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Contents

Lecture 1

Kleinian surface singularities

1.1 Finite subgroups of SL$(2, \mathbb{C})$

Let G be a finite subgroup of the group $\mathrm{GL}(2, \mathbb{C})$ of invertible 2×2-matrices. It acts naturally on the projective one-dimensional space $\mathbb{P}^1(\mathbb{C})$ of lines in $\mathbb{C}^2$. We use the projective coordinates $(t_0 : t_1)$ dual to the standard basis (e_1, e_2) of $\mathbb{C}^2$. The point $(0 : 1)$ is denoted by ∞ so we can identify $\mathbb{P}^1(\mathbb{C})$ with $\mathbb{C} \cup \{\infty\}$, where the affine coordinate on $\mathbb{C}$ is $z = t_1/t_0$. A matrix $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ acts on $\mathbb{P}^1(\mathbb{C})$ by sending a point $x = (\alpha : \beta)$ to $g \cdot x = (a\alpha + b\beta : c\alpha + d\beta)$. In affine coordinates this action is the Möbius (or fractional-linear) transformation $z \mapsto \frac{dz+c}{bz+a}$. Note that the matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} d & c \\ b & a \end{smallmatrix}\right)$ are conjugate by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. It is clear that under this action ∞ is mapped to d/b ($= \infty$ if $b = 0$) and the point $-a/b$ is mapped to ∞.

Let $Z(G)$ be the center of G consisting of scalar matrices, $\bar{G} = G/Z(G)$. The subgroup $Z(G)$ is the kernel of the action of G on $\mathbb{P}^1(\mathbb{C})$, so $\bar{G}$ acts naturally and faithfully on $\mathbb{P}^1(\mathbb{C})$.

Although it is not hard to classify all finite subgroups of $\mathrm{GL}(2, \mathbb{C})$ we will restrict ourselves to finite subgroups G of the group $\mathrm{SL}(2, \mathbb{C})$ which consists of invertible matrices with determinant 1. By taking the standard hermitian inner product on $\mathbb{C}^2$ defined by $\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{z} \cdot \bar{\mathbf{w}}$ and averaging it by the group G, we arrive at a hermitian inner product on $\mathbb{C}^2$ which is invariant with respect to G. This shows that G is conjugate to a finite subgroup of the special unitary group $\mathrm{SU}(2)$. Recall that $\mathrm{SU}(N)$ denotes the subgroup of $\mathrm{SL}(N, \mathbb{C})$ that consists of matrices A satisfying ${}^t\bar{A} = A^{-1}$. The averaging argument shows the classification of finite subgroups of $\mathrm{SL}(2, \mathbb{N})$ is equivalent to the classification of finite subgroups of $\mathrm{SU}(N)$.

There is a natural surjective homomorphism from $\mathrm{SU}(2)$ to the group $\mathrm{SO}(3)$ of orthogonal 3×3-matrices with determinant 1. This group is isomorphic to the group of rotations of the three-dimensional euclidean space $\mathbb{R}^3$. The simplest (and the most beautiful) way to define the homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$ is by using the algebra of quaternions $\mathbb{H}$.

We will write a quaternion $q = a + bi + cj + dk \in \mathbb{H}$ in the form $q = z_1 + z_2 j$, where $z_1 = a + bi, z_2 = c + di$ are complex numbers. We have

$$\begin{aligned} |q|^2 : &= a^2 + b^2 + c^2 + d^2 = |z_1|^2 + |z_2|^2, \\ \bar{q} : &= a - bi - cj - dk = \bar{z}_1 - z_2 j. \end{aligned} \tag{1.1}$$

A quaternion q is invertible if and only if $|q| \neq 0$. In this case $q^{-1} = \frac{1}{|q|}\bar{q}$.

Let $\mathbb{H}_1$ be the group of quaternions of norm 1. There is a natural isomorphism of groups

$$\mathbb{H}_1 \to \mathrm{SU}(2),\ z_1 + z_2 j \mapsto \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}.$$

Let us identify $\mathbb{R}^3$ with the space of "pure quaternions", i.e. quaternions of the form $q = bi + cj + dk$. The euclidean inner product coincides with the quaternion norm. Define the action of $\mathbb{H}_1 \cong \mathrm{SU}(2)$ on $\mathbb{R}^3$ by

$$q \cdot q_0 = q \cdot q_0 \cdot q^{-1},\ q \in \mathbb{H}_1, q_0 \in \mathbb{R}^3.$$

It is immediate that this defines a homomorphism $p : \mathbb{H}_1 \to \mathrm{O}(3)$.

Write $q \in \mathbb{H}_1$ in the form $q = \cos\phi + \sin\phi q_1$, where q_1 is a pure quaternion of norm 1, considered as a vector in $\mathbb{R}^3$ of norm 1. Then one directly checks that the action of q in $\mathbb{R}^3$ is a rotation about the angle ϕ with axis defined by q_1. Since the group $\mathrm{SO}(3)$ consists of such rotations, we see that the homomorphism p defines a surjective homomorphism $\mathbb{H}_1 \to \mathrm{SO}(3)$. Its kernel consists of unit quaternions contained in the center of $\mathbb{H}^*$. There are only two of them: ± 1. This defines a short sequence of groups

$$1 \to \{\pm 1\} \to \mathrm{SU}(2) \to \mathrm{SO}(3) \to 1 \tag{1.2}$$

Thus any finite subgroup G of $\mathrm{SU}(2)$ defines a finite subgroup $\bar{G}$ of rotations of $\mathbb{R}^3$. Conversely, every $\bar{G} \subset \mathrm{SO}(3)$ can be lifted to a finite subgroup G of $\mathrm{SU}(2)$ such that the kernel of $G \to \bar{G}$ is of order ≤ 2. We will see that it is always of order 2 except the case when $\bar{G}$ is a cyclic group of odd order.

The classification of finite subgroups of $\mathrm{SO}(3)$ goes back to the ancient time. As is well-known each such group can be realized as the group of rotation symmetries of one of a regular polyhedron, or of a dihedral (a prism based on a regular polygon), or as a cyclic group of rotations around the same axis.

We will classify finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ by first classifying finite subgroups of $\mathrm{Aut}(\mathbb{P}^1(\mathbb{C})) \cong \mathrm{PGL}(2, \mathbb{C})$ by using the exact sequence analogous to (1.2)

$$1 \to \{\pm 1\} \to \mathrm{SL}(2, \mathbb{C}) \to \mathrm{Aut}(\mathbb{P}^1(\mathbb{C})) \to 1 \tag{1.3}$$

Its advantage is that we can do it purely algebraically. In fact, the same argument gives the classification of finite subgroups G of $\mathrm{SL}(2, F)$, where F is an algebraically closed field of arbitrary characteristic p provided that the order of G is coprime to p.

Let us first classify all possible finite subgroups $\bar{G}$ of $\mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$.

An eigensubspace of any matrix g from $\mathrm{GL}(2,\mathbb{C})$ of finite order is either 2-dimensional, in which case g is a scalar diagonal matrix, or one-dimensional. In the first case g acts identically on $\mathbb{P}^1(\mathbb{C})$, in the second case g has 2 fixed points in $\mathbb{P}^1(\mathbb{C})$ corresponding to two distinct eigensubspaces of g. Set

$$Z = \{(x,g) \in \mathbb{P}^1(\mathbb{C}) \times \bar{G} \setminus \{1\} : g(x) = x\}.$$

Let $\mathcal{P}$ be the projection of this set to the first factor. Its elements are fixed points of some non-trivial element of g. Let $\mathcal{P} = O_1 \cup \ldots \cup O_k$ be the orbit decomposition of $\mathcal{P}$. For each $x \in O_i$ let e_i be the order of the stabilizer subgroup $\bar{G}_x$. Since all stabilizer subgroups of points in the same orbit are conjugate, this number is independent of a choice of x in O_i. We have $|O_i| = N/e_i$, where $N = |\bar{G}|$. Let us count the cardinality $|Z|$ in two ways by considering the two projections. We have

$$|Z| = 2(N-1) = \sum_{x\in\mathcal{P}}(|\bar{G}_x|-1) = \sum_{i=1}^{k}\sum_{x\in O_i}(|\bar{G}_x|-1)$$

$$= \sum_{i=1}^{k}\frac{N}{e_i}(e_i-1) = N\sum_{i=1}^{k}(1-\frac{1}{e_i}).$$

This gives

$$\sum_{i=0}^{k}\frac{1}{e_i} = k-2+\frac{2}{N}.$$

Since $e_i \geq 2$, the left-hand size is less or equal than $k/2$. This immediately gives $k = 2$ or 3.

Assume $k = 2$. We get $e_1 = e_2 = N$. Thus $\bar{G}$ fixes 2 points z_1, z_2 in $\mathbb{P}^1(\mathbb{C})$. Let $g \in \mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$ be a transformation which sends z_1 to 0 and z_2 to ∞. Then the conjugate group $g\bar{G}g^{-1}$ fixes 0 and ∞. It is generated by the transformation

$$g_1 : z \mapsto \epsilon_n z,$$

where ϵ_n is a primitive nth root of unity.

Assume $k = 3$. The equation

$$\frac{1}{e_1}+\frac{1}{e_2}+\frac{1}{e_3} = 1+\frac{2}{N}$$

has the following solutions $(e_1, e_2, e_3; N)$

- $(2,2,n;2n)$,
- $(2,3,3;12)$,

- $(2, 3, 4; 24)$,
- $(2, 3, 5; 60)$.

Case $(2, 2, n; 2n)$.

