

KUMMER SURFACES:200 YEARS OF STUDY

IGOR DOLGACHEV

ABSTRACT. This is a brief history of discovery and later study of Kummer surfaces. The article is based on the author's Oliver Club talk at Cornell University on October 10, 2019 delivered exactly 101 years since the first talk at the club given by John Hutchinson.

The fascinating story about the Kummer surface starts from discovery by Augustin-Jean Fresnel in 1822 of the equation describing the propagation of light in an optically biaxial crystal [8]. Let v denote the light speed in some matter and $n = c/v$, where c is the light speed in vacuum. It depends on the coordinates $x = (x_1, x_2, x_3)$ of a point and the unit direction vector $\xi = (\xi_1, \xi_2, \xi_3)$. The propagation of light is described by the function $n(x, \xi)$. The matter is homogeneous if n does not depend on x and isotropic if it does not depend on the direction. For example, Maxwell studied the fisheye and found that the light is inhomogeneous but isotropic. A biaxial crystal gives an example of homogeneous but anisotropic propagation. Fresnel found the equation of propagation of light in such a crystal of the form:

Augustin-Jean Fresnel:1822

$$(1) \quad \frac{\xi_1^2}{\frac{1}{a_1^2} - \frac{1}{n^2}} + \frac{\xi_2^2}{\frac{1}{a_2^2} - \frac{1}{n^2}} + \frac{\xi_3^2}{\frac{1}{a_3^2} - \frac{1}{n^2}} = 0,$$

where a_1, a_2, a_3 are constants describing the property of the crystal (principal refraction indices). In new coordinates $(x, y, z) = n(\xi_1, \xi_2, \xi_3)$ one can rewrite the equation (1) in homogeneous form as

$$(2) \quad \frac{a_1^2 x^2}{x^2 + y^2 + z^2 - a_1^2 w^2} + \frac{a_2 y^2}{x^2 + y^2 + z^2 - a_2^2 w^2} + \frac{a_3 z^2}{x^2 + y^2 + z^2 - a_3^2 w^2} = 0$$

After clearing the denominators, we find a homogeneous equation of degree 4.

In 1833 Sir William Hamilton discovered that the surface has four real singular points

William Hamilton:1833

$$\left(\pm a_3 \sqrt{\frac{a_1^2 - a_2^2}{a_1^2 - a_3^2}}, 0, \pm a_1 \sqrt{\frac{a_2^2 - a_3^2}{a_1^2 - a_3^2}}, 1 \right)$$

and also four real *trope-conics* cut out by the planes $\alpha x + \beta y + \gamma z + w = 0$, where

$$(\alpha, \beta, \gamma, 1) = \left(\pm \frac{a_3}{a_2} \sqrt{\frac{a_1^2 - a_2^2}{a_1^2 - a_3^2}}, 0, \pm \frac{a_1}{a_2} \sqrt{\frac{a_2^2 - a_3^2}{a_1^2 - a_3^2}}, 1 \right)$$

[12, p. 134]. In fact, over \mathbb{C} , it has additionally 12 nodes and 12 trope-conics such that the incidence configuration of 16 nodes and 16 trope-conics is of type (16_6) (this the famous *Kummer abstract configuration*). In 1849 Arthur Cayley proved that the Fresnel's wave surface is a particular case of a quartic surface. It contains four pairs of coplanar conics, the three vertices of the tetrahedron formed by these four planes lying in the same face are conjugate with respect to the both conics [5]. He gave the name *tetraedroid quartic surface* to such a quartic surface.

Arthur Cayley: 1848

He also discovered an important property of tetraedroid quartic surfaces: they are projectively *self-dual* (or *reciprocal*). The wave surfaces were the subject of study for many mathematicians of the 19th century. Among them were A. Cauchy, A. Cayley, J. Darboux, J. Mac Cullagh, J. Sylvester and W. Hamilton (see [25, pages 114-115]). A nice modern exposition of the theory of Fresnel's wave surfaces can be found in [19].

Projective equivalence classes of Fresnel wave surfaces depend on 2 parameters and as we shall see momentary Kummer surfaces depend on 3 parameters.

