \( D_5 \) and \( A_3 \) and one nodal fiber, the image of the boundary component of \( V \). The same effect makes the blowing down of the \((-1)\)-curve obtained from \( C \). The Mordell-Weil group of translations is isomorphic to \( \mathbb{Z}/4\mathbb{Z} \) and lifts to a cyclic group of automorphisms of order 4 of \( V \). The square of its generator is equal to \( g_2 \). This transformation does not originate from an automorphism of \( D_1 \) that switches the lines \( L_1, L_2 \) and fixes the node that it is contained in the branch curve. The surface also has an elliptic fibration with a reducible fiber of type \( \bar{A}_8 \). It has also a fiber which consists of the union of the \((-4)\)-curve and the \((-1)\)-curve \( C \). We can blow down \( C \) to obtain an Halphen surface with an elliptic fibration with singular fiber of type \( \bar{A}_8 \) and 3 irreducible nodal fibers. The Mordell-Weil group of translations of this fibration is isomorphic to \( \mathbb{Z}/3\mathbb{Z} \) and lifts to an automorphism of \( V \). Thus \[
\text{Aut}(V) \cong S_4.
\]

The root invariant is \( (D_9, \{0\}) \).

Since all effective irreducible roots are the classes of \((-2)\)-curves and we have the same crystallographic basis the \( R \)-invariant is the same as for these Enriques surfaces.

Note that all three examples of Coble surfaces with finite automorphism groups over a field of characteristic 0 are obtained as specialization of one-parameter families of Kondo surfaces which are necessarily of type I or II. Although, by Corollary 9.1.9, any Coble surface is obtained as a specialization of a one-parameter family of Enriques surfaces the general member of this family is expected to have an infinite automorphism group.

Let us now discuss the case of positive characteristic. If \( p \neq 2 \), we look at the construction of a surface of type V-VII where some \( p \) is excluded and show that in the excluded cases the construction gives a Coble surface instead of an Enriques surface. If \( p = 2 \), we use that some of the surfaces with finite automorphism group vary in a family and we try to find a Coble surface on the boundary of these families.

We will start assuming \( p \neq 2 \).

**Example 9.8.7.** We use equation (8.9.5)(Shigeyuki: I can not find this equation) for the double plane model of a surface with finite automorphism group of type V. When \( p = 3 \), it degenerates, after normalization, to a double plane model of Coble surface given by equation

\[
x_3^2 + x_1(x_1 - x_2)(x_1x_2 - x_0^2)(x_1(x_1 - x_2) + x_0^2 + x_0x_1) = 0
\]

(9.8.3)

We note that it is very similar to equation (9.8.6). In the plane model of the del Pezzo surface, it consists of a line \( \ell \), a conic \( K \) and a conic \( K' \in |2e_0 - e_2 - e_3| \) which is tangent to the line \( V(x_2) \) at one point \( q_3 = [1, 1, 0] \). The line and the conic intersect at two points \( q_1 = [1, 1, 1], q_2 = [-1, 1, 1] \). (Shigeyuki: this is not complete). The conic \( K' \) passes through \( q_2 \) and intersects \( \ell \) at \( q_3 = [0, 1, 1] \). It also intersects \( K \) with multiplicity 3 at \( p_2 \). This describes the branch curve of the double plane. We draw the picture of the branch curve on \( D \) blown up at the node \( Q \) corresponding to the line \( V(x_2) \) in Figure 9.6.
Since \( V(x_2) \) is tangent to \( K' \), it splits into two \((-3)\)-curves on the double cover. They intersect at one point \( Q' \) which is contained in the proper transform of \( K' \). After, we blow-up this point and resolve the singularities of the branch curve, we obtain a terminal Coble surface \( V \) with two boundary components \( C_1, C_2 \). The pre-images of the lines on \( D_1 \) that pass through the node \( Q \) are two \((-1)\)-curves \( E_1, E_2 \) on \( V \). They define two effective roots \( \alpha_i = 2E_i + \frac{1}{2}(C_1 + C_2) \) with \( \alpha_1 \cdot \alpha_2 = 2 \). Let \( E \) be the exceptional curve over \( Q' \). It intersects both \( C_1, C_2 \) and defines an irreducible effective root \( \alpha_3 = 2E + \frac{1}{2}(C_1 + C_2) \) with \( \alpha_3 \cdot \alpha_1 = \alpha_3 \cdot \alpha_2 = 2 \). When we blow down \( E_1 \) and \( E_2 \) if obtain an Halphen surface \( H \) of index 2 with two reducible simple fibers of types \( \tilde{D}_6 \) and \( \tilde{A}_1 \) and a double fiber of type \( \tilde{A}_1 \), the proper transform of the line \( V(x_1) \).

Now observe that the line \( \langle q_1, p_2 \rangle \) passes through \( q_3 \) and its proper transform on \( V \) is a \((-2)\)-curve that intersects \( E \) with multiplicity 1 and hence intersects \( \alpha_3 \) with multiplicity 2.

The line \( \langle q_2, p_2 \rangle \) also defines a \((-2)\)-curve on \( V \). It intersects \( E_1, E_2 \) with multiplicity 1 and hence intersects \( \alpha_1, \alpha_2 \) with multiplicity 2.

The line \( \langle q_2, p_4 \rangle \) is tangent to \( K' \) at the point \( q_2 \). It defines a \((-2)\)-curve on \( V \).

The line \( \langle q_3, q_2 \rangle \) is tangent to the conic \( K \) at \( q_2 \). It defines a \((-2)\)-curve on \( V \). It intersects \( E \) and hence intersects \( \alpha_3 \) with multiplicity 2.

The pre-image of the pencil \( |e_0 - e_1| \) defines a pencil (not relatively minimal) with one multiple fiber of type \( \tilde{A}_1 \) and two reducible fibers of type \( \tilde{D}_6 \) and one fiber equal to the sum \( \alpha_1 + \alpha_3 \). The pre-image of the pencil \( |2e_0 - e_2 - e_3 - e_4 - e_5| \) defines a pencil (not relatively minimal) with one multiple fiber of type \( \tilde{A}_1 \) and two reducible fibers of type \( \tilde{D}_6 \) and one fiber equal to the sum \( \alpha_2 + \alpha_3 \). The number of different irreducible components of these two pencil (we count each \( \alpha_i \) as one irreducible component) is equal to 16. Together with the four \((-2)\)-curves described in above, we obtain 20 irreducible effective roots that form a crystallographic basis of type \( V \) in \( \text{CM}(V) \).

It follows that the surface \( V \) has a finite automorphism group. As in the case of Enriques surfaces we obtain that \( \text{Aut}(V) \) contains an cohomologically trivial element \( \sigma \) of order 2 and the quotient \( \text{Aut}(V)/\langle \sigma \rangle \) is a subgroup of \( \text{Sym}(\Gamma) \). As in the case of Enriques surface (Proposition 8.9.21), \( \text{Aut}(V) \cong \mathbb{Z}/2\mathbb{Z} \times S_4 \).

Let us record this example.
Theorem 9.8.8. Under specialization to characteristic 3, an Enriques surface with a crystallographic basis of type V specializes to a Coble surface $V$ with two boundary components that contains a crystallographic basis of the same type of irreducible effective roots in its Coble-Mukai lattice. Its group of automorphism is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathfrak{S}_3$. Its canonical cover $\tilde{X}$ is a supersingular K3 surface with Artin invariant $\sigma = 1$ and the $R$-invariant is $(K, H) = (E_7 \oplus A_2 \oplus 2A_1, (\mathbb{Z}/2\mathbb{Z})^2)$.

Proof. We should prove the last assertion. One can see that there exists a quasi-elliptic fibration on $X$ with two singular fibers of type $\tilde{E}_6$, four singular fibers of type $\tilde{A}_2$ and with nine sections (in case of the Enriques surface of type $V$, the covering $K3$ surface contains an elliptic fibration with two singular fibers of type $\tilde{E}_6$ and two singular fibers of type $\tilde{A}_2$. Each of the preimages of two boundaries is a component of additional two singular fibers). The Shioda-Tate formula implies that the determinant of the Picard group of $X$ is equal to $-3^6/9^2 = -3^2$, that is, $X$ is the supersingular $K3$ surface with Artin invariant 1. For $K$, we find $E_7 \oplus A_2$ from $(-2)$-curves on $V$ as in the case of the Enriques surface and $2A_1$ from two boundaries. Since $V$ has two boundaries, the Coble-Mukai lattice $\Lambda$ is isomorphic $E_{10}$ (Example 9.2.5). By considering the map $\gamma : K/2K \to \Lambda/2\Lambda$ given in (9.2.5) and the fact $\dim \text{rad}(q_{K/2K}) = 3$, we have $H = (\mathbb{Z}/2\mathbb{Z})^2$. \hfill $\square$

Example 9.8.9. In Section 8.9 we considered Enriques surfaces with a crystallographic basis of type VI as the quotient of the Hessian surface of the Clebsch diagonal cubic surface by an involution $\sigma$ defined by cubic Cremona involution in the ambient space. We assumed in the construction that $p \neq 3, 5$ in order that the Cremona involution does not have fixed points. In this and the next example we consider the excluded cases $p = 3$ (resp. $p = 5$) and will show that the quotient becomes a Coble surface with five (resp. one) boundary components.

So let us assume that $p = 3$. The Hessian surface $H$ is still defined as a quartic surface but it is not isomorphic to the Hessian of any cubic surface. The surfaces still has 10 nodes with the same coordinates as in the case $p \neq 2, 3, 5$. The Cremona involution has 5 fixed points which form the $\mathfrak{S}_5$-orbit of the point $[-1, 1, 1, 1, 1]$. They are additional ordinary nodes of $H$. Let $X$ be a minimal resolution of $H$. The involution $\sigma$ lifts to an involution of $X$ that fixes pointwise the exceptional curves $R_i$ over the new nodes. The quotient surface is a Coble surface $V$ with 5 boundary components.

We refer for the notations of what follows to Section 8.9. The lines $L_{abc}(\omega)$ disappear and instead we have 10 lines $\ell_{ab} : x_a + x_b = 0, x_c = x_d = x_e$. Each such line passes through two of the new node. For example the line $\ell'_{12}$ passes through $[-1, 1, 1, 1, 1]$ and $[1, -1, 1, 1, 1]$. Their proper transforms on $X$ are ten $(-2)$-curves $\ell_{ab}$ invariant under $\sigma$ and intersecting two of the exceptional curves $R_i$. Their images in $V$ are ten $(-1)$-curves $E_{ab}$, each intersecting two boundary components. This defines irreducible effective roots $\alpha_{ab} = 2E_{ab} + \frac{1}{2}(C_a + C_b)$. We have $\alpha_{ab} \cdot \alpha_{cd} = 1$ if $\# \{a, b\} \cap \{c, d\} = 1$ and zero if the sets are disjoint. Thus, we see that each $\alpha_i$ intersects 6 other $\alpha_j$’s. They play the role of $E_{11}, \ldots, E_{20}$ in the crystallographic basis of type VI. The edges $L'_{ab}$ of the Sylvester pentahedron survive and define, as in the case of Enriques surfaces ten $(-2)$-curves $R_{ab}$ on $V$ whose intersection diagram is the Petersen graph. The ten curves play the role $E_1, \ldots, E_{10}$ in the crystallographic basis of type VI. Each $\ell_{ab}$ passes through one node and intersects the opposite edge. This shows that each $\alpha_i$ intersects one of intersects one $R_{ab}$ with multiplicity 2. So, we see that 20 effective roots $\alpha_{ab}, R_{ab}$ form the crystallographic basis of type VI.

Let us now give a construction of the Coble surface $V$ as the blow-up of 15 points in the plane.
9.8. COBLE SURFACES WITH FINITE AUTOMORPHISM GROUP

Recall that in Example 9.2.7 we constructed a Coble surface with 10 boundary components whose K3 cover was one of Vinberg’s most algebraic K3 surfaces. It was obtained by blowing up the intersection points of 10 lines on the del Pezzo surface $D_5$ of degree 5. This time we blow up one point on each line not equal to one of 15 intersection points. We index the lines by duads \( \{a, b\} \subset [1, 5] \). As a result we obtain a surface with ten \((-2)\)-curves $R_{ab}$ that form the Petersen graph.

We also have ten \((-1)\)-curves $E_{ab}$, the exceptional curves over the points $p_{ab}$ lying on one of the lines. Suppose we find 5 conics $C_i$ from different five pencils of conics on $D_5$, each passing through 4 points $p_{ab}$. Then the proper transform of $C_i$ would be a \((-4)\)-curve intersecting four $E_{ab}$’s and each $E_{ab}$ would intersect two of the $C_i$’s. It is easy to see that the curves $R_{ab}$ and the effective roots $2E_{ab} + \frac{1}{2}(C_a + C_b)$ form the crystallographic basis of type VI. It is an amazing fact that we can find such conics (uniquely) only if $p = 3$.

We use model of $D_5$ as the blow up of four points $p_1 = [1, 0, 0], p_2 = [0, 1, 0], p_3 = [0, 0, 1], p_4 = [1, 1, 1]$ in the plane. The 10 lines obtained as the proper transforms of the six lines $V(t_i)$ and $V(t_i - t_j)$, and four exceptional curves over these lines corresponding to the tangent directions at the points $p_i$. The five pencils of conics on $D_5$ come from the pencil of lines with base point $p_i$ and the pencil of conics with base points $p_1, p_2, p_3, p_4$. Thus our 5 conics originate from four lines

\[
V(a_1 x_0 + a_2 x_1), V(b_1 x_0 + b_2 x_2), V(c_1 x_1 + c_2 x_2), V(d_1(x_0 - x_2) + d_2(x_0 - x_1))
\]

and the conic $V(ax_0 x_1 + bx_1 x_2 + cx_0 x_2)$.

An easy computation shows that the conditions on the intersection points forces the conic to have the equation $x_0 x_1 + x_0 x_2 + x_1 x_2 = 0$ and the lines to have the equations $x_i + x_j = 0, x_0 + x_1 + x_2 = 0$. The nodal sextic is the image of the five curves $C_i$ in the plane is the union of the four lines and the conic. The set of its 14 double points consists of 10 points

\[
[0, 1, -1], [1, -1, 0], [1, 0, -1], [1, 1, -1], [1, -1, 1], [-1, 1, 1], [1, 0, 0], [0, 1, 0], [0, 0, 1], [1, 1, 1]
\]

and four infinitely near points to the last four ones that correspond to the tangent directions defined by the lines $V(t_i)$. By solving the linear equations we find this is possible only if $p = 3$.

It is known that $\text{Aut}(D_5) \cong \mathbb{S}_5$ (see [180], 8.5.4). The blowing-up morphism $V \to D_5$ is obviously $\mathbb{S}_5$-invariant and the lift of the action of $\mathbb{S}_5$ on $V$ coincides with the action of $\mathbb{S}_5$ descended from its action on $X$. By analyzing its action on the crystallographic basis of $V$ we find that $\text{Aut}(V) \cong \mathbb{S}_5$.

Let us record this example.

**Theorem 9.8.10.** Under specialization to characteristic 3, an Enriques surface with a crystallographic basis of type VI specializes to a Coble surface with five boundary components that contains a crystallographic basis of the same type of irreducible effective roots in its Coble-Mukai lattice. Its group of automorphism is isomorphic to $\mathbb{S}_5$. Its canonical cover $X$ is a supersingular K3 surface with Artin invariant $\sigma = 1$ and the $R$-invariant is $(K_H) = (E_6, \oplus D_5, \mathbb{Z}/2\mathbb{Z})$.

**Proof.** One can see that there exists a quasi-elliptic fibration on $X$ with three singular fibers of type $\tilde{E}_6$, one singular fiber of type $\tilde{A}_2$ and with three sections. The Shioda-Tate formula implies that the determinant of the Picard group of $X$ is equal to $-3^4/3^2 = -3^2$, that is, $X$ is again a
supersingular K3 surface with Artin invariant 1. For \( K \), we find \( E_6 \) from \((-2)\)-curves on \( V \) and \( D_5 \) from five boundaries. Since \( \text{rank } H \geq 1 \) and the dimension of the kernel of \( q_{K/2K} \) is 1, we have \( H = \mathbb{Z}/2\mathbb{Z} \).

**Remark 9.8.11.** This is follow-up of Remark 9.2.13. In this remark we discussed Coble surfaces with 5 boundary components obtained by blowing up 10 intersection points of five conics \( C_a \) from different pencils on \( D_5 \). This 5-dimensional family of Coble surfaces lives in any characteristic. The ten exceptional curves \( E_{ab} \) and 5 conics define ten irreducible effective roots \( \alpha_i = 2E_{ab} + \frac{1}{2}(C_a + C_b) \). Its intersection graph is isomorphic to the subgraph of the graph of Type VI formed by the vertices \( E_{11}, \ldots, E_{20} \). So we see our surface as a specialization to characteristic 3 of a 5-dimensional family of Coble surfaces with 5 boundary components whose general member has infinite automorphism group.

Note that, when we choose the conics to be a pair of lines, we obtain a non-terminal Coble surface with 10 boundary components. Blowing up 5 singular points, we obtain a terminal Coble surface whose canonical cover is Vinberg’s most algebraic surface.

**Example 9.8.12.** Now we specialize the same Enriques surface \( S \) of Type VI to characteristic 5. The point \( q = [1, 1, 1, 1, 1] \) is a unique fixed point of \( \sigma \) on the Hessian surface \( H \). This time \( H \) is the Hessian surface of the Clebsch diagonal cubic surface in characteristic 5. We have the same 30 lines \( L(a, b, c)(\omega) \) and \( L(a, b) \) as in the case \( p = 0 \). Also we have the exceptional curves \( R_{ab} \) over the 10 singular points \( p_{cde} \) and one additional \((-1)\)-curve \( E_q \).

Note that the residual component of the plane section \( x_i = x_j = 0 \) of \( H \) containing an edge of the pentahedron formed by the coordinate hyperplanes \( x_i = 0 \) is a cubic curve with a double point at \( q \). Its proper inverse transform on \( X \) is a \((-2)\)-curve \( C_{ij} \) intersecting the exceptional curve \( E_q \) at two points. The \( G_5 \)-orbit of \( E_{ij} \) consists of 10 disjoint curves on \( X \). Since \( \rho(X') = 20 \), we obtain that \( \rho(X) = 22 \), so that \( X \) is a supersingular K3 surface.

The quotient \( V = X/(\sigma) \) of the minimal resolution \( X \) of \( H \) is a Coble surface \( V \) with \( K_V^2 = -1 \) with an irreducible isolated curve \( R_0 \) in \( |-2K_V| \), the image of the curve \( E_q \).

The surface \( V \) contains 20 \((-2)\)-curves \( E_1, \ldots, E_{20} \) which form a crystallographic basis of type VI in \( K_V^2 \). The curves \( E_1, \ldots, E_{10} \) are the images of the edges of the pentahedron. Their intersection graph is the Peterson graph. The curves \( E_{11}, \ldots, E_{20} \) are the images of the curves \( L(a, b, c)(\omega) \). Their intersection graph is the incidence graph of the configuration \((10_6, 30_2)\). The images of \( C_{ij} \) are disjoint \((-1)\)-curves \( S_{ij} \) on \( V \). Pick up one of them, say \( C_{01} \) and blow it down to obtain a surface \( Y \) with \( K_Y^2 = 0 \). It is a straightforward computation that \( Y \) is an Halphen rational elliptic surface of index 2. The image of \( E_q \) on \( Y \) is an irreducible nodal fiber on \( Y \). The images of \( C_{ij} \neq C_{01} \) are 9 disjoint 2-sections. The fibration has 3 reducible fibers of type \( \tilde{A}_5, \tilde{A}_2, \tilde{A}_1 \). The fiber of type \( \tilde{A}_5 \) is formed by curves \( E_2, E_3, E_5, E_6, E_7, E_9 \). The fiber of type \( \tilde{A}_2 \) is formed by the curves \( E_{14}, E_{18}, E_{19} \). The fiber of type \( \tilde{A}_2 \) is formed by \( E_{10}, E_{16} \).

The automorphism group of \( V \) is again \( G_5 \).

**Theorem 9.8.13.** Under specialization to characteristic 5, an Enriques surface with a crystallographic basis of type VI specializes to a Coble surface with one boundary component that contains a crystallographic basis of the same type of irreducible effective roots in its Coble-Mukai lattice. Its
group of automorphism is isomorphic to \( \mathfrak{S}_5 \). Its canonical cover \( X \) is a supersingular K3 surface with Artin invariant \( \sigma = 1 \) and the \( R \)-invariant is \( (K, H) = (E_6 \oplus A_4, \{0\}) \).

**Proof.** Since the number of boundaries is 1, its \( \mathcal{R} \)-invariant does not change. We already know that \( X \) is a supersingular K3 surface. Note that \( \pi^*(\text{Num}(S)) \oplus K(2) \) has the determinant \( 2^{10} \cdot 3 \cdot 5 \) and its orthogonal complement \( T \) in \( \text{Pic}(X) \) has the rank 2. Since \( \text{Pic}(X) \) contains \( \pi^*(\text{Num}(S)) \oplus K(2) \oplus T \) as a sublattice of finite index, the Artin invariant should be 1.

**Example 9.8.14.** Consider the surface \( X' \) given by equations (8.9.10). If \( p = 5 \), the Cremona involution \( \sigma \) has a unique fixed point which is an ordinary node of \( X' \). The quotient is not an Enriques surface but rather a Coble surface.

We know that from Example 8.9.30 that an Enriques surface of type VII contains 15 curves \((-2)\)-curves \( L_\sigma \), the images of 15 pairs of lines \( \ell^\pm_\sigma \subset X' \) spanned by points \([1, -1, \pm i, \mp i, 0]\) and \([0, 0, 0, 0, 1]\), where \( \sigma \) is one of 10 even involutions in \( \mathfrak{S}_5 \). It also contains five \((-2)\)-curves \( K_\tau \), the images of the exceptional curves on \( X \) over the five nodes of \( X \). Together they form the diagram of type VII.

Consider a hyperplane in \( \mathbb{P}^4 \) spanned by two lines \( \ell^+_\sigma \) and the new singular point \( q = [1, \ldots, 1] \). The intersection of \( X' \) with this hyperplane is a curve of bidegree \((3, 3)\) on a quadric that contains its two lines intersecting at a point. The residual curve is of bidegree \((2, 2)\), and since it has a double point at \( q \), it must be a rational \((-2)\)-curve. Its proper transform on the K3 surface \( X \) is a new \((-2)\)-curve. So, we have additionally fifteen \((-2)\)-curves on \( X \). Each curve is invariant with respect to the Cremona involution \( \sigma \). We denote its image on the Coble surface \( V \) by \( R_\sigma \). This is a \((-1)\)-curve on \( V \). The images of each pair of lines \( \ell^\pm_\sigma \) is a \((-2)\)-curve \( L_\sigma \) on \( V \). The image of the exceptional curve over \( q \) on \( V \) is a \((-4)\)-curve \( C \in \left(-2K_V\right) \).

One check that two curves \( L_\sigma \) and \( L_\tau \) intersect if and only if they share a dual entering into \( \sigma \) and \( \tau \). Two curves \( R_\sigma \) and \( R_\tau \) intersect with multiplicity one if and only if \( \sigma \) and \( \tau \) do not fix the same element in \([1, 5]\). The curve \( L_\sigma \) intersects \( R_\tau \) with multiplicity 2 if and only if \( \sigma \) and \( \tau \) fix the same element in \([1, 4]\). Otherwise they intersect with multiplicity 1.

Consider the nine curves

\[
L_{12,35}, L_{35,24}, L_{24,13}, L_{13,25}, L_{25,14}, L_{14,23}, L_{23,45}, L_{12,45}.
\]

It follows from above that their sum is a nodal cycle \( L \) of type \( \tilde{A}_8 \). They can be taken to realize the curves \( E_1, \ldots, E_9 \), respectively, from the diagram of type VII. The disjoint \((-1)\)-curves \( R_{23,45}, R_{24,35}, R_{25,34} \) intersect the components \( E_1, E_4, E_7 \). Consider the blowing down \( V \to \mathbb{P}^2 \) of the disjoint exceptional configurations \( R_{23} + E_1 + K_1 + K_5, R_{24,35} + E_4 + E_5, \) and \( R_{25,34} + E_7 + E_8 \). The image of the \((-4)\)-curve is a curve of degree 6 with 10 double points \( p_4 \succ p_3 \succ p_2 \succ p_1, p_7 \succ p_6 \succ p_5, p_{10} \succ p_9 \succ p_8 \). The surface is obtained by blowing up the singular point of an irreducible fiber of an Halphen elliptic surface with one double fiber of type \( \tilde{A}_8 \). The images of the curves \( R_{23,45}, R_{24,35}, R_{25,34} \) are three disjoint bisections. The image of the nodal cycle \( L \) is the triangle of lines with vertices at the points \( p_1, p_5, p_8 \).

Note that the Halphen surface is defined uniquely as a torsor of a jacobian elliptic surface with a reducible fiber of type \( \tilde{A}_8 \). So, the peculiarity of the existence of a diagram of type VII on it is
explained by the fact that \( p = 5 \).

The surface \( V \) has the obvious symmetry with the group \( \mathfrak{S}_5 \). The only effective roots in the Mukai lattice \( C_{10} \) are the classes of \((-2)\)-curves. Using Theorem 9.8.1, we see that \( \text{Aut}(V) \cong \mathfrak{S}_5 \).

**Theorem 9.8.15.** Under specialization to characteristic 5, an Enriques surface with a crystallographic basis of type VII specializes to a Coble surface with one boundary component that contains a crystallographic basis of the same type of irreducible effective roots in its Coble-Mukai lattice. Its group of automorphism is isomorphic to \( \mathfrak{S}_5 \). Its canonical cover \( X \) is a supersingular K3 surface with Artin invariant \( \sigma = 1 \) and the \( R \)-invariant is \((K, H) = (A_0 \oplus A_1, \mathbb{Z}/2\mathbb{Z})\).

**Proof.** Since the number of boundaries is 1, its \( R \)-invariant does not change. The Picard number of the canonical cover is at least 21 (= 10 + 10 + 1) and hence \( X \) is a supersingular K3 surface. Since \( \pi^\ast(\text{Num}(S)) \oplus K(2) \) has the determinant \( 2^{22} \cdot 5 \) and its orthogonal complement in \( \text{Pic}(X) \) has the rank 2, the Artin invariant should be 1. \( \square \)

Next we give examples of Coble surfaces in characteristic 2 with finite automorphism group.

**Example 9.8.16.** In Theorems 9.2.8, 9.2.10 and in Remark 9.8.11 we discussed a Coble surface with ten boundary components that live in any characteristic. Here we show that this surface in characteristic 2 is a Coble surface with ten boundary components and with the crystallographic basis in \( \text{CM}(V) \) of type VII given in Figure 8.16.

In the proof of Theorem 9.2.10, we showed that the Coble-Mukai lattice contains 20 \((-2)\)-classes forming a crystallographic basis of type VII in which 15 classes are effective and the remaining 5 classes correspond to five involutions. Thus it suffices to prove that the five classes are represented by effective classes. We consider the projective plane \( \mathbb{P}^2(\mathbb{F}_4) \). We use the same coordinates as in Example 9.2.7 for the complete quadrangle and its vertices \( p_1, p_2, p_3, p_4 \) and the diagonal points \( q_1, q_2, q_3 \). As in this example we consider the surface obtained by blowing up the intersection points of 10 lines on the quintic del Pezzo surface \( D_5 \), the blow up of \( p_1, p_2, p_3, p_4 \). The specific of characteristic 2 is that the proper transforms of the following four conics \( K_i \) each passing through the set \( \Sigma_i = \{ p_1, p_2, p_3, p_4 \} \setminus \{ p_i \} \) and tangent at these points to the lines \( \langle p_i, p_j \rangle, j \in \Sigma_i \):

\[
K_1 : x^2 + yz = 0, \quad K_2 : y^2 + xz = 0, \quad K_3 : z^2 + xy = 0, \quad K_4 : xy + yz + xz = 0.
\]

We also let \( K_5 \) be the line \( x + y + z = 0 \) that passes through \( q_1, q_2, q_3 \). The proper transform of each \( K_i \) is a rational curve with self-intersection 1 that passes through 3 intersection points of the ten lines. Thus, their proper transforms on \( V \) are five \((-2)\)-curves which we will continue to denote by \( K_1, \ldots, K_5 \). Thus the twenty effective roots \( \alpha_{ab,cd} \), \( K_i \) form the crystallographic basis of type VII.

Note that three lines given in (9.2.6) degenerate to \( x + y + z = 0 \) in characteristic 2. Thus \((-2)\)-classes defining five reflections in Vinberg’s most algebraic K3 surface are now represented by effective curves and hence do not define an involution of the surface.

This example was obtained independently by S. Mukai.

As mentioned in Example 9.2.8, \( V \) dominates an elliptic surface with singular fibers of \( \tilde{A}_4, \tilde{A}_4, \tilde{A}_0, \tilde{A}_0^4 \) and with ten sections. Let \( \pi : X \to V \) be the canonical cover defined by the invertible sheaf
\( \mathcal{L} = \omega_{\mathcal{V}}^{-1} \) and the section of \( \mathcal{L}^{\otimes 2} \) with the zero scheme equal to the boundary \( \sum C_{ab} \). We have \( \omega_{\mathcal{V}} \cong \mathcal{O}_{\mathcal{V}} \) and Proposition 0.2.7 tells that \( \text{Sing}(\mathcal{V}) \) is a finite subscheme \( Z \) with \( h^0(\mathcal{O}_Z) = 12 \). Since \( \mathcal{V} \) has an ordinary double point over singular points of fibers of type \( \tilde{A}_9 \) and of \( \tilde{A}_9^* \), we infer that \( \mathcal{V} \) has exactly 12 double points and its minimal resolution \( \mathcal{Y} \) is a supersingular K3 surface. It has an elliptic fibration with singular fibers of type \( \tilde{A}_9, \tilde{A}_9, \tilde{A}_1, \tilde{A}_1 \) and with 10 sections, and the Shioda-Tate formula implies that the Artin invariant of \( \mathcal{Y} \) is equal to 1.

Next we show that the above Coble surface is a specialization of the Enriques surfaces of type VII given in §8.9, Example 8.10.8. We use the same notation given there. Recall that we considered the elliptic fibration \( p : \mathcal{Y} \to \mathbb{P}^1 \) on the minimal resolution \( \mathcal{Y} \) of the canonical cover \( X \) of \( S \) which is the minimal model of the elliptic surface defined by the Weierstrass equation

\[ y^2 + t^2 xy + y = x^3 + x^2 + t^2. \]

The fibration \( p \) has two reducible fibers of type \( \tilde{A}_9 \) over \( t = 1, \infty \) and two singular fibers \( F_\omega + E_\omega, F_\omega + E_\omega^2 \) of type \( A_1 \) over \( t = \omega, \omega^2 \). The elliptic surface \( f \) has 10 sections \( s_i, m_i \) (\( i = 0, 1, \ldots, 4 \)). To obtain Enriques surfaces, we considered the following derivation defined by

\[ \partial = \frac{1}{t+1} \left( (t+1)(t+\alpha)(t+\beta) \frac{\partial}{\partial t} + \frac{(1+t^2 x)}{t+1} \frac{\partial}{\partial x} \right) \]

with \( \alpha, \beta \in k, \alpha + \beta = \alpha \beta, \alpha^3 \neq 1 \).

Now we put \( \beta = \alpha/(\alpha+1) \) in \( \partial \), multiple \( \partial \) by \( \alpha + 1 \) and put \( \alpha = 1 \). Then we obtain the derivation \( \partial_0 = (t+1) \frac{\partial}{\partial t} \). A direct calculation shows the following Lemma.

**Lemma 9.8.17.** (i) The integral curves with respect to \( \partial_0 \) are \( E_{1,1}, E_{1,3}, E_{1,5}, E_{1,7}, E_{1,9}, E_{\infty,1}, E_{\infty,3}, E_{\infty,5}, E_{\infty,7}, E_{\infty,9}, E_\omega, E_{\omega^2}, s_0, s_1, s_2, s_3, s_4 \).

(ii) Let \( D \) be the divisor of \( \partial_0 \). Then

\[ D = F_1 + E_{1,2} + E_{1,4} + E_{1,6} + E_{1,8} + F_\omega + E_{\infty,2} + E_{\infty,4} + E_{\infty,6} + E_{\infty,8} + E_\omega + E_{\omega^2}. \]

(iii) \( D^2 = -24 \).

Let \( Y^{\phi_0} \) be the quotient of \( Y \) by \( \partial_0 \) and \( \pi : Y \to Y^{\phi_0} \) the canonical map. By the same argument as in the case of Enriques surfaces, \( Y^{\phi_0} \) is smooth. It follows from the canonical divisor formula that \( \pi^*K_{Y^{\phi_0}} = -D \). For a curve \( C \) on \( Y \), we denote by \( \bar{C} \) the image of \( C \) on \( Y^{\phi_0} \). It follows that the integral curves in Lemma 9.8.17, (i) are \(( -1)\)-curves and the other are \(( -4)\)-curves. Thus we have

\[ 2K_{Y^{\phi_0}} = -(\bar{F}_1 + \bar{E}_{1,2} + \bar{E}_{1,4} + \bar{E}_{1,6} + \bar{E}_{1,8} + \bar{F}_\infty + \bar{E}_{\infty,2} + \bar{E}_{\infty,4} + \bar{E}_{\infty,6} + \bar{E}_{\infty,8} + 2\bar{E}_\omega + 2\bar{E}_{\omega^2}). \]

By contracting \(( -1)\)-curves \( \bar{E}_\omega, \bar{E}_{\omega^2}, \) we obtain a non-singular surface \( V \) with \( -2K_V = \{ \bar{F}_1 + \bar{E}_{1,2} + \bar{E}_{1,4} + \bar{E}_{1,6} + \bar{E}_{1,8} + \bar{F}_\infty + \bar{E}_{\infty,2} + \bar{E}_{\infty,4} + \bar{E}_{\infty,6} + \bar{E}_{\infty,8} \} \). The images of five sections \( m_0, m_1, \ldots, m_4 \) are \(( -2)\)-curves forming a complete graph of five vertices with double edges. The images of 15 integral curves in Lemma 9.8.17, (i) are 15 \(( -1)\)-curves which define 15 irreducible effective roots. One can check that the dual graph of 20 roots is nothing but the one of type VII. Thus we have a Coble surface \( V \) with the dual graph of type VII and with ten boundary components.

Let us record the discussion in the previous example in the following.

**Theorem 9.8.18.** The surface \( V \) is a Coble surface in characteristic 2 with ten boundary components and with the crystallographic basis of type VII which is a specialization of the one dimensional
family \( \{S_{a,b}\} \) \( a, b \in k, \ a + b = ab, \ a^3 \neq 1 \) of Enriques surfaces given in Theorem 8.10.10. The automorphism group \( \text{Aut}(V) \) is isomorphic to \( S_5 \) induced from \( \text{Aut}(D_5) \).

Example 9.8.19. In this example we construct a Coble surface in characteristic 2 with two boundary components \( C_1, C_2 \) and with the same crystallographic basis as in the previous example. It is obtained as a specialization of the one dimensional family of Enriques surfaces of type VII given in Theorem 8.10.10.

We use the same notation from the previous example. Observed that the curves \( K_1, \ldots, K_5 \) have two points \( Q(\omega) \) in common with coordinates \([1, \omega, \omega^2], [1, \omega^2, \omega], \omega^3 = 1, \omega \neq 1 \). Looking at the Petersen graph we immediately locate two pentagons (e.g. the interior 5 vertices and the exterior five vertices) each representing a cycle of five \((-1)\)-curves on \( D_5 \). Each vertex of one pentagon is connected to one vertex of the second pentagon. We blow up the corresponding set of 5 intersection points \( P_1, \ldots, P_5 \) of the ten lines, we obtain a surface \( V' \) with two disjoint cycles of \((-2)\)-curves an 5 sections \( E_1, \ldots, E_5 \) coming from the exceptional curves over the five points. They define a structure of a minimal rational elliptic surface an \( V' \) with an elliptic fibration containing two fibers of type \( A_4 \). This pencil contains two irreducible nodal fibers with nodes \( Q(\omega) \) (identified with points on \( V' \)).

Note that \( \#D_5(F_4) = (\#\mathbb{P}^2(F_4) - 4) + 4\#\mathbb{P}^1(F_4) = 17 + 20 = 37 \). The set of 10 lines contains \( 15 + 10\#(\mathbb{P}^1(F_4) - 3) = 35 \) points. Since the points \( Q(\omega) \) belong to \( D_5(F_4) \) and do not lie on lines, we obtain a beautiful fact: the two points \( Q(\omega) \) do not depend on a choice of two disjoint pentagons and coincide with the set of singular points of two irreducible singular fibers of the elliptic fibrations defined by the two pentagons. Another equivalent fact is that any automorphism \( g \) of order 5 of \( D_5 \) has the same set of two fixed points.