Replacing G by a conjugate group we may assume that the orbit O_3 with stabilizer subgroup H of order n consists of points 0 and ∞. It is easy to see that the stabilizer subgroup of any point in $\mathbb{P}^1(\mathbb{C})$ is a cyclic group. Since the index of H in $\bar{G}$ is equal to 2, it is a normal subgroup. Thus H fixes both 0 and ∞. A fractional-linear transformation of order n which fixes these points is of the form $z \mapsto az$, where $a^n = 1$. Thus we may assume that H is generated by a transformation

$$g_1 : z \mapsto \epsilon_n z.$$

Choose $g_2 \in \bar{G}$ which transforms 0 to ∞. Since $\{0, \infty\}$ is an orbit, $g(\infty) = 0$. It is easy to see that this implies that

$$g_2 : z \mapsto \frac{a}{z}, \quad a \neq 0.$$

After conjugation of $\bar{G}$ by a transformation $z \mapsto \sqrt{a}z$, we may assume that $a = 1$. Obvioulsy, g_2 is of order 2. Since $g_2 \notin H$, the group $\bar{G}$ is generated by g_1, g_2. It is immediately checked that

$$g_2 g_1 g_2^{-1} = g_1^{-1}.$$

So, $\bar{G}$ is isomorphic to the dihedral group D_{2n} of order $2n$.

Case $(2, 3, 3; 12)$.

Let g_1 generate a stabilizer subgroup of order 2 of the orbit O_1 of cardinality 6. As above, replacing G by a conjugate subgroup, we may assume that g_1 fixes 0 and ∞ and hence is represented by the transformation

$$g_1 : z \mapsto -z.$$

Since there is only one orbit with stabilizers of order 2, the points $0, \infty$ belong to O_1. Let g_2 be an element from $\bar{G}$ such that $g_2(0) = \infty$. Then $g_2 g_1 g_2^{-1}$ fixes ∞ and hence coincides with g_1. This implies that $g_2(\infty) = 0$ and hence, as in the previous case, we may assume that

$$g_2 : z \mapsto \frac{1}{z}. \tag{1.4}$$

The elements g_1, g_2 generate a subgroup $\langle g_1, g_2 \rangle$ of $\bar{G}$ isomorphic to the direct sum of two cyclic groups of order 2. The fixed points of g_2 are $\{1, -1\}$ and the fixed points of $g_1 g_2$ are $i, -i$. Thus $O_1 = \{0, \infty, 1, -1, i, -i\}$. Let g_3 be an element of order 3 from $\bar{G}$ stabilizing a point from the orbit O_2. It cannot commute with any element from $\langle g_1, g_2 \rangle$ since otherwise we obtain an element of order 6 in $\bar{G}$ which fixes 2 points with stabilizer of order 6. Without loss of generality we may assume that $g_3g_1g_3^{-1} = g_2$. Thus $g_3(0) = 1, g_3(\infty) = -1$. This implies that

$$g_3 : z \mapsto \frac{\lambda z + 1}{-\lambda z + 1}, \quad \lambda \neq 0.$$

Since g_3 is of order 3, this gives $\lambda = \pm i$. Again, conjugating by the transformation $z \mapsto -z$, we may assume that

$$g_3 : z \mapsto \frac{iz + 1}{-iz + 1}. \tag{1.5}$$

Now it is easy to check that $\bar{G}$ is isomorphic to the tetrahedral group $T \cong A_4$ of rotation symmetries of a regular tetrahedron.

Case $(2,3,4;24)$.

Let g_1 be an element of order 4 which we may assume fixes the points $0, \infty$. Then

$$g_1 : z \mapsto iz.$$

Arguing as in the previous case, we see that an element from $\bar{G}$ which sends 0 to ∞ must be of the form (1.4). The subgroup $\langle g_1, g_2 \rangle$ is isomorphic to the dihedral group D_8 of order 8. Also, as in the previous case, we find that $\bar{G}$ contains an element of order 3 of the form (1.5). It is easy to check that $\bar{G} = \langle g_1, g_2, g_3 \rangle$ is isomorphic to the octahedral group of rotation symmetries of a regular octahedron (or a cube). It is also isomorphic to the symmetric group S_4.

Case $(2,3,5;60)$.

As above, we find that $\bar{G}$ contains an element g_1 of order 5 represented by

$$g_1 : z \mapsto \epsilon_5 z,$$

and element of order 2 represented by (1.4). To agree with classical books, we change it to

$$g_2 : z \mapsto -1/z.$$

By Sylow's Theorem, a group of order 60 contains a subgroup of order 4. In our case, it cannot be cyclic since no stabilizer subgroup is of order 4. So, it is the direct sum of cyclic groups of order 2. Let

$$g_3 : z \mapsto \frac{az + b}{cz + d} \tag{1.6}$$

be an element of order 2 which commutes with g_2. Direct computation shows that $a = -d, b = c$. Note that the icosahedron group is isomorphic to the alternating group A_5. It contains a subgroup of order 10 generated by a cyclic permutation of order 5 and a permutation of order 2 equal to the product of two transpositions. This is our subgroup $\langle g_1, g_2 \rangle$. Also it is easy to see that the product $g_1g_3g_1$ must be of order 3. This gives the additional condition on (a,b). Direct computation shows that

$$b/a = \epsilon + \epsilon^4 = \frac{\epsilon^2 - \epsilon^3}{\epsilon - \epsilon^4}.$$

Thus we may take

$$g_3 : z \mapsto \frac{(\epsilon - \epsilon^4)z + (\epsilon^2 - \epsilon^3)}{(\epsilon^2 - \epsilon^3)z - (\epsilon - \epsilon^4)}.$$

Let O_3 be the orbit of cardinality 12. It contains $0, \infty$. Let $f(z) = z^{10} + \ldots = 0$ be the equation with roots equal to the remaining points in the orbit. Since g_1 acts on this set, the equation can be written in the form $\phi(z^5)$, where $\phi(t) = t^2 + mt + n$. Since g_2 acts on the roots too, we see that $n = -1$. Since g_3 leaves invariant the set of zeroes of $f(z)$, the direct computations give $m = 11$.

One finds the 10 roots of $f(z)$ which together with $0, \infty$ are the vertices of a regular icosahedron inscribed in the sphere identified with $\mathbb{P}^1(\mathbb{C})$ by using the stereographic projection. The group $\bar{G}$ is the icosahedron group isomorphic to A_5.

Now, after we have found the structure of $\bar{G}$, it is easy to determine the structure of any $G \subset \mathrm{SL}(2, \mathbb{C})$. We know that $\mathrm{Ker}(G \to \bar{G})$ is trivial or equal to $\{\pm I_2\}$. If $\bar{G}$ contains a non-central element of order 2 (this occurs always if $\bar{G}$ is not a cyclic group of odd order), then the kernel is not trivial because the only element of order 2 in $\mathrm{SL}(2, \mathbb{C})$ is the matrix $-I_2$ which belongs to the center of the group. The group G is called a *binary polyhedral group*. If $\bar{G}$ is not a cyclic group of odd order, then $|G| = 2|\bar{G}|$, so G is either cyclic group, or binary dihedral group of order $4n$, binary tetrahedral group of order 24, binary octahedron group of order 48, or binary icosahedron group of order 120.

If G is a cyclic group C_n of odd order n , then it can be lifted to an isomorphic group in $\mathrm{SL}(2, \mathbb{C})$ or to the direct product $C_2 \times C_n$.

1.2 Grundformen

Let $F(t_0, t_1)$ be a homogeneous polynomial of degree d. Its set of zeroes in $\mathbb{C}^2$ is the union of lines taken with multiplicities. Its set of zeros in $\mathbb{P}^1(\mathbb{C})$ can be identified with an effective divisor of degree d. We denote it by $V(F)$. In a more sophisticated way, we identify F with a section of the invertible sheaf $\mathcal{O}_{\mathbb{P}^1}(d)$ and its set of zeroes with a closed subscheme $V(F)$ of $\mathbb{P}^1$ such that $h^0(\mathcal{O}_{V(F)}) = d$.

Let

$$V_d = \mathbb{C}[t_0, t_1]_d \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$$

be the space of homogeneous polynomials of degree d in variables t_0, t_1. We identify V_1 with the dual space of $\mathbb{C}^2$. Thus V_d is isomorphic to the dth symmetric power $S^d(V_1)$. The group $\mathrm{SL}(2, \mathbb{C})$ acts on V_d by $g : F \mapsto g^*(F) := f \circ g^{-1}$, or more explicitly,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : F(t_0, t_1) \mapsto F(dt_0 - bt_1, -ct_0 + at_1).$$

Here we switched to g^{-1} to define the linear action of the group G on the vector space V_d.

Since

$$g^*(F)(x) = 0 \Leftrightarrow F(g^{-1}(x)) = 0 \Leftrightarrow g^{-1}(x) \in V(F) \Leftrightarrow x \in g(V(F)),$$

we have

$$V(g^*(F)) = g(V(F)). \tag{1.7}$$

It is immediately checked that it also takes into account the multiplicities.

Definition 1.2.1. A homogeneous polynomial F is a *relative invariant* of G if

$$g(V(F)) = V(F).$$

It follows from (1.7) that F is a relative invariant if and only if for any $g \in G$,

$$g^*(F) = a_g F$$

for some nonzero $a_g \in \mathbb{C}$. It is immediately checked that $g \mapsto a_g$ defines a homomorphism of groups $\chi : G \to \mathbb{C}^*$. We call it the *character* of F.

Let O be an orbit of G in $\mathbb{P}^1(\mathbb{C})$ considered as a divisor $\sum_{x \in O} x$. For any relative invariant F of G its divisor of zeroes $V(F)$ is the sum of orbits. Thus F is the product of relative invariants with sets of zeroes equal to an orbit of G.

Definition 1.2.2. A *Grundform* is a relative invariant F with divisor of zeroes equal to an exceptional orbit (i.e. an orbit with a non-trivial stabilizer).