Adolph Göpel:1847

In 1847 Adolph Göpel using the transcendental theory of theta functions had found a relation of order four between theta functions of second order in two variables that expresses an equation of a general Kummer surface [9].

Let $T = \mathbb{C}^g/\Lambda$ be a compact g -dimensional torus, the quotient of \mathbb{C}^g by the group of translations Λ isomorphic to \mathbb{Z}^{2g} . There are no holomorphic functions on T because it is a compact complex manifold, instead one considers nonzero holomorphic sections of a holomorphic line bundle L on T . For general Λ no line bundle has enough sections to embed T into a projective space, i.e. T does not admit any ample (or positive) line bundle. However, in 1857 Bernard Riemann found a condition on Λ such that such L exists and $\dim \Gamma(T, L^n) = n^g$. Such complex tori are now called *principally polarized abelian varieties*. They depend on $\frac{1}{2}g(g+1)$ parameters. Holomorphic sections of L^n can be lifted to holomorphic functions on \mathbb{C}^g that are Λ -invariant up to some multiplicative factor. They are called *theta-functions* of order n . For $n = 1$, such a holomorphic function is the famous Riemann theta function $\Theta(z, \Lambda)$. One can modify Riemann's expression for $\Theta(z, \Lambda)$ to obtain n^{2g} theta functions $\Theta_{m, m'}(z, \Lambda)$ with characteristics $(m, m') \in (\mathbb{Z}/n\mathbb{Z})^g \oplus (\mathbb{Z}/n\mathbb{Z})^g$. They generate the linear space $\Gamma(T, L^n)$.

An example of a principally polarized abelian variety is the *Jacobian variety* $\text{Jac}(C)$ of a Riemann surface C of genus g . Here Λ is spanned by vectors $v_i = (\int_{\gamma_i} \omega_1, \dots, \int_{\gamma_i} \omega_g) \in \mathbb{C}^g$, $i = 1, \dots, 2g$, where $(\omega_1, \dots, \omega_g)$ is a basis of the linear space of holomorphic differential 1-forms and $(\gamma_1, \dots, \gamma_{2g})$ is a basis of $H_1(T, \mathbb{Z})$.

In the case $g = 2$ and $n = 2$, Göpel was able to find a special basis $(\theta_0, \theta_1, \theta_2, \theta_3)$ in the space $\Gamma(\text{Jac}(C), L^2) \cong \mathbb{C}^4$ such that the map

$$\Phi : T \rightarrow \mathbb{P}^3, (z_1, z_2) \mapsto (\theta_0(z), \theta_1(z), \theta_2(z), \theta_3(z))$$

satisfies $\Phi(-z_1, -z_2) = \Phi(z_1, z_2)$ and its image X is the set of zeros in \mathbb{P}^3 of a quartic polynomial

$$F = A(x^4 + y^4 + z^4 + w^4) + 2B(x^2y^2 + z^2w^2) + 2C(x^2z^2 + y^2w^2) + 2D(x^2w^2 + y^2z^2) + 4Exyzw,$$

where the coefficients (A, B, D, E) satisfy a certain explicit equation in terms of theta constants, the values of theta functions at 0. The images of sixteen 2-torsion points $\epsilon \in \frac{1}{2}\Lambda/\Lambda \in \mathbb{C}^2/\Lambda$ are singular points of X . The Abel-Jacobi map $C \rightarrow \text{Jac}(C), x \mapsto (\int_{x_0}^x \omega_1, \int_{x_0}^x \omega_2) \bmod \Lambda$ embeds C into $\text{Jac}(C)$ and the images of the curves $C + \epsilon$ are the 16 trope-conics of X .