Now we blow up the points \( Q(\omega) \) and obtain a Coble surface with two boundary components \( C_1 \) and \( C_2 \) equal to the proper transforms of the two irreducible singular fibers. We know that the rational elliptic surface \( V' \) is an extremal rational elliptic surface with the Mordell-Weil group isomorphic to \( \mathbb{Z}/5\mathbb{Z} \). Each of the curves \( E_i \) is a section intersecting two nodal fibers. This defines 5 irreducible effective roots

\[
\alpha_i = 2E_i + \frac{1}{2}(C_1 + C_2).
\]

We used in the previous example that the proper transforms of the curves \( K_1, \ldots, K_5 \) on \( D_5 \) are curves with self-intersection 1. Each of these curves contains one point \( P_i \) and hence its proper transform on \( V' \) is a curve of self-intersection 0 passing through the singular points of irreducible singular fibers and one singular point of each reducible fiber. The proper transforms of \( K_i \) on \( V \) are \((-2)\)-curves.

Now again have 15 classes of \((-2)\)-curves which are components of reducible fibers and five curves \( K_1, \ldots, K_5 \). Its intersection graph is the dual Petersen graph. We also have five classes of effective irreducible roots \( \alpha_i \) with \( \alpha_i \cdot \alpha_j = 2 \). As in the previous example we check the 20 divisor classes in CM(\( V \)) form the crystallographic basis of type VII.

Now we show that this Coble surface is a specialization to characteristic 2 of the Enriques surfaces of type VII given in §8.9, Example 8.10.8. We use the same notation given there. Recall that we considered the elliptic fibration \( p : Y \to \mathbb{P}^1 \) on the minimal resolution \( Y \) of canonical cover \( X \) of
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S which is the minimal model of the elliptic surface defined by the Weierstrass equation

\[ y^2 + t^2 xy + y = x^3 + x^2 + t^2. \]

The fibration \( p \) has two reducible fibers of type \( \tilde{A}_9 \) over \( t = 1, \infty \) and two singular fibers \( F_\omega + E_\omega, F_\omega^2 + E_\omega^2 \) of type \( A_1 \) over \( t = \omega, \bar{\omega} \). We consider the rational derivation \( \partial \) defined by

\[ \partial = \frac{1}{(t + 1)(t + \omega)(t + \omega^2)(1 + t^2 x)} \frac{\partial}{\partial t} \]

where \( \omega^3 = 1, \omega \neq 1 \). We can prove that the divisor \( D \) of \( \partial \) is equal to

\[ D = E_\omega + E_\omega^2 - \left( F_1 + F_\infty + \sum_{i=1}^{4} (E_{1,2i} + E_{\infty,2i}) \right) \]

(9.8.4)

where \( D \) is the divisor of \( \partial \) and \( F_\omega, F_\omega^2, F_1, F_\infty, E_{1,2i}, E_{\infty,2i} \) (\( 1 \leq i \leq 4 \)) are integral with respect to \( \partial \) (compare this with Lemma 8.10.9). By the same argument in Example 8.10.8, we can see that \( \partial \) has no isolated zeros, the quotient map \( \pi : Y \to Y^{\partial} \) is inseparable and \( Y^{\partial} \) is nonsingular. By contracting the \((-1)\)-curves which are images of \( E_{1,2i}, E_{\infty,2i} \) (\( 1 \leq i \leq 4 \)), we obtain a birational morphism \( \phi : Y^{\partial} \to V' \). Let \( \bar{E}_\omega, \bar{E}_\omega^2, \bar{F}_\omega, \bar{F}_\omega^2 \) be the images of \( E_\omega, E_\omega^2, F_\omega, F_\omega^2 \). Then \( \bar{E}_\omega^2 = \bar{E}_\omega^2 = -4, \bar{F}_\omega^2 = \bar{F}_\omega^2 = -1 \). Note that the non-singular surface \( V' \) is obtained by blowing ups of an elliptic surface at two singular points of two fibers of type \( \tilde{A}_9^* \). Then

\[ 0 = 2K_Y = \pi^*(2K_{V'}) + 2D = \pi^*(\varphi^*(2K_{V'}) + 2 \sum_{i=1}^{4} (E_{1,2i} + E_{\infty,2i})) + 2D. \]

Since \( E_\omega, E_\omega^2 \) are not integral, we have \( \bar{E}_\omega + \bar{E}_\omega^2 \in | - 2K_{V'} | \). Thus \( V' \) is a rational surface and hence a Coble surface with two boundary components. Let \( \bar{s}_i, \bar{m}_i \) (\( 0 \leq i \leq 4 \)) be the images of sections \( s_i, m_i \) respectively. Since \( s_i \cdot \bar{E}_\omega = s_i \cdot \bar{E}_\omega^2 = m_i \cdot E_\omega = m_i \cdot E_\omega^2 = 1, \bar{s}_i^2 = -2 \) and \( \bar{m}_i^2 = -1 \) (these imply that \( \bar{m}_i \) is integral and \( s_i \) is not as pointed out by T. Katsura). Now it is easy to see that the surface \( V' \) is isomorphic to the Coble surface \( V \) obtained above.

Thus we have the following theorem.

**Theorem 9.8.20.** The surface \( V \) is a Coble surface in characteristic 2 with two boundary components and with the crystallographic basis of type VII which is a specialization \( a = \omega, b = \omega^2 \) of the one dimensional family \( \{ S_\alpha, \beta \} \) (\( a, b \in \mathbb{K}, a + b = ab, a^3 \neq 1 \)) of Enriques surfaces given in Theorem 8.10.10. The automorphism group \( \text{Aut}(V) \) is isomorphic to \( \mathfrak{S}_5 \) induced from \( \text{Aut}(D_5) \).

**Remark 9.8.21.** In the above example, the canonical cover \( X \) of \( V \) is obtained by contracting twelve \((-2)\)-curves appeared in the divisor \( D \) given in (9.8.4). The induced derivation on \( X \) has two isolated zeros at the images of \( E_\omega, E_\omega^2 \) because \( \partial \) has zeros along \( E_\omega, E_\omega^2 \). We will explain the reason why we can get a Coble surface in Proposition 10.3.3.

We will give two examples of Coble surfaces in characteristic 3 with two and eight boundary components and with finite automorphism group in Examples 10.5.18,10.6.8.
Bibliographical notes

The notion of a Coble surface arises from work of A. Coble on the group of birational transformation of the projective plane that leaves a rational curve of degree 6 invariants [121], [123]. Although not expressed explicitly by Coble, the group of such automorphisms can be interpreted as the group of biregular automorphisms of the blow-up of the plane with center at the ten nodes of the plane sextic. Apparently the first study of such surfaces which were appropriately called Coble surfaces was undertaken in an unpublished notes by M. Miyanishi based on the joint work with R. Miranda and P. Murthy during his stay at the University of Chicago in 1980. The special role played by Coble surfaces in the theory of automorphisms of rational surfaces was explained in a paper of M. Gizatullin [243]. The first modern exposition of Coble’s theory was given in [167], [501] and [170]. A more general definition of a Coble surface as a rational surface $V$ with $|−K_V| = \emptyset$ and $|−2K_V| \neq \emptyset$ was given in [174]. This paper contains a complete classification of Coble surfaces. The classical Coble surfaces constructed from rational 10-nodal plane sextic were studied in [103] were they were characterized as the only rational surfaces whose automorphism group is represented in the corresponding hyperbolic Weyl group by a subgroup of finite index. In this chapter we study a special class of Coble surfaces which are related to the theory of K3 and Enriques surfaces. Over the field of complex numbers they are related to K3 surfaces with a $2$-elementary Picard lattice classified by V. Nikulin [533]. We discuss the moduli space of Coble surfaces as degenerations of Enriques surfaces in Chapter 5 of the Volume 1 of the book.

In section 9.1 we remind some general properties and construction of Coble surfaces from [174]. In the case when the boundary of a Coble surface consists of only one irreducible component the orthogonal complement of its canonical class is isomorphic to the Enriques lattice. This is one of the main ingredients of a connection between the theories of Enriques and Coble surfaces. The analog of this lattice in the case of more than one boundary components which we discuss in section 9.2 was communicated to us by S. Mukai.

In section 9.3 we extend the rational twist construction of Enriques surfaces from Lemma 4.10.9 to Coble surfaces.

In section 9.4 we reconstruct a work of R. Winger [704] that contains a classification of irreducible 10-nodal plane sextics admitting a non-trivial group of projective automorphisms. The Coble surfaces that constructed from some of these sextics have been studied in [189] and [188].

In section 9.5 we discuss the work of Coble about the relationship between rational plane sextics and quartic symmetroid presented in [121],[123]. Using this relationship we give examples of quartic symmetroids and nodal Enriques surfaces that admit some non-trivial projective symmetries.

In section 9.6 we prove that the automorphism group of general Coble and Enriques surfaces are isomorphic to the same group. Over complex numbers this fact can be proven by using the Global Torelli Theorem for K3 surfaces along the same lines as the proof of Corollary 5.5.6 from section 5.5 of Volume 1. Coble’s work suggests a proof that is characteristic free [103, Theorem 3.5]. We discuss the proof in section 8.3.

The group of automorphisms of a general Hessian quartic surface was first studied in [175]. It was proven there that the ten projection involutions descend to the quotient. The fact that these involutions generate the automorphism group of the quotient was proven by I. Shimada [642] and S. Mukai. The study of Coble surfaces of Hessian type was first undertaken in [189] and [188]. The automorphism groups of special Enriques and Coble surfaces of Hessian type with the octahedron symmetry were described in [9].

In section 9.8 we extend the classification of complex Enriques surfaces with finite automorphisms to Coble surfaces. It is based on an analogue of the Nikulin R-invariant introduced by S. Mukai. The classification of possible $R$-invariants of Coble surfaces with finite automorphism groups was given independently in an unpublished work of S. Mukai. Although we have no a complete classification of Coble surfaces with finite
automorphism group in positive characteristic we give plenty examples of such surfaces in this section. Some of them were communicated to us by S. Mukai.
Chapter 10

Supersingular K3 surfaces and Enriques surfaces

10.1 Supersingular K3 surfaces

Recall Definition 1.1.10: a K3 surface $Y$ is called Shioda-supersingular if $b_2(Y) = 22$. Also, in Remark 0.10.26, we defined a supersingular variety satisfying two equivalent conditions: the height of the formal Brauer group is infinite or the slopes of the Frobenius acting on the crystalline cohomology $H^2(Y/W)$ are equal to 1. The Igusa-Artin-Mazur equality (0.10.65) implies that a Shioda-supersingular K3 surface is always supersingular. Recently, it was proven the converse [478], [464] if $p \neq 2$ and [378] if $p = 2$. A partial explanation for the name supersingular is the following property (however weaker than the supersingularity) which defines supersingular elliptic curves and supersingular Enriques surfaces.

**Proposition 10.1.1.** The Frobenius endomorphism acts trivially on $H^2(Y, \mathcal{O}_Y)$.

**Proof.** It follows from Section 0.10 (after Example 0.10.18) that the Dieudonne module $\mathbb{D}(\Phi^2_{Y/k})$ of the formal Brauer group is isomorphic to $H^2(Y, \mathcal{O}_Y)$. Since $\Phi^2_{Y/k} \cong \mathbb{G}_a$, Example 0.1.19 shows that $\Phi(H^2(Y, \mathcal{O}_Y)) = 0$. In particular $\Phi(H^2(Y, \mathcal{O}_Y)) = 0$. □

Recall that in Section 0.10 we assigned to any variety $Y$ the crystalline cohomology $H^i(Y/W)$ and defined a structure of a crystal or a Dieudonne $W$-module. This structure consists of a $\sigma$-linear endomorphism $\Phi : H^i(Y/W) \to H^i(Y/W)$, i.e an endomorphism of abelian groups satisfying $\Phi(wx) = \sigma(w)\Phi(x)$, where $\sigma : W \to W$ is the automorphism of Frobenius acting on the ring of Witt vectors $W$.

**Definition 10.1.2.** A K3-crystal is a crystal $(H, B, \Phi)$, where $H$ is a $W$-module, $B : H \otimes H \to W$ is a symmetric bilinear form, and $\Phi : H \to H$ is a $\sigma$-linear endomorphism satisfying the following properties:

(i) $H$ is a free $W$-module of rank 22;
(ii) $B$ is a perfect pairing, i.e. the map $H \to H'$ defined by $B$ is an isomorphism.

(iii) For $x, y \in H$ we have $B(\Phi(x), \Phi(y)) = p^2(x, y)$.

(iv) The rank of $\Phi \otimes k : H \otimes k \to H \otimes k$ is equal to 1.

**Theorem 10.1.3.** Let $Y$ be a supersingular K3 surface over an algebraically closed field $k$. Then the crystalline cohomology $H^2(Y/W)$ is a K3-crystall.

**Proof.** (i) By universal coefficient formula (??), $H$ has no torsion. By (0.10.42), rank $H^2(Y/W) = b_2(Y) = 22$.

(ii) follows from the Poincaré Duality for crystalline cohomology (0.10.41).

(iii) This follows from the fact that the Frobenius acts on $H^4(Y/W)$ by multiplication by $p^2$ (see Example 0.10.10).

(iv) Since $H^2(Y/W)$ has no torsion,

$$H^2(Y/W) \otimes W \cong H^2_{DR}(Y/k).$$

Under this isomorphism, the map $\Phi \otimes k$ is the action of the Frobenius on the de Rham cohomology. We use the Hodge versus de Rham spectral sequence (0.10.35). It defines a filtration $0 \subset F_2 \subset F_1 \subset F_0 = H^2_{DR}(Y/k)$ with

$$F_2 = H^0(Y, \Omega^2_{Y/k}), \quad F_1/F_2 = H^1(Y, \Omega^1_{Y/k}), \quad F_0/F_1 = H^2(Y, \mathcal{O}_Y).$$

Since $F$ acts trivially on differential forms (because $F(da) = d(a^p) = 0$), we see that $F^*(F_1) = \{0\}$. This shows that the rank of $F^*$ is less than or equal to one. Suppose it is the zero map. This means that $\Phi(H) \subset pH$.

We know from (0.10.75) in Section 0.10 that the first Chern class map $c_1 : \text{NS}(Y) \otimes \mathbb{Z}_p \to H^2_{\text{ét}}(Y, \mathbb{Z}_p) \to H^2(Y/W)$ is injective and its image lies in the Tate module

$$T_H = \{ x \in H^2(Y/W) : F(x) = px \} \subset H^2(X, \mathbb{W}^\geq 1).$$

(The latter is a $\mathbb{Z}_p$-module not a $W$-module). Let us prove that

$$c_1(\text{NS}(Y) \otimes W) = T_H.$$

First of all, the rank of $T_H$ as a $\mathbb{Z}_p$-module cannot be greater than 22. In fact, suppose we have $n > 22$ linearly independent elements in $T_H$, then they must be linearly independent over $W$. Indeed, in any linear combination $\sum \alpha_i x_i = 0$ with coefficients in $W$, we may assume that there exists a coefficient, say $\alpha_1$, which is not divisible by $p$. Applying the Frobenius $F$ and subtracting, we obtain that a new linear combination with $\alpha_1 = 0$ and all coefficients divisible by $p$. Continuing in this way, we obtain that the coefficients must be equal to zero. Since the rank of $H^2(Y/W)$ is equal to 22, the rank of $T_H$ over $\mathbb{Z}_p$ must be equal to 22. Since we know that $H^2(Y/S)/c_1(\text{NS}(Y) \otimes \mathbb{Z}_p)$ has no torsion, we obtain the needed equality $c_1(\text{NS}(Y) \otimes W) = T_H$.

Let $\Phi' = p^{-1} \Phi : H \to H$. Restricting $\Phi'$ to $T_H$ we obtain the identity map. Since $T_H$ is of finite index in $H$, this would imply that $\Phi'$ is the identity on $H$ (see the proof of the next Proposition). Therefore $T_H = H$, and $c_1(\text{NS}(Y) \otimes W) = H$. It follows from property (ii) that $\text{NS}(Y)$ is a unimodular even lattice. This contradicts to Proposition 0.8.7 that claims that there are no hyperbolic unimodular lattices of rank 22.

\[
\square
\]
Recall that a non-degenerate quadratic lattice is called \( p \)-elementary if its discriminant group is a \( p \)-elementary abelian group.

**Proposition 10.1.4.** The Picard lattice \( \text{Pic}(Y) = \text{NS}(Y) \) is a \( p \)-elementary lattice with discriminant equal to \(-p^{2\sigma}\).

*Proof.* As we explained in Remark 0.10.29, the Poincaré Duality for \( l \)-adic cohomology implies that \( \text{NS}(Y) \otimes \mathbb{Q}_l = \{0\} \) for \( l \neq p \). Since \( \text{NS}(Y) \) is an even hyperbolic lattice of rank 22, we obtain that its discriminant is equal to \(-p^s\) for some \( s \). It follows from the proof of the previous Theorem that the image of the first Chern class map \( c_1 : \text{NS}(Y) \to H^2_H(Y, \mathbb{Z}_p) \to H^2(Y/W) \) is \( W \)-submodule of finite index \( d \) that coincides with the Tate module \( T_H \). This implies that \( \text{discr} \left( \text{NS}(Y) \otimes W \right) = d^2 \text{discr}(H^2(Y/W)) \). By Property (ii) of a K3-crystal, \( \text{discr}(H^2(Y/W)) = 1 \). Thus we obtain that \( \text{discr} \left( \text{NS}(Y) \otimes W \right) = \text{discr}(\text{NS}(Y)) \) is a square, hence \( s \) is even.

It remains to prove that \( \text{NS}(Y) \) is \( p \)-elementary. Since \( Y \) is supersingular, the Kummer sequence (0.10.61) shows that \( \text{NS}(Y) \otimes F_p = H^2_H(Y, \mathbb{Z}_p)/pH^2_H(Y, \mathbb{Z}_p) \). This implies that \( \text{NS}(Y)/p \text{NS}(Y) \cong H^2_H(Y, \mu_p) \).

Using (0.10.70), we have

\[
H^2_H(Y, \mu_p) \cong H^1_H(Y, O_Y^\vee / \mathcal{O}_Y^{p}) \xrightarrow{d\log} H^2(Y, \mathbb{Q}^\vee_{\overline{Y}}) \xrightarrow{} H^2_{DR}(Y/\overline{\mathbb{Q}}).
\]

The last inclusion follows from the Hodge versus de Rham spectral sequence (0.10.36). It follows from the inclusion \( \text{NS}(Y)/p \text{NS}(Y) \hookrightarrow H^2(Y/W)/pH^2(Y/W) \) that the quotient of \( H^2(Y/W, \mathbb{Z}_p) \) by \( \text{NS}(Y) \otimes \mathbb{Z}_p \) has no torsion.

Let \( K = \{ x \in H^2(Y/W) : \Phi^a(x) \in p^nH^2(Y/W) \} \subset H^2(Y/W) \). It contains \( T_H \) and hence contains \( c_1(\text{NS}(Y) \otimes \mathbb{Z}_p) \). Obviously, the rank of \( K \) over \( W \) is equal 22. Consider the endomorphism \( \phi = p^{-1} \circ \Phi \). We have 22 linearly independent elements \( x_i \in K \) coming from \( T_H \) such that \( \Phi(x_i) = x_i \). Suppose \( e_1, \ldots, e_{22} \) is a \( W \)-basis of \( K \) and let \( C \) be the matrix expressing \( x_i \) in terms of \( e_i \) and let \( A \) be the matrix of \( \phi \) in the basis \( e_1, \ldots, e_{22} \). Then \( C = \sigma(C) \cdot A \). We know that \( \sigma \) is an automorphism of \( W \), hence the \( (p) \)-adic evaluation of \( \det(C) \) and \( \det(\sigma(C)) \) are equal. This implies that \( \det(A) \in W^* \), hence \( A \in \text{GL}(22, W) \). Since the homomorphism \( r : \text{GL}(22, W) \to \text{GL}(n, \overline{\mathbb{Q}}) \) is surjective, we can find an invertible matrix \( C_0 \) such that \( r(A) = C_0 \cdot F(C_0)^{-1} \) and then lift it to an invertible matrix \( B \in \text{GL}(22, W) \) such that \( A = B \cdot \sigma(B)^{-1} \). This implies that the matrix of \( \phi = p^{-1} \Phi \) in the basis \( e_1, \ldots, e_{22} \) is the identity too, hence \( K = T_H = c_1(\text{NS}(Y) \otimes \mathbb{Z}_p) \).

Suppose we prove that \( \Phi^a(H) \subset p^{n-1}H \) for all \( n \geq 1 \). Then this would imply that \( pH \subset K \) and hence \( pH \subset \text{NS}(Y) \otimes W \). Passing to the dual \( W \)-module, we obtain \( \text{NS}(Y)^\vee \subset (pH)^\vee = p^{-1}H^\vee = p^{-1}H \). This implies that \( p \text{NS}(Y)^\vee \subset H \). Since \( p \text{NS}(Y)^\vee \) is invariant with respect to \( \Phi \) and belongs to \( K \), we obtain that \( p \text{NS}(Y)^\vee \subset \text{NS}(Y) \). This means that \( \text{NS}(Y) \) is \( p \)-elementary lattice.

So, it remains to prove that \( \Phi^a(H) \subset p^{n-1}H \) for all \( n \geq 1 \). The proof is by induction on \( n \). It is obviously true for \( n = 1 \). Suppose it is true for \( n = k \). Let \( \Phi = p^{-k+1} \Phi \). Then \( \Phi(x) = px \) if and only if \( \phi(x) = px \). This implies that \( \phi^N(x) = p^{N+1}x \) and taking \( N \) to go to \( \infty \), we see that \( \phi = \phi \otimes k \) is a nilpotent endomorphism of \( H \otimes k \). Since it commutes with \( \Phi = \Phi \otimes k \) which is of rank 1 by property (iv) of a K3-crystal, we obtain that \( \phi \circ \Phi = 0 \). Thus \( \phi \circ \Phi(x) = p^{-k+1} \Phi^{k+1}(x) \in pH \), hence \( \Phi^{k+1}(x) \in p^k H \) and the assertion is true for \( n = k + 1 \).

**Definition 10.1.5.** The Artin invariant of a supersingular K3 surface \( Y \) is the number \( \sigma \) such that...
$D(\text{NS}(Y)) \cong (\mathbb{Z}/p\mathbb{Z})^{2\sigma}$.

Thanks to the work of Nikulin [532],[533] and Rudakov and Shafarevich [602], [601], the classification of $p$-elementary hyperbolic lattices is known.

**Theorem 10.1.6.** Let $M$ be a $p$-elementary even hyperbolic lattice of rank $n > 2$.

1. If $p \neq 2$, the isomorphism class of $M$ is uniquely determined by the rank $n$ and the order $2^s$ of its discriminant group. Such a lattice exists if and only if

- $n \equiv 0 \mod 2$,
- $n \equiv 2 \mod 4$ if $s \equiv 0 \mod 2$,
- $p \equiv (-1)^{\frac{1}{2}(n-2)} \mod 4$ if $s \equiv 1 \mod 2$,
- $n > s > 0$ if $n \equiv 2 \mod 8$.

2. If $p = 2$, the isomorphism class of $M$ is uniquely determined by the rank $n > 4$, the order $2^s$ of its discriminant group and the type. A lattice is of type I if the discriminant quadratic form takes values in $\mathbb{Z}/2\mathbb{Z}$, (equivalently, $x^2 \equiv 0 \mod 2$ for any $x \in M^\vee$), and the remaining lattices are of type II. A lattice of type I exists if and only if

- $s \equiv 0 \mod 2, n \equiv 2 \mod 4$ (the discriminant quadratic space is isomorphic to a regular quadratic space $(\mathbb{P}_2^s, q)$, where $q$ is of even type);
- $n > s > 0$ if $n \not\equiv 2 \mod 8$.

A lattice of type II exists if and only if

- $s > 0, n \equiv s \mod 2$;
- if $s = 2$, then $n \not\equiv 6 \mod 8$,
- if $s = 1$, then $n \equiv 1 \mod 8$ or $n \equiv 3 \mod 8$.

**Corollary 10.1.7.** The Artin invariant $\sigma$ satisfies

$$1 \leq \sigma \leq 10.$$  

**Proof.** Since the rank of $\text{NS}(Y)$ is equal to 22, we have $\sigma \leq 11$. If $\sigma = 11$, then $\text{NS}(Y)(\frac{1}{p})$ is an even unimodular lattice. Since there are no even unimodular hyperbolic lattices of rank 22, we get a contradiction. For the same reason, we have $\sigma > 0$. □

**Proposition 10.1.8.** The Néron-Severi lattice of a supersingular K3 surface over a field of characteristic 2 is of type I.

**Proof.** Since there is only one isomorphism class of a 2-elementary hyperbolic lattice of given type, rank and discriminant, we can exhibit all of them which are of Type II. Since in our case $n = 22 \equiv 6 \mod 8$, it follows from Theorem 10.1.6 that $s \neq 2$ and $s = 2\sigma$ is even. The lattice

$$M = U \oplus A_1^{\oplus 2\sigma-2} \oplus D_{22-2\sigma}.$$  (10.1.2)
is 2-elementary with \( n = 22 \) and \( s = 2\sigma \neq 2 \). It must be of Type II and every 2-elementary hyperbolic lattice of type II must be isomorphic to \( M \). Let \( f \) be a primitive isotropic vector from the summand \( U \). Applying an element from \( W^{\text{mod}}_f \), we may assume that it is nef. Hence it defines a genus one fibration. The other summands show that the fibration has one reducible fiber of types \( \bar{D}_{22-2\sigma} \) and \( 2(\sigma - 1) \) fibers of type \( \bar{A}_1 \) or \( A_1^\vee \).

Suppose \( f \) is an an elliptic fibration. Adding up the Euler-Poincaré characteristics of fibers we obtain that the sum is greater than 24 if \( \sigma > 2 \). Thus we may assume that \( \sigma = 2 \). Recall that the isomorphism class of any even 2-elementary hyperbolic lattice is uniquely determined by the discriminant quadratic form and its type I or II (Nikulin [533], Theorem 4.3.2). This implies that \( M \cong U \oplus E_8^{\oplus 2} \oplus A_1^{\oplus 4} \). Again, by considering a primitive isotropic vector \( f \) in the summand \( U \), we have a genus one fibration with two fibers of type \( \bar{E}_8 \) and four fibers of type \( \bar{A}_1 \). We assume that this fibration is also an elliptic fibration. Then the sum of the Euler-Poincaré characteristics of fibers is greater than 24 which is a contradiction.

So, we may assume that \( f \) defines a quasi-elliptic fibration. Now we use a nice argument from [602, Proposition, §5]. Passing to the jacobian fibration, we may assume that it has a section which we fix. Let \( C \) be the curve of cusps. Then it intersects each fiber of type \( \bar{A}_1^\ast \) at its unique singular point. Write \( [C] = c_u + \sum_{i=1}^{2r-2} a_i [E_i] + c_d \), where \( c_u \in U, c_d \in D_{22-2\sigma}, a_i \in \mathbb{Z} \) and \( E_i \) is the component of the fiber of type \( \bar{A}_1^\ast \) that does not intersect the zero section. Intersecting both sides with \([E_1]\), we obtain \( 1 = -2a_1 \), a contradiction.

**Corollary 10.1.9.** Let \( Y \) be a supersingular K3 surface in characteristic 2 with Artin invariant \( \sigma \). Then \( \text{NS}(Y) \) is isomorphic to one of the following lattices

\[
1 \leq \sigma \leq 5 : \quad U \oplus \bigoplus_{i=1}^{\sigma} D_{4n_i}, \quad \sum_{i=1}^{\sigma} n_i = 5
\]

\[
10 > \sigma > 5 : \quad U \oplus \bigoplus_{i=1}^{10-\sigma} D_{4n_i}^\vee(2), \quad \sum_{i=1}^{10-\sigma} n_i = 5,
\]

\[
\sigma = 10 : \quad E_{10}(2) \oplus M_{12} \cong U(2) \oplus D_{20}^\vee(2).
\]

where \( M_{12} = \langle e_1, \ldots, e_{12}, 1/2(e_1 + \cdots + e_{12}) \rangle, \quad e_i \cdot e_j = -2\delta_{ij}. \)

**Remark 10.1.10.** Let \( M \) be a \( p \)-elementary lattice \( M \) of rank \( 2r \) and discriminant group of rank \( 2t \). Since the value of the quadratic form \( q : M \to \mathbb{Z} \) lie in \( p\mathbb{Z} \), for any \( v \in M^\vee \), we have \( q(v) = q(pv)/p^2 \in M(1/p) \). Thus \( M^\vee(p) \subset M \) is a quadratic \( p \)-elementary lattice. Let \( G_M \) be the Gram matrix of \( M \) so that \( G_M^{-1} \) is the Gram matrix \( G_{M^\vee} \) of \( M^\vee \). Then \( \det G_{M^\vee(p)} = p^r \det G_{M^\vee} = p^{2r-2t} \). Thus \( M^\vee(p) \) is a \( p \)-elementary lattice of rank \( r \) and discriminant group of rank \( 2r - 2t \).

Applying this to the case when \( M = \text{NS}(Y) \) for some supersingular K3 surface of Artin invariant \( \sigma \), we find a duality between families of supersingular K3 surfaces of Artin invariant \( \sigma \) and \( 11 - \sigma \) [408]. One of the features of this remarkable duality is that, if a primitive isotropic vector \( f \in \text{NS}(Y) \) defines a jacobian genus one fibration on \( Y \), then the corresponding vector in \( \text{NS}(Y)^\vee(2) \) defines a genus one fibration without a section on the dual K3 surface.\(^1\) For example, it is known that every genus one fibration on a supersingular K3 surface with Artin invariant \( \sigma = 1 \) is jacobian.

\(^1\)It follows from the computation of the Tate-Shafarevich group in Theorem ?? in Volume 1 that a non-jacobian genus one fibration always has a \( p \)-section.
[210]. Thus any genus one fibration on a supersingular K3 surface with Artin invariant $\sigma = 10$ has no sections. Passing to the jacobian fibration, we obtain a supersingular K3 surface with Artin invariant $9$ (see [446, Proposition 3.8])

Although we are not going to use it, let us mention the following fundamental fact due to Rudakov and Shafarevich [601].

**Theorem 10.1.11.** A supersingular K3 surface in characteristic 2 is unirational.

It is not known whether it is true in other positive characteristic (the proof of this claim in [446] contains a gap). In case $p > 2$, the unirationality is known only the following cases $p = 3, \sigma \leq 6$ [601] and $p = 5, \sigma \leq 3$ [572]).

Next we discuss the moduli problem of supersingular K3 surfaces.

We have already observed in Section 1.3 that a K3 surface has no nonzero regular vector fields. The theory of local deformations from Section 5.11 implies that the functor $\text{Def}_{Y/k}$ is pro-representable with tangent space $H^1(Y, \Theta_{Y/k}) \cong k^{20}$. The formal local universal deformation scheme is isomorphic to the algebra $R = k[[t_1, \ldots, t_{20}]]$. If we choose an invertible sheaf $L \not\cong O_Y$, then one can consider the local deformation functor $\text{Def}_{Y/k, L}$ for the pair $(Y, L)$ and prove that it is pro-representable and the forgetting morphism $\text{Def}_{Y/k, L} \to \text{Def}_{Y/k}$ is a closed embedding defined by one equation (the latter because $h^2(O_Y) = 1$). A theorem of Deligne from [151] asserts that this equation is non-trivial (i.e. it is not divisible by $p$) and the local versal deformation space is a formal local scheme of dimension 19. More generally, one can choose a finite set $\Sigma$ of invertible sheaves spanning a subspace of dimension $r$ in the image of $c_1 : N = \text{Pic}(Y) \to H^1(Y, \Omega^1_Y)$, defined by the first Chern class map $c_1 : \text{NS}(Y) \to H^2_{\text{DR}}(Y/k)$, and to construct the local versal deformation of the functor $\text{Def}_{Y/k, \Sigma}$. Its dimension is less than or equal than 20 $- r$ (see Section 5.11). Thus

$$20 - r = \dim H^1(Y, \Omega^1_Y)/c_1(N) \leq \dim H^1(Y/W)/c_1(N) \leq \dim H^2(Y/W)/c_1(N) - 1.$$  

However, for a 2-elementary lattice $N$ we have

$$\dim H^2(Y/W)/c_1(N) \otimes k = \dim H^2(Y/W)/c_1(N) \otimes W = \sigma.$$  

This gives $20 - r \leq \sigma - 1$.

More delicate argument using the formal Brauer groups proves that, in fact, we have the equality [602, §9].

**Theorem 10.1.12.** Let $X \to \text{Spec } R$ be the versal formal deformation of a supersingular K3 surface $Y$. Let $N$ be a $p$-elementary sublattice of $\text{NS}(Y)$ of rank 22 and discriminant $-p^{2\sigma}$. Then the formal versal deformation of the functor $\text{Def}_{X, \Sigma}$, where $\Sigma$ is a basis of $N$, is a smooth subscheme of $\text{Spec } R$ of dimension $\sigma - 1$.

Let us discuss the global moduli problem for supersingular K3 surfaces. First we fix one of $p$-elementary lattices $N$ (of type I if $p = 2$) and discriminant $-2^{2\sigma}$ and consider a lattice $N$ polarization of $Y$ as defined in Section 5.2. We also use the notion of an ample polarization and the notion of a family of lattice $N$ polarized K3 surfaces.

**Theorem 10.1.13.** Let $K_N$ be the moduli stack of families of lattice $N$ polarized K3 surfaces. Then $K_N$ is representable by a smooth locally separated algebraic space $K_N$ of dimension $\sigma - 1$. 

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Proof. Let \( \mathcal{K}_N \) be the moduli stack of families of lattice \( N \) polarized K3 surfaces. It is proven in [550, Proposition 2.3] that for any families \( (Y/T, \phi : N \to \text{Pic}(Y/T)) \) and \( (Y'/T, \phi' : N \to \text{Pic}(Y'/T)) \) of lattice \( N \) polarized K3 surfaces over an algebraic base space \( T \), the functor \( \text{Isom}_{(Y/T, \phi), (Y'/T, \phi')} \) is represented by a locally closed algebraic subspace of \( S \). Together with Lemma 8.1.8 from [556] this implies that \( \mathcal{K}_N \) is an algebraic (or Artin) stack. Since \( H^0(Y, \Theta_{Y/k}) = \{0\} \) and \( \text{Aut}_T((Y/T, \phi)) = \{1\} \) [602, §8, Proposition 3], we obtain that \( \mathcal{K}_N \) is represented by an algebraic space [556, Corollary 8.5.3]. Applying Theorem 10.1.12, we obtain that the stack \( \mathcal{K}_N \) is smooth and of dimension \( \sigma - 1 \).

So, now we may consider families of lattice \( N \) amply polarized supersingular K3 surfaces \( (f : Y \to T, j : N_T \to \text{Pic}_Y/T) \), where, by definition, \( j_r(N) \subset \text{Pic}_{X_t/k} \) always contains an ample class. This allows one to construct a Deligne-Mumford stack \( \mathcal{K}_N^0 \) of ample lattice \( N \) polarized supersingular K3 surfaces. We have seen already that \( \text{NS}(Y)_W := \text{NS}(Y) \otimes W \) is a free submodule of finite index \( p^\omega \) in \( H^2(Y/W) \). Let \( Y \) be a lattice \( N \)-polarized supersingular K3 surface. Let \( N_W = N \otimes W \cong W^{2\sigma} \). The inclusions \( N_W \subset \text{NS}(Y)_W \subset H^2(Y/W) \subset \text{NS}(Y)^W_W \subset N'_W \) imply (as in the case of quadratic lattices) that \( H^2(Y/W)/N_W \) is an isotropic subspace of dimension \( p^\sigma \) in the quadratic space \( D(N) \otimes \mathbb{k} \) of dimension \( 2\sigma \). This defines the period of lattice \( N \)-polarized supersingular K3 surface \( Y \), a point in the orthogonal Grassmann variety \( \text{OG}(\sigma, 2\sigma) \) of maximal isotropic subspaces of dimension \( p^\sigma \) in \( D(\text{NS}(Y)) \otimes \mathbb{k} \). We also have shown that \( F_{Y/W} \) defines a linear map \( H^2(Y/W) \otimes \mathbb{k} \to H^2(Y/W) \otimes \mathbb{k} \) of rank 1. This shows that the period \( L \) satisfies the property \( \dim L \cap F^*(L) = \sigma - 1 \). We denote by \( \Omega_N \) the open subset of \( \text{OG}(\sigma, 2\sigma) \) consisting of periods and call it the period space. The standard computations show that \( \dim \text{OG}(\sigma, 2\sigma) = \frac{1}{2}\sigma(\sigma - 1) \). The period space is a closed smooth subvariety of dimension \( \sigma - 1 \).