Let F be a relative invariant with set of zeroes $V(F)$ equal to a non-exceptional orbit O. Suppose F_1 and F_2 are Grundformen (we use the German plural) corresponding to exceptional orbits with cardinalities $|\bar{G}|/e_1$ and $|\bar{G}|/e_2$ and the characters χ_1, χ_2 satisfying

$$\chi_1^{e_2} = \chi_2^{e_1}. \tag{1.8}$$

Choose a, b such that $\Phi = aF_1^{e_1} + bF_2^{e_2}$ has a zero in O. Then the whole O will be the set of zeroes of Φ. Thus F is equal to Φ up to a multiplicative factor. We will use this argument to show that any relative invariant is a polynomial in Grundformen. We consider the case when G is a binary polyhedral group.

We choose a representative of a generator of $\bar{G}$ as a matrix with determinant 1. This give us generators of G.

From now on

$$\epsilon_n = e^{2\pi i/n}.$$

Case 1: G is a cyclic group of order n.

A generator is given by the matrix

$$g = \begin{pmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n^{-1} \end{pmatrix}.$$

The corresponding Moebius transformation is $\bar{g} : z \mapsto \epsilon_n^{-2} z$. So, if n is odd, then $\bar{g}$ is of order n. If n is even, then $\bar{g}$ is of order $n/2$.

The exceptional orbits are $\{0\}$ and $\{\infty\}$. The Grundformen are

$$\Phi_1 = t_0, \ \Phi_2 = t_1$$

with characters determined by

$$\chi_1(g) = \epsilon_n, \quad \chi_2(g) = \epsilon_n^{-1}.$$

Thus $\chi_1^n = \chi_2^n$ if n is odd, and $\chi_1^{n/2} = \chi_2^{n/2}$ if n is even. This shows that any relative invariant F with $|V(F)| = n$ if n is odd (resp. $|V(F)| = n/2$ if n is even) is equal to $\lambda t_0^n + \mu t_1^n$ (resp. $\lambda t_0^{n/2} + \mu t_1^{n/2}$) for some λ, μ. This shows that any relative invariant is a polynomial in t_0, t_1.

Case 2: G is a binary dihedral group of order $4n$.
Its generators are given by the matrices

$$g_1 = \begin{pmatrix} \epsilon_{2n} & 0 \\ 0 & \epsilon_{2n}^{-1} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Any exceptional orbit is the orbit of a fixed point of some element $g \in G$ different from $\pm I_2$. The fixed points of g_1 are $0, \infty$. Applying g_2, we see that they form one orbit of cardinality 2. The fixed points of g_2 are ± 1. Applying powers of g_1 we get two exceptional orbits. One is formed by nth roots of 1, the other one is formed by nth roots of -1. The Grundformen are

$$\Phi_1 = t_0^n + t_1^n, \quad \Phi_2 = t_0^n - t_1^n, \quad \Phi_3 = t_0 t_1. \tag{1.9}$$

The generators g_1 and g_2 act on the Grunforms with characters

$$\chi_1(g_1) = -1, \chi_1(g_2) = i^n, \; \chi_2(g_1) = -1, \chi_2(g_2) = -i^n,$$

$$\chi_3(g_1) = 1, \chi_3(g_2) = -1.$$

Since Φ_1, Φ_2 correspond to exceptional orbits with $e_1 = e_2 = 2$ and $\chi_1^2 = \chi^2$, the condition (3.12) is satisfied. Thus any relative invariant is a polynomial in the Grundformen.

Case 2: G is a binary tetrahedral group of order 24.
Its generators are given by the matrices

$$g_1 = \begin{pmatrix} \epsilon_4 & 0 \\ 0 & \epsilon_4^{-1} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad g_3 = \frac{1}{1-i} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

The fixed points of g_1 are $0, \infty$ permuted under g_2. Their stabilizer is of order 2. Thus orbit O_1 with $e_1 = 2$ consists of 6 points. Applying powers of g_3 to 0 and 1 we see that O_1 consists of the points $0, \infty, 1, -1, i, -i$. The fixed points of g_3 are $\frac{1-i}{2}(1 \pm \sqrt{3})$. Applying g_1 and g_2 to these points, we obtain two orbits of cardinality 4

$$O_2 = \{\pm \frac{1-i}{2}(1+\sqrt{3}), \pm \frac{1+i}{2}(1-\sqrt{3})\}, \; O_3 = \{\pm \frac{1-i}{2}(1-\sqrt{3}), \pm \frac{1+i}{2}(1+\sqrt{3})\}.$$

The Grundformen are

$$\Phi_1 = t_0 t_1 (t_0^4 - t_1^4), \Phi_2, \Phi_3 = t_0^4 \pm 2\sqrt{-3} t_0^2 t_1^2 + t_1^4.$$

Their characters are

$$\chi_1(g_1) = \chi_1(g_2) = \chi_1(g_3) = 1,$$

$$\chi_2(g_1) = \chi_2(g_2) = \chi_2(g_3) = \epsilon_3,$$

$$\chi_3(g_1) = \chi_3(g_2) = 1, \chi_3(g_3) = \epsilon_3^2.$$

We see that $\chi_2^3 = \chi_3^3$ so (3.12) is satisfied.

Case 3: G is a binary octahedral group of order 48.

Its generators are

$$g_1 = \begin{pmatrix} \epsilon_8 & 0 \\ 0 & \epsilon_8^{-1} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad g_3 = \frac{1}{1-i}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}.$$

The fixed points with stabilizers of order 4 are $0, \infty, 1, -1, i, -i$. They form the first orbit O_1. The fixed points with stabilizers of order 3 form the second orbit of order 8. It consists of the union of the orbits O_2 and O_3 from the previous case.

The group $\bar{G}$ isomorphic to the permutation group S_4. It has 2 conjugacy classes of elements of order 2. One of them belongs to the conjugacy class of an element of order 4, say of g_1. Another one is a stabilizer of an exceptional orbit O_3 of cardinality 12. It corresponds to a transposition in S_4. It can be realized by the product $g_1g_2 : z \mapsto i/z$. Its fixed points are $\pm\epsilon_8$. The corresponding orbit O_3 consists of

$$\epsilon_8^k, \quad \frac{i\epsilon_8^k + i}{\epsilon_8^k - 1}, \frac{i\epsilon_8^k - 1}{i\epsilon_8^k + 1}, \quad k = 1, 3, 5, 7.$$

Now it is easy to list the Grundformen. They are

$$\Phi_1 = t_0t_1(t_0^4 - t_1^4),\ \Phi_2 = t_0^8 + 14t_0^4t_1^4 + t_1^8 = (t_0^4 + 2\sqrt{-3}t_0^2t_1^2 + t_1^4)(t_0^4 - 2\sqrt{-3}t_0^2t_1^2 + t_1^4),$$

$$\Phi_3 = (t_0^4 + t_1^4)((t_0^4 + t_1^4)^2 - 36t_0^4t_1^4).$$

The characters are

$$\chi_1(g_1) = -1, \chi_1(g_2) = \chi_1(g_3) = 1,$$

$$\chi_2(g_1) = \chi_2(g_2) = \chi_2(g_3) = 1,$$

$$\chi_3(g_1) = -1, \quad \chi_3(g_2) = \chi(g_3) = 1.$$

We have $\chi_1^4 = \chi_3^2$, so (3.12) is again satisfied.

Case 4: G is a binary icosahedra group of order 120.

Its generators are

$$g_1 = \begin{pmatrix} \epsilon_{10} & 0 \\ 0 & \epsilon_{10}^{-1} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad g_3 = \frac{1}{\sqrt{5}}\begin{pmatrix} \epsilon_5 - \epsilon_5^4 & \epsilon_5^2 - \epsilon_5^3 \\ \epsilon_5^2 - \epsilon_5^3 & -\epsilon_5 + \epsilon_5^4 \end{pmatrix}.$$

Note that we can replace g_3 with a generator of order 3 equal to $g_1g_3g_1$. We already used that the orbit O_3 with stabilizer subgroups of $\bar{G}$ of order 5 are roots of the Grundform

$$\Phi_3 = t_0t_1(t_0^{10} + 11t_0^5t_1^5 - t_1^{10}).$$

Other Grundformen must be of degree 30 and 20. Recall that the hessian of a function in two variables is the determinant of the matrix of its partial derivatives of second order. The hessian of Φ_3 must be of degree 20. It is easy to see, using the chain rule, that the hessian of a relative invariant is a relative invariant. Thus the hessian of Φ_3 is a Grundform corresponding to the orbit O_2 with stabilizers of order 3. The direct computation gives

$$\Phi_2 = -(t_0^{20} + t_1^{20}) + 228(t_0^{15}t_1^5 - t_0^5t_1^{15}) - 494t_0^{10}t_1^{10}.$$

Next we need a Grundform of degree 30. Recall that the jacobian of functions f, g in two variables is the determinant of the matrix whose first row are partial derivatives of the first order of f and the second row is the same for g. The jacobian of Φ_2, Φ_3 must be of degree 30, and it is easy to see that it is a relative invariant. This gives us a Grundform of degree 30

$$\Phi_1 = t_0^{30} + t_1^{30} + 522(t_0^{25}t_1^5 - t_0^5t_1^{25}) - 10005(t_0^{20}t_1^{10} + t_0^{10}t_1^{20}).$$

Since $\bar{G} \cong A_5$ is a simple group and all Grundformen are of even degree, we see that the characters are trivial.

1.3 Algebras of invariants

Now we are ready to compute the algebras of invariants for binary polyhedral groups. It follows from the discussion given in section 1 and the computation of characters of Grundformen that any relative invariant is a polynomial in Grundformen. In particular, an invariant can be written as a polynomial in Grundformen.