Ernst Kummer:1864

In 1864 Ernst Kummer had shown that the Fresnel's wave surface represents a special case of a 3-parametrical family of 16-nodal quartic surfaces [21]. He proved that they must contain 16 trope-conics which together with 16 nodes form an abstract incidence configuration (16_6) . Kummer shows that any 16-nodal quartic surface has a tetrahedron with conic-tropes in the faces intersecting at two points on the edges. No vertex is a node. From this he deduced that there exists a quadric surface that contains the four trope-conics. Using this he finds an equation of a

general Kummer surface of the form

$$(x^2 + y^2 + z^2 + w^2 + a(xy + zw) + b(xz + yw) + c(xw + yz))^2 + Kxyzw = 0,$$

where $xyzw = 0$ is the equation of a chosen tetrahedron and $K = a^2 + b^2 + c^2 - 2abc - 1$. Since then any 16-nodal quartic surface became known as a *Kummer quartic surface*.

It took almost 30 years since Göpel's discovery to realize that the Göpel equation, after a linear change of variables, can be reduced to Kummer equation. This was done by Carl Borchardt in 1878 [4].

Carl Borchardt:1878

In fact, Göpel's discovery leads to a modern definition of the Kummer surface and its higher-dimensional version, the *Kummer variety*. One considers any g -dimensional complex torus T and divides it by the involution $z \mapsto -z$, the orbit space is called the *Kummer variety* of T and is denoted by $\text{Kum}(T)$. If T admits an embedding into a projective space, then $\text{Kum}(T)$ is a projective algebraic variety with $2^{2 \dim T}$ singular points. Moreover, if T is a principally polarized abelian variety, for example the Jacobian variety of a genus g algebraic curve, one can embed $\text{Kum}(T)$ into the projective space \mathbb{P}^{2^g-1} to obtain a self-dual subvariety of degree $2^{g-1}g!$ with 2^{2g} singular points and 2^{2g} trope hyperplanes, each intersects it with multiplicity 2 along a subvariety isomorphic to the Kummer variety of a $(g-1)$ -dimensional principally polarized abelian variety.

There is also a *generalized Kummer variety* of dimension $2r$ introduced by Arnaud Beauville [1]. It is a nonsingular compact holomorphic symplectic manifold birationally isomorphic to the kernel of the addition map $A^{(r+1)} \rightarrow A$, where A is an abelian variety of dimension 2 and $A^{(r+1)} = A^r / \mathfrak{S}_{r+1}$ is its symmetric product. It is one of a few known examples of complete families of compact holomorphic symplectic manifolds of dimension $2r$.

Arnaud
Beauville:1983

The first book entirely devoted to Kummer surfaces that combines geometric, algebraic and transcendental approaches to their study was published by Ronald Hudson in 1905 [11].

Ronald Hudson:1905

It seems that Kummer's interests to 16-nodal quartic surfaces arose from his pioneering study of 2-dimensional families (congruences) of lines in \mathbb{P}^3 [24]. In [23] (see also [22]) he gives a classification of quadratic line congruences (i.e. congruences of lines such that through a general point $x \in \mathbb{P}^3$ passes exactly two lines from the congruence). Let n be the class of the congruence, i.e. the number of lines of the congruence that lie in a general plane. Kummer had shown that $2 \leq n \leq 7$ and when $n = 2$ all the lines are tangent to a Kummer quartic surface at two points. There are 6 congruences like that whose lines are tangent to the same Kummer surface.

Ernst Kummer:1865

In 1870, Felix Klein in his dissertation develops a beautiful relationship between the Kummer surfaces and quadratic line complexes [17]. A quadratic line complex is the intersection $X = G \cap Q$ of the Grassmann quadric $G = \text{Gr}(2, 4)$ in the Plücker space \mathbb{P}^5 with another quadric hypersurface. For any point $x \in \mathbb{P}^3$ the set of lines passing through x is a plane σ_x contained in G . Its intersection with X is a conic. The locus of points x such that this conic becomes reducible is the *singular surface* of the complex, and Klein had shown that it is a Kummer surface if the intersection $G \cap Q$ is transversal. Its 16 nodes correspond to points where $\sigma_x \cap X$ is a double line. In fact, Klein shows that the set of lines in X is parameterized by $\text{Jac}(C)$, where C is the Riemann surface of genus 2, the double cover of \mathbb{P}^1 realized as the pencil of quadrics spanned by G and Q ramified along 6