Let \( V \) be a quadratic space over \( \mathbb{F}_p \) of dimension \( 2\sigma \) and rank \( 2\sigma \). A maximal isotropic subspace \( L \) of \( V \) is called a characteristic subspace if \( L \cap F^*(L) \) is a hyperplane in \( L \).

For any \( \mathbb{F}_p \)-algebra \( A \) let \( \mathcal{M}_V(A) \) be the set of direct summands \( L \subset V \otimes \mathbb{F}_p, A \) such that \( F^*(L) \cap L \) is a direct summand of \( L \) of rank \( \sigma - 1 \).

We refer for the proof of the next proposition to [551, Proposition 4.6], [602, §10, Theorem 1].

**Proposition 10.1.14.** The functor \( \mathcal{M}_V : A \mapsto \mathcal{M}_V(A) \) is representable by a smooth projective variety \( \mathcal{M}_V \) of dimension \( \sigma - 1 \) over \( \mathbb{F}_p \). The variety \( \mathcal{M}_V \otimes \mathbb{F}_p \) consists of two disjoint copies interchanged by the Frobenius isomorphism. Each copy is a unirational variety.

**Example 10.1.15.** If \( \sigma = 1 \), \( M_V \cong \text{Spec} \mathbb{F}_p^2 \). Hence \( M_V \otimes \mathbb{F}_p \) is isomorphic to the disjoint union of two copies of \( \text{Spec} \mathbb{F}_p \). If \( \sigma = 2 \), \( M_V \cong \mathbb{P}^1_{\mathbb{F}} \mathbb{F}_p^2 \) and \( M_V \otimes \mathbb{F}_p \) is isomorphic to the disjoint union of two copies of \( \mathbb{P}^1_{\mathbb{F}_p} \) (see [551, Examples 4.7]). The Frobenius morphism exchanges the two copies.

If \( \sigma = 3 \), \( M_V \) is isomorphic to the Fermat surface

\[
x^{p+1} + y^{p+1} + z^{p+1} + w^{p+1} = 0
\]

in \( \mathbb{P}^3_{\mathbb{F}_p^2} \) (see [602, §10]).

We apply this to our case and take \( V = N_0 \), where

\[
N_0 := pN V / N \cong D(N) \cong \mathbb{F}_p^{2\sigma}
\]
and set
\[ \Omega_N := M_{N_0}. \]
By above, this is a smooth projective subvariety (not geometrically connected) of dimension \( \sigma - 1 \) over \( \mathbb{F}_p \).

Next we discuss the extension of the Global Torelli Theorem for complex K3 surfaces to the case of supersingular K3 surfaces. Recall that the fine moduli space of complex marked K3 surfaces exists in the category of analytic spaces but not separated. The reason is that the marking may not be ample. The same reason explains that our space \( K_N \) is only locally separated. Since \( \Omega_N \) is separated we have to deal first with this problem.

Let \( W_N \) be the \((-2)\)-reflection group of \( N \) and let us choose an open fundamental domains \( C_N \) for the action of \( W_N \) on the positive convex cone \( V(N_\mathbb{R})^+ \) in \( N_\mathbb{Q} \). For each characteristic subspace \( K \) in \( N_0 \) we have an over-lattice \( N \subset N_K \) and \( C_K \) is equal to the union of fundamental chambers of \( N \). Since the number of over-lattices is finite, the number of such chambers in \( C_N \) is finite too. Now suppose we have a family of characteristic subspaces \( K_T \in M_{N_0}(T) \). Let
\[ \Omega(t) := N_{K(t)} := \{ x \in N_\mathbb{Q} : px \in N, \bar{px} \in K_t \} \]
be the over-lattice of \( N \) corresponding to \( K_t \), where bar denotes the coset of modulo \( pN \). Following Ogus, the ample cone of \( K_T \) is a choice \( \alpha(t) \) for each \( t \in T \) of a fundamental chamber of \( W_N(t) \) lying in \( C_N \). We require that \( \alpha(t) \subset \alpha(t') \) whenever \( t \) specializes to \( t' \).

Let us \( \tilde{\mathcal{M}}_{N_0} \) be the functor that assigns to each \( T \) the set of pairs \( (K_T, \sigma) \) that consist of a choice of \( K_T \in M_{N_0}(T) \) and an ample cone of \( K_T \).

We have the following Proposition (1.16) from [550]:

**Proposition 10.1.16.** The functor \( \tilde{\mathcal{M}}_{N_0} \) is represented by a \( \mathbb{K} \)-scheme \( \tilde{\Omega}_N \) which is locally of finite type and the natural forgetting map \( \tilde{\mathcal{M}}_{N_0} \to M_{N_0} = \Omega_N \) is étale and surjective.

Now suppose we have a family \( (\mathcal{Y}/T, \phi) \) of \( N \)-polarized K3 surfaces. For each \( t \in T \), we have an ample cone \( \text{Amp}(\mathcal{Y}_t) \) in \( \text{NS}(\mathcal{X}_t)_\mathbb{Q} = N(t)_\mathbb{Q} \). We define the period map
\[ p_N : K_N \to \tilde{\Omega}_N \]
by assigning to a family \( (\mathcal{Y}/T, \phi) \) the characteristic subspace \( K_T \in M_N(T) \) and its ample cone which is the image of \( \prod_{t \in T} \text{Amp}(\mathcal{Y}_t) \).

The following is a supersingular analog of the Global Torelli Theorem for Kähler K3 surfaces of Burns-Rapoport [101]. We refer for the proof to [551, §3] in the case \( p > 3 \) and in [91] in the case \( p = 3 \). We expect that the same theorem is true if \( p = 2 \) but as of today nobody has written up a proof.

**Theorem 10.1.17.** The period map \( p_N \) is an isomorphism.

**Corollary 10.1.18.** Let \( K^0_N \) be the subspace of ample lattice \( N \) polarized supersingular K3 surfaces. Then the composition of the restriction of the period map \( p_N \) to \( K^0_N \) with the projection \( \tilde{\Omega}_N \to \Omega_N \) defines an isomorphism
\[ p^0_N : K^0_N \cong \Omega^0_N, \]
where \( \Omega^0_N \) is an open subset of the period domain \( \Omega_N \).
Remark 10.1.19. Over the complex numbers, it follows from the Torelli type theorem that a K3 surface $X$ is the canonical cover of an Enriques surface if and only if the Picard lattice $\text{Pic}(X)$ contains $E_{10}(2)$ as a primitive sublattice and its orthogonal complement in $\text{Pic}(X)$ does not contain any $(-2)$-class. We have discussed that the finiteness of the number of Enriques surfaces covered by a given K3 surface up to isomorphisms and Ohashi’s estimate of the number of such Enriques surfaces in Vol. I, §5.3.

In positive characteristic, by using Ogus’ Torelli theorem, J. Jang [337] proved the same result for supersingular K3 surfaces in positive characteristic $p \neq 2$ and showed that if $p > 23$ or $p = 19$, a supersingular K3 surface admits a fixed point free involution if and only if its Artin invariant $\sigma$ is less than 6. Recently Behrens [54] claims that if a K3 surface $X$ is of finite height, then the number of Enriques quotiens of $X$ is finite by applying a result of Lieblich-Maulik [439]. He also extends Ohashi’s estimate for supersingular K3 surfaces in characteristic $p \neq 2$ based on Ogus’ Torelli theorem. For example, in case of $p = 3$ and $\sigma = 1$, there are exactly two Enriques surfaces which are Enriques surfaces with finite automorphism group of type III and of type IV. In characteristic $p = 2$ and $\sigma = 1$, the second author [406] shows that there are three types of Enriques surfaces (see Remark 10.6.15). We will give examples of such Enriques surfaces later.

10.2 Simply connected Enriques surfaces and supersingular K3 surfaces

Let $\pi : X \rightarrow S$ be the canonical cover of an Enriques surface. Assume that it is inseparable. This happens if and only if $p = 2$ and $\text{Pic}^0_{S/k}$ is $\mathbb{Z}/2\mathbb{Z}$ or $\alpha_2$, i.e. a unipotent group scheme of order $2^2$, or, equivalently, when $S$ is simply-connected.

Assume that $X$ is birationally isomorphic to a K3 surface, i.e. it has only double rational points as its singularities. We call such a surface a RDP-K3 surface. By Theorem 1.3.5, its minimal resolution $p : Y \rightarrow X$ is a Shioda-supersingular K3-surface. We denote by $\tilde{\pi} : Y \rightarrow S$ the composition of $\pi \circ p$.

We also know from the discussion in Section 4.9 that $S$ has no quasi-elliptic fibrations and simple fibers of any elliptic fibration are reduced. If the half-fibers are smooth, then the singular points of $X$ lie over singular points of fibers. They are locally isomorphic to $z^2 + f(x, y) = 0$ (because locally a principal cover is given by $z^2 = \epsilon(x, y)$, where $\epsilon$ a unit, changing $z$ to $z + \epsilon(0, 0)$ we get the claim). Such singularities are sometimes called Zariski singularities. If a fiber is of multiplicative type, then the singular points lying over its singular points are of type $A_1$. If a fiber is of additive type $A^*_1$ then the singular point over its singular point is of possible type $D_4^{(0)}$, $E_7^{(0)}$, or $E_8^{(0)}$. If a fiber is of additive type $A^*_2$ then the singular point over its singular point is of possible types $D_4^{(0)}$ or $E_7^{(0)}$.

Proposition 10.2.1. The singular locus of $X$ consists of rational double points of types $A_n$, $D_n^{(0)}$, $E_n^{(0)}$.

\footnote{In [209] such surfaces are called unipotent.}
with the total index (i.e. sum of the subscripts) equal to 12. \footnote{It does not coincide with the Milnor number which may not be defined but twice the index coincides with the Tyurina number \cite[Proposition 3.3]{612}.}

Proof. Since a minimal resolution of $X$ is a K3 surface, all singular points of $X$ are rational double points. Since $X$ is homeomorphic to $S$ in étale topology, we obtain that $e(Y) - e(X) = 12$ and clearly this number coincides with the total index of singularities. We have already observed that the singular points are Zariski singularities. It follows from Artin’s classification of rational double points in characteristic 2 given in Proposition 0.4.13 that all of them are of type $A_n, D_n(0), E_n(0)$. Note that these local computations are confirmed by the fact that the orthogonal complement of $\tilde{\pi}^*(\text{Num}(S))$ in $\text{Pic}(Y)$ is a 2-elementary lattice $E$ of rank 12 that contains a sublattice of finite index generated by the components of the exceptional divisors of $p : Y \to X$. This excludes singularities of type $D_{2k+1}, E_6$ and $A_n, n \not= 1$.

The classification of possible singularities on K3-covers $X$ of simply-connected Enriques surfaces birationally isomorphic to a K3 surface is due to Ekedahl, Hyland and Shepherd-Barron \cite[Corollary 6.16]{209} (see also \cite[Corollary 1.6]{474}).

\textbf{Theorem 10.2.2.} Under the previous assumptions,

- If $S$ is classical, then $\text{Sing}(X)$ is one of
  \begin{align*}
  12A_1, \ 8A_1 + D_4^{(0)}, \ 6A_1 + D_6^{(0)}, \ 5A_1 + E_7^{(0)}.
  \end{align*}

- If $S$ is an $\alpha_2$-surface, then $\text{Sing}(X)$ is one of
  \begin{align*}
  12A_1, \ 3D_4^{(0)}, \ D_4^{(0)} + D_8^{(0)}, \ D_4^{(0)} + E_8^{(0)}, \ D_{12}^{(0)}.
  \end{align*}

All such possibilities can be realized.

\textbf{Example 10.2.3.} Assume $\sigma = 10$. Then the sublattice $\text{Exc}(Y)$ generated by exceptional curves of $p : Y \to X$ is contained in $\tilde{\pi}^*(\text{NS}(S))^\perp \cong M_{12}$. The embedding of the lattices is defined by an isotropic subgroup $A$ of the discriminant group of $\text{Exc}(Y)$ such that $A^+ / A \cong D(M_{12}) \cong \langle 12 \rangle^{10}$. The classification of possible singularities shows that the only possibility is $\text{Exc}(Y) = A_1^{\oplus 12}$.

\textbf{Remark 10.2.4.} Schröer \cite{613} and Matsumoto \cite{474} gave examples of Enriques surfaces whose canonical cover has a non rational double point (an elliptic double point) of Arnold’s type $E_{12}$ (i.e. formally isomorphic to the singularity $z^2 + x^3 + y^7 = 0$). They do not occur if $S$ is a classical Enriques surface \cite[Proposition 3.2]{474}.

\textbf{Definition 10.2.5.} Let $S$ be an Enriques surface whose canonical cover is a RDP-K3 surface $X$. The image of a singular point of $X$ on $S$ is called a canonical point.

\textbf{Proposition 10.2.6.} Let $f : S \to \mathbb{P}^1$ be an elliptic fibration on $S$. Let $Z = (\omega)_0$ be the 0-cycle of zeroes of a non-zero regular 1-form $\omega$ generating $H^0(S, \Omega^1_S|_{\mathbb{P}^1})$. Then the support of $Z$ is equal to the set of canonical points and $i^*\mathcal{I}_Z = \text{adj} F$, where $i : F \hookrightarrow S$ is the inclusion morphism of a fiber or a half-fiber of $f$ and $\text{adj} F$ is the adjoint ideal of $F$. 

\footnote{It does not coincide with the Milnor number which may not be defined but twice the index coincides with the Tyurina number \cite[Proposition 3.3]{612}.}
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**Proof.** The first assertion follows from Proposition 1.3.8. Let $F$ be a fiber or a half-fiber of $f$. Since $X$ is birationally isomorphic to a K3 surface, it is reduced. The exact sequence of sheaves of differentials

$$0 \to \mathcal{I}_F/\mathcal{I}_F^2 \to \Omega^1_{S/k} \otimes_{\mathcal{O}_S} \mathcal{O}_F \to \Omega^1_{F/k} \to 0$$

defines a homomorphism of sheaves $\Omega^1_{S/k} \otimes_{\mathcal{O}_S} \mathcal{O}_F \to \Omega^1_{F/k}$ which induces an isomorphism of the spaces of global sections.

Let $\omega$ generate $H^0(S, \Omega^1_{S/k})$. By Proposition 0.2.7 it vanishes at a finite set of points which is the support of a 0-cycle $Z$ of degree 12 = $c_2(\Omega^1_{S/k})$. Let $i : F \hookrightarrow S$ be the closed embedding of a fiber or a half-fiber of $f$. The exact sequence of sheaves of differentials

$$0 \to \mathcal{I}_F/\mathcal{I}_F^2 \to i^*(\Omega^1_{S/k}) \to \Omega^1_{F/k} \to 0$$

defines a homomorphism $\mathcal{I}_F/\mathcal{I}_F^2 \to i^*(\Omega^1_{S/k}) \to \Omega^1_{F/k}$.

shows that the restriction map $H^0(S, \Omega^1_{S/k}) \to H^0(S, \Omega^1_{S/k} \otimes_{\mathcal{O}_S} \mathcal{O}_F) \to H^0(F, \Omega^1_{F/k})$ is an isomorphism and hence $i^*(\omega)$ generates $H^0(F, \Omega^1_{F/k})$. If $F$ is smooth, then $\Omega^1_{F/k} \cong \omega_F$ and $i^*(\omega)$ has no zeros. Thus $Z \cap F = \emptyset$. If $F$ is singular, $\Omega^1_{S/k}/\text{Torsion}$ is equal to $\text{adj}_F \otimes \omega_F$, where $\text{adj}_F$ is the adjoint ideal of $F$ generated locally at a point by the partial derivatives $\phi_u, \phi_v$ of the local equation $\phi = 0$ of $F$ at this point. This follows from considering a natural homomorphism $\Omega^1_{F/k} \otimes \mathcal{I}_F/\mathcal{I}_F^2 \to \Lambda^2 i^*(\Omega^1_{S/k})$ that defines a homomorphism

$$\Omega^1_{F/k} \to i^*(\Omega^1_{S/k}) \otimes (\mathcal{I}_F/\mathcal{I}_F^2)^{\vee} \cong \omega_F.$$ 

Its kernel is the torsion subsheaf $\mathcal{T}$ of $\Omega^1_{F/k}$ and its image is equal to $\text{adj}_F \otimes \omega_F$. The image of $i^*(\omega)$ in $H^0(F, \text{adj}_F \otimes \omega_F)$ is equal to zero, hence $i^*(\omega) \in H^0(S, \mathcal{T})$. It is known that $h^0(F, \mathcal{T}) = h^0(S, \mathcal{O}_S/J_F)$, where $J_F$ is the jacobian ideal of $F$ locally generated by $\phi, \partial_u, \partial_v$ [709]. Since singular points of $F$ satisfy $\phi \subset (\partial_u, \partial_v)$, the jacobian ideal coincides with the adjoint ideal, we obtain that $h^0(\mathcal{O}_{Z \cap F}) = h^0(F, \mathcal{T})$. This implies that $i^*\mathcal{I}_Z$ is equal to the adjoint ideal of $F$.

**Corollary 10.2.7.** Let $F$ be a fiber or a half-fiber of an elliptic fibration on $S$ over a point $t$. Then $X$ has a singular point over each singular point $x$ of $F$ and the sum of indices of these singular points is equal to $\nu_1(h)$, where $h$ is the discriminant of the jacobian fibration of $f$. In particular, if $F$ is of multiplicative type with $s$ irreducible components, then $X$ has $s$ singular points of type $A_1$.

**Proof.** It follows from the previous Proposition that the length of $F \cap (\omega)_0$ is equal to the sum of Milnor numbers of singular points of $F$. It is known that this number is equal to the total number of vanishing cycles of $f$ that coincides with the contribution $c(S) + \delta_1$ of $F$ in the formula (4.1.12) for the Euler-Poincaré characteristic of $S$ (see [150]). It remains to use that, by Corollary 4.4.10, this local contribution is equal to the order $\nu_2(h)$ of the discriminant $h$ of the jacobian fibration.

**Example 10.2.8.** We can use the classification of rational jacobian elliptic surfaces in characteristic 2 from [425] to determine possible singularities of the canonical cover of $S$. For example, the Weierstrass equation $y^2 + t^3y + x^3 + c_1tx^2 + c_2tx + tc_3 = 0$ defines a fibration with one singular fiber of type $A_1^3$. Since the $j$-invariant of this fibration is equal to 0, any Enriques torsor $S$ must be an $\alpha_2$-surface. We see that the canonical cover $X$ of $S$ has a singular point of type $D_{12}$ or $E_{12}$. We do not know whether the latter case can be excluded. Another example of a rational elliptic surface with one singular fiber (of type $A_0^{*2}$) is given by the Weierstrass equation $y^2 + ty + x^3 + c_1tx^2 + c_2tx + tc_3 = 0$. 


Lemma 10.2.9. Let $R$ be a $(-2)$-curve on $S$. Then $R$ contains two canonical points $x_1, x_2$ and its proper transform on $Y$ is a $(-2)$-curve that intersects one of the irreducible components of the exceptional curve over each point $x_1, x_2$ with multiplicity 1.

Proof. We use the intersection theory on normal algebraic surfaces [514]. Let $\phi : V' \to V$ be a minimal resolution of singularities of such a surface and $\tilde{C}$ the full pre-image of an irreducible curve $C$ on $V$. We can write it as $\tilde{C} = \bar{C} + \sum_{i \in I} m_i E_i$, where $(E_i)_{i \in I}$ is the set of irreducible components of the exceptional curve of $\phi$. Since $C \sim D$ for some Weil divisor on $V$ that does not contain singular points in its support, we have $E_i \cdot \bar{C} = 0$ for all $i \in I$. Since the matrix $A = (E_i \cdot E_j)$ is negative definite, we can solve for the coefficients $m_i$ in $\mathbb{Q}$ in terms of the integer numbers $\bar{C} \cdot E_i$ and compute $\tilde{C}^2$. By definition $C^2 = \bar{C}^2$. In the same way one defines the intersection numbers of any two Weil divisors, check that it depends only on its linear equivalence class and prove that the intersection theory satisfies the usual property for finite maps $f : V' \to V$ of degree $n$, i.e. $f^*(D) \cdot f^*(D') = n(D \cdot D')$. Let us now apply this to the case when $V = X$ and $V' = Y$ with $\phi = p : Y \to X$. Let $R$ be a $(-2)$-curve on $S$. Since the restriction of the principal cover $\pi : X \to S$ to $R$ is trivial, $\pi^*(R) = 2C$ for some curve $C$ on $X$. The intersection theory gives $C^2 = -1$, hence

$$p^*(C)^2 = C^2 = -1 = p^*(C) \cdot \bar{C} = \tilde{C}^2 + \sum_{i \in I} m_i \bar{C} \cdot E_i.$$  

This implies that $\tilde{C}^2$ must be negative, and since it is an irreducible curve, we get $\tilde{C}^2 = -2$, hence

$$\sum_{i \in I} m_i \bar{C} \cdot E_i = 1. \quad (10.2.1)$$

Since our singularities are double rational points, the inverse of the intersection matrix $A = (E_i \cdot E_j)$ has non-negative entries. Also, the classification of singularities on $X$ from Theorem 10.2.2 shows that these entries belong to $\frac{1}{2}\mathbb{Z}$. It follows that $m_i \geq 0$ and $m_i \in \frac{1}{2}\mathbb{Z}$. Thus the only solutions of (10.2.1) are

- $\# I = 2, m_1 = m_2 = \frac{1}{2}, \bar{C} \cdot E_1 = \bar{C} \cdot E_2 = 1$,

- $\# I = 1, m_1 = \frac{1}{2}, \bar{C} \cdot E_1 = 2$.

In the latter case the image of $\tilde{C}$ on $X$ is a singular curve that cannot be equal to $\pi^*(R)_{\text{red}} \cong R$. So, only the first equality holds that implies that $\tilde{C}$ intersects two irreducible exceptional components with multiplicity 1. If these components are blown down to one point, the image $R$ of $\tilde{V}$ is a singular curve. Thus the image of $\tilde{C}$ contains two canonical points. This proves the assertion. \hfill $\Box$

It follows from the previous Lemma that if a $(-2)$-curve contains a canonical point of type $D_{12}$, then its proper transform on $Y$ intersects one of the components of multiplicity 2. But then its image
cannot be a smooth rational curve. Hence $S$ is an unnodal Enriques surface in this case. The same is true if the singularity is of type $E_{12}$.

Let $\mathcal{M}_{\text{En},\text{CV}}$ be the stack of Cossec-Verra polarized Enriques surface. Recall from Section 5.11 that it is a smooth quasi-separated Deligne-Mumford stack over $\mathbb{k}$. It consists of two irreducible components $\mathcal{M}_{\text{En},\text{CV}}^{\mu_2}$ and $\mathcal{M}_{\text{En},\text{CV}}^{\alpha_2}$ both of which are smooth algebraic stacks of dimension 10. They intersect transversally along the 9-dimensional smooth stack $\mathcal{M}_{\text{En},\text{CV}}^{\alpha_2}$ of $\alpha_2$-surfaces. The geometric points of the complement $\mathcal{M}_{\text{En},\text{CV}}^{\mu_2} \setminus \mathcal{M}_{\text{En},\text{CV}}^{\alpha_2}$ classifies classical Enriques surfaces.

Let $N = E_{10}(2) \oplus M_{12}$ be the Picard lattice of supersingular K3 surfaces with Artin invariant $\sigma = 10$. Let $N_{0} = 2N^\vee / 2N \cong D(N) \cong \mathbb{F}_2^{10}$. Let $\Omega_N$ be the subvariety of Grassmann variety $G(10, N_0 \otimes \mathbb{k})$ that parameterizes totally isotropic subspaces $L$ of dimension 10 such that $\dim L \cap F(L) = 9$. It consists of two irreducible components of dimension 9.

Let $c_1 : \text{NS}(X) \to H^2(X/W)$ be the first Chern class homomorphism with values in crystalline cohomology. We proved in the previous section (10.1.1) that it is injective and its composition with the reduction modulo $p$ defines an injective map

$$\bar{c}_1 : \text{NS}(X)/p \text{NS}(X) \to H^2_{\text{DR}}(X/\mathbb{k}).$$

The following theorem is stated in [209, Theorem 6.12] (with reference to the Global Torelli Theorem for supersingular K3 surfaces whose proof in the case $p = 2$ is still not published).

**Theorem 10.2.10.** There is an algebraic space $K_N$ which is a fine moduli space of $N$ lattice polarized supersingular K3 surfaces. The period map $K_N \to \Omega_N$ is étale and surjective of degree one. It is an isomorphism on the open subspace $K_0$ of ample polarized surfaces and its complement is a divisor. Let $K_M \to K_N$ be the universal family and $K_0$ its pre-image over $K_0$. Then there is a contraction $K_0 \to X_N$ over $K_0$ such that each fiber $(X_N)_t$ over a geometric point is a K3 surface with rational double points of total index 12 such that $\text{Exc}((K_N)_t)$ is contained in the orthogonal complement of the image of $E_{10}$ in $\text{NS}((K_N)_t)$.

### 10.3 Quotients of a supersingular K3 surface by a vector field

We keep notations $p : Y \to X, \pi : X \to S$ from the previous section. In this section we always assume that the characteristic of the ground field $\mathbb{k}$ is equal to 2.

**Proposition 10.3.1.** Assume that $X$ is a RDP-K3 surface with Zariski singularities. Then the tangent sheaf $\Theta_X$ is free of rank 2.

**Proof.** First we show it is locally free. So, let $A$ be the local ring at a singular point of $X$. We know that it is locally isomorphic to $B/(f)$, where $f = z^2 + g(x, y) = 0$ and $B = \mathbb{k}[[x, y, z]]$. The standard exact sequence of modules of differentials gives an exact sequence

$$(f)/(f^2) \to \Omega^1_{B/\mathbb{k}} \otimes_B A \to \Omega^1_{A/\mathbb{k}} \to 0.$$

Passing to the duals, we get an exact sequence

$$0 \to \Theta_A/ \mathbb{k} \to A^3 \to A \to \text{Ext}^1_A(\Omega^1_{A/\mathbb{k}}, A) \to 0.$$
The assertion will follow if we show that the $A$-module $T^1_A := \text{Ext}^1_A(\Omega^1_{A/k}, A)$ is of finite projective dimension (it will be automatically less than or equal to 2). It follows from the exact sequence that $T^1_A/k = B/J$, where $J = (g_x, g_y, f)$. Since the singular point is isolated, it is a module of finite length (called the Tyurina number). We know that $A_0 = k[x, y] = A^G$, where $G = \mu_2$ or $\alpha_2$ that makes $A$ a flat $A_0$-module. Let $J_0 = (g_x, g_y)$. Obviously $J_0A = J$, and the projective dimension of $B$-module $A/J = A/J_0A = (A_0/J_0A_0) \otimes_{A_0} A$ coincides with the projective dimension of $A_0$-module $A_0/J_0$. By Auslander-Buchsbaum formula, it is equal to $\dim B - \text{depth}(A_0/J_0) \leq 2$. 

Note that by [209, Corollary 7.3] or [474, Theorem 1.4] all vector fields $\partial \in H^0(X, \Theta_X)$ are 2-closed.

**Theorem 10.3.2.** Let $X$ be the canonical cover of an Enriques surface $S$ which is a RDP-K3 surface. For any section of $\Theta_X$ that does not vanish at singular points, the quotient $X^\partial$ is an Enriques surface whose canonical cover is isomorphic to $X$.

**Proof.** Let $j : X' = X \setminus \text{Sing}(X) \hookrightarrow X$ and let $S'$ be the image of this set in $S$. The cover $Y' \to X'$ is a principal cover of smooth varieties. Applying (0.3.2), we get an exact sequence

$$0 \to \Theta_{X'/k} \to \Theta_X/k.$$ 

Since the kernel of $\pi^*(\Theta_{S'/k}) \to \Theta_{X'}$ is locally free and the determinant of $\pi^*(\Theta_{S'/k})$ is equal to $\pi^*\omega_{X'} \cong \Theta_{X'}$, we obtain an exact sequence

$$0 \to \Theta_{X'/k} \to \Theta_{X'}/k \to \Theta_{X'} \to 0.$$ 

Since $\Theta_{X'/k}$ is the restriction of a locally free sheaf $\Theta_{X/k}$ on a Cohen-Macaulay scheme $X$, we have $\Theta_{X/k} = j_\ast \Theta_{X'/k} = \Theta_X$ and $j_\ast \Omega_{X'} = \Omega_X$, $j_\ast \Omega_{X'} = 0$. Thus applying $j_\ast$ to the exact sequence, we get an exact sequence

$$0 \to \Theta_X \to \Theta_X/k \to \Theta_X \to 0.$$ 

It remains to show that it splits. But, $\text{Ext}^1(\Theta_X, \Omega_X) = H^1(X, \Theta_X) = 0$ because $X$ is K3-like.

Now we see that

$$H^0(X, \Theta_{X/k}) \cong k^2.$$ 

Since $S = X/G$, where $G$ is a group scheme of order 2, $S$ is isomorphic to the quotient of $X$ by a regular global vector field $\partial$. It is of additive (resp. multiplicative) type, i.e. $\partial^2 = 0$ (resp. $\partial^2 = \lambda \partial, \lambda \neq 0$) if and if $G \cong \alpha_2$ (resp. $\mu_2$). The surface $X^\partial$ is an Enriques surface if and only if the scheme of fixed points of $G$ is empty, or, equivalently, the projection $X \to X^\partial$ is a non-trivial principal $G$-cover.

**Proposition 10.3.3.** Suppose $\partial \in H^0(X, \Theta_X)$ is of multiplicative type and vanishes at $k$ singular points $p_1, \ldots, p_k$ of type $A_1$. Then the quotient $X^\partial$ is a rational surface whose minimal resolution is a Coble surface with $k$ boundaries.

**Proof.** Let $y_i$ be the image of the singular point $p_i$ on $X^\partial$. We may assume that $p_i$ is formally isomorphic to a singular point $xy + z^2 = 0$. Then the derivations $x \partial_x + y \partial_y + \partial_z$ form a basis
of the module of derivations [612, Proposition 2.3]. We lift these derivations to the derivations of \( k[[x, y, z]] \) that leave the ideal \((xy + z^2)\) invariant. According to Proposition 0.3.10, the subring \( k[[x, y, z]]^{\partial} \) is equal to the Veronese ring \( k[[x^2, y^2, z^2, xy, xz, yz]] \). It follows that \( \text{Spec } R^\partial \) is formally isomorphic to the singular point of the vertex of the affine cone over a hyperplane section of the Veronese surface \( \nu_2(\mathbb{P}^2) \subset \mathbb{P}^5 \). The exceptional curve of its minimal resolution of singularities is a smooth rational curve \( C_i \) with self-intersection \(-4\). Let \( V \) be the minimal resolution of \( X^\partial \).

The composition map \( Y \to X \to X^\partial \) extends to a map \( \phi : Y \to V \) equal to the projection map \( Y \to Y^D \), where \( D \) is a rational derivation of \( Y \) with divisor of zeroes equal to the union of exceptional curves over the zeros of \( \partial \). The formula for the canonical class of purely inseparable covers shows that \( 2K_V + C_1 + \cdots + C_k \sim 0 \). This shows that \( V \) is a Coble surface with \( k \) boundary components.

\[ \square \]

**Remark 10.3.4.** Note that any derivation \( \partial \in H^0(X, \Theta_X) \) of additive type does not vanish at isolated fixed points of type \( A_1 \) and hence the quotient \( X^\partial \) is an Enriques surface [209, Lemma 7.5].

This explains the fact that isomorphism classes of Coble surfaces lie in the closure of the moduli space of classical Enriques surfaces.

We have already used [612, Proposition 2.3] for the description of the module of differentials of the ring \( k[[x, y, z]]/(z^2 + xy) \). In fact, the description holds for any Zariski singularities defined by the ring \( k[[x, y, z]]/(z^2 + g(x, y)) \). According to Proposition 2.4 from loc.cit., the module of derivations has a basis formed by \( \partial \) and \( x \partial f_y + y \partial x \).

**Proposition 10.3.5.** Let \( \partial = u \partial_z + v(x \partial f_y^\prime + y \partial f_y^\prime) \) be a 2-closed derivation of \( R = k[[x, y, z]]/(z^2 + f(x, y)) \). Then the corresponding action of \( G \cong \mu_2 \) or \( \alpha_2 \) is free and \( R^\partial \) is a regular ring.

**Corollary 10.3.6.** Let \( X \) be the canonical cover of an Enriques surface \( S \) birationally isomorphic to a K3 surface. Then there exists a line \( \ell_{\text{sing}} \subset H^0(X, \Theta_X) \) such that \( \partial \in H^0(X, \Theta_X) \) defines a free action of the group scheme \( G \) if and only if it does not belong to \( \ell_{\text{sing}} \).

\[ \square \]

**Proof.** For any singular point \( x \) of \( X \), let \( r_x : H^0(X, \Theta_X) \to \text{Der}_k(\mathcal{O}_{X,x}) \cong \text{Der}_k k[[x, y, z]]/(z^2 + f(x, y)) \) be the restriction homomorphism. We define \( \ell_{\text{sing}} \) to be the linear span of the preimage of \( \partial \). One checks that this is well-defined and line does not depend on the choice of a singular point \( x \) (see the proof of [612, Proposition 2.5]).

Another line in \( H^0(X, \Theta_X) \) controls whether the quotient Enriques surface is classical or not. We refer for the proof of the following theorem to [474, Theorem 1.4]

**Theorem 10.3.7.** Let \( X \to S \) be a canonical cover of a simply-connected Enriques surface which is a birationally isomorphic to a K3 surface. Let \( \mathfrak{g} = H^0(X, \Theta_X) \). Assume that the singular points of \( X \) are either \( 12A_1 \), or \( 8A_1 + D_4^{(0)} \), or \( 6A_1 + D_4^{(0)} \), or \( 5A_1 + E_7^{(0)} \). Then the subset \( \ell_{\text{add}} \) of \( \partial \in \mathfrak{g} \) with \( \partial^2 = 0 \) is a line. In the remaining cases it coincides with the whole \( \mathfrak{g} \). Moreover,

(1) If all singular points are of type \( A_1 \), then \( \ell_{\text{add}} \neq \ell_{\text{sing}} \) and \( \partial \in \ell_{\text{add}} \) if and only if \( X^\partial \) is an \( \alpha_2 \)-Enriques surface.

(2) If the singular points are of other types and \( \ell_{\text{add}} \) is a line, then it coincides with the line \( \ell_{\text{sing}} \).
(3) If \( \ell \text{add} = g \), then all quotients \( X^\vartheta \) for \( \vartheta \notin \ell \text{sing} \) are \( \alpha_2 \)-Enriques surfaces.

If \( X \) is a normal rational surface, then it has a unique minimal elliptic double point of type \( E_{12} \) and all quotients \( X^\vartheta \) are \( \alpha_2 \)-Enriques surfaces.

Recall from Section 5.11 that we have a stack \( \mathcal{E}_{\text{uni}} : \text{Sch}/\mathbb{F}_2 \to \text{Groupoids} \) that assigns to each scheme \( B \) over \( \mathbb{F}_2 \) morphisms of algebraic spaces \( (S, \phi) \to B \) whose generic fibers are simply connected (=unipotent) Enriques surfaces with a marking \( \phi : E_{10, B} \to \text{Num}(S) \) such that \( \phi_b : E_{10} \to \text{Num}(S_b) \) maps the chamber \( D_0 \) in \( E_{10} \otimes \mathbb{R} \) defined by simple roots \( \alpha_0, \ldots, \alpha_9 \) into the ample cone of \( S_b \). Let \( \mathcal{E}_{\text{uni},K^3} \) be the open and dense substack of \( \mathcal{E}_{\text{uni}} \) of those Enriques surfaces whose canonical cover is birationally K3 surface.

Let \( K_{\text{EnrK3}} \) be the stack of families \( (f : X \to T, \phi) \) of algebraic spaces whose geometric fibers are RDP-K3 surfaces with singular points of total index 12 and trivial tangent bundle together with a marking \( \phi : E_{10}(2) \to \text{Pic}(X) \) such that \( \phi(D_0(2)) \) lies in the ample cone. It follows from Global Torelli Theorem (conditioned on its proof in characteristic 2) that the stack \( K_{\text{EnrK3}} \) has a fine moduli space \( K_{\text{EnrK3}} \) in the category of algebraic spaces which admits a universal \( E_{10}(2) \)-polarized K3 surface \( f : \mathcal{X} \to K_{\text{EnrK3}} \). Let \( \text{PK}_{\text{EnrK3}} = \mathbb{P}(f_* \Theta^\vee_{\mathcal{X}/K_{\text{EnrK3}}}) \) be the projective line bundle over \( K_{\text{EnrK3}} \) whose fibers are projectivized spaces of regular vector fields on the fibers of \( f \).