First we make a general remark. For any homomorphism $\chi : G \to \mathbb{C}^*$ the relative invariants of degree d with character χ form a linear subspace W_χ of V_d. Let $\Phi_i, i = 1, \ldots, k$, be nonzero elements from V_{χ_i}, where all χ_i's are distinct characters. Then $\Phi_1, \ldots, \Phi_k$ are linearly independent. The proof is standard. Assume that $A = c_1\Phi_1 + \ldots + c_k\Phi_k = 0$ with some nonzero coefficients. Without loss of generality we may assume that no subset of $\Phi_1, \ldots, \Phi_k$ is linearly dependent, in particular, all c_i's are nonzero. Let $g \in G$ such that $\chi_1(g) \neq \chi_2(g)$. We have

$$0 = g(A) - \chi_1(g)A = c_2(\chi_2(g) - \chi_1(g))\Phi_2 + \ldots + c_k(\chi_k(g) - \chi_1(g))\Phi_k = 0.$$

Since $c_2(\chi_2(g) - \chi_1(g)) \neq 0$, this contradicts the assumption that $\Phi_2, \ldots, \Phi_k$ are linearly independent.

This remark proves the following.

Lemma 1.3.1. *Suppose an invariant polynomial F is written as a linear combination $\sum c_i \Phi_i$, where each $c_i \neq 0$ and Φ_i are relative invariants corresponding to different characters. Then each Φ_i is invariant.*

Our second general remark is the following. The ring of invariants $A = \mathbb{C}[t_0, t_1]^G$ of a finite group is a finitely generated normal graded integral domain of dimension 2. This is a well-known fact which can be found, for example, in Chapter V of Bourbaki's"Commutative Algebra" or in my book "Lectures on invariant theory". Suppose we prove that A is generated by 3 homogeneous polynomials. Then the ideal of relations between the generators is a principal homogeneous ideal. This follows immediately from Krull's Hauptsatz Theorem. We will prove that this is the case for all finite subgroups G of $\mathrm{SL}(2, \mathbb{C})$ and hence

$$\mathbb{C}[t_0, t_1]^G \cong \mathbb{C}[x, y, z]/(R(x, y, z))$$

for some irreducible *weighted homogeneous polynomial* $R(x, y, z)$. Recall that this means that there is the following identity of polynomials in x, y, z, t

$$R(t^a x, t^b y, t^c z) = t^d R(x, y, z),$$

where $(a, b, c; d)$ are positive integers. The first three a, b, c are called the *weights* weights and the last one d is called the *degree*. Of course, in our case the weights must be the degrees of the homogeneous generators of the algebra of invariants. Our goal in this section is to prove this and find $R(x, y, z)$ explicitly.

Case 1: $G \cong C_n$ is a cyclic group of order n.

It is clear that each monomial is a relative invariant. Writing an invariant polynomial F as a sum of monomials and applying Lemma 1.3.1 we obtain that F is the sum of invariant monomials. A monomial $t_0^a t_1^b$ is invariant if and only if $\epsilon_n^{a-b} = 1$. Since $t_0^a t_1^b = (t_0 t_1)^a t_1^{b-a}$, where we assume without loss of generality that $a \leq b$, we see that n must divide $b - a$. Hence $t_0^a t_1^b$ is the product of a power of $t_0 t_1$ and t_1^n. This proves that the invariants

$$F_1 = t_0^n, F_2 = t_1^n, F_3 = t_0 t_1$$

generate the algebra of invariants. They satisfy the obvious relation

$$F_3^n = F_2 F_3.$$

Therefore we obtain

$$\mathbb{C}[t_0, t_1]^G \cong \mathbb{C}[x, y, z]/(xy + z^n). \tag{1.10}$$

Case 2: $G \cong \bar{D}_{2n}$ is a binary dihedral group of order $4n$.

Assume n is odd. It follows from the computation of the characters of the Grundformen that

$$F_1 = \Phi_3^2,\ F_2 = \Phi_1 \Phi_2,\ F_3 = \Phi_3 \Phi_2^2$$

are invariants. We will use that

$$\Phi_3\Phi_1^2 = t_0t_1(t_0^n - t_1^n)^2 = t_0t_1[(t_0^n + t_1^n)^2 - 4(t_0t_1)^n] = F_3 - 4F_1^{\frac{n+1}{2}}.$$

$$\Phi_2^4 = (t_0^n - t_1^n)^4 = t_0^{4n} + t_1^{4n} - 4t_0^nt_1^n(t_0^{2n} + t_1^{2n}) + 6t_0^{2n}t_1^{2n}$$
$$= (t_0^{2n} - t_1^{2n})^2 - 4t_0^nt_1^n(t_0^n - t_1^n)^2 = F_2^2 - 4F_3F_1^{\frac{n-1}{2}}.$$

Similarily,

$$\Phi_1^4 = (t_0^n + t_1^n)^4 = [(t_0^n - t_1^n)^2 + 4t_0^nt_1^n]^2$$
$$= (\Phi_2^2 + 4\Phi_3^n)^2 = \Phi_2^4 + 8\Phi_2^2\Phi_3^n + 16\Phi_3^{2n} = F_2^2 + 4F_3F_1^{\frac{n-1}{2}} + 16F_1^n.$$

Let us show that any invariant F is a polynomial of F_i's. Write F as a sum of monomials $\Phi_1^a\Phi_2^b\Phi_3^c$ in Grundformen. By Lemma 1.3.1, each monomial is an invariant.

If $c \geq 2$ we can factor a power of the invariant F_3 to assume that $c \leq 1$. Factoring out a power of F_2 we may assume that a or b is zero. This leaves us with monomials of type $\Phi_1^a, \Phi_2^a, \Phi_1^a\Phi_3, \Phi_2^a\Phi_3, \Phi_3$, where $a \neq 0$. Factoring out some powers of F_3, $\Phi_3\Phi_1^2$, Φ_1^4 and Φ_2^4, leaves us with monomials

$$\Phi_1^a, \Phi_2^a, \Phi_1\Phi_3, \Phi_2\Phi_3, \Phi_3,$$

where $a \leq 3$. It follows from the description of the characters of the Grundformen that none of these relative invariants is an invariant.

So, we have checked that any invariant can be written as a polynomial in F_1, F_2, F_3. Observe the relation

$$F_3^2 - F_1F_2^2 + 4F_3F_1^{\frac{n+1}{2}} = (F_3 + 2F_1^{\frac{n+1}{2}})^2 - 4F_2^{n+1} - F_1F_2^2 = 0.$$

Replacing F_3 with $F_3 + 2F_1^{\frac{n+1}{2}}$, and scaling the generators, we get

$$\mathbb{C}[t_0, t_1]^G \cong \mathbb{C}[x, y, z]/(z^2 + x(y^2 + x^n)). \tag{1.11}$$

Note that the ring is graded by the condition

$$\deg x = 4,\ \deg y = 2n,\ \deg z = 2 + 2n.$$

We leave to the reader to check that in the case when n is even any invariant is a polynomial in

$$F_1 = \Phi_3^2,\ F_2 = \Phi_2^2,\ F_3 = \Phi_1\Phi_2\Phi_3$$

and the algebra of invariants is isomorphic to the same ring as in the case of odd n.

Case 3: G is a binary tetrahedral group of order 24.

We check from the characters of Grundformen that

$$F_1 = \Phi_1,\ F_2 = \Phi_2\Phi_3,\ F_3 = \Phi_2^3 + \Phi_3^3$$

are invariants. Let $\Phi_1^a\Phi_2^b\Phi_3^c$ be a monomial entering in an invariant F. Then we can factor Φ_1^a to assume that $a = 0$. We factor a power of $\Phi_2\Phi_3$ to assume that $bc = 0$. We have

$$F_1^2 = t_0^2t_1^2(t_0^4 + t_1^4)^2 - 4t_0^6t_1^6 = (12\sqrt{-3})^{-1}(\Phi_2^3 - \Phi_3^3). \tag{1.12}$$

This shows that Φ_2^3 and Φ_3^3 can be expressed in terms of F_1, F_2, F_3. Since Φ_2^b or Φ_3^c is invariant only if b or c is divisible by 3, we see that any F is a polynomial in F_i's.

We use (1.12) to get the relation between the basic invariants

$$F_3^2 = F_1^4 + 4F_2^3.$$

This shows that

$$\mathbb{C}[t_0, t_1]^G \cong \mathbb{C}[x, y, z]/(z^2 + x^4 + y^3), \tag{1.13}$$

where we have scaled the basic invariants. Note that the ring is graded by the condition

$$\deg x = 6,\ \deg y = 8,\ \deg z = 12.$$

Case 4: G is a binary octahedral group of order 48.

We check from the characters of Grundformen that

$$F_1 = \Phi_1^2,\ F_2 = \Phi_2,\ F_3 = \Phi_3\Phi_1$$

are invariants. Notice that

$$\Phi_2^3 - \Phi_3^2 = 108\Phi_1^4.$$

This allows to express the invariant Φ_3^2 in terms F_1, F_2, F_3. Arguing as in the previous case by considering invariant monomials in Φ_1, Φ_2, Φ_3 we check that any invariant is a polynomial in F_1, F_2, F_3. Notice the relation

$$F_3^2 = \Phi_1^2\Phi_3^2 = F_1(F_2^3 - 108F_1^2).$$

This shows that

$$\mathbb{C}[t_0, t_1]^G \cong \mathbb{C}[x, y, z]/(z^2 + x(y^3 + x^2)), \tag{1.14}$$

where we have scaled the basic invariants. Note that the ring is graded by the condition

$$\deg x = 12,\ \deg y = 8,\ \deg z = 18.$$

Case 5: G is a binary icosahedral group of order 120.

In this case the Grundformen are invariants. Thus any invariant is a polynomial in Grundformen Φ_1, Φ_2, Φ_3. We find the relation

$$\Phi_1^2 + \Phi_2^3 = 1728\Phi_3^5.$$

This shows that

$$\mathbb{C}[t_0, t_1]^G \cong \mathbb{C}[x, y, z]/(x^2 + y^3 + z^5), \tag{1.15}$$

where we have scaled the basic invariants. Note that the ring is graded by the condition

$$\deg x = 30,\ \deg y = 20,\ \deg z = 6.$$

1.4 Exercises

1.1 Classify finite subgroups of $\mathrm{GL}(2,\mathbb{C})$ by using a surjective homomorphism $\mathbb{C}^* \times \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(2,\mathbb{C}), (c,A) \mapsto cA$.