Felix Klein:1870

points corresponding to 6 singular quadrics in the pencil. The Kummer surface is isomorphic to $\text{Kum}(\text{Jac}(C))$. By a simultaneous diagonalization of quadrics G and Q he shows that the Kummer surface admits a birational nonsingular model as a complete intersection of 3 quadrics

$$\sum_{i=1}^6 x_i^2 = \sum_{i=1}^6 a_i x_i^2 = \sum_{i=1}^6 a_i^2 x_i^2 = 0,$$

where $y^2 = (x - a_1) \cdots (x - a_6)$ is the genus 2 Riemann surface C from above. Instead of 16 nodes and 16 trope-conics we now have two sets of 16 skew lines that form a configuration (16_6) . Klein also shows that the Fresnel's wave surface is characterized by the condition that C is bielliptic, i.e. it admits an involution with quotient an elliptic curve. By degenerating the quadratic complex to a *tetrahedral quadratic complex* one obtains the wave quartic surface. Jessop's book [15] gives an exposition of the works of Kummer and Klein on the relationship between line geometry and Kummer surfaces.

Klein's equations of a Kummer surface exhibit obvious symmetry defined by changing signs of the unknowns. They form an elementary abelian 2-group 2^5 , the direct sum of 5 copies of the cyclic group of order 2. The quartic model also admits 16 involutions t_i defined by the projection from the nodes p_i (take a general point x , join it with the node p_i , and then define $t_i(x)$ to be the residual intersection point). In 1886 Klein asked whether the group of birational automorphisms $\text{Bir}(X)$ of a (general) Kummer surface is generated by the group 2^5 and the projection involutions [18].

Felix Klein:1886

In 1901 John Hutchinson, using the theory of theta functions, showed that a choice of one of 60 Göpel tetrad of nodes (means no trope-conics in its faces) leads to an equation of the Kummer surface of the form

$$q(x_1x_2 + x_3x_4, x_1x_3 + x_2x_4, x_1x_4 + x_2x_3) + cx_1x_2x_3x_4 = 0,$$

where q is a quadratic form in 3 variables [14]. He observed that the transformation $x_i \mapsto 1/x_i$ leaves this equation invariant, and hence defines a birational involution of the Kummer surface. He proves that these 60 transformations generate an infinite discontinuous group. However he did not address the question whether adding these new transformations to Klein transformations would generate the whole group $\text{Bir}(X)$.

Thomas Weddle:1850

In 1850 Thomas Weddle, correcting a mistake of M. Chasles, noticed that the locus of singular points of quadric surfaces passing through a fixed set of 6 general points in \mathbb{P}^3 is a quartic surface with singular points at the six points [32]. It contains the unique twisted cubic curve through the six nodes (it seems that Chasles asserted that there is nothing else). In 1861 Cayley gave an explicit equation of the Weddle surface [6]. The simplest equation was later given by Hutchinson [13]:

Arthur Cayley:1861

$$\det \begin{pmatrix} xyz & w & a & \alpha \\ yzw & x & b & \beta \\ xzw & y & c & \gamma \\ xyw & z & d & \delta \end{pmatrix} = 0$$

Here the six points are the reference points $(1 : 0 : 0 : 0)$, $(0 : 1 : 0 : 0)$, $(0 : 0 : 1 : 0)$, $(0 : 0 : 0 : 1)$ and two points with coordinates given in the last two columns of the matrix. If one inverts the coordinates, the equation does not change. In 1889 Schottky proved that a Weddle surface is birationally isomorphic to a Kummer

Friedrich Schottky:1889

Schottky

surface [29]. This implies that the groups of birational automorphisms of a Weddle surface and a Kummer surface are isomorphic. For example, it is immediate to see that the same inversion transformation used by Hutchinson leaves the Weddle surface invariant. In 1911 Virgil Snyder gave many geometric constructions of involutions of the Weddle surface that give corresponding involutions of the Kummer surface [31]. He somehow does not mention *Hutchinson involution*.¹ In a paper of 1914, Francis Sharpe and Clide Craig pioneered the new approach to study birational automorphisms of algebraic surfaces based on Francesco Severi's Theory of a Basis. It consists of representing an automorphism of an algebraic surface as a transformation of the group of algebraic cycles on the surface. This gave an easy proof of Hutchinson's result that 60 Göpel-Hutchinson involutions generate an infinite group [30].