Suppose \( (X, \phi) \in K_{\text{EnrK3}}(\text{Spec } k) \) and let \( \pi : Y \to X \) be its minimal resolution. Then \( \tilde{\phi} := \pi^* \circ \phi : E_{10}(2) \to \text{NS}(Y) \) defined a lattice \( E_{10} \) polarization. It is obviously non-ample since the exceptional curves of \( \pi \) lie in the orthogonal complement of \( \tilde{\phi}(E_{10}(2)) \). Let us extend the lattice to get an ample polarization. We set \( N = E_{10}(2) \oplus M_{12} \) to be the 2-elementary lattice isomorphic to the Néron-Severi lattice of a supersingular surface with Artin invariant \( \sigma = 10 \). Let \( e_1, \ldots, e_{12} \) be the natural generators of \( M_{12} \) and \( \rho_{12} = \frac{1}{2}(e_1 + \cdots + e_{12}) \). Note that \( \rho \) coincides with the half-sum of positive roots in \( M_{12} \) that coincides with the Weyl vector, i.e. the vector whose inner product with each positive root is equal to \(-1\). We had encountered with such vectors in the Appendix to Chapter V in Volume 1.

The following Lemma follows immediately by applying the Borel-de Siebenthal-Dynkin algorithm to embed the root lattice \( A_{\ell}^{\text{opt}} \) into an irreducible root lattice of type \( A, D, E \). We leave its proof to the reader (see also [209, Lemma 6.5]).

**Lemma 10.3.8.** Let \( M \) be one of the root lattices of type \( D_{2n}, E_7, E_8 \). Then \( M \) contains \( r = \text{rank} \) \( M \) orthogonal positive roots such that their sum is equal to the sum of simple roots taken with coefficients indicated in the following diagrams:

![Diagram](https://via.placeholder.com/150)

**Corollary 10.3.9.** Let \( \text{Exc}(Y)_s, s = 1, \ldots, k, \) be an orthogonal direct summand of \( \text{Exc}(Y) \) gener-
ated by the divisor classes of a connected component of the exceptional curve of \( \pi : Y \to X \). The set of generators \( e_1, \ldots, e_{12} \) of the lattice \( M_{12} \) can be split into disjoint subsets \( I_1, \ldots, I_k \), one for each component \( \text{Exc}(Y)_s \), such that there exists a lattice embedding \( j_s : \bigoplus_{i \in I_s} \mathbb{Z} e_i \cong A_1^{\oplus I_s} \hookrightarrow \text{Exc}(Y)_s \) with the images of \( e_i \) being orthogonal roots in \( \text{Exc}(Y)_s \) whose sum is indicated in the diagrams from the previous Lemma.

Let
\[
j = \bigoplus_{s=1}^k j_k : \bigoplus_{s=1}^k A_1^{\oplus I_s} \cong A_1^{\oplus 12} \hookrightarrow \text{Exc}(Y) = \bigoplus_{s=1}^k \text{Exc}(Y)_s
\]
be the lattice embedding obtained from the previous corollary. Note that \( \sum_{i \in I_s} j_k(e_i) \) intersects each simple root with multiplicity \( \pm 2 \), therefore \( \frac{1}{2} \sum_{i \in I_s} j_k(e_i) \in \text{Exc}(Y)^{\vee}_s \). This defines an embedding
\[
j : M_{12} \hookrightarrow \text{Exc}(Y)^{\vee}.
\]

We know that \( Y \) is a specialization of a supersingular K3 surface \( Y' \) with Artin invariant \( \sigma = 10 \). Thus the lattice \( \text{NS}(Y') \cong \text{E}_{10}(2) \oplus M_{12} \) admits an embedding into the lattice \( \text{NS}(Y) \).

**Definition 10.3.10.** Let \( N = \text{E}_{10}(2) \oplus M_{12} \). A \( N \)-polarization of a supersingular K3 surface \( Y \) is a lattice embedding \( \phi : N \hookrightarrow \text{NS}(Y) \) such that \( \phi(\text{E}_{10}(2)) \) is a primitive sublattice and the restriction of \( \phi \) to \( M_{12} \) coincides with the embedding (10.3.2).

Let \( \phi : N \hookrightarrow \text{NS}(Y) \) be a lattice \( N \) polarization. Fix a root chamber \( \mathcal{C} \) in the positive cone \( V(\mathbb{R}_N)^+ \) of \( N \). The Weyl group \( W(N) \) acts transitively on the set of chambers. We may choose \( \mathcal{D} \) such that the intersection of its closure with \( \text{E}_{10}(2)_{\mathbb{R}} \) contains the fundamental chamber \( \mathcal{D} \) in \( \text{E}_{10} \) spanned by the fundamental weights \( \omega_0, \ldots, \omega_9 \) dual to the simple roots \( \alpha_0, \ldots, \alpha_9 \).

We say that a lattice polarization \( \phi : N \to \text{NS}(Y) \) is **ample** if \( \phi(N) \) meets the ample cone of \( Y \). Following [209, §6], we say that \( (Y, \phi) \) is **good** if \( \text{NS}(Y) \) does not contain a root lying in \( \phi(\text{E}_{10}(2)^{\vee}) \). It is clear that the embedding \( \text{E}_{10}(2) \cong \pi^*(\text{NS}(S)) \hookrightarrow \text{NS}(Y) \) is good since \( \pi^*(\text{NS}(S)) \) is equal to the orthogonal complement of \( \text{Exc}(Y) \).

Suppose \( r \) is the class of a \( (-2) \)-curve on \( Y \). Since \( \text{NS}(Y) \subset \phi(N)^{\vee} \), we can write \( r = r_1 - r_2 \), where \( r_1 \in \phi(\text{E}_{12}(2)^{\vee}) \cong \text{E}_{12}(\frac{1}{2}) \) and \( r_2 \in M_{12} \cong \text{E}_{12}(\mathbb{Z}) \). Since \( r_1^2 = 2 \), we have \( r_1^2 = r_2^2 = -1 \) or \( r_1^2 = -2 \) and \( r_2 = 0 \). So, if \( (Y, \phi) \) is good, the latter case does not occur. Suppose \( Y \) is realized as a minimal resolution of a canonical cover \( \pi : X \to S \) and we take \( N \)-polarization such that the restriction of \( \phi \) to \( \text{E}_{10}(2) \) is equal to the composition \( \pi^* \circ j \), where \( j : \text{E}_{10} \to \text{Num}(S) \) is an ample \( \text{E}_{10} \)-lattice polarization of \( S \). Assume that \( r \not\in \text{Exc}(Y) \). Then \( 2r_1 = \pi^*(R) \), where \( R \) is a \( (-2) \)-curve on \( S \), so that \( \pi^*(R) = 2r + 2r_2 \). It follows from Lemma 10.2.9 that \( 2r_2 \) is the sum of the classes of two disjoint irreducible components of the minimal resolution. If we choose a vector \( \eta \) in \( \text{E}_{10} \) such that \( j(\eta) \) is an ample class, we obtain \( \phi(\eta) = \pi^*(j(\eta)) \) intersects positively \( r \), hence \( \phi(N) \) meets the ample cone of \( Y \). The converse is also true since \( \pi^*(\text{Pic}(S)) = p^*(\text{Pic}(X)) \) is equal to the orthogonal complement of \( \text{Exc}(Y) \) and hence it coincides with its primitive closure.

This proves the following.

**Proposition 10.3.11.** Let \( \phi : N \hookrightarrow \text{NS}(Y) \) be a lattice \( N \) polarization of the canonical cover birationally isomorphic to a K3 surface \( Y \) of a simply-connected Enriques surface in characteristic 2. Then the polarization is ample.
CHAPTER 10. SUPERSINGULAR K3 SURFACES AND ENRIQUES SURFACES

Let $\mathcal{E}_{\text{uni}}$ be the moduli stack of simply connected (= unipotent) Enriques marked surfaces $(S, j)$ such that the image of the cone $D_0 \subset (E_{10})_\mathbb{R}$ spanned by the fundamental weights $\omega_0, \ldots, \omega_9$ lies in the ample cone. We introduced this moduli space in the last section of Chapter 5. The image of the fundamental weight $\omega_1$ of square-norm 4 corresponds to a Cossec-Verra polarization. There is a forgetting map

$$\mathcal{E}_{\text{uni}} \to \mathcal{M}_{\text{Enr, CV}}$$

to the Deligne-Mumford stack of Enriques surfaces with Cossec-Verra polarization (not necessary ample). Its image lies in the closure of the component $\mathcal{M}_{\text{Enr, CV}}^!$ parameterizing classical Enriques surfaces.

The next proposition complements Theorem 10.2.10 whose notations we will keep and its proof can be found in [209, Theorem 6.1.2 (5)].

**Proposition 10.3.12.** There are invertible sheaves $\mathcal{L}_0, \ldots, \mathcal{L}_9$ in $\text{Pic}(X_N)$ that are equal to the images of $\omega_0, \ldots, \omega_9$ under the polarization map.

Let $E_{\text{uni}}^0$ be substack of $E_{\text{uni}}$ parameterizing simply connected Enriques surfaces whose canonical cover is a RDP K3-surface. Let $E_{\text{uni}}^0 \to K_N$ be the map of stacks defined by taking the minimal resolution of the canonical cover. We know that its image lies in $K_0^N$. Also we know that the lattice $N$ polarization on the images is good. By [209, Lemma 6.11], the complement is a divisor.

Let $\Theta_{X_N/K_N}$ be the relative tangent sheaf. Let $K_{00}^0$ be the open subset of $K_0^N$ where this sheaf is trivial. By taking the canonical cover we get a map $E_{\text{uni}}^0 \to K_{00}^0$ whose images are families of RDP K3-surfaces with trivial tangent bundle. Let $\mathbb{P}(\Theta_{K_{00}^0/K_N}) \cong K_{00}^0 \times \mathbb{P}^1$ be the projectivization of the relative tangent sheaf and $\mathbb{P}(\Theta_{K_{00}^0/K_N}^0)$ be the open subset whose fibers correspond to free derivations. By taking the corresponding free $\mu_2$ or $\alpha_2$ action, we can reconstruct the Enriques surface. This defines an isomorphism of moduli functors

$$E_{\text{uni}}^0 \to \mathbb{P}(\Theta_{K_{00}^0/K_N}^0).$$

The composition of this map with the projection to $K_{00}^0$ defines a map

$$\Phi : E_{\text{uni}}^0 \to K_{00}^N.$$

By Corollary 10.3.6, its fibers are isomorphic to affine lines.

Let $K_{0,N,R}^0$ be the subset of $K_0^N$ such the sublattice $\text{Exc}((K_N)_t)$ of the Néron-Severi lattice of the fibers generated by the exceptional curves is isomorphic to one of the lattices $\mathcal{R}$ from Theorem 10.2.2. For example, $K_{N,12A_1}$ parameterizes fibers of $X_N \to K_N^0$ with 12 ordinary nodes. Let $E_{\text{uni},R} \subset E_{\text{uni}}$ be its pre-image under $\Phi$.

Let $E_{\text{uni},\alpha_2}$ be the locus of $\alpha_2$-Enriques surfaces. Applying Theorem 10.3.7, we see that

- $\Phi : E_{\text{uni},12A_1} \to K_{N,12A_1} \times \mathbb{A}^1$ is bijective.
- $\Phi(E_{\text{uni},12A_1} \cap E_{\text{uni},\alpha_2})$ is a section of $K_{N,12A_1} \times \mathbb{A}^1 \to K_{N,12A_1}$.
- If $\mathcal{R} \cong \mathbb{A}^8_1 \oplus D_4$, or $\mathbb{A}^5_1 + E_7$, or $\mathbb{A}^6_1 + D_6$, then $E_{\text{uni},R} \cap E_{\text{uni},\alpha_2} = \emptyset$. 
10.4. THE CREMONA-RICHMOND POLYTOPE

• If \( R \) is one of the remaining lattices then \( E_{\text{uni}, R} \subset E_{\text{uni}, \alpha_2} \) and \( \Phi : E_{\text{uni}, R} \rightarrow K_{N, \mathbb{R}} \times \mathbb{A}^1 \) is bijective.

Let \((Y, \phi)\) be a lattice \( N \) polarization of a supersingular K3 surface \( Y \) such that the embedding \( M_{12} \hookrightarrow \phi(\text{E}_{10}(2))^\perp \) is a natural isomorphism if all singular points of the corresponding RDP K3-surface \( X \) are of type \( A_1 \) or else given by one of the diagrams from Lemma 10.3.8.

The following Lemma is proven in [209, Lemma 6.3].

**Lemma 10.3.13.** Suppose there exists a 2-closed vector field \( \partial \) of \( X \) that does not vanish at a singular point whose exceptional curve defines an irreducible root lattice of type \( A_1, D_n, E_n \). Then there exists a rational derivation \( D \) of \( Y \) with the divisor of poles equal by the corresponding diagram from Lemma 10.3.8.

### 10.4 The Cremona-Richmond polytope

In this section we recall one of the most fascinating objects in classical algebraic geometry: the Cremona-Richmond abstract symmetric configuration \((15_3)\) and its various geometric realizations. We will show that it defines a convex polytope of finite volume with 40 facets in the hyperbolilc space \( \mathbb{H}^9 \) associated with \( (\text{E}_{10})_R \). It will appear several times in our constructions of Enriques surfaces as quotients of a supersingular K3 surface with Artin invariant 1 in characteristic 2 which we will study in the next section.

An **abstract configuration** is a triple \( \{A, B, R\} \), where \( A, B \) are non-empty finite sets and \( R \subset A \times B \) is a relation such that the cardinality of the set \( R(a) = \{b \in B : (a, b) \in R\} \) (resp. the set \( R(b) = \{a \in A : (a, b) \in R\} \) does not depend on \( a \in A \) (resp. \( b \in B \)). Elements of \( A \) are called **points**, elements of \( B \) are called **blocks**. If \( a \in R(b) \), we say that \( a \) belongs to \( b \). If

\[
u = \#A, \quad v = \#B, \quad r = \#R(a), \quad s = \#R(b),
\]

then a configuration is said to be an \((v, r)\)-configuration. A **symmetric configuration** is a configuration with \( \#A = \#B \), and hence \( r = s \). It is said to be a \((v, s)\)-configuration.

Replacing the relation \( R \subset A \times B \) with the dual relation \( R^* \subset B \times A \) we obtain the definition of the **dual abstract configuration** of type \((v_s, u_r)\).

Note that we consider only a special case of the notion of an abstract configuration studied in combinatorics. Ours are **tactical configurations** [304].

Each abstract configuration \((u_r, v_s)\) defines a bipartite graph whose vertices is the union of the sets \( A \) and \( B \) where each point \( a \) from \( A \) is joined by an edge to a block \( b \) from \( B \) if \((a, b) \in R\). This graph is called the **Levi graph** of the configuration.

Let \([1, 6] = \{1, 2, 3, 4, 5, 6\}\). A subset of two elements is a **duad**, a partition of \([1, 6]\) in three subsets is a **syntheme**. A **total** is a set of 5 synthemes that contains all duads. There are 6 totals with a bijection to \([1, 6]\) such that a duad \((ab)\) corresponds to a common syntheme in the corresponding totals. If one views a duad as a transposition in the permutation group \( \mathfrak{S}_6 \) and a syntheme as the products of three commuting transpositions, then the map from the set of duads to the set of
synthemes defined by 5 totals defines an outer automorphism of the group $S_6$. It is known that the quotient of the group of outer automorphisms by the group of inner automorphisms is a cyclic group of order 2. Thus the group $S_6$ acts transitively on the set of sixtuples of totals, each can be identified with an outer automorphism.

Following [340], we choose an outer automorphism $\iota \in \text{Out}(S_6)$ satisfying the property

$$\iota((ab)) = (ij, kl, mn) \leftrightarrow (ab) = \iota((ij)) \cap \iota((kl)) \cap \iota((mn)).$$  

It gives the following bijection between duads and synthemes:

<table>
<thead>
<tr>
<th>(12)</th>
<th>(15,26,34)</th>
<th>(23)</th>
<th>(16,23,45)</th>
<th>(35)</th>
<th>(14,26,35)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(13)</td>
<td>(13,25,46)</td>
<td>(24)</td>
<td>(14,25,36)</td>
<td>(36)</td>
<td>(15,24,36)</td>
</tr>
<tr>
<td>(14)</td>
<td>(16,24,35)</td>
<td>(25)</td>
<td>(13,24,56)</td>
<td>(45)</td>
<td>(15,23,46)</td>
</tr>
<tr>
<td>(15)</td>
<td>(12,36,45)</td>
<td>(26)</td>
<td>(12,35,46)</td>
<td>(46)</td>
<td>(13,26,45)</td>
</tr>
<tr>
<td>(16)</td>
<td>(14,23,56)</td>
<td>(34)</td>
<td>(12,34,56)</td>
<td>(56)</td>
<td>(16,25,34)</td>
</tr>
</tbody>
</table>

Table 10.1: Outer bijection between duads and synthemes

To reconstruct a total from this Table we collect all synthemes which correspond to five duads sharing a common number from $[1, 6]$. The six totals obtained in this way are the totals $(R, Q, Q', P, P', T')$ in the list of totals from [43, Vol. II, Note II, p. 221] and the totals $(T_2, T_3, T_4, T_5, T_1, T_6)$ in the list of totals from [180], 9.4.3)

It is clear that the sets of 15 duads and 15 synthemes is an abstract configuration of type $(15_3)$. It is called, by the reason which will become clear later, the Cremona-Richmond configuration.

The Levi graph of the Cremona-Richmond configuration is known as the Tutte-Coxeter graph.

Note that there are many non-isomorphic abstract configurations of type $(15_3)$, ours is of symmetry type 1, i.e. its points and blocks represent one orbit with respect to the symmetry group of the configuration (see [272]).

The set $\mathcal{D}$ of duads together with the empty set can be naturally endowed with the structure of a symplectic linear space over $\mathbb{F}_2$. In fact, the set of subsets of $[1, 6]$ can be identified with the set $2^{[1,6]}$ with the addition law defined by the symmetric sum. The subspace of sets of even cardinality modulo the one dimensional subspace spanned by the set $[1, 6]$ is isomorphic to $\mathbb{F}_2^4$. Its elements can be identified with subsets of even cardinality modulo taking the complementary subset. Choosing a
representative of cardinality 2 or 0, we obtain a structure of a linear space on \( \mathfrak{D} = \mathfrak{D} \cup \emptyset \) isomorphic to the symplectic space \( J_2 = \mathbb{P}^4_2 \). The symplectic structure can be defined by
\[
\langle ab, cd \rangle = \# \{a, \emptyset \} \cap \{c, \emptyset \} \mod 2.
\]
In this way \( \mathfrak{D} \) is identified with non-zero elements of \( \bar{\mathfrak{D}} \) and the set \( \mathcal{S} \) of synthemes is identified with the set of isotropic planes with the zero vector deleted. This shows that the Cremona-Richmond \((15_3)\)-configuration is isomorphic to the configuration where \( X \) is the set of non-zero vectors in a symplectic space \( \bar{\mathfrak{D}} \) and \( Y \) is the set of its isotropic planes with the zero vector deleted.

Recall that the group of symplectic automorphisms of \( J_2 \) is denoted by \( \text{Sp}(4, \mathbb{F}_2) \). It is isomorphic to the permutation group \( S_6 \) via the natural action of the latter on the set of maps from \([1, 6]\) to \( \mathbb{F}_2 \). This shows that the Cremona-Richmond configuration \((15_3)\) has the group \( S_6 \) as its group of its symmetries. An outer automorphism of a configuration \((\mathcal{A}, \mathcal{B}, R)\) of type \((15_3)\) is a bijective map \( \alpha : \mathcal{A} \to \mathcal{B} \) such that \((x, y) \in R\) if and only if \((\alpha^{-1}(y), \alpha(x)) \in R\). The bijection between the set of duads and the set of synthemes defined by the outer automorphism of \( S_6 \) is the outer automorphism of the configuration \((\mathfrak{D}, \mathfrak{S})\). Thus the group \( \text{Aut}(S_6) \) is realized as the group generated by the automorphisms and outer automorphisms of the Cremona-Richmond configuration.

We have already seen a geometric realization of the Cremona-Richmond configuration \((15_3)\) over the field of 2 elements. Let us give an example of its self-dual geometric realization over any field.

**Example 10.4.1.** Let \( (p_1, \ldots, p_6) \) be an ordered set of 6 points in \( \mathbb{P}^2(\mathbb{F}_2) \) with no three points are collinear. Recall that the linear system of plane cubic curves passing through the six points define a birational map \( f : \mathbb{P}^2 \to \mathbb{P}^3 \) from the plane to a cubic surface \( S \) in \( \mathbb{P}^3 \). If not all of the points lie on a conic, then the map is an isomorphism from the blow-up \( V \) of the six points. Otherwise, the map blows down the proper transform of the conic to the unique ordinary double point of \( S \).

Let \( \ell_{ab} = \langle p_a, p_b \rangle \) be the line spanned by the points \( p_a, p_b \). Its image under the map \( f \) is a line on \( S \) not passing through the node if \( S \) has a node. Thus, we have 15 lines in \( \mathbb{P}^3 \). This is our set \( \mathcal{A} \). The lines \( \ell_{ab}, \ell_{cd}, \ell_{ef} \) whose index sets form a syntheme define a plane in \( \mathbb{P}^3 \) that cuts \( S \) along the three lines \( \langle ab \rangle, \langle cd \rangle, \langle ef \rangle \). It is called a tritangent plane. The set of these 15 tritangent planes is our set \( \mathcal{B} \). The relation \( R \) is of course the incidence relation. This realization of the configuration \((15_3)\) appears in the study of cubic surfaces by L. Cremona.

Next we give another self-dual geometric realization of a Cremona-Richmond configuration (see [43, Vol. IV, Chapter V]).
Let $W$ be a 6-dimensional linear space over $\mathbb{k}$ with a basis $(u_1, \ldots, u_6)$ and coordinates $(x_1, \ldots, x_6)$. Let $E$ be the hyperplane in $W$ given by the equation $x_1 + \cdots + x_6 = 0$. For any linear subspace $L \subset W$, we have a natural identification of its dual space $L^\vee$ with the quotient space $W^\vee/L^\perp$. In particular, we can identify $E^\vee$ with $W^\vee/[x_1 + \cdots + x_6]$. The projection $W^\vee \to W^\vee/[x_1 + \cdots + x_6]$ can be split by choosing the hyperplane defined by the equation $u_1 + \cdots + u_6 = 0$. We will identify $E^\vee$ with this subspace of $W^\vee$. Let $x_1 = 0, \ldots, x_6 = 0$ be the coordinate hyperplanes in $W$, they cut out $E$ along the hyperplanes

$$H_i : x_i = x_1 + \cdots + x_6 = 0, \quad i = 1, \ldots, 6. \quad (10.4.2)$$

The dual $H_i^\perp$ of $H_i$ in $W^\vee$ is the plane spanned by the unit vector with 1 at the $i$th spot and $(1, 1, \ldots, 1)$. Its images in $E^\vee$ is represented by the line spanned by the vectors

$$P_1 = [−5, 1, 1, 1, 1, 1], \quad \ldots, \quad P_6 = [1, 1, 1, 1, 1, −5].$$

They are called the fundamental points. For any subset $I \subset \{1, 6\}$, let

$$F_I = \text{projective linear span } (\{P_i, i \in I\}) \text{ in } |E^\vee|.$$

Obviously, $\dim F_I = \# I - 1$ and $F_I \subset F_J$ if $I \subset J$. The union of the hyperplanes $F_I, \# I = 4$, is called the fundamental hexahedron. The subsets $F_I, \# I = k + 1$, are its $k$-faces. The 0-faces $F_i = P_i$ are its vertices, 1-dimensional faces are its edges.

The intersection of an edge $F_i$ with the face $F_J$ defined by the complementary subset $I^C$ is a point on $F_J$. It is called, depending on the classical literature, a diagonal point, a cross-point, a Cremona point or a principal point. We choose the first name. We have $15 = \binom{6}{2}$ 1-faces, hence 15 diagonal points. They can be indexed by duplas $(ab)$ in $[1, 6]$, so that we denote them by $P_{ab}$. The coordinates of the point $P_{12}$ are $[−2, −2, 1, 1, 1, 1]$, other points have similar coordinates.

For any syntheme $(ab, cd, ef)$, the hyperplanes $P_{abed}, P_{abef}, P_{cdef}$ contain the diagonal points $P_{ab}, P_{cd}, P_{ef}$, hence they intersect along the line containing these points. These lines are called transversal lines. They can be indexed by synthemes and denoted by $t_{ab,cd,ef}$. There are 15 transversal lines. The equations of the transversal line $t_{ab,cd,ef}$ is

$$u_a - u_b = u_c - u_d = u_e - u_f. \quad (10.4.3)$$

It is obvious that the configuration of diagonal points and transversal lines is a geometric realization of the Cremona-Richmond configuration and it exists over any field.

The dual geometric realization of the Cremona-Richmond configuration in $|E|$ consists of hyperplanes $P_{ab}^\perp$ (diagonal hyperplanes) and planes (transversal planes) $t_{ab,cd,ef}^\perp$. Using the outer bijection between duads and synthemes, we will often interchange duads and synthemes in the notation of diagonal hyperplanes $P_{ab}^\perp$ or the diagonal planes $t_{ab,cd,ef}^\perp$. It follows from (10.4.1) that

$$P_{ij}^\perp \subset t_{ab,cd,ef}^\perp \iff (ij) \in \{(ab), (cd), (ef)\}.$$

This shows that our geometric realization is self-dual.

Two transversal lines $t_{ab,cd,ef}$ and $t_{ij,kl,mn}$ are skew if and only if they do not contain a common diagonal point. In the dual notation this is equivalent to the property that the corresponding duads have a common element. Thus for any $a, b, c$ we have three skew lines $t_{ab}, t_{ac}, t_{bc}$. We also have three skew transversal lines $t_{ef}, t_{de}, t_{ef}$ defined by the complementary set. The union of these six lines has 9 intersection points, all of them are diagonal points. If we take one pair of skew lines from
the first set and one pair of skew lines from the second set, we would have 4 intersection points that span a hyperplane. Each other line intersects this hyperplane at two points, hence is contained in it. This shows that the set of six lines lies in a hyperplane. It is called a cardinal hyperplane and it is denote by $\mathcal{C}(abc)$. Obviously $\mathcal{C}(abc) = \mathcal{C}(def)$, hence we have 10 cardinal hyperplanes. It is easy to check that they are given by equations $u_a + u_b + u_c = u_1 + \cdots + u_6 = 0$.

It is now time to introduce a triad, it is a subset of $[1, 6]$ of cardinality 3, up to the complementary set.

It is an elementary fact that three skew lines in a 3-dimensional projective space lie on a unique nonsingular quadric surface. They belong to one of its rulings. Any line that intersects these three lines must be contained in the quadric. Thus, we see that our set of six transversal lines lie on a unique quadric in the cardinal hyperplane. They are called the cardinal quadric surfaces and denoted by $Q(abc)$.

Let $c(abc)$ be the points in the dual space corresponding to cardinal hyperplanes. They are called cardinal points. Each such point is contained in 6 transversal planes $\Sigma_{ab}, \Sigma_{ac}, \Sigma_{bc}, \Sigma_{ef}, \Sigma_{de}, \Sigma_{df}$. Each transversal plane $\Sigma_{ab}$ contains 4 cardinal points $c(abc), c(abd), c(abe), c(abf)$.

- The sets of transversal lines and cardinal hyperplanes is a geometric realization of an abstract configuration $(10, 6, 15, 4)$. The sets of cardinal points and transversal planes is a geometric realization of the dual abstract configuration.

A cardinal hyperplane $\mathcal{C}(abc)$ contains 9 diagonal points $P_{ij}$, where $i \in \{a, b, c\}, j \in \{d, e, f\}$. We can put them in the matrix

$$
\begin{pmatrix}
P_{ad} & P_{ae} & P_{af} \\
P_{bd} & P_{be} & P_{bf} \\
P_{cd} & P_{ce} & P_{cf}
\end{pmatrix}
$$

(10.4.4)

The triples of entries defining one of the six terms in the determinant define six transversal lines contained in the hyperplane.

Another way to code this is by the incidence graph of 9 points and 6 lines that looks as in Figure 10.3 below.

![Figure 10.3: Cardinal hyperplane $\mathcal{C}(abc)$](image)

Here the vertical and horizontal lines are the six transversal lines.

The incidence graph of the remaining 6 diagonal points and 9 transversal lines is a bipartite graph connecting any vertex from the set $\{P_{ab}, P_{ac}, P_{bc}\}$ to all vertices from the set $\{P_{ef}, P_{df}, P_{dg}\}$. 
One of the most frequently modern references to the Cremona-Richmond configuration appears in the following context.

The Segre cubic is a hypersurface $S_3$ of degree 3 in $|E|$ given by equations

$$x_1 + \cdots + x_6 = x_3^3 + \cdots + x_6^3 = 0.$$  \hspace{1cm} (10.4.5)

For any syntheme $(ab, cd, ef)$, let $(ab)(cd)(ef)$ denote the product of the three minors formed by the columns indexed by $(a, b), (c, d)$ and $(e, f)$ of the matrix

$$
\begin{pmatrix}
  t_1^{(1)} & t_1^{(2)} & t_1^{(3)} & t_1^{(4)} & t_1^{(5)} & t_1^{(6)} \\
  t_1^{(1)} & t_1^{(2)} & t_1^{(3)} & t_1^{(4)} & t_1^{(5)} & t_1^{(6)} \\
  & & & & & \\
  & & & & & \\
  & & & & & \\
  & & & & & \\
  & & & & & \\
  & & & & & \\
  & & & & & \\
  & & & & & \\
  & & & & & \\
  & & & & & \\

g\end{pmatrix}
$$

whose entries are independent variables. Given 6 points $p_1, \ldots, p_6$ in projective line $\mathbb{P}^1$ with coordinates $[a_0^{(1)}, a_1^{(1)}]$ and so on, we can take the value of $(ab)(cd)(ef)$ at $(p_1, \ldots, p_6) \in (\mathbb{P}^1)^6$ by plugging in the coordinates in the variables $(t_1^{(1)}, t_1^{(2)})$ and taking the product of the minors. For any total $T_i$ of 5 synthemes we define the function $U_i$ as the sum of the products $(ab)(cd)(ef)$ corresponding to synthemes entering in the total. Although the values if $F_i$ depend on a choice of representatives of the coordinates of points $p_i$, the rational map

$$J : (\mathbb{P}^1)^6 \rightarrow \mathbb{P}^5, (p_1, \ldots, p_6) \rightarrow [U_1(p_1, \ldots, p_6), \ldots, U_6(p_1, \ldots, p_6)]$$

(called the Joubert map) is well-defined. One can show (see [180], 9.4.4) that the function $U_i$, called the Joubert functions, satisfy the relations $\sum U_i = \sum U_i^3 = 0$, hence the image of the map $J$ is contained in the Segre cubic $S_3$. The map is obviously invariant with respect to the diagonal action of the group $SL_2$ on the source. One can show that $J$ defines an isomorphism

$$P^6_1 \rightarrow S_3,$$

where $P^6_1$ is the GIT-quotient of $(\mathbb{P}^1)^6$ by the diagonal action of $SL_2$ (with respect to the democratic linearization) (see [180], Theorem 9.4.10). If one considers the projective representation of $\mathfrak{S}_6$ associated to the irreducible linear representation on $E^\vee$ defined by the partition $(1, 5)$ of $[1, 6]$, then the Joubert map $J$ is $\mathfrak{S}_6$-equivariant.

The Segre cubic $S_3$ contains ten singular points. They represent one $\mathfrak{S}_6$-orbit of the point with coordinates $[1, 1, 1, -1, -1, -1]$. They are naturally indexed by triads $(i, j, k, l, m, n)$. We easily observe the following fact.

- The ten singular points of the Segre cubic $S_3$ are cardinal points. The fifteen planes in $S_3$ are transversal planes.

One can show that the pre-images of the cardinal points under the Joubert map $J$ in $P^6_1$ are the points representing minimal orbits of strictly semi-stable points, i.e. ordered sets of points of the form $(p, p, p, q, q, q)$. The pre-images of the cardinal planes are orbits of stable ordered sets of points of the form $(p, p, p_3, p_4, p_5, p_6)$.

- Three planes span a hyperplane if and only if their indices have a common duad. There are 15 such hyperplanes $H_{ab}$ given by equations $x_a + x_b = x_1 + \cdots + x_6 = 0$. 


Each hyperplane $H_{ab}$ intersect the cubic $S_3$ along the union of three planes. For this reason we call $H_{ab}$ a tritangent hyperplane.

- Two tritangent hyperplanes $H_{ab}$ and $H_{cd}$ have a common plane if and only if \( \{a, b\} \cap \{c, d\} = \emptyset \). Otherwise they intersect along a plane.

Let $p_i = [t_0^{(i)}, t_1^{(i)}, t_2^{(i)}]$ be the projective coordinates of 6 points in $\mathbb{P}^2$ no three of which are collinear. Consider the determinant of the $3 \times 3$-matrix with columns equal to the coordinates of $p_a, p_b$, and the coordinates $[t_0, t_1, t_2]$ of a general point $p$ in the plane. This is a linear function in $t_0, t_1, t_2$ which we denote by $(abt)$. For any syntHEME $(ab, cd, ef)$ consider the product $(abt)(cde)(ef)t)$. For any total $T_i$ consider the sum of these functions corresponding to syntemes entering in the total (taken with appropriate signs, see [180], p.474). This is a homogeneous cubic polynomial in $t_0, t_1, t_2$ which we denote by $F_i$. One can show that the polynomials $(F_1, \ldots, F_6)$ satisfy $\sum F_i = \sum F_i^3 = 0$, and hence define a rational map $\Phi : \mathbb{P}^2 \dashrightarrow S_3$ (10.4.6).

(see [180], Theorem 9.4.13). The image of this map is a hyperplane section $\sum a_ix_i = 0$ of $S_3$. The coefficients $a_i$ of this hyperplane sections are functions $\bar{U}_i$ of the point sets that are similar to the Joubert functions and correspond again to totals. In fact, they define a map $(\mathbb{P}^2)^6 \rightarrow |E^\vee|$ that factors through the GIT-quotient of $P_2^6$ of $(\mathbb{P}^2)^6$ by the group $SL_3$ and a choice of a democratic linearization $C : P_2^6 \rightarrow |E^\vee| \cong \mathbb{P}^4$.

This map is not an isomorphism but a finite map of degree 2. Its branch locus is a quartic hypersurface, which we call it in [180], the Castelnuovo-Richmond quartic, denoted by $CR_4$ (also known in current literature as the Igusa quartic). If one considers the projective representation of $S_6$ associated to the irreducible linear representation on $E^\vee$ defined by the partition $(3, 3)$ of [1, 6], then the map $C$ is $S_6$-equivariant. The equation of $CR_4$ is

$$\sum_{i=1}^{6} u_i^2 - 4 \sum_{i=1}^{6} u_i = 0. \quad (10.4.7)$$

The ramification divisor of the map $C$ is the hypersurface in $P_2^6$ isomorphic to the GIT-quotient of sixtuples points $(p_1, \ldots, p_6)$ that lie on a conic.

- The quartic hypersurface $CR_4$ is singular along the 15 transversal lines.

- The cardinal hyperplanes intersect $CR_4$ along the cardinal quadric surface taken with multiplicity 2. The equation of a cardinal quadric cut out by the cardinal hyperplane $u_1 + u_2 + u_3 = 0$ is

$$u_1 + u_2 + u_3 = u_1u_2 + u_1u_3 + u_2u_3 - u_4u_5 - u_4u_6 - u_5u_6 = 0. \quad (10.4.8)$$

- There is an isomorphism of the dual hypersurface of the Segre cubic to the Castelnuovo-Richmond quartic hypersurfaces. It is given by partial derivatives of the equation of the Segre cubic. Under this isomorphism, the images of the transversal planes are the transversal lines, and the images of the exceptional divisors over the singular points are the cardinal quadric surfaces.
We have already seen a modular interpretation of the Segre cubic hypersurface as a GIT-quotient of the space of ordered six points in $\mathbb{P}^1$. The Castelnuovo-Richmond quartic has also a modular interpretation due to J. Igusa (see [238]).

- The Castelnuovo-Richmond quartic $\text{CR}_4$ is a compactification of the moduli space $\mathcal{A}_2(2)$ of principally polarized abelian surfaces with a full level 2-structure (see [180], Proposition 9.4.8).

- The complement of the cardinal hyperplanes is the Jacobian locus in $\mathcal{A}_2(2)$.

- The union of 15 transversal lines is the boundary of $\mathcal{A}_2(2)$.

Note that both $S_3$ and $\text{CR}_4$ can be naturally considered as compactifications of the moduli space of curves of genus 2 together with an order on the set of its six Weierstrass points (equivalently, a 2-level structure on its Jacobian variety).

Let $(abc)$ be a triad, we denote by $\mathcal{D}(abc)$ (resp. $\mathcal{G}(abc)$) the set of six duads (resp. six synthemes) not appeared in the entries (resp. determined by the determinantal terms) of the matrix (10.4.4). The subconfiguration of the Cremona-Richmond configuration formed by these sets is of type $(6_0, 6_0)$.