1.2 Let $G = \langle g \rangle$ be a cyclic subgroup of $\mathrm{GL}(2,\mathbb{C})$.

(i) Show that the ring of invariants $\mathbb{C}[x,y]^G$ is generated by two elements if and only if one of the eigenvalues of g is equal to 1.

(ii) Suppose G does not contain non-trivial elements with eigenvalue equal to 1. Show that $\mathbb{C}[x,y]^G$ is generated by three elements if and only if G is a subgroup of $\mathrm{SL}(2,\mathbb{C})$.

1.3 Let D_{2n} be the dihedral group realized as a subgroup of $\mathrm{GL}(2,\mathbb{C})$ generated by matrices $\begin{pmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n^{-1} \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Show that its ring of invariants is freely generated by two polynomials.

1.4 Let G be a finite group of automorphisms of a noetherian ring A. Fix a homomorphism $\chi : G \to \mathbb{C}^*$ and consider the set A_χ^G of elements $a \in A$ such that $g(a) = \chi(g)a$.

(i) Show that A_χ^G is a finitely generated module over the subring of invariants A^G, considered as a submodule of A over its subring A^G.

(ii) Let $[G,G]$ be the commutator subgroup of G. Show that $A^{[G,G]}$ is a subalgebra of the A^G-algebra A and, considered as a module over A^G, it is isomorphic to the direct sum of the modules A_χ^G.

(iii) For each finite subgroup G of $\mathrm{SL}(2,\mathbb{C})$ and each possible $\chi : G \to \mathbb{C}^*$ describe explicitly the module $\mathbb{C}[x,y]_\chi^G$ (in terms of generators and relations).

(iv) For each finite subgroup G of $\mathrm{SL}(2,\mathbb{C})$ find its commutator $[G,G]$ and the corresponding ring of invariants. Use (iii) to check (ii).

1.5 Let F be an algebraically closed field of characteristic $p \neq 2$.

(i) Show that $\mathrm{SL}(2,F)$ contains non-abelian subgroups of order $2p^n(p^{2n}-1)$ for any $n > 0$.

(ii) Let $g \in \mathrm{SL}(2,F)$ be the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Find the ring of invariants $F[x,y]^{\langle g \rangle}$.

Lecture 2

Intersection theory on surfaces

2.1 Intersection pairing

Let S be a regular irreducible scheme of dimension 2 and Z be an effective reduced Cartier divisor on S with support proper over a field k. the following will be our most important examples.

(A) S is a smooth projective surface over a field k and Z is any effective divisor on S.

(B) S is a regular scheme of dimension 2 and Z is a reduced fibre of a proper morphism $f : S \to T$, where T is a regular one-dimensional scheme.

(C) Z is a reduced exceptional fibre of a resolution of singularities $f : S \to T$ of a normal two-dimensional scheme T.

Note that we do not exclude the case when Z is the zero divisor. In this case S must be proper over k and $Z = S$.

Let $\mathrm{Div}(S)$ be the group of *Cartier divisors* on S. By definition of a regular scheme, all local rings $\mathcal{O}_{S,s}$ of S are regular local rings (i.e. their maximal ideals are generated by a system of local parameters). A regular ring is a factorial integral domain ([Zariski-Samuel] or [Matsumura], Commutative Algebra), and hence the group of Cartier divisors coincides with the group of *Weil divisors*, the free abelian group generated by irreducible closed subschemes of S of codimension 1. Let $\mathrm{Div}_Z(S)$ be the subgroup of $\mathrm{Div}(S)$ which consists of divisors on S whose support is contained in the support of Z. We denote by $\mathrm{Div}(S)^+$ the submonoid of $\mathrm{Div}(S)$ which consists of effective divisors and let $\mathrm{Div}_Z(S)^+ = \mathrm{Div}(S)^+ \cap \mathrm{Div}_Z(S)$. We identify effective divisors on S with closed (not necessary reduced) subschemes of S defined by the ideal sheaf $\mathcal{O}_S(-D)$.

Let $E \in \mathrm{Div}_Z(S)^+$. For any invertible sheaf $\mathcal{L}$ on E we set

$$\deg \mathcal{L} = \chi(\mathcal{L}) - \chi(\mathcal{O}_E), \tag{2.1}$$

where, for any coherent sheaf $\mathcal{F}$ on a proper scheme Z over a field k

$$\chi(\mathcal{F}) = \sum (-1)^i \dim_k H^i(Z, \mathcal{F})$$

denotes the Euler characteristic of Z with coefficient in $\mathcal{F}$.

If $\mathcal{L} \subset \mathcal{O}_E$ is the ideal sheaf of a Cartier divisor on E, then the exact sequence

$$0 \to \mathcal{L} \to \mathcal{O}_E \to \mathcal{O}_E/\mathcal{L} \to 0$$

gives

$$\deg \mathcal{L} = -\chi(\mathcal{O}_E/\mathcal{L}). \tag{2.2}$$

If E is irreducible and nonsingular, then we see that $\deg \mathcal{L} = 0$ if $\mathcal{L} = \mathcal{O}_E$ and

$$\deg \mathcal{L} = \deg D = -\sum_{x \in E} m_x \deg(x),$$

where $D = -\sum m_x x$ is the Weil divisor on E corresponding to the invertible sheaf $\mathcal{L} \cong \mathcal{O}_E(-D)$, and $\deg(x) = [\kappa(x) : k]$.

The equality (2.1) is called the *Riemann-Roch Theorem* on E. In Mumford's approach (see [Mumford], Lectures on curves on algebraic surface) which we follow, it is the definition of the degree of an invertible sheaf. The non-trivial facts are the following properties of the degree

$$\begin{aligned} \deg(\mathcal{L} \otimes \mathcal{L}') &= \deg \mathcal{L} + \deg \mathcal{L}, \\ \deg \mathcal{L}^{-1} &= -\deg \mathcal{L}. \end{aligned} \tag{2.3}$$

For any divisor D on S and any $E \in \mathrm{Div}_Z^+(S)$, we set

$$E \cdot D = \deg \mathcal{O}_E(D), \tag{2.4}$$

where

$$\mathcal{O}_E(D) = \mathcal{O}_S(D) \otimes_{\mathcal{O}_S} \mathcal{O}_E.$$

Recall that two Cartier divisors D, D' are linear equivalent if $D - D'$ is a principal divisor, i.e. the divisior $\mathrm{div}(\Phi)$ of a rational function Φ on S (an element of the residue field $\kappa(\eta)$ of the generic point η of S). We write $D \sim D'$ if this happens. We have $D \sim D'$ if and only if there is an isomorphism of invertible sheaves $\mathcal{O}_S(D) \cong \mathcal{O}_S(D')$ ([Hartshorne], Chapter III, Prop;. 6.13).

Proposition 2.1.1. *The pairing*

$$\mathrm{Div}_Z(S)^+ \times \mathrm{Div}(S), \ (E, D) \mapsto E \cdot D$$

satisfies the following properties

(i) $D \mapsto (E, D)$ *is a homomorphism of abelian groups.*

(ii) $E \cdot D = E \cdot D'$, *if* $D \sim D'$.

(iii) *If* $E, D \in \mathrm{Div}_Z(S)^+$, *then*

$$E \cdot D = \chi(\mathcal{O}_D \otimes_{\mathcal{O}_S} \mathcal{O}_E) - \chi(\mathcal{T}or_1^{\mathcal{O}_S}(\mathcal{O}_D, \mathcal{O}_E)), \tag{2.5}$$

where we view the sheaves as coherent sheaves on D *or* E.

(iv) *If* E *and* D *are effective divisors without common irreducible components, and* $E \in \mathrm{Div}_Z(S)^+$, *then*

$$E \cdot D = \chi(\mathcal{O}_E \otimes_{\mathcal{O}_S} \mathcal{O}_D) = \sum_{x \in E \cap D} \dim_{\kappa(x)} \mathcal{O}_{S,x}/(a_x, b_x) \deg(x),$$

where $a_x = 0, b_x = 0$ *are local equations of* E, D *at* x.

(v) $E \mapsto E \cdot D$ *is a homomorphism of semi-groups* $\mathrm{Div}_Z(S)^+ \to \mathbb{Z}^+$.

(vi) *Let* $\phi : S' \to S$ *be a proper surjective morphism of regular surfaces. Let* $Z' = \phi^{-1}(Z)_{red}$ *be the reduced pre-image of* Z. *For any* $E \in \mathrm{Div}_Z(S)$ *and* $D \in \mathrm{Div}(S)$, *we have*

$$\phi^*(E) \cdot \phi^*(D) = \deg(\phi) E \cdot D,$$

where $\deg(\phi)$ *is the* degree *of* f, *i.e. the rank of the sheaf* $\phi_*(\mathcal{O}_{S'})$ *at the general point of* S.

Proof. Property (i) follows from the definition of $E \cdot D$ and (2.3).

(ii) Applying (i), it is enough to show that $E \cdot D = 0$ if $D = \mathrm{div}(\Phi)$ is a principal divisor. But $\mathcal{O}_S(\mathrm{div}(\Phi)) \cong \mathcal{O}_S$ and hence $\mathcal{O}_E(D) \cong \mathcal{O}_E$. Now the equality $E \cdot D = 0$ follows from the definition.