Virgil Snyder:1911

Francis Sharpe:1914

Clide Craig:1914

The Hutchinson involutions act freely on a nonsingular model of the Kummer surface and the quotient by these involutions are *Enriques surfaces*. In their paper of 2013 under the title 'Enriques surfaces of Hutchinson-Göpel type and Mathieu automorphisms' Shigeru Mukai and Hasinori Ohashi study the automorphism group of these Enriques surfaces [26].

Shigeru Mukai:2013

Hasinori Ohashi:2013

John Hutchinson also discovered the following amazing fact. The linear system of quadric surfaces through a *Weber hexad* of nodes on a Kummer surface defines a birational isomorphism to the *Hessian surface* of a nonsingular cubic surface $F_3(x, y, z, w) = 0$ (the Hessian surface defined by the hessian matrix of $F_3(x, y, z, w)$) [13]. The Hessian surface can be also defined as the locus of singular *polar quadrics* of the surface (quadrics given by linear combinations of partial derivatives of F_3). The rational map that assigns to a singular quadric its singular point defines a fixed-point-free involution on a nonsingular model of the Hessian surface with quotient isomorphic to an Enriques surface. There are 120 Weber hexads, each defines a birational involution of a Kummer surface, called the *Hutchinson-Weber involution*. The automorphism groups as well as complex dynamics of the Hessian quartic surfaces are the subjects of study of several recent papers.

John Hutchinson:1899

It is not surprising that none of the classical geometers could decide whether a given finite set of transformations generates the group $\text{Bir}(X)$. It had to wait until the end of the century for new technical tools to arrive.

A nonsingular birational model of a Kummer surface is an example of a K3 surface. By definition, a K3 surface Y is a compact analytic simply-connected surface with trivial first Chern class $c_1(Y)$. Its second Betti number is equal to 22. By a theorem of John Milnor, its homotopy type is uniquely determined by the quadratic form expressing the cup-product on $H^2(Y, \mathbb{Z}) \cong \mathbb{Z}^{22}$. It is a unique unimodular even quadratic lattice of signature $(3, 19)$ isomorphic to $E_8(-1)^{\oplus 3} \oplus U$, where $E_8(-1)$ is an even unimodular negative definite lattice of rank 8 and U is the integral hyperbolic plane. All K3 surfaces are diffeomorphic and realize the homotopy type defined by this lattice. Any complex K3 surface admits a Kähler metric and, by Yau's theorem it also admits a *Ricci-flat metric*. It is not known how to write it explicitly (a question of great importance for physicists). The Kummer surface is flat in this metric outside the singular points, but still one does not know how to extend this flat metric to a Ricci-flat metric on Y .

¹John Hutchinson, Virgil Snyder, Francis Sharpe and Clide Craig were on the faculty of the Cornell Mathematics Department.

The quadratic lattice of cohomology $H^2(Y, \mathbb{Z})$ of an algebraic K3 surface contains a sublattice $S_Y = H^2(Y, \mathbb{Z})_{alg} \cong \mathbb{Z}^\rho$ of algebraic 2-cycles. It contains $c_1(L)$, where L is an ample line bundle. The signature of S_Y is equal to $(1, \rho - 1)$ and, in general, it is not unimodular. The group $\text{Bir}(X)$ is isomorphic to the group $\text{Aut}(Y)$ of biregular automorphisms of Y and it admits a natural representation $\rho : \text{Aut}(Y) \rightarrow \text{O}(H^2(Y, \mathbb{Z}))$ in the orthogonal group of $H^2(Y, \mathbb{Z})$ that leaves S_Y invariant.