**Definition 10.4.2.** We define the extended double Cremona-Richmond diagram $\Gamma_{\text{CR}}$ as follows. Its set of vertices is the union of the set $\mathcal{D}$ of 15 duads $(ab)$, the set $\mathcal{G}$ of 15 synthemes $(ij, kl, mn)$ and the set $\mathcal{I}$ of 10 triads $(abc)$. Their edges are described as follows.

- **The subgraph with vertices $\mathcal{D}$ or $\mathcal{G}$ is the dual graph of the complete graph $K(6)$.**

- **Each vertex from $\mathcal{D}$ is connected by a double edge with a vertex from $\mathcal{G}$ and vice versa forming the Levi graph of the Cremona-Richmond configuration where all edges are doubled.**

- **Two vertices from $\mathcal{I}$ are joined by a double edge.**

- **Each vertex $(abc)$ from $\mathcal{I}$ is joined by a double edge with the set of vertices from $\mathcal{D}(abc)$ and $\mathcal{G}(abc)$.**

Recall from Section 0.8 that a finite set of vectors $v_i, s_i \in I$ of square norm $-2$ in $V = (E_{10})_\mathbb{R}$ defines a polytope $\Pi$ in the hyperbolic space $\mathbb{H}^9$ associated to the quadratic vector space $V$ with the Sylvester signature $(1, 9)$. It is the closure of a fundamental domain for the reflection group $\Gamma$ generated by reflections with respect to the vectors $v_i$. It is equal to a connected component of the complement of the set of the hyperplanes $H(v_j)$ orthogonal to the vectors $v_j$. The group $\Gamma$ is a Coxeter group generated by the reflections $r_{v_j}$ with the Coxeter matrix $(m_{ij})$, where $m_{ij} = \infty$ if $v_i \cdot v_j \geq 2$, or 3 if $v_i \cdot v_j = 1$, or 2 if $v_i \cdot v_j = 0$. We assume that there exists a set of forty $(-2)$-vectors in $V$ forming the diagram $\Gamma_{\text{CR}}$. We call the polytope $\Pi$ in $\mathbb{H}^9$ defined by these forty vectors the Cremona-Richmond polytope.

**Theorem 10.4.3.** The Coxeter-Richmond polytope in $\mathbb{H}^9$ is of finite volume. Its symmetry group is isomorphic to the group of automorphisms of $\mathcal{S}_6$ isomorphic to $\mathcal{S}_6 \cdot 2$. 

10.5. **QUOTIENTS OF K3 SURFACES WITH ARTIN INVARIANT 1: TYPE MI**

**Proof.** Any symmetry of $\Gamma_{CR}$ must be a symmetry of the Cremona-Richmond configuration. The group of such symmetries is obviously $\text{Aut}(G_6)$. We see immediately that it extends to the group of symmetries of the whole diagram.

The polytope is of finite volume in $\mathbb{H}^9$ if and only if the intersection matrix $(v_s,v_{s'})$ satisfies Vinberg’s criterion from Theorem 0.8.22. To check this we exhibit all parabolic subdiagrams of $\Gamma_{CR}$ of maximal rank. They are

\[ \tilde{A}_2 + \tilde{A}_2 + \tilde{A}_2 + \tilde{A}_2, \quad \tilde{A}_3 + \tilde{A}_3 + \tilde{A}_1 + \tilde{A}_1, \quad \tilde{A}_4 + \tilde{A}_4, \quad \tilde{A}_5 + 2 \tilde{A}_1 + \tilde{A}_2. \tag{10.4.9} \]

For example, take the parabolic diagram of type $\tilde{A}_1$ defined by two vectors $c_{abc}, c_{a'b'c'}$. Without loss of generality we may assume that the triads are $(123)$ and $(124)$. We check that they are not connected to the vertices corresponding to duads $(15), (16), (25), (26), (34)$ and the vertices corresponding to synthemes

\[ (12, 36, 45), (14, 23, 56), (13, 24, 56), (12, 35, 46), (12, 34, 56). \]

The first four vertices in each set define two disjoint parabolic subdiagrams of type $\tilde{A}_3$ and the pair $(34), (12, 34, 56)$ defines another parabolic subdiagram of type $\tilde{A}_1$.

We refer for the rest of the proof to [360, Lemma 4.2].

**Remark 10.4.4.** Let us identify a duad $(ab)$ with a transposition $(ab)$, a syntheme $(ij,kl, mn)$ with the product of three transposition, and a triad $(abc)$ with the product or two commuting cyclic subgroups of order 3. Then one easily checks the following.

Two duads or symtures do not commute if and only if they are incident in the extended Cremona-Richmond diagram.

A duad or a syntheme centralizes a triad if and only if they are incident in the extended Cremona-Richmond diagram.

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10.5 **Quotients of K3 surfaces with Artin invariant 1: Type MI**

In this section we will construct a family of Enriques surfaces $S$ and a unique Coble surface $V$ that realize the extended Cremona-Richmond diagram as the intersection diagram of 40 divisor classes in the lattice $\text{Num}(S)$ or $\text{CM}(V)$ with square-norm $-2$. We will refer to these surfaces as $\textit{surfaces of Type MI}$. Their canonical covers have 12 ordinary double points, and their minimal resolution are the supersingular K3 surface with Artin invariant 1.

Next we give another realization of the Cremona-Richmond configuration over a field of characteristic 2.

We start with the finite plane $\mathbb{P}^2(\mathbb{F}_4)$. The finite plane has 21 points and 21 lines. Each line contains 5 points and each point is contained in 5 lines. This gives an example of a symmetric configuration $(21_5)$.

Let us fix a $6$-arc $\mathcal{P} = \{p_1, \ldots, p_6\}$ in $\mathbb{P}^2(\mathbb{F}_4)$ (an $n$-arc in a finite plane is a subset of $n$ points no
three of which are collinear). It is known that there are 168 6-arcs and the group $\text{PGL}_3(\mathbb{F}_4)$ acts transitively on the set of 6-arcs with the stabilizer subgroup isomorphic to $\text{A}_6$ (see [129, L3(4)]).

We have 15 lines in $\mathbb{P}^2(\mathbb{F}_4)$ joining pairs of these points. Each point in $\mathcal{P}$ is contained in 5 such lines, this shows that there are 6 lines $T_1, \ldots, T_6$ which do not contain any points from $\mathcal{P}$. None of the points in $T_i$ belong to $\mathcal{P}$. Let $\mathcal{P}^c$ be the set of 15 intersection points of these lines in $\mathbb{P}^2(\mathbb{F}_4)$ and let $\pi_\mathcal{P} : V(\mathcal{P}) \to \mathbb{P}^2$ be the blow-up of this set. Each line $\ell_{ij} = \langle p_i, p_j \rangle$ contains 3 points from the set $\mathcal{P}^c$, hence its proper transform on $V(\mathcal{P})$ is a $(-2)$-curve which we denote by $R_{ij}$. On the other hand, each line $T_i$ contains 5 points from $\mathcal{P}^c$, hence its proper transform is a $(-4)$-curve $C_i$. Thus we see that $V(\mathcal{P})$ is a Coble surface with 6 boundary components.

**Remark 10.5.1.** Let $C$ be the blow-up of any 6-arc $\mathcal{P}$ in the finite plane $\mathbb{P}^2(\mathbb{F}_4)$. It is isomorphic to a smooth cubic surface in $\mathbb{P}^3$. Since the stabilizer subgroup of $\mathcal{P}$ on $\text{PGL}_3(\mathbb{F}_4)$ is isomorphic to $\text{A}_6$, we see that $\text{A}_6 \subset \text{Aut}(C)$. The classification of possible automorphism groups of smooth cubic surfaces from [187] shows that $C$ is isomorphic to the Fermat cubic.

Every triad $(abc)$ defines two triangles with sides $\ell_{ab}, \ell_{ac}, \ell_{bc}$ and $\ell_{de}, \ell_{df}, \ell_{ef}$. The triangles intersect at 9 points in $\mathcal{P}^c$. We denote the complementary set of 6 points in this set by $Q(abc)$ (as before $Q(abc) = Q(def)$). We call it a cardinal set of 6 points. Let $c_{abc}$ be the divisor class $2e_0 - \sum_{p \in Q(abc)} e_p$, where we use the standard notation for a geometric basis of a blow-up of a set of points in the plane. We have $c_{abc}^2 = -2$. Since a 6-arc does not lie on a conic (necessary defined over $\mathbb{F}_4$), $c_{abc}$ is not an effective class.

**Theorem 10.5.2.** Let $\mathcal{E}$ be the union of the set of fifteen classes $\alpha_{ab}$ of the $(-2)$-curves $R_{ab}$, fifteen effective roots $\alpha_{ij,kl,mn} = 2E_p + \frac{1}{2}(C_q + C_r)$, where $p$ corresponds to the syntheme $(ij, kl, mn)$ that is common to the totals $T_q$ and $T_r$, and the set of 10 divisor classes $c_{abc}$. Then the intersection diagram of the 40 classes in the Coble-Mukai lattice $\text{CM}(V(\mathcal{P}))$ is the extended Cremona-Richmond diagram.

**Proof.** Every line $\ell_{ab}, a, b \in \mathcal{P}$, contains 3 points from $\mathcal{P}^c$, each point is the intersection of two lines $T_i$. In this way each duad $(ab)$ defines a syntheme $(ij, kl, mn)$. Since each $T_i$ contains 5 points in $\mathcal{P}^c$, we see each $T_i$ can be taken as a total. It is immediately seen, that the set of 15 lines $\ell_{ab}$ and 15 points $\mathcal{P}^c$ form a symmetric $(15_3)$-configuration invariant with respect to the group $\text{S}_6$ permuting the points in $\mathcal{P}$. Thus they form a Cremona-Richmond configuration with the set $\mathcal{A} = \mathcal{P}^c$ of points associated to synthemes and the set $\mathcal{B}$ of lines associated to duads. We have also defined a bijection $\iota : \mathcal{B} \to \mathcal{A}$. It is immediate to check that the intersection diagram of the divisor classes $\alpha_{ab}$ (resp. $\alpha_{ij,kl,mn}$) corresponding to duads (resp. synthemes) is equal to the Levi graph of the Cremona-Richmond configuration.

First of all, $c_{abc} \cdot c_{a'b'c'} = 2$ for different triads $(abc)$ and $(a'b'c')$. Indeed, replacing $(abc)$ with the complementary set, we may assume that $a, b, c$ and $a', b', c'$ have one common element. Without loss of generality we can take $(abc) = (123)$ and $(a'b'c') = (145)$, and check that they define two common synthemes $(16, 25, 34)$ and $(16, 24, 35)$ from $Q(abc)$. This shows that $c_{123} \cdot c_{145} = 2$.

It follows from the definition of a cardinal set $Q(abc)$ that the 6 lines $\ell_{ij}, i,j \in \{a, b, c\}$ or $\{d, e, f\}$ do not contain any point from this set. Thus $c_{abc} \cdot \alpha_{ij} = 2$. Any other line $\ell_{ij}$ passes through two
points in \( P \), each from different triangle defined by \((abc)\). It also passes through the intersection point of the opposite sides of the triangles. It follows that \( \ell_{ij} \) contains 2 points from the set \( Q(abc) \), and hence \( c_{abc} \cdot \alpha_{ij} = 0 \). This agrees with our definition of the extended Cremona-Richmond diagram.

Each point in \( A \setminus Q(abc) \) is the intersection point of two lines \( \ell_{ab} \), hence it is the intersection point of 2 lines \( T_i \) and one line \( \ell_{ij} \) passing through the vertex of the opposite side of the triangle containing this point. This defines the incidence correspondence of type \((9_2,6_3)\). It follows that each \( T_i \) passes through 3 points not in \( Q(abc) \), hence passes through 2 points from this set. This shows that \( c_{abc} \cdot C_i = 0 \) (as expected because \( c_{abc}^2 = -2 \)). Now \( c_{abc} \) intersects with multiplicity 1 each \( E_p, p \in Q(abc) \), and hence it intersects the corresponding effective root with multiplicity 2. All other such roots it does not intersect.

We leave to the reader (for example, by taking \((abc) = (123)\) and using Table 10.1) to check that the synthemes are \( \{ai,bj,ck\} \), where \( \{i,j,k\} = \{d,e,f\} \).

Finally, if \( \ell_{ab} \) corresponds to the duad \((ab)\), then it intersects all lines corresponding to totals (in 3 pairs intersecting at one point on the line). This shows that \( \alpha_{ab} \) intersects three effective roots \( \alpha_{ij,kl,mm} \) with multiplicity 2, where the three synthemes correspond to \((ab)\) in the Cremona-Richmond configuration. Similarly, each point corresponding to a syntheme \((ij,kl,mm)\) lies on three lines \( \ell_{ij}, \ell_{kl}, \ell_{mm} \), and hence \( \alpha_{ij,kl,mm} \) intersect \( \alpha_{ij}, \alpha_{kl}, \alpha_{mm} \). These are the only non-zero intersections between the divisor classes corresponding to duads and synthemes.

All of this agrees with our definition of the Cremona-Richmond diagram.

\[ \square \]

**Remark 10.5.3.** Note that the divisor classes \( c_{abc} \) are never effective. In fact, any irreducible conic in plane \( \mathbb{P}(F_4) \) contains exactly 5 points in this plane, and hence neither \( P \) nor \( Q(abc) \) lie on a conic.

We know that each triad \((abc)\) defines a pair of triangle of lines with vertices \( a, b, c \) and \( d, e, f \) intersecting at the set \( B \) of 9 points from \( P^c \). We also know that each point in \( B \) lies on two lines \( T_i \) and each line passes through two points in \( B \). This defines a symmetric configuration. The only possible Levi graphs of this configuration is a hexagon with vertices in \( Q(abc) \) or the disjoint sum of two triangles. Thus we see that a choice of \((abc)\) defines two triangles with vertices in \( Q(abc) \).

**Example 10.5.4.** Choose \((abc) = (123)\). Then \( Q(123) \) consists of 6 points corresponding to synthemes \((14)(25)(36), (14)(26)(35), (15)(24)(35), (15)(26)(34), (16)(25)(34), (16)(24)(35)\). They define two triangles with sides \((T_1,T_2,T_4)\) and \((T_3,T_5,T_6)\).

It follows from above that a choice of a triad \((abc)\) defines 4 triangles of lines in \( \mathbb{P}^2(F_4) \), considered as plane cubic in \( \mathbb{P}^2 \) each pair intersecting at the same set of points \( B \). As we know this defines a unique Hesse pencil of cubic curves in \( \mathbb{P}^2 \) with base point \( B \). It is defined uniquely, up to projective transformation, over any field containing 3 distinct third roots of unity. So, we can choose coordinates to assume that its equation is

\[ \lambda(x^3 + y^3 + z^3) + \mu xyz = 0. \quad (10.5.1) \]

We can order the coordinates of the nine base points of the Hesse pencil as follows:

\[
\begin{align*}
p_1 &= [0, 1, 1], & p_2 &= [0, 1, \epsilon], & p_3 &= [0, 1, \epsilon^2], \\
p_4 &= [1, 0, 1], & p_5 &= [1, 0, \epsilon^2], & p_6 &= [1, 0, \epsilon], \\
p_7 &= [1, 1, 0], & p_8 &= [1, \epsilon, 0], & p_9 &= [1, \epsilon^2, 0].
\end{align*}
\quad (10.5.2)
\]
where $\epsilon$ is a primitive 3rd root of unity.

The four triangles are $\Delta_{\infty} = V(xyz)$ and the triangles $\Delta_{\epsilon^i}, i = 0, 1, 2$, with sides $x + \epsilon^a y + \epsilon^b z, a + b \equiv i \mod 3$. They correspond to members of the pencil with $t = \mu/\lambda = \infty, 1, \epsilon, \epsilon^2$.

Let $L_{0,i}, L_{1,i}$, resp. $L_{2,i}$, be the lines that join $[1, 0, 0]$ (resp. $[0, 1, 0]$, resp. $[0, 0, 1]$) with three base points $[0, 1, \epsilon^i]$ (resp. $[1, 0, \epsilon^i]$, resp. $[1, \epsilon^i, 0]$). Three of the lines $L_{0,i}, L_{1,i}, L_{2,i}$ intersect at a point $p_{i,j,k}$ if and only if $i + j + k \equiv 0 \mod 3$. Thus we have a symmetric configuration of type $(9_3)$ of 9 lines and 9 intersection points $p_{i,j,k}$. It is a special case of the Ceva configuration Ceva(n) with $n = 3$.

![Figure 10.4: 9 sections and 9 bisections on $H$](image)

For any base point $p_i$, the polar conic of a general member of the Hesse pencil with pole at $p_i$ is equal to the union of the tangent line at this point and a line $\ell_i$ that does not depend on the parameter. It is classically known as a harmonic line of the Hesse pencil (see [17]). In our case when $p = 2$, the 9 harmonic lines coincide with the nine lines $L_{0,i}, L_{1,i}, L_{2,i}$. The nine points $p_{i,j,k}$ are the vertices of the triangles $\Delta_{\epsilon^i}$. So, we see that each harmonic line $\ell_i$ passes through 4 vertices, one in each triangle and intersect the opposite sides at the base point $p_i$ of the pencil. Note that this is specific to the characteristic 2 case. In other characteristics (also different from 3), the harmonic lines intersect the opposite side of each triangle at a point different from the base point.

We denote by $\pi_{abc} : H \to \mathbb{P}^2$ the blow-up of the set $B$. Let $f_{abc} : H \to \mathbb{P}^1$ be the corresponding relatively minimal rational elliptic surface. It is one of the extremal rational elliptic surface in characteristic 2 which we classified in Section 4.8. We have the blowing down morphism $p_{abc} : V(P) \to H$ that blows up the set of six singular points of two fibers whose image in the plane is the union of six lines $T_i$ corresponding to the totals.

Each base point defines a section of the elliptic fibration on the surface $H$ obtained by blowing up the set $B$ of base points. It is the exceptional curve $E_p, p \in B$. Fixing such a section $E_p$, we have the standard Bertini involution whose restriction to a general fiber is the negation involution. Its set of fixed points consists of the section and the harmonic line corresponding to the base point $p$. Each harmonic line defines an inseparable bisection of the pre-image of the Hesse pencil on $H$. It intersects each singular fiber at one of its singular points.

Let $H'$ be the blow-up of the 12 vertices of the four triangles in $H$. This is of course the same as the
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blow-up of the set $\mathbb{P}^2(\mathbb{F}_4)$ in $\mathbb{P}^2$, or the blow-up of the set $\mathcal{P}$ on the Coble surface $V(\mathcal{P})$. The pre-image of the Hesse pencil is an elliptic pencil on $H'$ (not relatively minimal). It has 4 reducible fibers formed by a hexagon of smooth rational curves. Three mutually non-incident vertices correspond to $(-1)$-curves, they enter with multiplicity 2. The other such set corresponds to $(-4)$-curves. The pre-image of a harmonic line is a $(-4)$-curve intersecting the double component of a fiber. The pre-image of a section on $H$ is a $(-1)$-curve that intersects the simple component of a fiber.

The Frobenius base change of the Hesse fibration $f_{abc} : H \to \mathbb{P}^1$ defines a surface $X$ with 12 singular points lying over the singular points of the fibers of $f_{abc}$. Locally, at each such point the cover $X \to H$ is given by $z^2 = xy$, hence all twelve points are ordinary double points. Let $\sigma' : Y \to X$ be a minimal resolution of singularities. The proper transform of an irreducible component of singular fibers of the Hesse pencil on $Y$ are $(-2)$-curves, as follows from the intersection theory on a normal surface. This shows that the pre-image of the Hesse pencil on $Y$ is a relatively minimal elliptic fibration $f : Y \to \mathbb{P}^1$ with 4 fibers of type $A_5$. The pre-image of any section of the Hesse pencil is a $(-2)$-curve which is a section of $f$. The formula for the canonical class of an elliptic surfaces tells us that $Y$ is a K3 surface.

Let $\sigma_{abc} : H' \to H$ be the blow-up of singular points of the fibers. The composition $H' \to H \to \mathbb{P}^2$ is the blow-up of the subset $\mathbb{P}^2(\mathbb{F}_4)$ of $\mathbb{P}^2$. It is easy to see that the surface $Y$ is a minimal resolution of singularities of the split inseparable cover $\mathbb{P}^2$ define by the sheaf $\mathcal{L} = \mathcal{O}(3)$ and a section of $\mathcal{L}^{\otimes 2}$ defined by the union of any two members of the Hesse pencil. By Proposition 0.2.7 the inseparable cover has 21 ordinary double, all points from $\mathbb{P}^2(\mathbb{F}_4)$. The surface $Y$ is a minimal resolution of singularities if an inseparable finite map of degree 2 of the surface $H'$ (see [181]). It is a supersingular surface with Artin invariant 1. We refer to loc. cit. for different birational models of $Y$.

It follows from above that the four reducible fibers $F_\alpha$ of $f : Y \to \mathbb{P}^1$ over the points $\alpha = 1, \epsilon, \epsilon^2, \infty$ are the unions of two sets $F_\alpha^+, F_\alpha^-$, each consists of three disjoint components. We may assume that $F_\alpha^+$ consists of exceptional curves over the singular points of the Hesse pencil, and $F_\alpha^-$ consists of the proper transforms of fibers of the Hesse pencil. In our duad-syntheme notation, $F_\alpha^+$ can be indexed by synthemes, and $F_\alpha^-$ by duads. In the Figure 10.5, the components $F_\alpha^+$ (resp. $F_\alpha^-$) are drawn in blue (resp. red).

The proper transforms of sections of $H \to \mathbb{P}^1$ are of course sections of $f : Y \to \mathbb{P}^1$. They intersect the components $F_\alpha^+$. In the Figure 10.5, they are drawn in red. The proper transforms of harmonic lines of the Hesse pencil are also sections. They intersect the components $F_\alpha^-$. They are drawn in blue. All together we obtain 18 sections which generate the Mordell-Weil group isomorphic to $(\mathbb{Z}/3\mathbb{Z})^{\otimes 2} \oplus \mathbb{Z}/2\mathbb{Z}$.

The unique element of order 2 in the Mordell-Weil group of $f : Y \to \mathbb{P}^1$ translates the normal subgroup of index 2 to its coset. Thus it sends the blue sections corresponding to base points of the Hesse pencil to red sections corresponding to harmonic lines. It also translates the 12 blue components of reducible fibers to 12 red components. The involution of $Y$ defined by the order 2 section is called a switch [181]. Also it defines a correlation between $\mathbb{P}^2(\mathbb{F}_4)$ and the dual finite plane.

Remark 10.5.5. It is shown by Mukai that the surface $Y$ admits a birational model isomorphic to
the intersection of two divisors of type (2, 1) and (1, 2) in \( \mathbb{P}^2 \times \mathbb{P}^2 \):

\[
x_0^2 y_0 + x_1^2 y_1 + x_2^2 y_2 = y_0^2 x_0 + y_1^2 x_1 + y_2^2 x_2 = 0.
\]

(see [181]). The switch involution is induced by the interchanging the factors in \( \mathbb{P}^2 \times \mathbb{P}^2 \). Its set of fixed points is its intersection with the diagonal which is a supersingular elliptic curve in characteristic 2. It is the unique smooth supersingular fiber of the elliptic fibration of \( f : Y \to \mathbb{P}^1 \) arising from the Hesse pencil on which the translation by the section of order 2 acts identically.

We know that a harmonic line passes through one of the base points of the Halphen pencil. It follows that the corresponding blue and red sections intersect are tangent at one point. Again, this is special to positive characteristic, since it follows from Proposition 4.2.1 that sections of finite order prime to the characteristic do not intersect.

Let \( \pi : X \to V(\mathcal{P}) \) be the canonical cover defined by the invertible sheaf \( \mathcal{L} = \omega_{V(\mathcal{P})}^{-1} \) and the section of \( \mathcal{L}^{\otimes 2} \) with the zero scheme equal to the boundary \( C_1 + \cdots + C_6 \). We have \( \omega_X \cong \mathcal{O}_X \) and Proposition 0.2.7 tells that \( \text{Sing}(X) \) is a finite subscheme \( Z \) with \( h^0(\mathcal{O}_Z) = 12 \). Since \( X \) has an ordinary double point over singular points of fibers of type \( \tilde{A}_2 \), we infer that \( X \) has exactly 12 double points and its minimal resolution \( Y \) is a supersingular K3 surface. It has an elliptic fibration with 4 fibers of type \( \tilde{A}_5 \), and the Shioda-Tate formula implies that the Artin invariant of \( Y \) is equal to 1.

Applying Theorem 10.3.2, we obtain the following.

**Theorem 10.5.6.** The canonical cover of the Coble surface \( V(\mathcal{P}) \) has 12 ordinary double points and its minimal resolution is the supersingular K3 surface \( Y \) with Artin invariant 1.

The Coble surface \( V(\mathcal{P}) \) is called the Coble surface of Type MI.

Next we show that the surface \( Y \) admits a family of rational vector field \( \partial_{\alpha, \beta} \), where \( \alpha + \beta = \alpha \beta = \alpha \) such that the quotient by the derivation is birationally isomorphic an Enriques surface if \( \alpha \neq 1 \) and to the Coble surface \( V(\mathcal{P}) \) otherwise.

In Section 4.8 we have found the Weierstrass equation of the Hesse pencil. If we set \( s = \mu/\lambda \), the equation is

\[
y^2 + sxy + y + x^3 + 1 + s^3 = 0 \tag{10.5.3}
\]

Its discriminant is equal to \( (1 + s^3)^3 \).
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We will construct the surfaces \( S_{\alpha,\beta} \) as the quotients of \( Y \) by a rational vector field and will describe a certain configuration of thirty \((-2)\)-curves on \( S_{\alpha,\beta} \) that is equal to the image of the double Hesse configuration of 42 \((-2)\) curves on \( Y \). The construction of the vector field is similar to the constructions in Section 8.10 and we will omit the details.

Replacing \( s \) with \( t^2 \) in (10.5.3), we obtain the Weierstrass equation of the pre-image of the Hesse pencil on \( \bar{Y} \):

\[
y^2 + t^2 xy + y + x^3 + 1 + t^6 = 0 \tag{10.5.4}
\]

Following Example 8.10.8, we consider the rational derivation on \( Y \)

\[
\partial_{\alpha,\beta} = \frac{1}{t + 1} \left( (t + 1)(t + \alpha)(t + \beta) \frac{\partial}{\partial t} + (1 + t^2 x) \frac{\partial}{\partial x} \right),
\]

where \( \alpha, \beta \in \mathbb{k}, \alpha + \beta = \alpha \beta, \alpha^3 \neq 1 \). The choice of the Weierstrass model defines a choice of an irreducible component in each singular fiber that is intersected by the zero section. If \( t \neq \infty \), it does not enter in the divisor of poles of \( \partial_{\alpha,\beta} \) on \( Y \) and it does enter if \( t = \infty \). A direct calculation following Lemma 10.3.13 shows that the extension of \( \partial_{\alpha,\beta} \) to \( \bar{Y} \) will have poles on blue components.

**Lemma 10.5.7.**

(i) \( \partial_{\alpha,\beta}^2 = \alpha \beta \cdot \partial_{\alpha,\beta} \), hence, \( \partial_{\alpha,\beta} \) is 2-closed. Moreover, it is of additive type if \( \alpha = \beta = 0 \) and of multiplicative type otherwise.

(ii) On the surface \( Y \), the divisor \( D \) of \( \partial \) is given by

\[
D = -(F^{+}_\epsilon + F^{+}_{\epsilon 2} + F^{-}_{1} + F^{-}_\infty),
\]

and \( D^2 = -24 \).

(iii) The integral curves with respect to \( \partial_{\alpha,\beta} \) are the smooth fibers over \( t = \alpha, \beta \) (in case \( \alpha = \beta = 0 \), the smooth fiber over \( t = 0 \)) and \( F^{-}_{1}, F^{-}_\infty, F^{+}_{\epsilon}, F^{+}_{\epsilon 2} \).

Since \( K_Y \) is trivial and \( D^2 = -24 \), it follows formula (0.3.4) that

\[
24 = c_2(Y) = \deg(Z) - (K_Y, D) - D^2
\]

where \( Z \) is the scheme of non-divisorial zeros of \( \partial_{\alpha,\beta} \). Therefore \( \deg(Z) = 0 \) and \( \partial \) has no isolated zeros. Thus the quotient surface \( Y^0 \) by \( \partial_{\alpha,\beta} \) is non-singular. Denote by \( \pi' : Y \to Y^0 \) the quotient map. It is a finite inseparable map of degree 2.

The canonical bundle formula from Proposition 0.3.14 gives

\[
K_Y = \pi'^* K_{Y^0} + D.
\]

If a \((-2)\)-curve \( C \) is integral with respect to \( \partial \), then applying Proposition 0.3.19, we obtain that \( \pi'(C) \) is a \((-1)\)-curve on \( Y^0 \).

Let \( \sigma' : Y^0 \to S_{\alpha,\beta} \) be the contraction of the sum \( \mathcal{E} \) of the twelve \((-1)\)-curves. Note that \( \sigma'^* (\mathcal{E}) = -D \). We have \( K_{Y^0} = \sigma'^* (K_{S_{\alpha,\beta}}) + \mathcal{E} \), and hence

\[
0 = D + \pi'^* (\sigma'^* (K_{S_{\alpha,\beta}})) + \pi'^* (\mathcal{E}) = D + \pi'^* (\sigma'^* (K_{S_{\alpha,\beta}})) - D = \pi'^* (\sigma'^* (K_{S_{\alpha,\beta}})).
\]

Thus \( \pi'^* (\sigma'^* (K_{S_{\alpha,\beta}})) \) is numerically trivial. Since \( \pi \) is finite and purely inseparable, \( b_2(Y) = 22 \), and hence \( b_2(S) = 10 \). Thus \( S_{\alpha,\beta} \) is an Enriques surface. Let \( \pi' \circ \sigma' = \sigma \circ \pi \), where \( \sigma : Y \to X \) is a birational morphism and \( \pi : X \to S_{\alpha,\beta} \) is a finite inseparable

cover. We immediately see $\sigma$ blows down the integral components of $D$ to ordinary double points and $\pi : X \to S_{\alpha,\beta}$ is the canonical cover of $S_{\alpha,\beta}$.

We will refer to the surface $S_{\alpha,\beta}$ as an Enriques surface of Type MI in characteristic 2.

To summarize we have the following commutative diagram.

![Diagram](image)

where $F$ is the Frobenius morphism.

It follows from Lemma 10.5.7 that the integral curves in fibers over $1, \infty$ (resp. $\epsilon, \epsilon^2$) originate from lines $\ell_{ab}$ (resp. synthemes $(ij, kl, mn)$), they are drawn in red (resp. blue) in Figure 10.5. When we blow down their images in $Y^{\partial}$, we obtain the following picture of fibers of the Hesse type elliptic fibration on $S_{\alpha,\beta}$.

![Figure 10.6: Fibers of the Hesse type elliptic fibration on $S_{\alpha,\beta}$](image)

There are two fibers for each type. All in all we see thirty $(-2)$-curves. Twelve are components of fibers and divided into two sets of 6, blue and red. The rest are 18 special bisections divided into two sets of 9, blue and red. The set of 30 curves is the union of a set $A$ of blue curves and the set $B$ of red curves. The incidence relation between $A$ and $B$ is the Cremona-Richmond configuration $(15_3)$. However, the intersection diagram has the double edges of the corresponding Levi graph. If we identify the vertices of the blue triangles with the set $[1, 6]$ and the vertices of the red triangles with the set of totals, then we see that the curves from the set $A$ (resp. resp. $B$) intersecting at $a \in [1, 6]$ (resp. at $T_i$), correspond to duads $(ab), b \neq a,$ (resp. the five synthemes entering in $I_i$). This shows that the intersection graph of $A$ and $B$ is the dual of the complete graph $K(6)$.

Let $c_{ijk}$ be the divisor class on the surface $H'$ equal to $2e_0 - \sum_{p \in Q(ijk)} e_p$ (we identify it with the corresponding class on $V(P)$). We immediately check that its pre-image in $Y$ is a divisor class.
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Figure 10.7: Six points (six totals) and fifteen duads (fifteen synthemes)

with self-intersection $-4$ which is orthogonal to the 12 components of fibers that are blown down to singular points of $X$ under the map $\sigma$. Its image $\varepsilon_{abc}$ in $S_{\alpha,\beta}$ is a divisor class of square norm $-2$ that intersects 6 red curves (corresponding to the points from $Q(abc)$) and 6 blue curves (corresponding to harmonic lines) with multiplicity 2.

There are twelve canonical points on $S_{\alpha,\beta}$ which are the images of $F_1^-, F_2^-, F_1^+, F_\infty$. These twelve points are indexed by the set of six points and the set of six totals. In the following Figure 10.7, the black (resp. white) circles are canonical points corresponding to six points (resp. six totals). Each line passing through two black circles (resp. white circles) is the $(-2)$-curve corresponding to the duad (resp. syntheme) containing two points (resp. contained in the two totals).

This shows that the intersection diagram of the set of 40 divisor classes is the same as the extended Cremona-Richmond diagram.

**Theorem 10.5.8.** The Enriques surface $S_{\alpha,\beta}$ contains the set of thirty $(-2)$-curves that is the union into two sets $A$ and $B$ of 15 divisor classes, and it also contains a set $C$ of 10 divisor classes of square norm $-2$. The set $A$ (resp. $B$) is the union of two subsets $A_{\text{fib}}$ and $A_{\text{bis}}$ (resp. $B_{\text{fib}}$ and $B_{\text{bis}}$) of cardinalities 6 and 9.

- The intersection graph of the sets $A$ and $B$ are the dual graphs of the complete graph $K(6)$.
- Each element from $A_{\text{fib}}$ (resp. $B_{\text{fib}}$) intersects three elements from the set $B_{\text{bis}}$ (resp. $A_{\text{bis}}$) at one point with multiplicity 2. Each element from $A_{\text{bis}}$ (resp. $B_{\text{bis}}$) intersects two elements from the set $B_{\text{fib}}$ (resp. $A_{\text{fib}}$) at one point with multiplicity 2 and also intersects one element from $B_{\text{bis}}$ (resp. $B_{\text{bis}}$) at one point with multiplicity 2.
- The incidence relation between the sets $A$ and $B$ defines a symmetric configuration $(15_3)$ isomorphic to the Cremona-Richmond configuration.
- The divisor classes from $C$ intersect each other with multiplicity 2.
- Each divisor class from $C$ intersects 6 divisor classes from $B$ and 6 divisor classes from $A$ with multiplicity 2.
- The surface contains 10 elliptic fibrations of Hesse type indexed by the set of triads (up to complementary set). The curves from the sets $A_{\text{bis}}$ and $B_{\text{bis}}$ are inseparable special bisections.
Each fibration has 18 inseparable special bisections divided into two subsets $A_s$ and $B_s$ of nine of the sets $A$ and $B$.

- The set of 12 singular points of singular fibers of the fibration is the set of canonical points on $S_{\alpha,\beta}$.
- There is a bijection between the set $A$ and the set of duads and the set $B$ with the set of synthemes.

The intersection diagram of the total set fo 40 divisor classes is the extended Cremona-Richmond diagram.

We have 10 elliptic fibrations $2F_{abc}$ of Hesse type corresponding to 10 triads. The next lemma follows easily from Theorem 10.5.8 and we leave its proof to the reader.

**Lemma 10.5.9.** Let $2F_{abc}$ be the ten elliptic fibrations on $S_{\alpha,\beta}$ of Hesse type corresponding to 10 triads.

(i) Each of 45 pairs of the fibrations form a non-degenerate $U$-pair. In particular, it defines a bielliptic involution.

(ii) If we fix one of the 45 pairs of Hesse fibrations, then each fiber of the first one has a unique common component with some fiber from the second one.

(iii) A component of a reducible fiber of one fibration either does not intersect a fiber of another fibration, or it is tangent to a component of its reducible fiber, or it passes through a singular point of its reducible fiber.

It follows from the lemma that 45 pairs of the elliptic fibrations define 45 bielliptic maps. Since the surface $S_{\alpha,\beta}$ does not admit a quasi-elliptic fibrations, this map is separable and hence defines an involution of the surface.

**Proposition 10.5.10.** Let $\tau_{ij}$ be the bielliptic involution defined by a pair of Hesse fibrations. Then it fixes pointwise the four common components of two fibers and permutes by pairs the other components in the same fiber. It acts on the set of thirty $(-2)$-curves as the product of two transpositions.