(iii) Tensoring the exact sequence

$$0 \to \mathcal{O}_S(-D) \to \mathcal{O}_S \to \mathcal{O}_D \to 0 \tag{2.6}$$

with $\mathcal{O}_E$ we obtain an exact sequence

$$0 \to \mathcal{T}or_1^{\mathcal{O}_S}(\mathcal{O}_D, \mathcal{O}_E) \to \mathcal{O}_E(-D) \to \mathcal{O}_E \to \mathcal{O}_D \otimes \mathcal{O}_E \to 0. \tag{2.7}$$

By property (i), $-E \cdot D = E \cdot (-D)$. Passing to cohomology we obtain

$$-E \cdot D = \chi(\mathcal{O}_E(-D)) - \chi(\mathcal{O}_E) = -\chi(\mathcal{O}_D \otimes \mathcal{O}_E) + \chi(\mathcal{T}or_1^{\mathcal{O}_S}(\mathcal{O}_D, \mathcal{O}_E)). \tag{2.8}$$

It remains to apply property (iii).

(iv) Since $\dim D \cap E = 0$, for any $x \in E \cap D$, the pair (a_x, b_x) is a regular sequence in the local ring $\mathcal{O}_{S,x}$. Thus

$$\mathcal{T}or_1^{\mathcal{O}_S}(\mathcal{O}_D, \mathcal{O}_E)_x = Tor_1^{\mathcal{O}_{S,x}}(\mathcal{O}_{S,x}/(a_x), \mathcal{O}_{S,x}/(b_x)) = 0.$$

From (iii), we get

$$E \cdot D = \chi(\mathcal{O}_D \otimes_{\mathcal{O}_S} \mathcal{O}_E) = \dim_k H^0(\mathcal{O}_D \otimes_{\mathcal{O}_S} \mathcal{O}_E)$$

$$= \sum_{x \in E \cap D} \dim_k \mathcal{O}_{S,x}/(a_x, b_x) = \sum_{x \in E \cap D} \dim_k \mathcal{O}_{S,x}/(a_x, b_x) \deg(x).$$

(v) Suppose D and E have a common irreducible component E_1. In some open affine set U the component E_1 is given by a local equation $\Phi = 0$. We can choose U and Φ such that $\mathrm{div}(\Phi)$ does not contain any other component of E. Replacing D with $D - \mathrm{div}(\Phi)$, and using property (ii), we may assume that D and E have no common irreducible components. Using the additivity of $E \cdot D$ in D, we may assume that D is effective with no common components with E. Now we can apply property (iv).

Let $E = \sum_{i=1}^r n_i E_i$, where E_i are irreducible components of E. Let $a_i = 0$ be a local equation of E_i at a point x and b_x be a local equation of D at x. Then $a_x = a_1^{n_1} \cdots a_r^{n_r} = 0$ is a local equation of E at x. It is easy to check that

$$\dim_k \mathcal{O}_{S,x}/(a_x, b_x) = \sum_{i=1}^r \dim_k \mathcal{O}_{S,x}/(a_i^{n_i}, b_x) = \sum_{i=1}^r n_i \dim_k \mathcal{O}_{S,x}/(a_i, b_x)$$

(see [Eisenbud], Commutative Algebra, p. 260). Applying property (iv), we obtain $E \cdot D = \sum n_i E_i \cdot D$. This proves (v).

(vi) Since $f : S' \to S$ is a proper morhisms, the map $f^*(Z) = S' \times_S Z \to Z$ is proper. Thus $f : Z' \to Z$ is proper and hence Z' is proper over k. This implies that the sheaf $\phi_*(\mathcal{O}_{S'})$ is a coherent sheaf of $\mathcal{O}_S$-algebras. The morphism ϕ factors into the composition $\phi' \circ g$, where $g : S' \to \bar{S}'$ is of degree 1 and $g_*(\mathcal{O}_{S'}) = \mathcal{O}_{\bar{S}'}$ and $\phi' : \bar{S}' \to S$ is a finite morphism of degree equal to $\deg(\phi)$ (the *Stein factorization*, see [Hartshorne], Chap. III, Corollary 11.5). Therefore it is enough to check the assertion in two cases: (a) ϕ is a finite morphism and (b) $\phi_*(\mathcal{O}_{S'}) = \mathcal{O}_S$.

In case (a) $\phi_*(\mathcal{O}_{S'})$ is a locally free sheaf on S of some rank m (locally it corresponds to a finite module of depth 2 over a regular ring of dimension 2, hence a free module). We may assume, as before, that E and D are effective divisors without common irreducible components. It follows from property (iv) that

$$\phi^*(E) \cdot \phi^*(D) = \chi(\mathcal{O}_{\phi^*(E)} \otimes \mathcal{O}_{\phi^*(D)}) = \chi(\phi^*(\mathcal{O}_E) \otimes \phi^*(\mathcal{O}_D)).$$

By the projection formula and vanishing of $R^i\phi_*(\mathcal{F}), i > 0$, for any coherent sheaf $\mathcal{F}$ and a finite morphism f ([Hartshorne], Chap. 3, Corollary 11.2), we have

$$\chi(\phi^*(\mathcal{O}_E \otimes \mathcal{O}_D)) = \chi(\mathcal{O}_E \otimes \mathcal{O}_D \otimes \phi_*\mathcal{O}_{S'}).$$

Since $\mathcal{O}_E \otimes \mathcal{O}_D$ is a sky-scrapper sheaf, the restriction of $\phi_*\mathcal{O}_{S'}$ to $E \cap D$ is a free sheaf of rank m. The additivity of the Euler characteristic gives

$$\phi^*(E) \cdot \phi^*(D) = m\chi(\mathcal{O}_E \otimes \mathcal{O}_D) = mE \cdot D.$$

In case (b), the fibres of ϕ are connected and, since S and S' are regular schemes of the same dimension, the morphism ϕ is an isomorphism over an open Zariski subset U of S. To see this choose U to be an open subset such that $V = \phi^{-1}(U) \to U$ is a homeomorphism. Since ϕ is proper and has finite fibres, it is a finite morphism ([Hartshorne], Chap. III, Exercise 11.2). Thus $\mathcal{O}_{S'}(V)$ is a finite algebra over $\mathcal{O}_S(U)$ of rank 1. Since U is regular, it is normal. This implies that $V \to U$ is an isomorphism.

Now replace D with linear equivalent divisor which does not contain the finite set of points in S such that the map ϕ^{-1} is not defined. Then we can repeat the arguments from the previous case to obtain

$$\phi^*(E) \cdot \phi^*(D) = E \cdot D.$$

□

Extending $(E, D) \mapsto E \cdot D$ by linearity to not necessary effective divisiors E, we obtain a bilinear form

$$\mathrm{Div}_Z(S) \times \mathrm{Div}(S) \to \mathbb{Z}. \tag{2.9}$$

It follows immediately from property (iii) that its restriction to $\mathrm{Div}_Z(S) \times \mathrm{Div}_Z(S)$ is a symmetric bilinear form. We call the bilinear form (2.9) the *intersection form* on S (relative to $f : S \to T$).

Example 2.1.2. Let X be a nonsingular projective curve over a field k and $\mathcal{L}$ be an invertible sheaf on X. Recall that the *line bundle* associated to $\mathcal{L}$ is the scheme $\mathbb{V}(\mathcal{L}) = \mathrm{Spec}\ S^\bullet(\mathcal{L})$ over X, where $S^\bullet(\mathcal{L})$ is the symmetric algebra of $\mathcal{L}$. The pre-image of an open affine subset U of X such that $\mathcal{L}|U \cong \mathcal{O}_U$ is isomorphic to Spec $\mathcal{O}(U)[x] \cong \mathbb{A}^1_U$. Thus S is a smooth surface over k but not a projective surface.

A section of the natural projection $\pi : \mathbb{V}(\mathcal{L}) \to X$ is defined by a homomorphism of sheaves of $\mathcal{O}_X$-modules $\mathcal{L} \to \mathcal{O}_X$, or equivalently, by sections of the dual sheaf $\mathcal{L}^{-1}$. The section s_0 corresponding to the zero homomorphism is called the *zero section*. Locally, in the notation from above, it corresponds to the homomorphism $\mathcal{O}(U)[t] \to \mathcal{O}(U)$ which sends x to 0. This implies that $\pi^*(\mathcal{L})$ locally isomorphic to the ideal $(t) \subset \mathcal{O}(U)[x]$. The image $E = s_0(X)$ is locally given by this ideal. Globally, E is given by the ideal sheaf $\pi^*(\mathcal{L})$. Since $\pi \circ s_0 = \mathbf{id}_X$, we obtain

$$\mathcal{O}_E(-E) \cong s_0^*(\mathcal{O}_S(-E)) \cong s^*(\pi^*(\mathcal{L})) \cong \mathcal{L}.$$

Thus

$$E \cdot E = -\deg \mathcal{L} \tag{2.10}$$

For example, take $\mathcal{L} = \Omega^1_X$, the sheaf of regular differential 1-forms on X. Its sheaf of sections is $\Theta_X = (\Omega^1_X)^*$, the tangent sheaf of X. We know that $\deg(\Omega^1_X) = 2g - 2$, where g is the genus of X. Thus the zero section of the line bundle $\mathbb{V}(\Omega^1_X)$ (the *tangent bundle* of X) has self-intersection equal to $2 - 2g$.

Remark 2.1.3. Assume S is of finite type over a field k and let E be a smooth projective curve on S. Then we have the exact sequence

$$0 \to \mathcal{I}_E/\mathcal{I}_E^2 \to \Omega^1_{S/k} \otimes \mathcal{O}_E \to \Omega^1_{E/k} \to 0$$

([Hartshorne], p.178). We have

$$\mathcal{I}_E/\mathcal{I}_E^2 \cong \mathcal{O}_S(-E)/\mathcal{O}_S(-2E) \cong \mathcal{O}_E(-E).$$

Passing to the dual sheaves we can interpret the sheaf $\mathcal{O}_E(E)$ as the sheaf of sections of the *normal line bundle* of E in S. In particular, we see that the degree of the normal bundle of the zero section of the tangent bundle is equal to $2-2g$. This corresponds to a well-known fact: the number of zeroes of a nonzero holomorphic vector field on a Riemann surface is equal to the Euler-Poincaré characteristic of the surface (see [Griffiths-Harris]).