The fundamental *Global Torelli Theorem* of I. Pyatetsky-Shapiro and I. Shafarevich [28] allows one to describe the image of homomorphism ρ as the set of all isometries of $H^2(Y, \mathbb{Z})$ that leave (after complexification) $H^{2,0}(Y, \mathbb{C})$ invariant and also leave invariant the semi-group of cohomology classes of holomorphic curves on Y . The group of automorphisms $\text{Aut}(Y)$ acts naturally on the lattice S_Y and on the *hyperbolic space* $\mathbb{H}^{\rho-1}$ associated with the linear space $S_Y \otimes \mathbb{R}$ of signature $(1, \rho - 1)$. In this way it is realized as a discrete group of motions of a hyperbolic space. By using an isometric embedding of S_Y into the unimodular even lattice $II_{1,25}$ of signature $(1, 25)$ isomorphic to the orthogonal sum of the *Leech lattice* and the hyperbolic plane U , Richard Borcherds introduced a method that in some cases allows one to compute the automorphism group of a K3 surfaces using the isometries of $II_{1,25}$ defined by the reflections into *Leech roots* [3]. In 1998, based on Borcherds' ideas, Shigeyuki Kondo proved that the group of birational automorphisms of a general Kummer surface (i.e. the Jacobian surface of a general curve of genus 2) is generated by the group 2^5 , 16 projection involution, 60 Hutchinson-Göpel involutions and 120 Hutchinson-Weber involutions [20].² The group of birational automorphisms of an arbitrary Kummer surface is still unknown.

We refer to some modern expositions of the theory of Kummer surfaces to [2], [7] and [10]. We apologize for the brevity of our discussion that omits many important contributions to the study of Kummer surfaces in the past and in the present. Note that a search of the data base of Math.Sci.Net under Title:Kummer surface gives 130 items (70 of them in this century).

REFERENCES

- [1] A. Beauville, *Variétés Kähleriennes dont la première classe de Chern est nulle*. J. Differential Geom. **18** (1983), 755–782
- [2] Ch. Birkenhake, H. Lange, *Complex tori*. Progress in Mathematics, **177**. Birkhuser Boston, Inc., Boston, MA, 1999.
- [3] R. Borcherds, *Automorphism groups of Lorentzian lattices*, J. Algebra, **111** (1097), 133–153.
- [4] C. Borchardt, *Ueber die Darstellung der Kummerische Fläche vierter Ordnung mit sechzehn Knoten durch die Göpelsche biquadratische Relation zwischen vier Thetafunktionen mit zwei Variablen*, **83** (1877), 234–244.
- [5] A. Cayley, *Sur la surface des ondes*, J. Math. Pures et Appl. **11** (1846), 291–206.
- [6] A. Cayley, *Sur les cones du second ordre qui passent par six points donnés*, Comptes Rendus Acad. Sci. de Paris, **52** (1861), 1216–1218.
- [7] I. Dolgachev, *Classical algebraic geometry. A modern view*. Cambridge University Press, Cambridge, 2012.
- [8] A.-J. Fresnel, *Second supplément au mémoire sur la double réfraction* (signed 31 March 1822, submitted on 1 April 1822), in Oeuvres Complètes d'Augustin Fresnel, ed. H. de Sénarmont, É. Verdet, et L. Fresnel, vol. 2 (1868), pages 369–442.
- [9] A. Göpel, *Theoriae transcendentium Abelianarum primi ordinis adumbratio levis*, Journ. die pure und appl. Mathematik, **35** (1847), 277–312.

²In fact, Kondo includes in the set of generators some new automorphisms of infinite order discovered by JongHae Keum [16] instead of Hutchinson-Weber involutions. It was later observed by Ohashi that one can use Hutchinson-Weber involutions instead of Keum's automorphisms [27]