**Proof.** Without loss of generality we may assume that our pair of Hesse elliptic fibrations corresponds to triads $(123)$ and $(124)$. The four fibers of the first fibration correspond to two triples of duads and two triples of synthemes:

- $(12), (23), (13)$
- $(15, 26, 34), (16, 23, 45), (13, 25, 46)$
- $(45), (46), (56)$
- $(15, 23, 46), (13, 26, 45), (16, 25, 34)$

The four fibers of the second fibration correspond to two triples of duads and two triples of synthemes:

- $(12), (24), (14)$
- $(15, 26, 34), (14, 25, 36), (16, 24, 35)$
- $(35), (36), (56)$
- $(14, 26, 35), (15, 24, 36), (16, 25, 34)$. 
The four common components $R_1, \ldots, R_4$ are $(-2)$-curves corresponding to duads $(12), (56)$ and synthemas $(15, 26, 34), (16, 25, 34)$. The bielliptic map $\phi : S_{\alpha, \beta} \to D$ blows them down to nonsingular points in $D$ ($D = D_1$ or $D_3$, depending whether $S_{\alpha, \beta}$ is a classical or a supersingular surface). We may assume that the image of a fiber is a line in the plane model of $D$. The proper transform of this line splits in the cover into two components of the fiber that different from $R_i$. This shows that the bielliptic involution $\tau$ acts on the set of 20 components of the two fibrations as the involution $(12)(56)$.

At the end of the previous section we listed the remaining 10 curves from the set $\mathcal{A} \cup \mathcal{B}$. They are $(15), (16), (25), (26), (34), (12, 36, 45), (14, 23, 56), (13, 24, 56), (12, 35, 46), (12, 34, 56)$. This set is invariant with respect to the involution $(12)(56)$. Each of these curves intersects the half-fibers $F_{123}$ and $F_{124}$ of the elliptic fibrations $|2F_{123}|$ and $|F_{124}|$ with multiplicity 1. Thus, the bielliptic map defines a bijective map from each curve to a rational quartic curve in $\mathbb{P}^4$. Its image under the involution splits into two curves from the set of ten curves from above. This shows $\tau$ leaves the set $\mathcal{A} \cup \mathcal{B}$ invariant and acts on this set as the involution $(12)(56)$. \hfill \Box

The products of two commuting transpositions generate the subgroup $\mathfrak{A}_6$ of $\mathfrak{S}_6$. We see the group $G$ generated by the ten involution acts on the set $\mathcal{A} \cup \mathcal{B}$ as a group isomorphic to $\mathfrak{A}_6$. Since the thirty $(-2)$-curves generate $\text{Num}(S_{\alpha, \beta})$, we obtain the kernel of $G \to \mathfrak{A}_6$ consists of numerically trivial automorphisms.

**Proposition 10.5.11.** Let $S$ be an Enriques or a Coble surface of Type MI. Then $\text{Aut}_{\text{nt}}(S)$ is trivial.

**Proof.** A numerically trivial automorphism preserves any of 10 Hesse pencils on $S$. It leaves all their fibers invariant. Thus it acts identically on the pencil and hence acts identically on each of the 18 inseparable bisections (or 9 sections if $S$ is a Coble surface). This shows that it fixes two many points on the generic fiber, and hence acts identically on it, and hence identically on the surface. \hfill \Box

**Corollary 10.5.12.** The group of automorphisms of $S_{\alpha, \beta}$ generated by 45 bielliptic involutions $\tau_{ij}$ is isomorphic to $\mathfrak{A}_6$.

Recall from Remark 10.5.1 that the Coble surface of Type MI is obtained from the Fermat cubic surface $C$ by blowing up the set of 9 points, the base points of a Hesse pencil $f_{abc} : V(P) \to \mathbb{P}^1$. Here we fix an isomorphism from $C$ to the blow-up of the set $Q(abc)$ of 6 points.

It is known that the group $\text{Aut}(C)$ is isomorphic to the subgroup $\text{PSU}_4(2)$ of $\text{PGL}_3(\mathbb{F}_2)$ that leaves invariant the Hermitian form $x^3 + y^3 + z^3 + w^3 = 0$ over $\mathbb{F}_4$. It contains the group $\mathfrak{S}_6$ as a maximal subgroup of index 36 isomorphic to the stabilizer subgroup of a double-sixer on the cubic surface. The subgroup $W(E_6) \cdot W(E_6)$ generated by the product of two simple reflections is of index 2 and it is isomorphic to the group $\text{PSU}_4(2)$. All elements in the coset are realized on the surface as the compositions of an automorphism and the Frobenius endomorphism.

The orbits of $\text{Aut}(C)$ on linear subspaces of $\mathbb{P}^3(\mathbb{F}_4)$ containing the set $C(\mathbb{F}_4)$ are known [187, Lemma 5.3]. One orbit is the set $C(\mathbb{F}_4)$ that consists of 45 points and coincides with the set of Eckardt points on $C$. Its stabilizer subgroup is a maximal subgroup of $W(E_6) \cdot W(E_6)$. There is also an orbit of 40 planes, each cuts out a smooth supersingular elliptic curve $E$ on $C$. Its stabilizer
subgroup is one of the two maximal subgroups of $W(E_6)^+$ of index 40 (another one is the stabilizer of a point in $\mathbb{P}^3(\mathbb{F}_4) \setminus C(\mathbb{F}_4)$). It is isomorphic to an extension of the binary tetrahedral group with the normal subgroup isomorphic to the Heisenberg group of order $3^3$. The quotient by the center is isomorphic to the Hesse group $G_{216}$ of automorphisms of a Hesse pencil. In our case it acts on the base of the fibration as the affine group $A^1(\mathbb{F}_4)$ and the kernel is isomorphic to the group $(\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$ of translations and the Bertini involution.

In fact, the plane contains 9 Eckardt points lying on the cubic, and the blowing up this set we obtain our Coble surface $V(\mathcal{P})$. The subgroup $3^2 \rtimes S_3$ of $G_{216}$ of order 54 is realized as a subgroup of $\text{Aut}(\mathbb{C})$ preserving one of 10 Hesse fibrations. We know that $\mathfrak{S}_6$ has the orbit of 10 Hesse fibrations on $V(\mathcal{P})$, thus the intersection of $G_{216}$ with $\mathfrak{S}_6$ is a subgroup of order 36.

The group $\text{Aut}(\mathbb{C})$ contains two conjugacy classes of involutions, type 2A and type 2B in terminology of the ATLAS. The Hesse subgroup contains an involution of type 2A, it acts on the Hesse pencil fixing pointwisely its unique smooth supersingular fiber. In our situation, it is realized by an involution from $\mathfrak{S}_6$ that preserves the triad $(abc)$, for example $(ab)(de)$. For such triad there exists a unique other triad $(ijk)$ that is invariant with respect to the same involution. For example, if $(abc) = (123)$ and the involution is as above, then $(ijk) = (126)$. This shows that all involutions of type 2A that occur in $\text{Aut}(V(\mathcal{P}))$ originate from the bielliptic involutions on the Coble surface similar to ones we studied in Proposition 10.5.10.

An involution $g$ of Type 2B is realized on $\mathbb{C}$ as an involution that leaves invariant a double-sixer and switches its sixers. If $p : \mathbb{C} \to \mathbb{P}^2$ is the blowing down of one sixer, then $g$ descends to a Cremona transformation $\tilde{g}$ that blows down six conics through 5 of its six fundamental points to the fundamental points. Note if we order the fundamental points $p_1, \ldots, p_6$ and denote by $C_k$ the conic through all points except $p_k$, then $\tilde{g}(C_k) = p_{\sigma(k)}$, where $\sigma$ is the product of three transpositions. The involution $\tilde{g}$ is given by the homaloidal linear system of plane quintics with double points at the fundamental points. In the standard geometric basis $g$ acts as the composition of the involution with respect to $2e_0 - e_1 - \cdots - e_6$ and the product of three commuting transpositions of the $e_i$'s. Its set of fixed points is a line on $\mathbb{C}$. Note the composition of two commuting involutions of the same type do not commute but the composition of two commuting involutions of types 2A and 2B is an involution of type 2A.

The known properties of the Fermat cubic in characteristic 2 show that each line on $\mathbb{C}$ contains 5 Eckardt points and we have altogether 45 Eckardt points. In our case, the set of 27 lines consists of nine lines $(ad), (ae), (af), (bd), (be), (bf), (cd), (ce), (cf)$, six lines $T_1, \ldots, T_6$ six conics through five points in $Q(abc)$ and six exceptional curves $E_p, p \in Q(abc)$.

Recall that an element of the Weyl group $W(E_6)$ that switches the ordered sixers in a double-sixer is a reflection $s_\alpha$ with respect to the vector $\alpha = 2e_0 - \sum_{p \in A} e_p$. Since this reflection does not belong to $W(E_6)^+$, it is never realized by an automorphism. However, it composition with an odd permutation from the subgroup $S_6$ of $W(E_{10})$ could be realized by an automorphism. In other words, a cubic surface may contain an automorphism that switches the sixers but does not preserve the orders of the set of six lines. Since $\text{Aut}(\mathbb{C}) \cong W(E_6)^+$, we see that in our case such an automorphism exists. It is of type 2B in terminology of the ATLAS. Its fixed locus is a line on the cubic surface. As a Cremona transformation it is given by a homaloidal linear systems of quintics with double points at points from $A$. 
We apply this by taking for $A$ the set $Q(abc)$. Since $\mathfrak{A}_6$ is a subgroup of $\text{Aut}(C)$, a choice of the quintic Cremona involution is unique up to composition with an even involution from $\mathfrak{A}_6$ that commutes with it.

**Example 10.5.13.** We take for $A$ the set

$$p_1 = [1, 0, 0], \quad p_2 = [0, 1, 0], \quad p_3 = [0, 0, 1], \quad p_4 = [1, 1, 1], \quad p_5 = [1, \epsilon, \epsilon^2], \quad p_6 = [1, \epsilon^2, \epsilon],$$

the set of vertices of two triangles $V(x_0x_1x_2)$ and $V((x_0 + x_1 + x_2)(x_0 + \epsilon x_1 + \epsilon^2 x_2)(x_0 + \epsilon^2 x_1 + \epsilon x_2))$. Consider the following quintic Cremona involution $g$:

$$[x_0, x_1, x_2] \mapsto [x_0^3x_1^2 + x_0^2x_1x_2 + x_0x_1^2x_2 + x_1^2x_2^2, x_0^3x_1x_2 + x_0^2(x_1^3 + x_2^3) + x_0x_1^2x_2 + x_0^2x_1x_2 + x_0^2x_2^2 + x_0x_1x_2^2 + x_1x_2^2].$$

We leave to the reader to verify that it is an indeed a Cremona involution. Let $C_i$ be the conic through the points $p_1, \ldots, p_6$ except $p_i$. Then $g$ blows down $(C_1, \ldots, C_6)$ to $(p_1, p_2, p_3, p_4, p_5, p_6)$. Its action on $\text{Pic}(C)$ in the geometric basis defined by the ordered 6-arc $A$ is given by the matrix

$$
\begin{pmatrix}
5 & 2 & 2 & 2 & 2 & 2 \\
-2 & -1 & 0 & -1 & -1 & -1 \\
-2 & 0 & -1 & -1 & -1 & -1 \\
-2 & -1 & -1 & 0 & -1 & -1 \\
-2 & -1 & -1 & -1 & 0 & -1 \\
-2 & -1 & -1 & -1 & -1 & 0
\end{pmatrix}.
$$

We see that its trace is equal to $-3$, and this agrees with the fact this transformation defines an involution of the cubic surface of type 2B. Its acts as the composition of two reflections with respect to the roots $2e_0 - e_1 - \cdots - e_6$ and $e_1 - e_2$.

We check that it switches the sides $V(x_0)$ and $V(x_1)$ of the first triangle and leaves its remaining side pointwise fixed. It leaves invariant the sides of the second triangle. It leaves invariant the 10 base points of the Hessian pencil spanned by the two triangles. It transforms the remaining 6-arc $\mathcal{P}$ of points $[\epsilon^i, 1, 1], [1, \epsilon^i, 1], [1, 1, \epsilon^i], i = 1, 2$ to itself by switching 3 pairs $[\epsilon, 1, 1], [\epsilon^2, 1, 1], \text{etc.}$.

The permutation is odd because $g$ does not come from a projective automorphism of the plane. The set $\mathcal{P}$ is the set of vertices of the remaining two triangles of the Hesse pencil. The image of the Hesse pencil under the involution is a Hesse pencil. The images of each of the two triangles with vertices at $\mathcal{P}$ is the union of 3 plane quintic curves with double points at $A$. The other two triangles are invariant with respect to $g$.

The centralizer of $g$ in $\text{Aut}(C)$ is the subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ generated by the even involutions of the points $p_3, p_4, p_5, p_6$. The product of $g$ with any such transformation is an involution of type 2A of the cubic surface $C$.

**Remark 10.5.14.** Over the complex numbers, S. Mukai obtained an Enriques surface which contains 30 $(-2)$-curves with the same dual graph as the above example. The canonical cover of the Mukai’s example is the intersection of three quadrics given by the equations:

$$
\begin{align*}
x^2 - (1 + \sqrt{3})yz &= u^2 - (1 - \sqrt{3})vw, \\
y^2 - (1 + \sqrt{3})xz &= v^2 - (1 - \sqrt{3})uw, \\
z^2 - (1 + \sqrt{3})xy &= w^2 - (1 - \sqrt{3})uv.
\end{align*}
$$

See [509, Remark 2.7]. We discussed this surface in Example 8.8.5.
The following seems to be a natural conjecture.

**Conjecture.**

\[ \text{Aut}(S_{\alpha,\beta}) \cong \mathfrak{A}(10) \rtimes \mathfrak{A}_6, \]

where \( \mathfrak{A}(10) \) is the universal Coxeter group with 10 generators identifies with the sets of triads and the group \( \mathfrak{A}_6 \) acts on this group as its natural action on the sets of triads.

The group of automorphisms of the K3 surface \( Y \) was described in [181]. It is generated by the group \( \text{PSL}_3(\mathbb{F}_4) \) that acts on \( Y \) via its action on \( \mathbb{P}^2(\mathbb{F}_4) \), 168 involutions that act on \( \text{Pic}(Y) \) as a lift of a quintic Cremona involution associated to 6-arc, and a switch. It is known that \( \text{PSL}_3(\mathbb{F}_4) \) (denoted by \( L_3(4) \) in [129]) contains 3 conjugacy classes of maximal subgroups isomorphic to \( \mathfrak{A}_6 \) (there is one conjugacy class in its extension \( \text{PGL}_3(\mathbb{F}_4) \)). It also contains one conjugacy class of maximal subgroup of order 72, the subgroups of index 3 of the Hesse group \( G_{216} \) of automorphisms of the Hesse pencil, or, equivalently, the automorphism group of the rational elliptic surface \( H \). Another maximal subgroup is the Klein group \( \text{PSL}_2(\mathbb{F}_7) \) of order 168. There are 3 conjugacy classes of such subgroups. One of these subgroups is \( \text{PSL}_3(\mathbb{F}_2) \). It acts transitively on the set of 6-arcs in the finite plane \( \mathbb{P}^2(\mathbb{F}_2) \) with the stabilizer subgroup of order 24.

Note that there are other types of elliptic fibrations on \( S_{\alpha,\beta} \). We have listed in (10.4.9) all maximal parabolic subdiagrams of rank 8 of \( \Gamma_{\text{CR}} \). Some of them contains a connected component which contains a vertex represented by a non-effective root, so this connected component does not represent a reducible fiber. Taking this into account, it is easy to prove the following.

**Proposition 10.5.15.** There exist exactly four types of elliptic fibrations on \( S_{\alpha,\beta} \) as follows:

\[ A_2 + A_2 + A_2 + A_2, \quad A_3 + A_3 + 2A_1^*, \quad A_4 + A_4 + A_0^* + A_0^*, \quad A_5 + 2A_1 + A_2. \]

Finally in this section we show that the Coble surface of Type MI is the specialization of surfaces \( S_{\alpha,\beta} \) of Type MI when \( (\alpha, \beta) \) becomes the excludes pair equal to \( (\epsilon, \epsilon^2) \).

We use this rational derivation \( \partial_{\epsilon,\epsilon^2} \). By a direct calculation we can check that the divisor of \( \partial_{\epsilon,\epsilon^2} \) is given by

\[ D = F_\epsilon^+ + F_\epsilon^+ - F_1^- - F_\infty^- \]

and the integral curves on the fibers over \( t = \epsilon, \epsilon^2 \) are \( F_\epsilon^+, F_\epsilon^- \). Compare these with Lemma 10.5.7. By contracting twelve \((-2)\)-curves appeared as components of \( D \), we have a surface \( X \) with twelve rational double points of type \( A_1 \). The induced derivation has isolated zeros at six singular points which are the images of \( F_\epsilon^-, F_\epsilon^- \). By applying Proposition 10.3.3, we can see that the quotient surface of \( Y \) by \( \partial_{\epsilon,\epsilon^2} \) is non-singular and by contracting six exceptional curves on the fibers over the points \( t = 1, \infty \) we obtain the Coble surface \( V(\mathcal{P}) \) discussed above.

**Theorem 10.5.16.** The Coble surface \( V(\mathcal{P}) \) of Type MI is a specialization \( \alpha = \epsilon, \beta = \epsilon^2 \) of the one-dimensional family \( \{ S_{\alpha,\beta} \} \) \( (\alpha, \beta \in \mathbb{K}, \alpha + \beta = \alpha \beta, \alpha^3 \neq 1) \) of Enriques surfaces of Type MI. It has thirty effective roots and ten non-effective \((-2)\)-classes. To each non-effective \((-2)\)-class we associate an involution induced from a quintic Cremona transformation. The automorphism group \( \text{Aut}(V) \) is isomorphic to \( \mathfrak{A}(10) \rtimes \mathfrak{A}_6 \).

**Example 10.5.17.** We will give a Coble surface in characteristic 3 with 2 boundaries whose au-

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4The existence of this example was kindly communicated to us by S. Mukai. See [511].
tomorphism group is finite. This surface contains forty effective roots whose dual graph is the extended Cremona-Richmond diagram.

Let \( X \) be the Fermat quartic surface
\[
x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0.
\]

The equation (10.5.6) is defined by a Hermitian form over \(\mathbb{F}_9\) and hence the unitary group \(\text{PGU}(4,\mathbb{F}_9)\) acts on \( X \) as projective automorphisms. It is known that \( X \) contains 112 lines. Let \( \ell, \ell' \) be two skew lines on \( X \). Let \( p \in X \) not lying on \( \ell \cup \ell' \). Then there exists a unique line \( \ell'' \) on \( X \) containing \( p \) and meeting \( \ell, \ell' \). Let \( q \in X \) satisfying \( \ell'' \cap X = \{ p, q, \ell'' \cap \ell, \ell'' \cap \ell' \} \). By associating \( q \) with \( p \), we have a birational involution \( s_{\ell, \ell'} \) of \( X \) which can be extended to a regular automorphism of \( X \) by the minimality of K3 surfaces (Segre [620], §16, Barth [45], §4). The fixed point set of \( s_{\ell, \ell'} \) is the union of \( \ell \) and \( \ell' \), and the quotient surface \( V \) of \( X \) by \( s_{\ell, \ell'} \) is a Coble surface with two boundary components.

The surface \( V \) is also obtained as follows (e.g. [359]). Let \( Q = \mathbb{P}^1(\mathbb{F}_9) \times \mathbb{P}^1(\mathbb{F}_9) \) and let \([u_0, u_1], [v_0, v_1]\) be homogeneous bi-coordinates on \( Q \). Denote by \( \zeta \) a primitive eighth root of unity with \( \zeta^2 = \sqrt{-1} \). Consider the following curves in \( Q \) defined by
\[
B : u_0 v_0^3 = u_1 v_1^3, \quad B' : u_0^3 v_0 = u_1^3 v_1.
\]

The curves \( B \) and \( B' \) meet at ten points
\[
p_1 : [(1, \zeta^i), [1, -\zeta^i]], \quad p_2 : [[1, \zeta^2], [1, \zeta^2]], \quad p_3 : [[1, \zeta^3], [1, -\zeta^3]],
\]
\[
p_4 : [(1, -1), [1, -1]], \quad p_5 : [[1, -\zeta], [1, \zeta]], \quad p_6 : [[1, -\zeta^2], [1, -\zeta^2]],
\]
\[
p_7 : [[1, -\zeta^3], [1, \zeta^3]], \quad p_8 : [[1, 1], [1, 1]], \quad p_9 : [[1, 0], [0, 1]], \quad p_{10} : [[0, 1], [1, 0]].
\]

Let
\[
C_i : u_1 = \zeta^i u_0 \quad (1 \leq i \leq 8), \quad C_9 : u_1 = 0, \quad C_{10} : u_0 = 0,
\]
\[
C_i' : v_1 = \zeta^i v_0 \quad (1 \leq i \leq 8), \quad C_9' : v_1 = 0, \quad C_{10}' : v_0 = 0.
\]

Then \( C_i \) (resp. \( C_i' \)) meets \( C \) (resp. \( C_i' \)) at one of the above ten points with multiplicity 3. Also there are 30 curves of bidegree \( (1, 1) \) passing through four points from the above ten points.

Let \( V \) be the surface obtained by blowing ups \( Q \) at ten points and let \( E_i' \) be the exceptional curve over the point \( p_i \). We denote by the same symbols the proper transforms of the above curves. Then \( B^2 = B'^2 = -4, C_i^2 = C_i'^2 = -1 \) and 30 curves have the self-intersection number \(-2\). By taking the double cover of \( V \) branched along \( B + B' \) we obtain a non-singular surface which is nothing but the Fermat quartic surface \( X \). The preimages of \( B, B' \) are two skew lines and those of \( E_i' \) are ten lines meeting the two skew lines. The preimage of \( C_i \) (resp. \( C_i' \)) splits into two \((-2)\)-curves and each of the above 30 \((-2)\)-curves splits into disjoint two \((-2)\)-curves. Thus we have 112 \((= 2 + 10 + 40 + 60)\) \((-2)\)-curves on \( X \) which are lines on \( X \).

There are 30 \((-2)\)-curves on \( V \) which are identified with the duads and synthemes in the following way. In the bracket we indicate four points among ten points to which the curve passes.

\begin{align*}
(12) & : u_0 v_0 - u_1 v_1 = 0 \quad (p_4, p_8, p_9, p_{10}), \\
(13) & : u_0 v_0 - \zeta^3 (u_0 v_1 + u_1 v_0) + \zeta^2 u_1 v_1 = 0 \quad (p_2, p_3, p_4, p_7), \\
(14) & : u_0 v_0 + \zeta (u_0 v_1 + u_1 v_0) - \zeta^2 u_1 v_1 = 0 \quad (p_1, p_2, p_5, p_8),
\end{align*}
We can also identify 10 exceptional curves $E'_i$ with ten elements in $S_6$ as follows:

(15) : $u_0v_0 + \zeta^2u_1v_0 - u_1v_1 = 0 \ (p_3, p_5, p_6, p_9),$
(16) : $u_0v_0 + \zeta^2u_0v_1 - u_1v_1 = 0 \ (p_1, p_6, p_7, p_{10}),$
(23) : $u_0v_0 - \zeta(u_0v_1 + u_1v_0) - \zeta^2u_1v_1 = 0 \ (p_1, p_4, p_5, p_6),$
(24) : $u_0v_0 + \zeta^3(u_0v_1 + u_1v_0) + \zeta^2u_1v_1 = 0 \ (p_3, p_6, p_7, p_8),$
(25) : $u_0v_0 - \zeta^2u_1v_0 - u_1v_1 = 0 \ (p_1, p_2, p_7, p_9),$
(26) : $u_0v_0 - \zeta^2u_0v_1 - u_1v_1 = 0 \ (p_2, p_3, p_5, p_{10}),$
(34) : $u_0v_0 + u_1v_1 = 0 \ (p_2, p_6, p_9, p_{10}),$
(35) : $u_0v_0 + u_0v_1 + u_1v_1 = 0 \ (p_1, p_3, p_8, p_{10}),$
(36) : $u_0v_0 + u_1v_0 + u_1v_1 = 0 \ (p_5, p_7, p_8, p_9),$ 
(45) : $u_0v_0 - u_0v_1 + u_1v_1 = 0 \ (p_4, p_5, p_7, p_{10}),$ 
(46) : $u_0v_0 - u_1v_0 + u_1v_1 = 0 \ (p_1, p_3, p_4, p_9),$
(56) : $u_0v_1 - u_1v_1 = 0 \ (p_2, p_4, p_6, p_8),$
(12, 34, 56) : $u_0v_1 + u_1v_0 = 0 \ (p_1, p_3, p_5, p_7),$
(12, 35, 46) : $u_0v_0 + u_0v_1 - u_1v_0 + u_1v_1 = 0 \ (p_2, p_5, p_6, p_7),$ 
(12, 36, 45) : $u_0v_0 - u_0v_1 + u_1v_0 + u_1v_1 = 0 \ (p_1, p_2, p_3, p_6),$ 
(13, 24, 56) : $u_0v_0 - \zeta^2u_1v_1 = 0 \ (p_1, p_5, p_9, p_{10}),$
(13, 25, 46) : $u_0v_0 - \zeta^3u_0v_1 + \zeta^2u_1v_1 = 0 \ (p_5, p_6, p_8, p_{10}),$
(13, 26, 45) : $u_0v_0 - \zeta^3u_1v_0 + \zeta^2u_1v_1 = 0 \ (p_1, p_6, p_8, p_9),$ 
(14, 23, 56) : $u_0v_0 + \zeta^2u_1v_1 = 0 \ (p_3, p_7, p_9, p_{10}),$
(14, 25, 36) : $u_0v_0 + \zeta u_0v_1 - \zeta^2u_1v_1 = 0 \ (p_3, p_4, p_6, p_{10}),$
(14, 26, 35) : $u_0v_0 + \zeta u_1v_0 - \zeta^2u_1v_1 = 0 \ (p_4, p_6, p_7, p_9),$
(15, 23, 46) : $u_0v_0 - \zeta u_0v_1 - \zeta^2u_1v_1 = 0 \ (p_2, p_7, p_8, p_{10}),$
(15, 24, 36) : $u_0v_0 + \zeta^3u_0v_1 + \zeta^2u_1v_1 = 0 \ (p_1, p_2, p_4, p_{10}),$
(15, 26, 34) : $u_0v_0 - \zeta^2(u_0v_1 - u_1v_0) - u_1v_1 = 0 \ (p_1, p_4, p_7, p_8),$
(16, 23, 45) : $u_0v_0 - \zeta u_1v_0 - \zeta^2u_1v_1 = 0 \ (p_2, p_5, p_8, p_9),$
(16, 24, 35) : $u_0v_0 + \zeta^3u_1v_0 + \zeta^2u_1v_1 = 0 \ (p_2, p_4, p_5, p_9),$ 
(16, 25, 34) : $u_0v_0 + \zeta^2(u_0v_1 - u_1v_0) - u_1v_1 = 0 \ (p_3, p_4, p_5, p_8).$
Now we define ten effective roots by

\[ E_i = \frac{1}{2}B + \frac{1}{2}B' + 2E'_i. \]

The dual graph of \( E_1, \ldots, E_{10} \) is the complete graph with double edges. It is now easily to see that the dual graph of 40 effective roots \( \{(ij), (ij, kl, mn), E_i\} \) is the same as that of 40 \((-2)\)-divisors on Enriques surfaces given in Example A. Since 40 \((-2)\)-classes are effective, by Vinberg’s theorem 10.8.2.2, \( \text{Aut}(\mathcal{V}) \) is finite. The symmetry group of this dual graph is the automorphism group \( \text{Aut}(\mathcal{S}_6) \cong \mathbb{S}_6 \cdot \mathbb{Z}/2\mathbb{Z} \) of the symmetric group \( \mathbb{S}_6 \). On the other hand, a subgroup of \( \text{Aut}(\mathcal{Q}) \) of order 1440 generated by \( \text{PGL}(2, \mathbb{F}_9) \) and an involution \( [[[u_0, u_1], [v_0, v_1]] \to [[v_0, v_1], [u_0, u_1]] \) preserves \( B, B' \) and hence acts on \( \mathcal{V} \) as automorphisms. Thus we have \( \text{Aut}(\mathcal{V}) \cong \text{Aut}(\mathcal{S}_6) \). We now have the following theorem.

**Theorem 10.5.18.** The surface \( \mathcal{V} \) is a Coble surface of type MI in characteristic 3 with two boundary components which is a double quadric model of the Fermat quartic surface. It has 30 effective \((-2)\)-curves forming duads and synthemes, and ten effective roots. The automorphism group \( \text{Aut}(\mathcal{V}) \) is isomorphic to \( \text{Aut}(\mathcal{S}_6) \). The \( R \)-invariant \( (K, H) \) is \( (2A_5 \oplus 2A_1, (\mathbb{Z}/2\mathbb{Z})^3) \).

**Proof.** We prove the last assertion. From 30 \((-2)\)-curves, we have 2\( A_5 \) and from two boundaries we have 2\( A_1 \). Since the Coble-Mukai lattice is isomorphic to \( E_{10} \) (Example 9.2.5), \( \dim H \geq 3 \). Since the dimension of the kernel of \( q_{K/2K} \) is 3, we obtain \( H = (\mathbb{Z}/2\mathbb{Z})^3 \).

### 10.6 Quotients of the K3 surface with Artin invariant 1: Type MII

The surface \( Y \) admits elliptic fibrations of other types. All genus one fibrations on the supersingular K3 surface \( Y \) with Artin invariant 1 are classified into 18 types (see [210], [408]).

The following is the list of reducible singular fibers of genus one fibrations on \( Y \):

**Elliptic fibrations:**
\[
\tilde{A}_5 + \tilde{A}_5 + \tilde{A}_5 + \tilde{A}_5, \quad \tilde{A}_7 + \tilde{A}_7 + \tilde{D}_5, \quad \tilde{A}_9 + \tilde{A}_9 + \tilde{A}_1 + \tilde{A}_1, \quad \tilde{A}_{11} + \tilde{D}_7, \\
\tilde{A}_{11} + \tilde{A}_3 + \tilde{E}_6, \quad \tilde{E}_6 + \tilde{E}_6 + \tilde{E}_6, \quad \tilde{A}_{15} + \tilde{D}_5, \quad \tilde{A}_{17} + \tilde{A}_1 + \tilde{A}_1 + \tilde{A}_1.
\]

**Quasi-elliptic fibrations:**
\[
\tilde{D}_4 + \tilde{D}_4 + \tilde{D}_4 + \tilde{D}_4, \quad \tilde{D}_6 + \tilde{D}_6 + \tilde{D}_6 + \tilde{A}_1^* + \tilde{A}_1^*, \quad \tilde{D}_4 + \tilde{D}_8 + \tilde{D}_8, \quad \tilde{D}_{10} + \tilde{E}_7 + \tilde{A}_1^* + \tilde{A}_1^* + \tilde{A}_1^*, \\
\tilde{D}_6 + \tilde{E}_7 + \tilde{E}_7, \quad \tilde{E}_8 + \tilde{D}_{12}, \quad \tilde{D}_4 + \tilde{D}_{16}, \quad \tilde{D}_{12} + \tilde{E}_8, \quad \tilde{D}_4 + \tilde{E}_8 + \tilde{E}_8, \quad \tilde{D}_{20}.
\]

Moreover it is shown in [210] that these 18 types are unique up to automorphism of the surface. Any genus one fibration on \( Y \) has a section. The existence of a quasi-elliptic fibration with a singular fiber of type \( \tilde{D}_{20} \) shows that \( \text{Pic}(Y) \) is isomorphic to \( U \oplus D_{20} \).

Recall that in the previous section we have constructed a rational vector field \( \partial_{\alpha, \beta} \) with quotient birationally isomorphic to an Enriques surface \( S_{\alpha, \beta} \) by choosing one of these fibrations, namely the (double) Hesse type with four singular fibers of type \( \tilde{A}_5 \). The surface \( S_{\alpha, \beta} \) has 12 canonical points, and its canonical cover has 12 ordinary double points. A natural question is whether we can
construct other examples by choosing different fibration. It turns out that only two more fibrations work, namely types $\tilde{A}_7 + \tilde{A}_7 + \tilde{D}_5$ and $\tilde{A}_9 + \tilde{A}_9 + \tilde{A}_1 + \tilde{D}_1$ [406]. The last case leads to a construction of an Enriques surface of type VII in characteristic 2 which we have already discussed in Section 8.10. In this section we will discuss the remaining construction.

We start with a construction of an elliptic fibration of type $\tilde{A}_7 + \tilde{A}_7 + \tilde{D}_5$ on $Y$.

Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$, considered as a quadric over $\mathbb{F}_4$. The set $Q(\mathbb{F}_4)$ consists of 25 points $(q_i, q_j)$, where $(q_1, q_2, q_3, q_4, q_5) = (0, 1, \epsilon, \epsilon^2, \infty)$. Let $F_{i,a} = \pi_i^{-1}(a)$, where $\pi_i : Q \to \mathbb{P}^1$ are the two projections and $a \in \mathbb{P}^1(\mathbb{F}_4)$. Let $P = (\infty, \infty)$. The quadric contains 85 = $\# \mathbb{P}^3(\mathbb{F}_4)$ conics, among them 25 are reducible and correspond to tangent hyperplanes at points in $Q(\mathbb{F}_4)$. We consider irreducible conics passing through $P$. There are $12 = \# \mathbb{P}^2(\mathbb{F}_4) - 9$ of them that correspond to planes passing through $P$ and not containing $F_{1,\infty}$. Since there are 3 tangent directions defined over $\mathbb{F}_4$ and different from $F_{1,\infty} = F_{2,\infty}$, they are divided into the union of 3 sets, two conics from the same set are tangent at $P$. Among these 12 conics one is the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$ that passes through $(q_i, q_i)$. All other are obtain from this conic by applying elements of $\text{PGL}_2(\mathbb{F}_4) \times \text{PGL}_2(\mathbb{F}_4)$ that fix $(\infty, \infty)$. We may assume that a conic passes through the points $(q_1, q_{\sigma(1)}), \ldots, (q_4, q_{\sigma(4)})$, where $\sigma$ is a permutation if $\{1, 2, 3, 4\}$. Since elements of the $\text{PGL}_2(\mathbb{F}_4)_{\infty}$ define even permutations of the point 0, 1, $\epsilon$, $\epsilon^2$, we see that the conic correspond to elements of $\mathfrak{A}_4$.

We say that the union of two fibers of $\pi_1$ and two fibers of $\pi_2$ is a quadrangle. Fix two disjoint quadrangles $T_1$ and $T_2$. Let $B = \{(q_i, q_j) : 1 \leq i, j \leq 4\}$. The blow-up $V \to Q$ of $B$ is a Coble surface with 8 boundary components. The Coble surface $V$ is called the Coble surface of Type MII in characteristic 2.

Besides the boundary it contains 28 curves, the proper transforms of the conics and the exceptional curves. We associate to each exceptional curve $E_{ij}$ over $(q_i, q_j) \in B$, the corresponding effective root $2E_{ij} + \frac{1}{2}(F_{1,i} + F_{2,j})$. We also have 12 non-effective roots

$$\alpha_{i_1,i_2,i_3,i_4} = h - e_{b_{1,i_1}} - e_{b_{2,i_2}} - e_{b_{3,i_3}} - e_{b_{4,i_4}},$$

where $h$ is the divisor class of a hyperplane section and $b_{k,i_k} = F_{1,k} \cap F_{2,i_k} \in B$. Since the roots are not effective $(i_1, i_2, i_3, i_4)$ is an odd permutation of $\{1, 2, 3, 4\}$.

Let us denote the classes of 12 conics by $\alpha_\sigma, \sigma \in \mathfrak{A}_4$, the classes of 12 non-effective roots by $\beta_\sigma, \sigma \in \mathfrak{S}_4 \setminus \mathfrak{A}_4$, and the classes of irreducible effective roots by $\alpha_{ij}, 1 \leq i, j \leq 4$.

Let $\Gamma_{MII}$ be the intersection diagram of the 40 divisor classes. It is determined by the following properties:

- $\beta_\sigma \cdot \beta_\tau = 1$ if $(\sigma^{-1} \circ \tau)^3 = 1$ and 2 if otherwise.
- $\beta_\sigma \cdot \alpha_{ab} = 2$ if $\sigma = (ab)$ or $(\sigma \circ (ab))^3 = 1$, 0 otherwise.
- $\beta_\sigma \cdot \alpha_r = 2$ if $(\sigma \circ \tau)^2 \neq 1$ and 0 otherwise.
- $\alpha_{ab} \cdot \alpha_\sigma = 2$ if $(\sigma \circ (ab))^2 = 1$ and 0 otherwise.
- $\alpha_{ab} \cdot \alpha_{ij} = 1$ if $\#(ab) \cap (ij) = 1$ and 0 otherwise.
- $\alpha_\sigma \cdot \alpha_r = 2$ if $(\sigma \circ \tau)^3 = 1$ and 1 otherwise.
Note that the subgraph with 16 vertices $\alpha_{ab}$ is dual to the complete bipartite graph $BK(4)$ on two sets of cardinality 4:

![Figure 10.8]

As we remarked earlier, the 12 conics are grouped in subsets such that conics in the same subsets are tangent at $P$ and conics from different subsets intersect with multiplicity 1 at $P$. In our notation in terms of substitutions, we may assume that the three groups are

\[ \{1, (12)(34), (13)(24), (14)(23)\} \times \{(123), (134), (142), (243)\}, \{(132), (124), (143), (234)\}. \]

The subgraph with vertices at the vectors $\alpha_{\sigma}$ is the union of two 3 complete graphs $K(4)$ with double edges, each vertex of one subgraph is joined to all other vertices from two remaining subgraphs.