2.2 Cartan matrices

Let $E_1,\ldots,E_s$ be irreducible components of a divisor $D \in \mathrm{Div}_Z(S)$. The symmetric matrix

$$A = (a_{ij})_{1\le i,j\le n}, \quad a_{ij} = E_i \cdot E_j \tag{2.11}$$

is called the *intersection matrix* of the divisor D. Let $\mathbb{Z}^{\mathrm{Ir}(D)}$ be the free abelian group generated by the set $\mathrm{Ir}(D)$ of irreducible components of D. The intersection matrix defines a symmetric bilinear form on $\mathbb{Z}^{\mathrm{Ir}(D)}$. We call it the *intersection form* of D. It is equal to the restriction of $\mathrm{Div}_Z(S)\times\mathrm{Div}_Z(S)\to\mathbb{Z}$ to the subgroup $\mathbb{Z}^{\mathrm{Ir}(D)}$ of $\mathrm{Div}_Z(S)$.

We use the notation (t_+,t_-,t_0) for the Sylvester signature of a real quadratic form (we drop t_0 if it is zero). A symmetric integer matrix A of size r defines a symmetric bilinear form on a free abelian group $M\cong\mathbb{Z}^s$ of rank r. We use the same matrix to extend the form to the linear space $M_\mathbb{Q} = M\otimes\mathbb{Q}\cong\mathbb{Q}^s$ or $M_\mathbb{R} = M\otimes\mathbb{R}\cong\mathbb{R}^s$. By definition, the signature of the form on M or on $M_\mathbb{Q}$ is the signature of the corresponding real bilinear form.

Recall that the signature can be computed by using the Jacobi Theorem. The number t_0 is equal to the dimension of the nullspace of A. It defines the *radical* of the quadratic form, the subspace of vectors v such that $(v,w)=0$, for all w. Writing A as the block matrix of the zero matrix and a nonsingular symmetric matrix we may assume that $\det A\neq 0$. It is always possible to choose a basis in the vector space such to assume that all corner matrices $A_s = (a_{ij})_{1\le i,j\le k}$ have nonzero determinants Δ_i. Then the Jacobi Theorem says that there exists a basis such that the bilinear form is given by the diagonal matrix with $b_{ii} = \Delta_{i-1}/\Delta_i$, where $\delta_0 = 1$. This implies that t_- is equal to the number of the sign changes in the sequence of numbers $1,\Delta_1,\ldots,\Delta_r$. For all this see any good text-book in linear algebra (if it does not have this theorem, then, by definition, it is not good). Examples of good text-books are [Gelfand], Lectures on Linear Algebra or [Bourbaki], Algebra.

Assume that S is a projective smooth surface over a field k. Let $\mathrm{Div}_0(S)$ be the subgroup of $\mathrm{Div}(S)$ that consists of divisors D such that $D \cdot D' = 0$ for all divisors D' on S (the radical of the intersection form on S). Divisors from $\mathrm{Div}^0(S)$ are called *numerically trivial*. If $D - D' \in \mathrm{Div}_0(S)$ we write $D \equiv D'$ and say that D is numerically equivalent to D'. The quotient group

$$\mathrm{Num}(S) = \mathrm{Div}(S)/\mathrm{Div}_0(S)$$

is known to be a free abelian group of some finite rank ρ, called the *Picard number* of S.

The intersection form on S induces the intersection form on $\mathrm{Num}(S) \cong \mathbb{Z}^\rho$.

Recall the following theorem (see [Hartshorne], Chapter V, Theorem 1.9), known as the *Hodge Index theorem*.

Theorem 2.2.1. *The signature of the intersection form on* $\mathrm{Num}(S)$ *is equal to* $(1, \rho - 1)$.

This theorem implies that for any divisor D on S its intersection form signatures equal to $(1, t_+, t_0)$ or $(0, t_+, t_0)$. Also, the number $t_0 \leq t + 1$, where t is the rank of the kernel of the projection of $\mathbb{Z}^{\mathrm{Ir}(D)}$ to $\mathrm{Num}(S)$.

We are able to say more the cases (B) and (C) from the beginning of the chapter. The following key lemma is due to D. Mumford.

Lemma 2.2.2. *Let $E_1, \ldots, E_s$ be irreducible components of a divisor $D \in \mathrm{Div}_Z(S)^+$. Let Φ be a rational function on S such that*

$$\mathrm{div}(\Phi) = \sum_i n_i E_i + Z,$$

where Z and D have no common irreducible components. For any $a_1, \ldots, a_s \in \mathbb{Q}$,

$$(\sum a_i n_i E_i)^2 = -\sum_{i<j} (a_i - a_j)^2 n_i n_j (E_i \cdot E_j) - \sum a_i^2 n_i E_i \cdot Z.$$

Proof. Put $D_i = n_i E_i$ to simplify the computations. Using the properties of the intersection form on S, we get

$$(\sum_i a_i D_i)^2 = \sum_i a_i (D_i \cdot \sum_j a_j D_j) = \sum_i a_i D_i \cdot (-a_i \mathrm{div}(\Phi) + \sum_j a_j D_j)$$

$$= \sum_i a_i D_i \cdot (-a_i (Z + \sum_j D_j) + \sum_j a_j D_j) = \sum_i a_i D_i \cdot \sum_j ((a_j - a_i) D_j - a_i Z)$$

$$= \sum_{i,j} a_i (a_j - a_i) D_i \cdot D_j - \sum_i a_i^2 D_i \cdot Z = -\sum_{i<j} (a_i - a_j)^2 D_i \cdot D_j - \sum_i a_i^2 D_i \cdot Z.$$

□

Recall that the scheme theoretical fibre S_t of $f : S \to T$ over a closed point $t \in T$ is the closed subscheme of S defined as the base change $S \times_T \mathrm{Spec}\, \kappa(t) \to \mathrm{Spec}\, \kappa(t)$. It follows from the definition that for any point $x \in f^{-1}(t)$

$$\mathcal{O}_{S_t,x} \cong \mathcal{O}_{S,x}/f_x^*(\mathfrak{m}_{T,t}),$$

where $f_x^* : \mathcal{O}_{T,t} \to \mathcal{O}_{S,x}$ is the homomorphism of local rings defined by the morphism f. Since f is surjective, f_x^* is injective and the image of $\mathfrak{m}_{T,t}$ generates an ideal $f_x^*(\mathfrak{m}_{T,t})$ contained in $\mathfrak{m}_{S,x}$. If the radical of $f_x^*(\mathfrak{m}_{T,t})$ is equal to $\mathfrak{m}_{S,x}$, then we can find a regular sequence of 2 elements in $\mathcal{O}_{S,x}$ contained in $f_x^*(\mathfrak{m}_{T,t})$. This shows that $\dim \mathcal{O}_{S_t,x} = 0$ and $\dim_x S_t = 0$. Otherwise, any associated prime ideal of $f_x^*(\mathfrak{m}_{T,t})$ is of codimension 1 (Krull's Haupsatz Theorem) and $\dim \mathcal{O}_{S_t,x} = 1$. Also in a regular ring A an ideal I of codimension 1 is principal. Thus the fibre S_t is given by a local equation $\phi = 0$ in an affine neighborhood U of x, and hence it is an effective Cartier divisor in U. By using the Stein factorization, it is easy to see that all connected components of the fibre are of the same dimension.

Corollary 2.2.3. *(K. Kodaira) Let $f : S \to T$ be a proper surjective morphism of S to a regular scheme of dimension 1. Let t be a closed point of T and*

$$S_t = \sum_{i=1}^{s} m_i E_i$$

be the scheme-theoretical fibre over a closed point considered as an effective divisor from $\mathrm{Div}_Z(S)$. Assume that S_t is connected. Then the signature of the intersection form of S_t is $(0, s-1, 1)$. The radical of the intersection form is generated by the vector $\mathfrak{f} = 1d(m_1, \ldots, m_s)$, where $d = g.c.d.(m_1, \ldots, m_s)$.

Proof. In the previous lemma take $\Phi = f^*(u)$, where u is a local parameter at t. Then $\mathrm{div}(\Phi) = f^{-1}(t) + Z$, where Z is disjoint from $f^{-1}(t)$. So it follows from property (ii) of the intersection pairing that $(m_1, \ldots, m_s)$ is in the radical of the intersection form of S_t. Any rational combination of the divisors E_i can be wriitten in the form $\sum_i a_i m_i E_i$. Hence

$$(\sum a_i m_i E_i)^2 = -\sum_{i<j} (a_i - a_j)^2 m_i m_j (E_i, E_j).$$

This proves that $t_+ = 0$. Assume the equality holds. Since $f^{-1}(t)$ is connected, we may assume that $E_1 \cdot E_2 \neq 0$. Then $a_1 = a_2$. If $s = 2$, we are none. If $s > 2$, we may assume that E_3 intersects either E_1 or E_2. This implies that $a_1 = a_2 = a_3$. Continuing in this way, we prove that all a_i' are equal to some number a. Thus the radical is one-dimension and generated (over $\mathbb{Q}$) by $\mathfrak{f}$ □

Corollary 2.2.4. *(D. Mumford) Let $f : S \to T$ be a birational surjective morphism, where $\dim T = 2$. Assume that the fibre S_t over a closed point t is of dimension 1. Then the intersection form of the divisor S_t is negative definite.*

Proof. Since S is normal, the morphism factors into $S \to T' \to T$, where $\pi : T' \to T$ is the normalization morphism. Then the $f^{-1}(t)$ is the union of fibres over points in the finite set $\pi^{-1}(t)$. The intersection matrix of S_t is the direct sum of the intersection matrices of these fibres. Thus we may assume that T is normal. Hence the fibre is connected and as we explained above, f is an isomorphism over an open Zariski subset of T. Replacing T by an affine neighborhood of t we may assume that f is an isomorphism over $T \setminus \{t\}$. Take $\Phi = f^*(u)$, where $u \in \mathcal{O}(T)$ which vanishes at t. By Krull's Hauptsatz Theorem, each associated prime ideal of (u) is of codimension1. This shows that $\mathrm{div}(\Phi) = \sum n_i E_i + Z$, where Z is the proper inverse transform of some one-dimensional subscheme in T passing through t. Obviously, Z intersects some component E_i and hence Lemma (2.2.2) gives $(\sum_i a_i n_i E_i)^2 < 0$ for any rational a_i. □

We will see that for any finite subgroup G of $\mathrm{SL}(2, \mathbb{C})$ the surface $T = \mathbb{C}^2/G$ admits a resolution of singularities $f : S \to T$ such that all irreducible components R_i of the exceptional fibre satisfy $R_i^2 = -2$. Thus the intersection matrix of the exceptional fibre is negative definite and satisfies

$$a_{ii} = -2,\ a_{ij} = a_{ji} \geq 0,\ i \neq j.$$

All such matrices can be described explicitly.