Ilya Pyatetsky-Shapiro:1977
Igor Shafarevich:1977

Richard Borcherds:1997

Shigeyuki Kondo:1998

- [10] P. Griffiths, J. Harris, *Principles of algebraic geometry*. Reprint of the 1978 original. Wiley Classics Library, John Wiley and Sons, Inc., New York, 1994.
- [11] R. Hudson, *Kummer's quartic surface*, Cambridge University Press, 1905 [reprinted in 1990 with a foreword by W. Barth].
- [12] W. Hamilton, *Third supplement to an essay on the theory of systems of rays*, Trans. Royal Irish Academy, **17**, part 1 (1837), 1–144.
- [13] J. I. Hutchinson, *The Hessian of the cubic surface*, Bull. Amer. Math. Soc. **5**(6) (1899), 282–292, .
- [14] J. I. Hutchinson, *On some birational transformations of the Kummer surface into itself*, Bull. Amer. Math. Soc., **7**(5) (1901), 211–217.
- [15] C. Jessop, *A treatise of the line complex*, Cambridge University Press, 1903 [reprinted by Chelsea Publ. Co., New York, 1969].
- [16] J. Keum, *Automorphisms of Jacobian Kummer surfaces*. Compositio Math. **107** (1997), 269–288.
- [17] F. Klein, *Zur Theorie der Liniencomplexe des ersten und Zwieter Grades*, Math. Ann. **2** (1870), 198–226.
- [18] F. Klein, *Ueber Configurationen, welche der Kummerschen Fläche zugleich eingeschrieben und umgeschriben sind*, Math. Ann. **27** (1886), 106–142.
- [19] H. Knörrer, *Die Fresnelsche Wellenfläche*, Arithmetik und Geometrie, Math. Miniaturen, 3, Basel, Boston, Berlin: Birkhäuser, 1986, pp. 115–141,
- [20] S. Kondo, *The automorphism group of a generic Jacobian Kummer surface*, J. Algebraic Geom., **7**(3) (1998), 589–609.
- [21] E. Kummer, *Ueber die Flächen vierten Grades mit sechszehn singulären Punkten*, Monatsberichte Berliner Akademie, **6**, (1864), 246–260.
- [22] E. Kummer, *Ueber Strahlensysteme, deren Brehnflächen Flächen vierten Grades mit sechszehn singulären Punkten Sind*, Monatsberichte Berliner Akademie, **6**, (1864), 495–499,
- [23] E. Kummer, *Ueber algebraische Strahlensysteme, in's Besondere die der ersten und der zweiten Ordnung*, Monatsberichte Berliner Akademie, **7**, (1865), 288–293,
- [24] E. Kummer, *Ueber die algebraische Strahlensysteme insbesondere die erste und zweiten Ordnung*, Monatsberichte Berliner Akademie, **8**(1866), 1–120.
- [25] G. Loria, *Il Passato ed il Presente delle principali Theorie Geometriche*, Torino, Carlo Clausen, 1896.
- [26] S. Mukai, H. Ohashi, *Enriques surfaces of Hutchinson-Göpel type and Mathieu automorphisms*. Arithmetic and geometry of K3 surfaces and
- [27] H. Ohashi, *Enriques surfaces covered by Jacobian Kummer surfaces*, Nagoya Math. J., **95** (2009), 165–186.
- [28] I. Pyatetskii-Shapiro, I. Shafarevich, *A Torelli theorem for algebraic surfaces of type K3*, Izv. Akad. Nauk SSSR, ser. math., **35** (1971), 503–572
- [29] F. Schottky, *Ueber die Beziehungen zwischen den sechzehn Thetafunktionen von zwei Variablen*, Jour. für die reine und angew. Mathematik, **105** (1889), 233–249.
- [30] F. Sharpe, C. Craig, *An application of Severi's theory of a basis to the Kummer and Weddle surfaces*, Trans. Amer. Math. Soc. **15** (1914), 236–244.
- [31] V. Snyder, *An application of a (1,2) quaternary correspondence to the Weddle and Kummer surfaces*, Trans. Amer. Math. Soc. **12** (1911), 354–366.
- [32] T. Weddle, *On theorems in space analogous to those of Pascal and Brianchon in a plane*, Cambridge and Dublin Quarterly Math. Journal, **5** (1850), 58–69.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, 525 EAST UNIVERSITY AVENUE,
ANN ARBOR, MI 48109-1109 USA
E-mail address: idolga@umich.edu