**Remark 10.6.1.** The reader may be more comfortable to see the previous picture after we project the quadric from the point $P$. The lines $F_{1,\infty}$ and $F_{2,\infty}$ are blown down to two points $p_1, p_2$, and the image of the exceptional curve at $P$ is the line $\ell$ joining these points. Denote by $p_1, \ldots, p_5$ the five $\mathbb{F}_4$-rational points on $\ell$. For $i = 1, 2$, let $\ell_{ij}$ ($j = 1, \ldots, 4$) be the four lines through $p_i$ except $\ell$. The images of the 12 conics are the lines passing through $p_3, p_4, p_5$. The image of the boundary are two sets of lines $\ell_{ij}$ ($i = 1, 2$) (see Figure 10.9).

![Figure 10.9]

As in the previous section, we can consider the reflection polytope $\Pi_{MII}$ in $\mathbb{H}^0$ associated with the diagram $\Gamma_{MII}$. The group of symmetries of $\Pi_{MII}$ is generated by $\mathfrak{S}_4 \times \mathfrak{S}_4$ acting as the product
of the affine groups of $\mathbb{P}^1(\mathbb{F}_4)$ and the switching the factors in $\mathbb{P}^1(\mathbb{F}_4) \times \mathbb{P}^1(\mathbb{F}_4)$. We have
\[ \text{Sym} (\Pi_{\text{MII}}) \cong (\mathfrak{A}_4 \times \mathfrak{A}_4) \rtimes \mathbb{Z}/2\mathbb{Z}. \]
All these symmetries are effective in the sense that they come from automorphisms of the Coble surface.

Note that the surface $V_\sigma$ obtained by blowing up $B(\sigma)$ is a special del Pezzo surface of degree 4, the blow-up of 5 points in $\mathbb{P}^2$. The group of automorphism of a del Pezzo surface in characteristic 2 is described in [187, Theorem 3.1]. There is a unique quartic del Pezzo surface $D_4$ with automorphism group $2^4 \times \mathfrak{A}_4$. Recall that the elementary abelian subgroup $2^4$ is always present. It acts on the set of 5 conic pencils on the surface identically. The image of $\text{Aut}(V_\sigma)$ in the group of permutation of 5 pencils is what distinguishes the automorphism group of quartic del Pezzo surfaces. In characteristic 2, there are three possibilities for the image of $\text{Aut}(V_\sigma)$ in $\mathfrak{S}_5$: trivial, $(\mathbb{Z}/2\mathbb{Z})^3 \rtimes \mathfrak{A}_4$. In our case the group $\mathfrak{A}_4$ clearly acts on $V_\sigma$ by permuting the set $B(\sigma)$, so our surface is isomorphic to the surface $D_4$.

In the model of $D_4$ as the blow-up of 5 points in the plane, our surface is the blow-up of a 5-arc in $\mathbb{P}^2(\mathbb{F}_4)$. Completing it to a 6-arc, we obtain that the blow-up of one point on $D_4$ is isomorphic to the Fermat cubic surface. Equivalently, we see that our del Pezzo surface is obtained by blowing up one line on the Fermat cubic $C$. It agrees with the fact that the stabilizer subgroup of a line in $\text{Aut}(C) \cong W(\mathbb{E}_6)^+$ is a maximal subgroup isomorphic to $2^5 \times \mathfrak{A}_5$. However, none of the 16 points $q_{i,j}$ which we blow-up to get our Coble surface complete $B(\sigma)$ to a set of 5 points that are projected to a 6-arc in the plane. So, our Coble surface is not the blow-up of the Fermat cubic as in the previous case.

The equation of the anti-canonical model of the quartic del Pezzo surface with automorphisms group $2^4 \rtimes \mathfrak{A}_5$ can be found in [187, §3].

\[ \epsilon x_1^2 + \epsilon^2 x_2^2 + x_1x_3 + x_2x_4 = \epsilon^2 x_0^2 + \epsilon x_1^2 + x_1x_3 + x_2x_4 = 0, \quad (10.6.1) \]
where $\epsilon^3 = 1, \epsilon \neq 1$. Let us record what we discussed before.

**Proposition 10.6.2.** The Coble surface $V$ of type MIII is isomorphic to the blow-up of 12 points on a unique del Pezzo surface $D_4$ of degree 4 given in equation (10.6.1) with the group of automorphisms $2^4 \rtimes \mathfrak{A}_5$.

Let us see that one can also associate an automorphism of $V$ to any non-effective roots $\beta_\sigma, \sigma \in \mathfrak{S}_4 \setminus \mathfrak{A}_4$. To do this, we consider the linear system $|3h - p_{1,\sigma(1)} - p_{2,\sigma(2)} - p_{3,\sigma(3)} - p_{4,\sigma(4)}|$ of curves of bi-degree $(3, 3)$ with double points at the four points in $B$. One can choose a basis in this linear system such that this linear system defines a birational automorphism $\phi$ of $Q$. The map blows down the four conics that pass through three of the fundamental points to a subset of four points defining another root $\beta_\tau$. Composing it with the automorphism of $Q$ coming from $\text{Sym} (\Pi_{\text{MII}})$, we may assume that the image of each conic is a fundamental point. This shows that $\phi$ lifts to the blow-up of $B(\sigma) = \{p_{1,\sigma(1)}, p_{2,\sigma(2)}, p_{3,\sigma(3)}, p_{4,\sigma(4)}\}$ and hence defines an automorphism $g_\sigma$ of $V$.

Let $g_\sigma^0$ be the action of $g_\sigma$ on $\text{Pic}(V)$. We choose a geometric basis formed by the divisor classes $h_1, h_2$ of two rulings and the classes $e_b$ of the exceptional curves $E_b, b \in B$ (ordered in some way).
Then
\[ g_\sigma^*(h) = s_{\beta_\sigma}(h) = 3h - 2 \sum_{b \in B(\sigma)} e_b, \]
\[ g_\sigma^*(h_1) = s_{\beta_\sigma}(h_1) = 2h_1 + h_2 - \sum_{b \in B(\sigma)} e_b, \]
\[ g_\sigma^*(h_2) = s_{\beta_\sigma}(h_2) = h_1 + 2h_2 - \sum_{b \in B(\sigma)} e_b. \]

However, \( s_{\beta_\sigma}(e_b) = h + e_b - \sum_{b \in B(\sigma)} e_b \) may differ from \( g_\sigma^* \) by a permutation \( \tau \) of \( B \) leaving invariant \( B(\sigma) \) coming from \((\mathbb{A}_4 \times \mathbb{A}_4) \times 2 \subset Aut(V)\). Since \( g \) leaves invariant \( 2h - \sum_{b \in B(\sigma)} e_b = -K_{D_4} \), it comes from an automorphism of the quartic del Pezzo surface \( D_4 \) that together with the subgroup \( \mathbb{A}_4 \) generated \( \mathbb{A}_5 \). This shows that we can also choose \( \tau \) to be an odd involution of the set \( B(\sigma) \) such that its composition with \( s_{\beta_\sigma} \) is an even involution. Thus we can always choose \( g_\sigma \) to be an involution.

We now conclude the following.

**Theorem 10.6.3.** Let \( V \) be the Coble surface of type MII in characteristic 2 with eight boundary components. It has twelve \((-2)\)-curves, sixteen effective roots and twelve non-effective \((-2)\)-classes satisfying the condition in Theorem 0.8.22. The automorphism group \( Aut(V) \) is generated by \((\mathbb{A}_4 \times \mathbb{A}_4) \times \mathbb{Z}/2\mathbb{Z}\) and the group \( G(12) \) generated by twelve involutions \( g_\sigma, \sigma \in \mathbb{S}_4 \setminus \mathbb{A}_4 \).

Let \( Z \to V \) be the canonical cover of \( V \) defined by the invertible sheaf \( \omega_V^{-1} \) and the section of \( \omega_V^{-2} \) whose scheme of zeros is the boundary \( C_1 + \cdots + C_8 \). Applying Proposition 0.2.7, we find that it is expected to have 4 ordinary double points. Since \( \omega_Z = \mathcal{O}_Z \) and \( b_2(Z) = b_2(V) = 18 \), we see that \( Z \) birationally isomorphic to a supersingular K3 surface \( Y \). We can see this in another way. Consider the pencil of quartic elliptic curves on the quadric spanned by the two quadrangles of lines on the quadric. It contains another reducible member, the union of two conics tangent at \( P \) and passing through the set of 8 base points. The proper transforms of this pencil on \( Y \) is an elliptic pencil of type \( A_7 + A_7 + D_5 \). It follows from the Shioda-Tate formula that the surface \( Y \) is a supersingular K3 surface with Artin invariant 1. This agrees with the known list of possible elliptic fibrations on such surface which we reproduced in the beginning of this section.

So, we have now a situation similar to one we discussed in the previous section. It suggests an existence of a family of Enriques surfaces \( S \) which are quotients of \( Y \) by a rational vector field. We now proceed to do this. Unfortunately \( V \) is not a specialization of this family.

Recall that the supersingular K3 surface \( Y \) with Artin invariant 1 is the minimal resolution of a purely inseparable double cover of \( \mathbb{P}^2 \). We use the same notation as in Remark 10.6.1. Let \( L, L_{ij} \) the proper transforms of \( \ell, \ell_{ij} \) on \( Y \). Also denote by \( E_3, E_4, E_5 \) the exceptional curves over the points \( p_3, p_4, p_5 \). Let \( X \) be the surface obtained by contracting \( L_{ij}, L, E_3, E_4, E_5 \) which has eight rational double points of type \( A_1 \) and one rational double point of type \( D_4 \). We shall construct a classical Enriques surface \( S \) whose canonical cover is \( X \). The Enriques surface \( S \) contains 28 \((-2)\)-curves. Sixteen of them are the images of the sixteen exceptional curves \( E_{ij} \) on \( Y \) over the sixteen intersection points of \( \ell_{11} \) and \( \ell_{2j} \), and twelve of them are the images of the twelve lines on \( \mathbb{P}^2(F_4) \) through \( p_3, p_4, \) or \( p_5 \). The sixteen \((-2)\)-curves are the straight lines on the left hand side in Figure 10.8 and the eight black circles are the eight canonical points which are the images of eight singular
points of type $A_1$. The twelve $(-2)$-curves are twelve curves on the right hand side which pass the black circle corresponding to the other canonical point, i.e. the image of a singular point of type $D_4$. These twelve $(-2)$-curves are divided into three groups each of which consist of four $(-2)$-curves touching each other at the point corresponding to the black circle. Thus the canonical cover of the desired Enriques surface $S$ has $8A_1 + D_4$ as singularities and hence $S$ is classical (Theorem 10.2.2).

To construct $S$ we consider a rational elliptic surface defined by

$$y^2 + xy + s(s + 1)y = x^3 + s(s + 1)x^2$$  \hfill (10.6.2)

which has two singular fibers of type $\tilde{A}_3$ over $s = 0, 1$ and a singular fiber of type $\tilde{A}_2^*$ over $s = \infty$ (see Table 4.7). By taking the Frobenius base change $s = t^2$, we have an elliptic fibration on $p : Y \to \mathbb{P}^1$ defined by

$$y^2 + xy + t^2(t + 1)^2y = x^3 + t^2(t + 1)^2x^2.$$  \hfill (10.6.3)

The fibration $p$ has two singular fibers of type $\tilde{A}_7$ over $t = 0, 1$ and a singular fiber of type $\tilde{D}_5$ over $t = \infty$. In the following, we may assume that the fibration defined by the linear system

$$|L_{11} + E_{11} + L_{21} + E_{21} + L_{12} + E_{22} + L_{22} + E_{12}|,$$

induced from $|\ell_{11} + \ell_{21} + \ell_{12} + \ell_{22}|$, which has another fiber $L_{13} + E_{33} + L_{23} + E_{43} + L_{14} + E_{44} + L_{24} + E_{34}$ of type $\tilde{A}_7$ and a fiber of type $\tilde{D}_5$ consisting of $L, E_3, E_4, E_5, F_1, F_2$ as its components, where $F_1, F_2$ are the pull backs of lines on $\mathbb{P}^2(\mathbb{P}^1)$ passing through a point among $p_3, p_4, p_5$.

Now let us consider a rational derivation $\partial_{\alpha,\beta}$ on $Y$ induced by

$$\frac{1}{t(t + 1)} \left( t(t + 1)(t + \alpha)(t + \beta) \frac{\partial}{\partial t} + \alpha \beta (x + t^2(t + 1)^2) \frac{\partial}{\partial x} \right)$$  \hfill (10.6.4)

where $\alpha, \beta \in \mathbb{K}^*$, $\alpha + \beta = 1$. A direct calculation shows the following (also see Lemmas 10.3.13, 10.3.8).

**Lemma 10.6.4.**

(i) $\partial_{\alpha,\beta}^2 = \partial_{\alpha,\beta}$, namely, $\partial_{\alpha,\beta}$ is 2-closed and of multiplicative type.

(ii) On the surface $Y$, the divisor $D$ of $\partial_{\alpha,\beta}$ is given by

$$D = -(L_{11} + L_{12} + L_{21} + L_{22} + L_{13} + L_{14} + L_{23} + L_{24} + 2(L + E_3 + E_4 + E_5))$$

and $D^2 = -24$.

(iii) The integral curves with respect to $\partial_{\alpha,\beta}$ in the fibers of $p : Y \to \mathbb{P}^1$ are the following:

the smooth fibers over the points $t = \alpha, \beta$ and


Now by the same argument as in the previous example, the quotient surface $Y^{\partial_{\alpha,\beta}}$ has 11 exceptional curves of the first kind which are the images of the above integral curves. By contracting these curves and then contracting the image of $L$ we get a smooth Enriques surface $S = S_{\alpha,\beta}$. The elliptic fibration $p : Y \to \mathbb{P}^1$ induces an elliptic fibration $\pi : S_{\alpha,\beta} \to \mathbb{P}^1$ which has two singular fibers of type $\tilde{A}_3$ and a singular fiber of type $\tilde{A}_2^*$ consisting of the images of $F_1$ and $F_2$. Since the images of two smooth integral elliptic curves over $t = \alpha, \beta$ are double fibers of the elliptic fibration, $S_{\alpha,\beta}$ is a classical Enriques surface. It is not difficult to see that $S_{\alpha,\beta}$ contains 28 $(-2)$-curves as in the Figure 10.8.
We will refer to the surface $S_{α,β}$ as an *Enriques surface of Type MII in characteristic 2* because it has a polytope $Π_{MII}$ (see Remark 10.6.7).

**Theorem 10.6.5.** There exists a 1-dimensional family $⎨S_{α,β}⎬$ of classical Enriques surfaces $S_{α,β}$ of type MII where $α,β ∈ k^*$, $α + β = 1$. The canonical cover $X$ has eight nodes and one rational double point of type $D_4$. Its resolution of singularities is the supersingular K3 surface $Y$ with Artin invariant 1. The surface $S_{α,β}$ contains $28 (−2)$-curves as in the Figure 10.8.

**Proposition 10.6.6.** There exist exactly five types of elliptic fibrations on $S_{α,β}$ as follows:

\[ \tilde{A}_3 + \tilde{A}_5 + 2\tilde{A}_4^*, \tilde{A}_5 + \tilde{A}_3 + 2\tilde{A}_4^*, 2\tilde{A}_2^* + 2\tilde{A}_5^* + \tilde{A}_7 + \tilde{A}_1^* . \]

It follows that $S_{α,β}$ exists only as a classical Enriques surface.

**Remark 10.6.7.** Let $W$ be the subgroup of $W_{nod}$ generated by reflections associated with $28 (−2)$-curves as mentioned above. Then $W$ is not of finite index in $O(Num(S_{α,β}))$. In fact the automorphism group $Aut(S_{α,β})$ is infinite. However there exist 12 non effective $(−2)$-classes in $Num(S_{α,β})$ such that the classes of $28 (−2)$-curves and these 12 $(−2)$-classes satisfy the Vinberg’s theorem 0.8.22, that is, the subgroup generated by reflections associated with these $40 (−2)$-classes is of finite index in $O(Num(S_{α,β}))$. These 12 $(−2)$-classes bijectively correspond to 12 $(−4)$-classes among 168 $(−4)$-classes which are perpendicular to $A_1^{⊙8} ⊕ D_4$ generated by exceptional curves over 8 nodes and a rational double point of type $D_4$ on the canonical cover of $S$. The symmetry group of the dual graph of $40 (−2)$-classes is $(Q_4 × Q_4) × Z/2Z$ (see Figure 10.9). This remarkable diagram of $(−2)$-classes was first discovered by Shigeru Mukai in case of an Enriques surface over the complex number covered by the Kummer surface associated with a curve of genus 2 (see Example 10.7.12).

**Example 10.6.8.** This is an example of a Coble surface in characteristic 3 with 40 effective irreducible roots forming a polytope $Π_{MII}$. In particular this Coble surface has a finite automorphism group. The example was communicated to us by S. Mukai ([511]).

Let $E$ be a curve defined by $y^2 = x^3 + x$ which is a supersingular elliptic curve in characteristic 3. Let $Kum(E × E)$ be th Kummer surface associated with the product of $E$. It is known that $Kum(E × E)$ is the supersingular K3 surface with Artin invariant 1. The involution $1_E × (−1_E)$ of $E × E$ descends to an involution $σ$ of $Kum(E × E)$ which fixes eight $(−2)$-curves pointwisely. The quotient of $E × E$ by the group $⎨1_E × (−1_E), (−1_E) × 1_E⎬(≅ (Z/2Z)^2)$ is isomorphic to $P^1 × P^1$. Then there is a morphism from the quotient of $Kum(E × E)$ by $σ$ to $P^1 × P^1$ which contracts sixteen $(−1)$-curves on $Kum(E × E)/⟨σ⟩$. Let $π : V → P^1 × P^1$ be the blow-up the sixteen $F_3$-rational points of $P^1 × P^1$ and let $E_{ij}$ be the exceptional curve over $(p_i, p_j)$. Let $p_1 = (1, 0)$, $p_2 = (1, 1)$, $p_3 = (1, −1)$, $p_4 = (0, 1)$ be $F_3$-rational points of $P^1$ and let $F_i, G_i$ be the proper transforms of $P^1 × (p_i), (p_i) × P^1 (1 = 1, ..., 4)$, respectively. Then

\[ −2K_V = \left\{ \sum_{i=1}^4 (F_i + G_i) \right\} . \]

Thus $V$ is a Coble surface with eight boundaries $∑_{i=1}^4 (F_i + G_i)$ and isomorphic to $Kum(E × E)$. We have sixteen effective roots

\[ e_{ij} = \frac{1}{2}F_i + \frac{1}{2}G_j + 2E_{ij} \quad (1 ≤ i, j ≤ 4). \]
Theorem 10.6.9. The surface $V$ is a Coble surface of type MII in characteristic 3 with eight boundary components whose double cover is the supersingular $K3$ surface with Artin invariant 1. The surface $V$ has 24 effective $(-2)$-curves and sixteen effective roots. The automorphism group $\text{Aut}(V)$ is isomorphic to $(\mathcal{S}_4 \times \mathcal{S}_4) \cdot \mathbb{Z}/2\mathbb{Z}$. The $R$-invariant $(K, H)$ is $(D_8 \oplus 2A_2, (\mathbb{Z}/2\mathbb{Z})^2)$.

Proof. We should prove the last assertion. We obtain $D_8$ from 8 boundaries and two $A_2$ from 24 $(-2)$-curves. Since $\text{rank} \, H \geq 2$ and the dimension of the kernel of $q_{K/2K}$ is 2, we see $H = (\mathbb{Z}/2\mathbb{Z})^2$. \qed
Remark 10.6.10. The Enriques surfaces of type MII in characteristic 2 have \((16 + 12)\) \((-2)\)-curves and 12 non-effective roots. The above Coble surface \(V\) contains 16 effective roots and \((12 + 12)\) \((-2)\)-curves. These difference make the difference of the symmetries \((\mathcal{A}_4 \times \mathcal{A}_4) : \mathbb{Z}/2\mathbb{Z}\) and \((\mathcal{S}_4 \times \mathcal{S}_4) : \mathbb{Z}/2\mathbb{Z}\).

We have defined the notion of \(R\)-invariants for Enriques surfaces which are étale quotients of K3 surfaces (see Section 6.4). In the following we extend this to the case that the canonical covers of Enriques surfaces have only rational double points. Let \(S\) be a classical or an \(\alpha_2\)-Enriques surface and let \(\pi : X \to S\) be its canonical cover. Assume that \(X\) has only rational double points. Denote by \(\phi : Y \to X\) the minimal resolution. By Theorem 1.3.5, \(Y\) is a supersingular K3 surface. Put \(\tilde{\pi} = \pi \circ \phi\). Let Pic\((Y)^+\) and Pic\((Y)^-\) be the orthogonal complement of Pic\((Y)^+\) in Pic\((Y)\). We denote by \(R\) the sublattice of Pic\((Y)^-\) generated by exceptional curves \(E_1, \ldots, E_{12}\) of the minimal resolution \(\phi : Y \to X\). Then \(R\) is a root lattice. Note that \(R\) is of finite index in Pic\((Y)^-\).

Define

\[
h^\pm = \{ \delta^\pm \in \text{Pic}(Y)^\pm : \exists \delta^\pm \in \text{Pic}(Y)^\mp, (\delta^\pm)^2 = -4, \frac{\delta^+ + \delta^-}{2} \in \text{Pic}(Y) \}. \tag{10.6.5}\]

Let \(\langle h^- \rangle\) be the sublattice of Pic\((Y)^-\) generated by \(h^-\). Then \(\langle h^- \rangle = K(2)\) where \(K\) is a root lattice. We have a homomorphism

\[
\gamma : K(2)/2K(2) \to \text{Pic}^+(Y)/2\text{Pic}^+(Y) \tag{10.6.6}
\]

defined by

\[
\gamma(\delta^- \mod 2) = \delta^+ \mod 2.
\]

We define a subgroup \(H\) of \(K/2K \cong K(2)/2K(2)\) by the kernel of \(\gamma\). We call the pair \((K, H)\) the \(R\)-invariant of \(S\) as in the case of Section 6.4.

Example 10.6.11. Let \(C\) be a \((-2)\)-curve that passes through the images \(\bar{p}_1, \ldots, \bar{p}_k\) of singular points \(p_1, \ldots, p_k\) of \(X\). We assume that all these points are of type \(A_1\) and denote by \(E_i \subset Y\) the exceptional curve over \(p_i\). Since the restriction of the K3 cover to \(C\) is trivial, \(\pi^*(C) = 2C'\). By Proposition 0.4.19, \(X\) is \(\mathbb{Q}\)-factorial. Applying the intersection theory of \(\mathbb{Q}\)-factorial surfaces, we find that \(C'^2 = -1\). Let \(\tilde{C}\) be the strict transform of \(C'\) in \(Y\). Then the full transform of \(2C' \in \text{Pic}(X)\) in Pic\((Y)\) is equal to \(D = 2\tilde{C} + \sum_{i=1}^{k} a_i E_i\), where \(D \cdot E_i = 0\). Since \(C'\) is nonsingular, \(\tilde{C}\) intersects each \(E_i\) with multiplicity 1. This gives \(a_i = 1\), and \(-4 = (2C')^2 = D^2 = -8 + 4k - 2k = -2(4 - k)\). This gives \(k = 2\). Thus

\[
\pi^*(C) = \phi^*(2C') = 2\tilde{C} + E_1 + E_2.
\]

Thus we may take \(\delta^+ = \tilde{\pi}^*(C)\), and \(\delta^- = -(E_1 + E_2)\) or \(\delta^- = E_1 - E_2\) to obtain that \(\frac{1}{2}(\delta^+ + \delta^-) = \tilde{C} \in \text{Pic}(Y)\) or \(\frac{1}{2}(\delta^+ - \delta^-) = \tilde{C} + E_1 \in \text{Pic}(Y)\). Thus \(K(2)\) contains \(A_1(2) \oplus A_1(2)\) generated by \(E_1 + E_2\) and \(H\) contains \(\mathbb{Z}/2\mathbb{Z}\) generated by \(2E_1 \mod 2K(2)\).

Example 10.6.12. Let \(p \in S\) and let \(\bar{p} \in X\) with \(\pi(\bar{p}) = p\). Assume that \(X\) has a rational double point of type \(D_4\) at \(\bar{p}\), and denote by \(E_i \subset Y\) (\(i = 0, 1, 2, 3\)) the exceptional curves over \(p\) with \(E_0 \cdot E_i = 1\), \(i = 1, 2, 3\). Let \(C\) be a \((-2)\)-curve on \(S\) through \(p\). As before, let \(C'\) be the reduced pre-image of \(C\) in \(X\) and let \(\tilde{C}\) be the proper inverse transform of \(C'\) in \(Y\). By Proposition 0.4.19, \(2C' \in \text{Pic}(X)\), so \(\phi^*(2C') = 2\tilde{C} + \sum a_i E_i \in \text{Pic}(Y)\). Since \(C'\) is nonsingular, \(\tilde{C}\) intersects only one component with multiplicity 1. Suppose this component is \(E_0\). Since \(\phi^*(2C') \cdot E_i = 0\), we obtain \(-2a_i + a_0 = 0, i \neq 0\) and \(2 - 2a_0 + a_1 + a_2 + a_3 = 0\), and hence \(\phi^*(2C') = \).
2\(\tilde{C} + 2(2E_0 + E_1 + E_2 + E_3)\). This gives \(-4 = \pi^*(2C')^2 = -8 - 8 + 16 = 0\), a contradiction. So, we may assume that \(\tilde{C} \cdot E_1 = 1\). Similar computations to above show that
\[
\pi^*(C) = 2\tilde{C} + 2E_0 + 2E_1 + E_2 + E_3,
\]
with agreement that \(\pi^*(C)^2 = -4\). This shows that we may take \(\delta^+ = \pi^*(C)\) and \(\delta^- = -(E_2 + E_3)\) or \(\delta^- = E_2 - E_3\) to obtain \(\frac{1}{2} \delta^+ + \delta^- = \tilde{C} + E_0 + E_1 \in \text{Pic}(Y)\) or \(\frac{1}{2} \delta^+ + \delta^- = \tilde{C} + E_0 + E_1 + E_2 \in \text{Pic}(Y)\). Thus \(K(2)\) contains \(A_1(2) \oplus A_1(2)\) generated by \(E_2 + E_3\) and \(E_2 - E_3\), and \(H\) contains \(\mathbb{Z}/2\mathbb{Z}\) generated by \(2E_2 \pmod{2K(2)}\).

Example 10.6.13. Suppose \(S\) has no \((-2)\)-curves. Then \(h^+ = 0\), hence \(h^- = 0\) and the \(R\)-invariant is equal to \(\{0\}\). On the other hand, suppose that \(S\) is a general nodal surface. By Theorem 6.5.4, there exists an elliptic fibration with irreducible fibers. Since the lattice \(\pi^* \text{Pic}(S)^\perp\) is 2-elementary, all singular fibers are nodal curves and the number of them is equal to 12. All singular points of \(X\) are ordinary double points. By Example 10.6.11, any \((-2)\)-curve \(R\) passes through the projections \(\bar{p}, \bar{q}\) of two singular points \(p, q\) of \(X\). By Theorem 6.5.4, any other \((-2)\) curve \(R'\) is \(f\)-equivalent to \(R\), i.e. there exists a sequence \(R_1, \ldots, R_k\) of \((-2)\)-curves with \(R \cdot R_1 = R_1 \cdot R_2 = \cdots = R_{k-1} \cdot R_k = R_k \cdot R' = 2\). Since \(R + R_1\) is a fiber of an elliptic fibration, there are two singular points in \(X\) over the intersection points \(R\) with \(R_1\). Since \(R_1\) passes through only two such points, we see that \(R \cap R_1 = \{\bar{p}, \bar{q}\}\). Continuing in this way we see that all \(R \cap R_i = R \cap R' = \{\bar{p}, \bar{q}\}\). Thus we get a remarkable fact that the 12 singular points have a distinguished subset of 2 points. We will find an explanation of this in Section 8.6 by realizing \(S\) as a Reye congurence of lines in \(\mathbb{P}^3\). It follows from Example 10.6.11 that the \(R\)-invariant \((K, H)\) is equal to \((A_1 \oplus A_1, \mathbb{Z}/2\mathbb{Z})\), contrary to \((A_1, \{0\})\) in the case \(p \neq 2\).

In the following we calculate \(R\)-invariants of Enriques surfaces in characteristic 2 of type MI, MII and of type VII.

Proposition 10.6.14. Let \(S\) be an Enriques surface of type VII, of type MI or of type MII. Then its \(R\)-invariant is \((D_{10} \oplus A_1 \oplus A_1, (\mathbb{Z}/2\mathbb{Z})^3), (D_6 \oplus D_6, (\mathbb{Z}/2\mathbb{Z})^3)\) or \((D_8 \oplus A_3, (\mathbb{Z}/2\mathbb{Z})^3)\), respectively.

Proof. In case of type VII, there exist 9 nodal curves in Figure 8.18 passing 10 canonical points and forming a dual graph of type \(A_9\). On the other hand, there exists a nodal curve passing the remaining two canonical points. From these we can get 12 \((-4)\)-vectors in \(h^-\) generating \((D_{10} \oplus A_1^{\oplus 2})(2)\) as mentioned in Example 10.6.11. We know that there are no \((-2)\)-curves passing through a point among the first 10 canonical points and a point among the second two canonical points. This root lattice has the maximal rank 12 and there are no root lattices of the same rank containing \(D_{10}\) or \(A_1^{\oplus 2}\), we have \(R = D_{10} \oplus A_1^{\oplus 2}\). Since \((E_1 + \cdots + E_{12})/2 \in \text{Pic}(Y)\) (Ekedahl, Hyland and Shepherd-Barron [209], Lemma 3.14, Lemma 6.5), by combining the calculation in Example 10.6.11, we have \(H = (\mathbb{Z}/2\mathbb{Z})^3\).

In case of type MI, for example, consider the ten \((-2)\)-curves on \(S\) corresponding to the following five duads and five synthemes
\[
(12), (23), (34), (45), (56), (14, 25, 36), (15, 26, 34), (14, 23, 56), (15, 24, 36), (14, 26, 35).
\]
As mentioned in Example 10.6.11, these give 12 \((-4)\)-vectors in \(h^-\) which generate \((D_6 \oplus D_6)(2)\). There are no \((-2)\)-curves passing through a canonical point corresponding to a number in \(\{1, \ldots, 6\}\) and a point corresponding to a total (e.g. [406, Lemma 3.3, (2)]). Since this lattice has the maximal
rank 12 and there are no root lattices of the same rank containing $D_9$, we have $R = D_6 \oplus D_6$. Since 
$(E_1 + \cdots + E_{12})/2 \in \text{Pic}(X)$, the $R$-invariant of $S$ is $(D_6 \oplus D_6, (\mathbb{Z}/2\mathbb{Z})^3)$.

In case of type MIII, there are 7 nodal curves among 16 nodal curves on the left hand side in Figure 10.8 passing 8 canonical points and forming a dual graph of type $A_7$ (see Example 10.6.11). On the other hand, there are 2 nodal curves among 12 nodal curves on the right hand side in Figure 10.8 passing one canonical point $p_0$ and forming a dual graph of type $A_2$ (see Example 10.6.12). Thus we have $D_8 \oplus A_3$ which has the maximal dimension 11 because we get at most a root lattice of rank 3 from $p_0$ (see Example 10.6.12). Note that there are no $(-2)$-curves passing through $p_0$ and another canonical point. Root lattices of rank 11 containing $D_8 \oplus A_3$ properly is only $E_8 \oplus A_3$. Assume that $R = E_8 \oplus A_3$. Then $E_8(2)$ is contained in $A_1^{\oplus 8}$, but both have the determinant $2^8$ which is a contradiction. Thus we have $R = D_8 \oplus A_3$. Since the sum of the eight components of $A_1^{\oplus 8}$ is divisible by 2 (Ekedahl, Hyland and Shepherd-Barron [209], Lemma 3.14, Lemma 6.5), we have $H = (\mathbb{Z}/2\mathbb{Z})^3$. \hfill \Box

Remark 10.6.15. Any Enriques surface $S$ in characteristic 2 covered by the supersingular K3 surface $Y$ with Artin invariant 1 has the same dual graph of $(-2)$-curves as the one of the three Enriques surfaces of type MI, MII and of type VII (see [406]). All three examples have $20$ or $40$ $(-2)$-classes which generate the reflection group of finite index in $O(\text{Num}(S))$ (by Vinberg’s theorem 0.8.22). Enriques surfaces with finite automorphism group are nothing but the ones with crystallographic basis of $(-2)$-curves. We will discuss these examples in Section 10.7 from the point of view of the Leech lattice.

Finally we calculate $R$-invariants of the Coble surfaces $V$ given in Theorems 9.8.18, 9.8.20, 10.5.16, 10.6.3. We employ the definition of $R$-invariants for Coble surfaces in characteristic 2 by mixing the definition for Coble surfaces in characteristic $p \neq 2$ and the above one for classical and $\alpha_2$-surfaces.

Proposition 10.6.16. Let $V$ be the Coble surface of type VII with ten boundaries, of type VII with two boundaries, of type MI or of type MIII in characteristic 2. The $R$-invariant $(K, H)$ of $V$ is $(D_{10} \oplus 2A_1, (\mathbb{Z}/2\mathbb{Z})^3)$, $(D_{10} \oplus 2A_1, (\mathbb{Z}/2\mathbb{Z})^3)$, $(D_6 \oplus D_6, (\mathbb{Z}/2\mathbb{Z})^3)$ or $(D_8 \oplus A_3, (\mathbb{Z}/2\mathbb{Z})^3)$, respectively.

Proof. The situation is very similar to the case of Enriques surfaces in Proposition 10.6.14. The minimal resolutions of the canonical covers are the supersingular K3 surfaces $Y$ with Artin invariant 1, and we consider the same elliptic fibration on $Y$ with singular fibers $A_0, A_9, A_1, A_1$, with singular fibers $A_5, A_5, A_5, A_5$, or with singular fibers $A_7, A_7, D_5$. The only difference is that we are replacing some of the canonical points with boundaries. Thus the proof of Proposition 10.6.14 works well in this case, too. \hfill \Box

### 10.7 Enriques surfaces and the Leech lattice

In this section we will discuss a relationship between the automorphism group of an Enriques surface and the Leech lattice. For this purpose we first recall the case of K3 surfaces.

Let $\Lambda$ be the Leech lattice, that is, $\Lambda$ is an even negative definite unimodular lattice of rank 24 without $(-2)$-vectors. Let $L$ be an even unimodular lattice of signature $(1, 25)$. Then $L$ and $U \oplus \Lambda$
are isomorphic where $U$ is a hyperbolic plane. We fix an isomorphism

$$L = U \oplus \Lambda$$

and for $x \in L$ we denote by $x = (m, n, \lambda)$ where $m, n \in \mathbb{Z}$, $\lambda \in \Lambda$ and $x^2 = 2mn + \lambda^2$. We fix a vector $\rho = (1, 0, 0) \in L$ called a Weyl vector. Obviously $\rho^2 = 0$. Note that there are no $(-2)$-vectors in $L$ perpendicular to $\rho$ because $\Lambda$ contains no $(-2)$-vectors. A $(-2)$-vector $r \in L$ is called a Leech root if $\langle r, \rho \rangle = 1$. We denote by $h$ the set of all Leech roots. Note that there is a bijective correspondence between $\Lambda$ and $h$ given by

$$\Lambda \ni \lambda \mapsto (-1 - \frac{1}{2}\lambda^2, 1, \lambda) \in h.$$ 

The following theorem says that Leech roots are nothing but simple roots.

**Theorem 10.7.1.** (Conway [128]) The set

$$C = \{x \in P^+(L \otimes \mathbb{R}) : \langle x, r \rangle > 0 \text{ for any } r \in h\}$$

is a fundamental domain of the reflection group of $L$.

Borcherds [80] studied the reflection groups of even hyperbolic lattices which can be embedded into $L$ whose orthogonal complements in $L$ are roots lattices generated by some Leech roots. We can apply this method to the case of K3 or Enriques surfaces.

The first example was given by the second author [399] in the case of a generic Kummer surface associated with a genus 2 curve. This method is powerful and many examples have been obtained.

**Example 10.7.2.** Let $R = 6A_1 + A_3$. Then there is an embedding of $R$ in $L$ whose image is generated by Leech roots and its orthogonal complement in $L$ is isomorphic to the Picard lattice of a generic Kummer surface $X = \text{Kum}(J(C))$ associated with a curve $C$ of genus 2 (Note: $R$ is not primitive in $L$, but an overlattice of $R$ of index 2 is primitive). Denote by $S_X$ the Picard lattice of $X$. The primitive embedding of $S_X$ in $L$ induces an embedding of the positive cone $P^+(S_X \otimes \mathbb{R})$ in $P^+(L \otimes \mathbb{R})$. Let $\mathcal{D}(X)$ be the restriction of $C$ to $P^+(S_X \otimes \mathbb{R})$ and $\mathcal{D}(X)$ its closure. The condition that $R$ contains a Leech root implies that $\mathcal{D}(X)$ is a finite polyhedron. We call a face of $\mathcal{D}(X)$ of codimension 1 a facet. If a hyperplane perpendicular to a vector of norm $-h$, then we call it a $(-k)$-hyperplane. By using a geometry of the Leech lattice, we have the following:

**Proposition 10.7.3.** The facets of $\mathcal{D}(X)$ consist of 32 $(-2)$-, 32 $(-4)$-, 60 $(-4)$-, 192 $(-12)$-hyperplanes. The projection $\rho_0$ of the Weyl vector $\rho$ is in $\mathcal{D}(X)$ with $\rho_0^2 = 8$.

These facets can be interpreted in terms of a geometry of Kummer surface $\text{Kum}(J(C))$ associated with a curve of genus 2. First we recall the Kummer surface $\text{Kum}(J(C))$. Let $C$ be a curve of genus 2 given by

$$y^2 = \prod_{i=0}^{5} (x - \xi_i).$$

Let $\mu_i$ ($0 \leq i \leq 5$), $\mu_{ij}$ ($1 \leq i < j \leq 5$) be the set of 2-torsion points in the Jacobian $\text{Pic}^0(C) = J(C)$. Let $\Theta$ be the theta divisor and let $\Theta_i = \Theta + \mu_i$ ($0 \leq i \leq 5$), $\Theta_{ij} = \Theta + \mu_{ij}$ ($1 \leq i < j \leq 5$). The incident relation between $\{\mu_i, \mu_{ij}\}$ and $\{\Theta_i, \Theta_{ij}\}$ is as follows:

$$\mu_i \in \Theta_0 \ (0 \leq i \leq 5), \ \mu_0, \mu_i, \mu_{ij} \in \Theta_i \ (j \neq 0, i), \ \mu_i, \mu_j, \mu_{ij}, \mu_{kl} \in \Theta_{ij} \ (k, l \neq i, j).$$
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\[ \mu_0 \in \Theta_i \ (0 \leq i \leq 5), \ \mu_i \in \Theta_0, \Theta_i, \Theta_{ij} \ (j \neq 0, i), \ \mu_{ij} \in \Theta_i, \Theta_j, \Theta_{ij}, \Theta_{kl} \ (k, l \neq i, j). \]

This incident relation is a \((16_6)\)-configuration. We use symbols \(\alpha, \beta, \ldots\) for \(i, ij\) sometimes. By taking the quotient of \(J(C)\) by the inversion \(-1_{J(C)}\) we obtain a quartic surface \(X \subset \mathbb{P}^3\), called the Kummer quartic surface, which is the image of the morphism from \(J(C)\) defined by the linear system \([30]\). It has sixteen nodes \(n_{\alpha}\) which are the images of sixteen 2-torsions \(\mu_{\alpha}\), and contains a conic \(Q_{\alpha}\) which is the image of \(\Theta_{\alpha}\). This conic is classically called a trope. The hyperplane containing \(Q_{\alpha}\) tangents to \(X\) along \(Q_{\alpha}\). The minimal resolution of \(X\), denoted by \(Kum(J(C))\), is the Kummer surface associated with \(C\). We denote by \(N_{\alpha}\) the exceptional curve over \(n_{\alpha}\) and by \(T_{\alpha}\) the proper transform of \(Q_{\alpha}\). The both sets \(\{N_{\alpha}\}\) and \(\{T_{\alpha}\}\) consist of sixteen disjoint \((-2)\)-curves and each member of one set meets exactly six members in other set, that is, they form a \((16_6)\)-configuration. The Kummer surface \(Kum(J(C))\) can be embedded in \(\mathbb{P}^5\) whose image is given by the intersection of three quadrics:

\[ \sum_{i=1}^{6} X_i^2 = \sum_{i=1}^{6} \xi_i X_i^2 = \sum_{i=1}^{6} \xi_i^2 X_i^2 = 0, \quad (10.7.3) \]

where \([X_1, \ldots, X_6]\) is homogeneous coordinates of \(\mathbb{P}^5\) and \(\xi_i\) is given in \((10.7.2)\). Note that \((\mathbb{Z}/2\mathbb{Z})^5\) acts on \(Kum(J(C))\) as projective transformations

\[ [X_1, X_2, X_3, X_4, X_5, X_6] \rightarrow [X_1, \pm X_2, \pm X_3, \pm X_4, \pm X_5, \pm X_6]. \quad (10.7.4) \]

Consider a set \(\{\alpha, \beta, \gamma, \delta\}\) of four nodes. It is called a Göpel tetrad if any trope does not contain three nodes in this set. It is called a Rosenhain tetrad if any three nodes in this set are contained in a trope. For example, \(\{n_0, n_1, n_{25}, n_{34}\}\) is a Göpel tetrad and \(\{n_0, n_{13}, n_{15}, n_{35}\}\) is a Rosenhain tetrad. There exist sixty Göpel tetrads and eighty Rosenhain tetrads. A set of six nodes is called a Weber hexad if it is the symmetric difference of a Göpel tetrad and a Rosenhain tetrad. For example, \(\{n_1, n_{13}, n_{15}, n_{25}, n_{34}, n_{35}\}\) is a Weber hexad which is the symmetric difference of \(\{n_0, n_1, n_{25}, n_{34}\}\) and \(\{n_0, n_{13}, n_{15}, n_{35}\}\). There exist 192 Weber hexads (see [302], [175]).

Now we return to Proposition 10.7.3. One can identify 32 \((-2)\)-vectors with 32 \((-2)\)-curves \(N_{\alpha}, T_{\alpha}\) on \(Kum(J(C))\) forming \((16_6)\)-configuration, 32 \((-4)\)-vectors with \(H - 2N_{\alpha}\) and \(\sigma(H - 2N_{\alpha})\), sixty \((-4)\)-vectors with \(H - N_{\alpha} - N_{\beta} - N_{\gamma} - N_{\delta}\), 192 \((-12)\)-vectors with \(3H - 2 \sum_{\alpha \in W} N_{\alpha}\) respectively, where \(H\) is the pullback of the hyperplane section of the Kummer quartic surface, \(\sigma\) is an isometry changing \(\{N_{\alpha}\}\) and \(\{T_{\alpha}\}\) corresponding to a self-dual map of \(Kum(J(C))\) called a switch, \(\{n_{\alpha}, n_{\beta}, n_{\gamma}, n_{\delta}\}\) is a Göpel tetrad and \(W\) is a Weber hexad. The projection \(\rho_0\) of \(\rho\) into \(S_X \otimes \mathbb{Q}\) coincides with the class \(\frac{1}{6} \sum_{\alpha}(N_{\alpha} + T_{\alpha})\) which is the hyperplane section of \(Kum(J(C)) \subset \mathbb{P}^5\) given by the equation \((10.7.3)\). It is classically known that there are sixteen involutions associated with sixteen projections from sixteen nodes of the Kummer quartic surface which are bijectively correspond to the reflections associated with \(H - 2N_{\alpha}\), sixteen involutions (called correlations) associated with sixteen projections from 16 nodes of the dual of Kummer quartic surface, sixty Cremona transformations associated with sixty Göpel tetrads (see [310]) and 192 Cremona transformations associated with 192 Weber hexads (see [309]). The symmetry group of the finite polytope \(D(Kum(J(C)))\) is isomorphic to \((\mathbb{Z}/2\mathbb{Z})^5 \cdot \mathbb{S}_6\) where the group \((\mathbb{Z}/2\mathbb{Z})^5\) is induced from the projective transformations given in \((10.7.4)\) and is generated by involutions \(t_{\alpha}\) associated with sixteen translations of \(J(C)\) by sixteen 2-torsions \(a \in J(C)_2\) and sixteen switches \(\sigma \circ t_{\alpha} (a \in J(C)_2)\), and \(\mathbb{S}_6\) is the symmetric group of degree 6 acting six Weierstrass points of \(C\).

We now conclude:
Theorem 10.7.4. The automorphism group of a generic Kummer surface $\text{Kum}(J(C))$ associated with a curve of genus 2 is generated by sixteen translations, a switch $\sigma$, sixteen projections, sixteen correlations, sixty Cremona transformations associated with sixty Göpel tetrads and 192 Cremona transformations associated with 192 Weber hexads.

Mukai [506] observed that there exist three kinds of fixed-point-free involutions of $\text{Kum}(J(C))$ and then Ohashi [553] proved the following theorem.

Theorem 10.7.5. Assume that $\text{Kum}(J(C))$ is generic. Then there are exactly $31 (= 10 + 15 + 6)$ fixed-point-free involutions up to conjugacy in $\text{Aut}(\text{Kum}(J(C)))$, that is, ten switches associated with even theta characteristics, fifteen Cremona involutions associated with Göpel tetrads and six Cremona involutions associated with Weber hexads.

Later we will discuss Enriques surfaces obtained from $\text{Kum}(J(C))$ by taking the quotients by these involutions.

Example 10.7.6. Let $R = \mathbb{D}_4$. Then there is a primitive embedding of $R$ in $L$ whose image is generated by Leech roots and its orthogonal complement in $L$ is isomorphic to the Picard lattice of the supersingular K3 surface $Y$ with the Artin invariant 1 in characteristic 2 mentioned in §10.5. Denote by $S_Y$ the Picard lattice of $Y$. The primitive embedding of $S_Y$ in $L$ induces an embedding of the positive cone $P^+(S_Y \otimes \mathbb{R})$ in $P^+(L \otimes \mathbb{R})$. Let $D(Y)$ be the restriction of $C$ to $P^+(S_Y \otimes \mathbb{R})$. In this case we have the following:

Proposition 10.7.7. The facets of $D(Y)$ consist of 42 $(-2)$-s, 168 $(-4)$-hyperplanes.

Recall that $Y$ is obtained as the minimal resolution of a purely inseparable double cover of $\mathbb{P}^2$ (§10.5). The inseparable double cover has 21 nodes over 21 points of $\mathbb{P}^2(\mathbb{F}_4)$, and thus $Y$ contains 42 $(-2)$-curves divided into two sets $A$ and $B$ both of which consist of 21 disjoint curves. A curve in $A$ (resp. in $B$) is an exceptional curve over a node (resp. a proper transform of a line on $\mathbb{P}^2$) whose image is in $\mathbb{P}^2(\mathbb{F}_4)$. The incidence relation of 42 curves is a $(21)_3$-configuration. One can identify 42 $(-2)$-vectors with 42 $(-2)$-curves in $A \cup B$, and 168 $(-4)$-vectors with $2\ell - N_1 - \cdots - N_6$ respectively, where $\ell$ is the pullback of the class of a line on $\mathbb{P}^2$ and $\{N_1, \ldots, N_6\}$ are $(-2)$-curves in $A$ and the corresponding 6 points in $\mathbb{P}^2(\mathbb{F}_4)$ are in general position. The symmetry group of the finite polytope $D(Y)$ is isomorphic to $\text{PGL}_3(\mathbb{F}_4) \cdot (\mathbb{Z}/2\mathbb{Z})^2$ where $(\mathbb{Z}/2\mathbb{Z})^2$ is generated by the switch $\sigma$ and the involution $\iota$ induced from the Frobenius automorphism of $\mathbb{F}_4$ mentioned in section 10.5. Thus we have the following theorem mentioned as in section 10.5.

Theorem 10.7.8. The automorphism group of the supersingular K3 surface with the Artin invariant 1 in characteristic 2 is generated by $\text{PGL}_3(\mathbb{F}_4)$, the switch and 168 Cremona transformations associated with 168 sets of six points in $\mathbb{P}^2(\mathbb{F}_4)$ in general position.

Very recently Brandhorst and Shimada [92] have found the following theorem with computer calculation. Recall that $E_{10}$ be the even unimodular lattice of signature $(1, 9)$ and $E_{10}(2)$ is the lattice obtained from $E_{10}$ by multiplying the bilinear form by 2.

Theorem 10.7.9. There exist exactly 17 primitive embeddings of $E_{10}(2)$ in $L$. 
<table>
<thead>
<tr>
<th>No.</th>
<th>name</th>
<th>facets</th>
<th>root</th>
<th>Enriques surface</th>
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<tr>
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<td>12</td>
<td>D₈</td>
<td>I</td>
</tr>
<tr>
<td>2</td>
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<td>12</td>
<td>A₇</td>
<td>II</td>
</tr>
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<td>20</td>
<td>D₁₊D₅</td>
<td>V</td>
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<td>III</td>
</tr>
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<td>VII in char 2 (Example 8.10.8)</td>
</tr>
<tr>
<td>6</td>
<td>20D</td>
<td>20</td>
<td>A₃₊A₄</td>
<td>VII</td>
</tr>
<tr>
<td>7</td>
<td>20E</td>
<td>20</td>
<td>5A₁₊A₅</td>
<td>VI</td>
</tr>
<tr>
<td>8</td>
<td>20F</td>
<td>20</td>
<td>2A₃</td>
<td>IV</td>
</tr>
<tr>
<td>9</td>
<td>40A</td>
<td>40</td>
<td>4A₁₊2A₃</td>
<td>MIII (Example 10.7.10)</td>
</tr>
<tr>
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<td>40B</td>
<td>40</td>
<td>8A₁₊2D₄</td>
<td>MIII in char 2 (Theorem 10.6.5)</td>
</tr>
<tr>
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<td>40C</td>
<td>40</td>
<td>6A₁₊A₃</td>
<td>MIII (Example 10.7.12)</td>
</tr>
<tr>
<td>12</td>
<td>40D</td>
<td>40</td>
<td>12A₁₊D₄</td>
<td>MI in char 2 (Theorem 10.5.8)</td>
</tr>
<tr>
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<td>40E</td>
<td>40</td>
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<td>MI (Remark 10.5.14)</td>
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<tr>
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<td>96</td>
<td>8A₁</td>
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<td></td>
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<td>96</td>
<td>4A₁</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>infty</td>
<td>infty</td>
<td></td>
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</tr>
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Table 10.2: 17 polytopes

In the Table 10.2 we give 17 primitive embeddings of $E_{10}(2)$ in $L$ and 17 polytope obtained by restricting of Conway’s fundamental domain $\mathcal{C}$ to the positive cone of the Enriques lattice. The name is the symbol of the polytope by the paper Brandhorst and Shimada [92]. The facets means the number of facets of the polyhedron. It is remarkable that all facets are defined by $(-2)$-vectors. For example, 12A and 12B are two polytopes both of them have 12 facets. The root means the root lattice which is the sublattice generated by $(-2)$-vectors in the orthogonal complement of $E_{10}(2)$ in $L$. Among 17 polytopes, in cases of No.5 and No.6, No.10 and No.11, No.12 and No.13, and No.15 and No.16, their polytopes coincide, but the embeddings are different. The final column means that the polytope is realized geometrically on the corresponding Enriques surface. The symbols I, ..., VII mean the types of Enriques surfaces with finite automorphism group (Kondo’s type I, ..., VII) and VII, MI, MII in char 2 are type VII, type MI, type MII given in §10.5, §10.6. The complex Enriques surfaces of type MII and MIII are obtained by Mukai as the quotient of the Kummer surface associated with a curve of genus 2 by a switch and by a Cremona transformation associated with a Göpel tetrad respectively (see the following Examples 10.7.10, 10.7.12). The complex Enriques surface of type MI is obtained by Mukai as the quotient of the K3 surface mentioned in Remark 10.5.14. In the following we explain more details.

In cases of No.1, ..., 4, 6, 7, 8 in the Table 10.2, the polytope coincides with the one of Enriques surface.

---

3Each of 168 $(-4)$-vectors defines a reflection of the Picard lattice $S_Y$ of $Y$. However the action of the corresponding Cremona transformation on $S_Y$ does not coincide with the one of the reflection. It coincides with the composite of the actions of the Cremona transformation and $\iota$. This fact was pointed out by S. Mukai. See also Example 10.5.13.
surfaces defined over the complex numbers with finite automorphism group of the same type. In case of No. 5, the root lattice $10A_1 + D_6$ contains a sublattice $12A_1 + D_4$ of index 2. Recall that the orthogonal complement of $D_4$ in $L$ is isomorphic to the Picard lattice of the supersingular K3 surface $Y$ with the Artin invariant 1 in characteristic 2. The remaining $12A_1$ in $12A_1 + D_4$ can be considered as the exceptional curves of 12 $A_1$-singularities of the canonical covering of the Enriques surfaces in characteristic 2 of type VII. Thus the polytope No.5 is obtained as a restriction of the polytope $D(Y)$ given in Proposition 10.7.7. Similarly the cases of No.10 and No.12, the root lattices contain $D_4$ as a direct summand and the remaining summands are $12A_1$ or $8A_1 + D_4$ which correspond to $12A_1$- or $8A_1 + D_4$-singularities of the canonical covers of Enriques surfaces in characteristic 2 of type MI or of type MII. The polytopes No.10, No.12 are obtained as restrictions of the polytope $D(Y)$ given in Proposition 10.7.7. For forty $(-2)$-vectors defining the facets of the polytopes in these cases, see Theorem 10.5.8, Remark 10.6.7.

Example 10.7.10. This example is due to S. Mukai. In case of No.9, the polytope is obtained as a restriction of the polytope $D(X)$ of a generic Kummer surface $X = Kum(J(C))$ given in Proposition 10.7.3. Let $G = \{n_{\alpha_0}, n_{\beta_0}, n_{\gamma_0}, n_{\delta_0}\}$ be a Göpel tetrad and $c_G$ the associated Cremona transformation. Then the Enriques surface is the quotient surface $Kum(J(C))/(c_G)$. It is known that the action of $c_G$ on $\{H, N_\alpha, T_\beta\}$ (Keum [372]). The forty facets of the polytope of No.9 are divided into four sets of facets consisting of the hyperplanes defined by 12, 8, 12, 8 $(-2)$-vectors respectively which are described as follows. For the given Göpel tetrad $G$ there exist exactly twelve tropes which contain exactly two nodes in $G$ and the remaining four tropes do not contain any node in $G$. We denote by $\{T_{0\alpha_0}, T_{0\beta_0}, T_{0\gamma_0}, T_{0\delta_0}\}$ the four $(-2)$-curves corresponding to the remaining four tropes. Then $c_G$ preserves 24 $(-2)$-curves $N_\alpha (\alpha \neq \alpha_0, \beta_0, \gamma_0, \delta_0)$, $T_\alpha (\alpha \neq \alpha_0, \beta_0, \gamma_0, \delta_0)$.

The first 12 $(-2)$-vectors are represented by twelve $(-2)$-curves corresponding to
\[
\{N_\alpha + c_G(N_\alpha)\}_{\alpha \neq \alpha_0, \beta_0, \gamma_0, \delta_0}, \{T_\alpha + c_G(T_\alpha)\}_{\alpha \neq \alpha_0, \beta_0, \gamma_0, \delta_0}.
\]
The second 8 $(-2)$-vectors are represented by eight $(-2)$-curves corresponding to
\[
\{N_\alpha + c_G(N_\alpha)\}_{\alpha = \alpha_0, \beta_0, \gamma_0, \delta_0}, \{T_\alpha + c_G(T_\alpha)\}_{\alpha = \alpha_0, \beta_0, \gamma_0, \delta_0}.
\]
The third 12 $(-2)$-vectors corresponds to the set of classes
\[
H - N_\alpha - N_\beta - N_\gamma - N_\delta.
\]
perpendicular to $H - N_{\alpha_0} - N_{\beta_0} - N_{\gamma_0} - N_{\delta_0}$.

The fourth one corresponds to the set of projections and correlations
\[
\{H - 2N_\alpha, \sigma(H - 2N_\alpha)\}_{\alpha = \alpha_0, \beta_0, \gamma_0, \delta_0}.
\]
For example, if we take $\{n_0, n_1, n_{25}, n_{34}\}$ as $G = \{n_{\alpha_0}, n_{\beta_0}, n_{\gamma_0}, n_{\delta_0}\}$, then $\{\alpha'_0, \beta'_0, \gamma'_0, \delta'_0\} = \{23, 24, 35, 45\}$ and $c_G$ acts on $N_\alpha, T_\alpha$ as follows:

$N_2 \leftrightarrow N_{15}, N_3 \leftrightarrow N_{14}, N_4 \leftrightarrow N_{13}, N_5 \leftrightarrow N_{12}, N_{23} \leftrightarrow N_{45}, N_{24} \leftrightarrow N_{35};$

$T_0 \leftrightarrow T_1, T_2 \leftrightarrow T_5, T_3 \leftrightarrow T_4, T_{12} \leftrightarrow T_{15}, T_{13} \leftrightarrow T_{14}, T_{25} \leftrightarrow T_{34};$

$N_0 \leftrightarrow H - N_1 - N_{25} - N_{34}, N_1 \leftrightarrow H - N_0 - N_{25} - N_{34},$

$N_{25} \leftrightarrow H - N_0 - N_1 - N_{34}, N_{34} \leftrightarrow H - N_0 - N_1 - N_{25},$

and $c_G$ acts on $T_\alpha (\alpha \in \{23, 24, 35, 45\})$ as the reflection associated with a $(-4)$-vector $H - N_0 - N_1 - N_{25} - N_{34}$. The 12 vectors perpendicular to $H - N_0 - N_1 - N_{25} - N_{34}$ are corresponding to
12 Göpel tetrads which have exactly two common nodes with $G$. The suffixes of such Göpel tetrads are given by:

\[ \{0, 1, 24, 35\}, \{0, 1, 23, 45\}, \{0, 2, 15, 34\}, \{0, 3, 14, 25\}, \{0, 4, 13, 25\}, \{23, 25, 34, 35\}, \{24, 25, 34, 35\}, \{1, 5, 12, 25\}, \{1, 2, 15, 25\}, \{1, 4, 13, 34\}, \{1, 3, 14, 34\} \]

If $C$ is a curve of genus 2 defined by $y^2 = x(x^4 - 1)$, then Mukai showed that the third ones are also represented by $(-2)$-curves. By this description, Mukai and Ohashi [512] proved the following theorem.

**Theorem 10.7.11.** Assume $C$ is given by $y^2 = x(x^4 - 1)$. Then $\text{Aut}(\text{Kum}(C)/\langle c_G \rangle)$ is isomorphic to $UC(S) \rtimes H_{192}$ where the first factor is the free product of 8 cyclic groups of order 2 generated by 8 projections and correlations, and $H_{192}$ is a finite group $(\mathbb{Z}/2\mathbb{Z})^3 \rtimes \mathfrak{S}_4$ of order 192. The group $H_{192}$ is a subgroup of the automorphism group of the polytope of type $\text{MIII}$ of index 4.

**Example 10.7.12.** This example is also due to S. Mukai. Consider a switch $\sigma$ associated with an even theta characteristic which is a fixed-point-free involution (Theorem 10.7.5). There exist 10 switches. In this case the restriction of the polytope of $\text{Kum}(J(C))$ given in Proposition 10.7.3 coincides with the one of No.11. For example, we can take a switch $\sigma$ associated with $\mu_{12}$ satisfying

\[ \sigma(N_i) = T_{i+12}, \quad \sigma(N_{ij}) = T_{ij+12}. \]

Thus the Enriques surface $\text{Kum}(J(C))/\langle \sigma \rangle$ contains 16 $(-2)$-curves whose dual graph coincides with the dual graph of 16 $(-2)$-curves given in Figure 10.8 (the left hand side 16 curves). Moreover there are 24 Göpel tetrads (e.g. $G = \{0, 25, 14, 3\}$) such that the associated the $(-4)$-divisors are invariant under the action of $\sigma$ and hence descend to $(-2)$-divisors on the Enriques surface. Among 24 Göpel tetrads, the following twelve give 12 $(-2)$-classes forming the dual graph in Figure 10.8 (the right hand side 12 curves):

\[ \{0, 4, 13, 25\}, \{1, 3, 14, 34\}, \{12, 15, 23, 35\}, \{1, 4, 15, 45\}, \{0, 5, 14, 23\}, \{2, 3, 25, 35\}, \{12, 13, 24, 34\}, \{1, 4, 15, 45\}, \{0, 3, 15, 24\}, \{12, 14, 25, 45\}, \{1, 4, 13, 34\}, \{2, 5, 23, 35\}. \]

The remaining 12 $(-2)$-classes form the same dual graph. The dual graph of the 40($= 16 + 24$) $(-2)$-classes coincides with the polytope $\Pi_{\text{III}}$ (i.e. No.11 (40C)).

**Example 10.7.13.** Finally consider a Cremona transformation $c_W$ associated with a Weber hexad $W$ which is a fixed-point-free involution (Theorem 10.7.5). In this case the restriction of the polytope $\mathcal{D}(X)$ of $X = \text{Kum}(J(C))$ given in Proposition 10.7.3 coincides with No.7. Let $W$ be a Weber hexad $W = \{\alpha_1, \ldots, \alpha_6\}$. Then the linear system

\[ |2H - \sum_{\alpha \in W} N_\alpha| \]

gives an another projective model

\[ \bar{X}_W : s_1 + \cdots + s_5 = 0, \quad \frac{\lambda_1}{s_1} + \cdots + \frac{\lambda_5}{s_5} = 0, \]

where $\lambda_1, \ldots, \lambda_5$ are non zero constants (see the equation 6.4.15). The quartic surface $\bar{X}_W$ has 10 nodes

\[ p_{ijk} : s_i = s_j = s_k = 0 \quad (1 \leq i < j < k \leq 5) \]
The Cremona transformation of type VI, all twenty
In case that the cubic surface specializes the Clebsch diagonal cubic surface (see Enriques surface whose nef cone is tessellated by the polytope given type. It would be interesting to give a geometric
Table 10.2 there exists a singular K3 surface, that is, a complex K3 surface with Picard number 20, remaining
invariant 1. The Picard lattice of
the canonical cover of the example of Coble surface
For example, consider No.4. In this case, a polytope of type VII appears. The minimal resolution of
the canonical cover of the example of Coble surface is isomorphic to the orthogonal complement of
D4 in U ⊕ Λ. The root lattice 10A1 + D6 contains 12A1 + D4 as a sublattice of index 2. We can consider the
remaining 12A1 of 12A1 + D4 as two exceptional curves of two nodes of the canonical cover of
and contains 10 lines
\[ l_{ij} : s_i = s_j = 0 \text{ (} 1 \leq i < j \leq 5 \text{).} \]
Note that \( X_W \) is the hessian quartic surface of the cubic surface defined by
\[ s_1 + \cdots + s_5 = 0, \quad \lambda_1 s_1^3 + \cdots + \lambda_5 s_5^3 = 0. \]
The Cremona transformation
\[ (s_1, \ldots, s_5) \rightarrow \left( \frac{\lambda_1}{s_1}, \ldots, \frac{\lambda_5}{s_5} \right) \]
induces a fixed-point-free involution \( c_W \) of \( \text{Kum}(J(C)) \) which interchanges the 10 exceptional curves \( E_{ijk} \) over the 10 nodes \( p_{ijk} \) and 10 proper transforms \( L_{ij} \) of 10 lines \( l_{ij} \). The quotient surface \( \text{Kum}(J(C))/\langle c_W \rangle \) is an Enriques surfaces containing 10 \((-2)\)-curves whose dual graph is the Petersen graph 6.1. Moreover the restriction of the polytope \( D(X) \) coincides with the nef cone of the Enriques surfaces with finite automorphism group of type VI. For example, if we take \( \{n_0, n_{14}, n_{15}, n_{23}, n_{25}, n_{34} \} \) as a Weber hexad \( W \), then \( c_W \) exchanges the following 20 \((-2)\)-curves:
\[ N_1 \leftrightarrow T_{23}, \quad N_2 \leftrightarrow T_{14}, \quad N_3 \leftrightarrow T_{15}, \quad N_4 \leftrightarrow T_{25}, \quad N_5 \leftrightarrow T_{34}, \]
\[ N_{12} \leftrightarrow T_5, \quad N_{13} \leftrightarrow T_4, \quad N_{24} \leftrightarrow T_3, \quad N_{35} \leftrightarrow T_2, \quad N_{45} \leftrightarrow T_1. \]
We can easily see that the dual graph of the images of these curves is the Petersen graph 6.1. The remaining ten \((-2)\)-vectors correspond to the following \( c_W \)-invariant \((-4)\)-classes:
\[ H - N_0 - N_1 - N_{25} - N_{34}, \quad H - N_0 - N_2 - N_{15} - N_{34}, \quad H - N_0 - N_3 - N_{14} - N_{25}, \]
\[ H - N_0 - N_4 - N_{15} - N_{23}, \quad H - N_0 - N_5 - N_{14} - N_{23}, \quad H - N_{12} - N_{14} - N_{23} - N_{34}, \]
\[ H - N_{13} - N_{15} - N_{23} - N_{25}, \quad H - N_{14} - N_{15} - N_{24} - N_{25}, \quad H - N_{14} - N_{15} - N_{34} - N_{35}, \]
\[ H - N_{23} - N_{25} - N_{34} - N_{45}. \]
In case that the cubic surface specializes the Clebsch diagonal cubic surface (see Enriques surface of type VI), all twenty \((-2)\)-vectors defining the polytope are represented by \((-2)\)-curves, that is, the polytope is the nef cone of the Enriques surfaces with finite automorphism group of type VI.

Remark 10.7.14. The following Table gives examples of Coble surfaces with finite automorphism group.

For example, consider No.4. In this case, a polytope of type VII appears. The minimal resolution of
the canonical cover of the example of Coble surface \( V \) is a supersingular K3 surface \( Y \) with Artin invariant 1. The Picard lattice of \( Y \) is isomorphic to the orthogonal complement of \( D_4 \) in \( U \oplus \Lambda \). The root lattice \( 10A_1 + D_6 \) contains \( 12A_1 + D_4 \) as a sublattice of index 2. We can consider the
remaining \( 12A_1 \) of \( 12A_1 + D_4 \) as two exceptional curves of two nodes of the canonical cover of
and the preimages of ten boundaries. On the other hand, the Picard lattice of our Vinberg’s K3 surface is isomorphic to the orthogonal complement of \( D_6 \) in \( U \oplus \Lambda \). Thus we can understand the remaining \( 10A_1 \) of the root \( 10A_1 + D_6 \) as ten boundaries.

Brandhorst and Shimada with computer calculation claim that in each case except No.17 in the
Table 10.2 there exists a singular K3 surface, that is, a complex K3 surface with Picard number 20, whose nef cone is tessellated by the polytope given type. It would be interesting to give a geometric
10.7. ENRIQUES SURFACES AND THE LEECH LATTICE

<table>
<thead>
<tr>
<th>No.</th>
<th>Nef Cone</th>
<th>Coble surface</th>
<th>Boundaries</th>
<th>Root</th>
<th>Root-Inv</th>
<th>Aut(V)</th>
<th>p</th>
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<td>A₇</td>
<td>(D₀, {0})</td>
<td>G₂</td>
<td>p ≠ 2</td>
</tr>
<tr>
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<td>Z/2Z × G₂</td>
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</tr>
<tr>
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<td>2</td>
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<td>(A₀ ⊕ A₁, Z/2Z)</td>
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<td>G₅</td>
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<td>Ex. 10.5.18</td>
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<td>(D₈ ⊕ 2A₂, (Z/2Z)²)</td>
<td>(G₄ × G₄) × 2</td>
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<tr>
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<td>40C</td>
<td>Ex. 10.6.8</td>
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<td>2A₁ + 2A₂</td>
<td>(2A₀ ⊕ 2A₁, (Z/2Z)³)</td>
<td>Aut(G₆)</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 10.3: Coble surfaces with finite automorphism group in characteristic p ≥ 0

Construction of Enriques and Coble surfaces with the polytope in cases of No. 14, 15, 16 as in the cases No. 1–13 mentioned as in Tables 10.2, 10.3.

Added in the proof: The second author [407] has proved that Coble surfaces with finite automorphism group in characteristic p ≠ 2 are given in Table 10.3 (except No. 4, 4'). Each type is unique. A brief outline of the proof is as follows. By using the same method explained in Section 8.9 we determine all possible crystallographic basis of effective irreducible roots on a Coble surface with finite automorphism group. Those are of type I, II, . . . , VII, MI and MII. In Enriques surfaces, Martin [469, Lemma 10.12] showed that there are no special elliptic fibrations with singular fibers of type A₅ + 2A₂ + 2A₁ or of type 2A₂ + 2A₂ + 2A₂ + A₂. Note that the crystallographic basis of type MI contains a parabolic subdiagram of type A₅ + 2A₂ + 2A₁ and the one of type MII contains a parabolic subdiagram of type 2A₂ + 2A₂ + 2A₂ + A₂. Thus both type MI and MII are excluded in the classification of Enriques surfaces with finite automorphism group. On the other hand, a Coble surface with such a crystallographic basis exists. One can easily determine the number n of boundary components by using the fact that boundary components are obtained by blowing-up all singular points of one or two fibers of type A₅⁺, A₆⁺, A₇⁺, A₈⁺, A₉⁺ (n ≥ 0) of any genus one fibration on a Coble surface. For example, the crystallographic basis of type I contains genus one fibrations with singular fibers of type E₈ + A₅⁺ (or E₈, E₈ + A₀, 2E₈ + A₀ + A₀), of type E₇ + A₅⁺ (or E₇ + A₁ + A₀), of type D₈ + A₀ + A₀, and of type A₇ + A₁ + A₀ + A₀, and hence n = 1 or 2. For the characteristic of the base field the Picard number of the covering K3 surface X, except of type I and of type II, is at least 20 plus 1, that is, X is supersingular, because the covering K3 surfaces of Enriques surfaces of type III, . . . , VII, MI, MII in characteristic 0 have already the Picard number 20. Now it is not difficult to determine the determinant −p²ⁿ of Pic(X) and hence possible characteristics p.

In case of characteristic two, Katsura and the second author has also determined the crystallographic basis of Coble surfaces with finite automorphism group as in the following Table 10.4.

The proof depends on a similar argument as above, and additionally the classification of conductrices due to Ekedahl, Shepherd-Barron [208] and the argument in Katsura, Kondo, Martin [363]. They also show that Coble surfaces of each type form an irreducible family. In the table, “Type” means the type of crystallographic basis given in Theorem 8.10.7, “boundaries” the number of boundary components, and “dim” the dimension of the family of each type. All cases are specializations of
classical Enriques surfaces with the same crystallographic basis.

### Bibliographical notes

The first example of a supersingular K3 surface was given by J. Tate [668] in 1965. The terminology is due to T. Shioda [647] who determined which Kummer surfaces are supersingular. In [29] M. Artin gave another definition of a supersingular K3 surfaces based on the notion of the formal Brauer group. He proved that the two definitions coincide if the surface carries an elliptic pencil. The fact that they are indeed equivalent was proven recently by M. Lieblich and D. Maulik in characteristic $p \neq 2$ [439] and by W. Kim and K. Madapusi Pera in characteristic 2 [378]. The fact that all supersingular K3 surfaces in characteristic 2 are unirational was first proven by A. Rudakov and I. Shafarevich [601]. Rudakov and Shafarevich also classified all possible Picard lattices of supersingular K3 surfaces [602].

The relationship between simply connected Enriques surfaces in characteristic 2 and supersingular K3 surfaces was first studied by W. Lang [418]. He was the first to ask the questions about possible singularities and possible Artin invariant of the canonical cover of an Enriques surface. Lang also proved that the Artin invariant is equal to 10 if the surface is general.

The first systematic study of supersingular covers of Enriques surfaces was undertaken in preprint of T. Ekedahl, J. Hyland, and N. Shepherd-Barron and [209]. Some of their results have been reproved and extended by S. Schröer [613]. In particular, Schröer gave the first example of an Enriques surface whose K3 cover is normal and has one non-rational singular point.

A construction of algebraic surfaces in characteristic $p > 0$ as quotients by rational vector fields was pioneered by T. Katsura and Y. Takeda [358]. A systematic study of inseparable maps defined by quotients by a rational vector field was done in a paper of A. Rudakov and I. Shafarevich [600]. The constructions of Enriques surfaces as quotients of supersingular K3 surfaces with Artin invariant 1 which we discuss in this chapter is based on the work of the second author and T. Katsura [360],[361] [406]. It was also successfully used in the classification of Enriques surfaces in characteristic 2 with finite automorphism group which we discussed in Section 8.9.


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