Lemma 2.2.5. *(E. Cartan) Let $A = (a_{ij})$ be a symmetric integer $n \times n$ matrix with $a_{ii} = -2$ and $a_{ij} \geq 0$ for $i \neq j$. Suppose the quadratic form defined by A is negative definite, or negative semi-definite with one-dimensional radical. Also assume that A is not a block-matrix of submatrices satisfying the same properties as A. Then, either $n = 2$ and $a_{12} = 2$ (we say that A is of type $\tilde{A}_1$), or $a_{ij} \leq 1$ for $i \neq j$ and the matrix $A + 2I_n$ is the incidence matrix of one of the following graphs Γ, having n vertices for A negative definite and $n+1$ vertices for A semi-definite.*

Proof. Let M be the incidence matrix of a graph from the list and $A = M - 2I_r$ be the corresponding symmetric matrix with -2 at the diagonal. Here r is the number of vertices in the graph. Applying the Jacobi Theorem we get that the quadratic form defined by the matrix A is negative definite (resp. semi-definite) if and only if $\Delta_k = (-1)^k \det((a_{ij})_{1 \leq i,j \leq k} > 0$ (resp. ≥ 0) for $k = 1, \ldots n$. It is directly checked that the matrices corresponding to the graphs in the first column are negative definite, and the matrices corresponding to the graphs in the second column are negative semi-definite with radicals spanned by the following vectors $\mathfrak{f}$

- $\tilde{A}_n$ $\mathfrak{f} = e_1 + \ldots + e_n$;
- $\tilde{D}_n$ $\mathfrak{f} = e_1 + e_2 + e_{n-1} + e_n + 2\sum_{i \neq 1,2,n-1,n} e_i$, where e_1, e_2, e_{n-1}, e_n correspond to the vertices joined to the vertices of valency 3 from the left and from the right;
- $\tilde{E}_6$ $\mathfrak{f} = e_1 + 2e_2 + e_3 + 2e_4 + 3e_5 + 2e_6 + e_7$, where e_1, e_2 correspond to the lower vertices, and the rest are numbered from the left to the right.

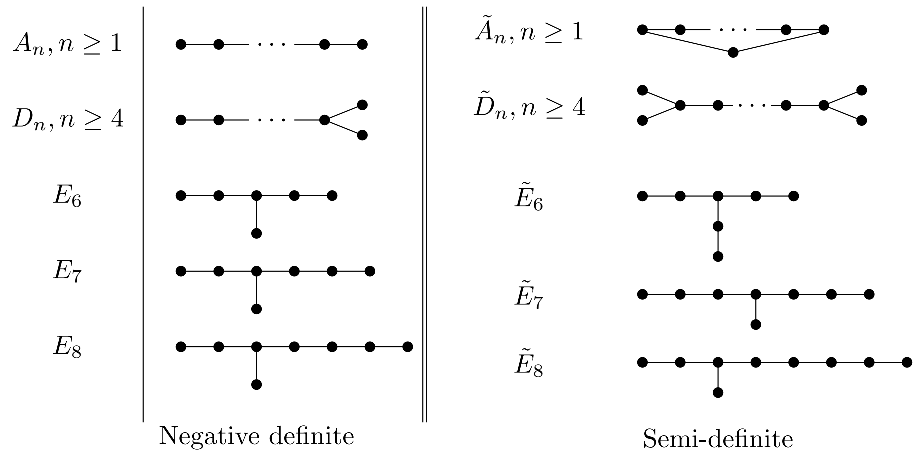


Table 2.1: Dynkin diagrams

$\tilde{E}_7$ $\mathfrak{f} = 2e_1 + e_2 + 2e_3 + 3e_4 + 4e_5 + 3e_6 + 2e_7 + e_8$, where e_1 corresponds to the lower vertex, and the rest are numbered from the left to the right.

$\tilde{E}_8$ $\mathfrak{f} = 3e_1 + 2e_2 + 4e_3 + 6e_4 + 5e_5 + 4e_6 + 3e_7 + 2e_8 + e_9$, where e_1

corresponds to the lower vertex, and the rest are numbered from the left to the right.

Note if we find that our matrix defines the graph Γ which contains a subgraph Γ' equal to one from the right column list, then $\Gamma = \Gamma'$. This immeduately follows from the assumption that Γ is connected and the radical of the quadratic form is of dimension one, hence coincides with the radical of the quadratic form defined by Γ'.

If $r = 1$, we get the graph A_1. Assume $r \geq 2$. Let $a_{ij} \neq 0$. After permuting the columns and the rows simultaneously (this corresponds to reordering the vertices) we may assume that $i = 1, j = 2$. Thus $\Delta_2 = a_{11}a_{22} - a_{12}^2 = 4 - a_{12}^2 > 0$ (resp. ≥ 0). Since a_{ij} is a non-negative integer, we obtain $a_{12} \leq 1$ (resp. ≤ 2). If the equality takes place, the subgraph formed by the vertices v_1, v_2 is of type $\tilde{A}$. By the above remark Γ is of type $\tilde{A}_1$.

From now on we assume that $a_{ij} \leq 1, i \neq j$, and $r \geq 2$. Thus $M = A + 2I_n$ is the incidence matrix of a connected graph Γ.

Assume that A is negative definite. We claim that Γ is a tree unless Γ is of type $\tilde{A}_n$. In fact, if Γ is not a tree, we can find a sequence $i_1 < \ldots < i_k$ such that $a_{i_1 i_2} = \ldots = a_{i_k i_1} = 1$. Then the vertices $v_{i_1}, \ldots, v_{i_k}$ form a subgraph of type $\tilde{A}_{k-1}$. Thus Γ is of type $\tilde{A}_{k-1}$.

From now we assume that Γ is a tree. Next we claim that any vertex v_i of Γ is incident to at most 3 vertices unless the graph Γ is of type $\tilde{D}_4$. In fact, suppose we find the indices i, j_1, j_2, j_3, j_4 such that $a_{ij_s} = 1, s = 1,2,3,4$. The the subgraph with these vertices is of type $\tilde{D}_4$ and we conclude as above.

Suppose we have two vertices v_i and $v_{i'}$ incident to three edges each. We leave to the reader to find a subgraph of Γ of type $\tilde{D}_k$ with $k \geq 5$.

If Γ does not have vertices incident to 3 edges, then we get the graph of T_{pqr}-shape We assume that $p \leq q \leq r$. It is easy to check that the solutions

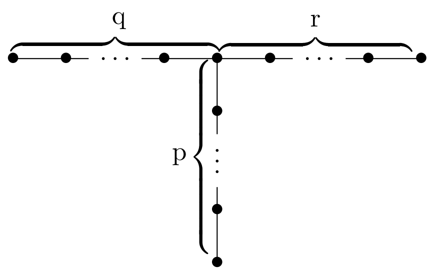


Figure 2.1:

$(1, q, r), (2, 2, r), (2, 3, 3), (2, 3, 4), (2, 3, 5)$ of

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1, \tag{2.12}$$

correspond to our graphs in the left column. The solutions $(3, 3, 3), (2, 4, 6), (2, 3, 6)$ of

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1, \tag{2.13}$$

correspond to the graphs in the right column. All other triples (p, q, r) satisfy either $p = 2, q = 3, r \geq 6$ or $p = 3, q = 4, r \geq 4$. They all contain a subgraph of type $T_{p'q'r'}$, where (p', q', r') satisfy (2.13).

In fact, by straightforward computation one checks that the graphs T_{pqr} with p, q, r not satisfying (2.12) or (2.13) define nondegenerate symmetric bilinear forms of signature $(1, p + q + r - 3)$. $\square$

Definition 2.2.1. A symmetric matrix is called a *Cartan matrix* if it satisfies the assumptions of Lemma 2.2.5.

2.3 Canonical class

Define the *canonical class* $k_{S/T}$ as a function on $\mathrm{Div}_Z(S)^+$

$$k_{S/T} : \mathrm{Div}_Z(S)^+ \to \mathbb{Z},\ E \mapsto -E \cdot E - 2\chi(\mathcal{O}_E). \tag{2.14}$$

Lemma 2.3.1.

$$k_{S/T} : \mathrm{Div}_Z(S)^+ \to \mathbb{Z}$$

is a homomorphism of semigroups.

Proof. Write $E = E' + R$, where R is an irreducible component of E. Tensoring the exact sequence

$$0 \to \mathcal{O}_S(-R) \to \mathcal{O}_S \to \mathcal{O}_R \to 0$$

by $\mathcal{O}_S(-E')$, we obtain

$$\mathcal{O}_S(-E')/\mathcal{O}_S(-E) \cong \mathcal{O}_R(-E').$$

Together with the exact sequence

$$0 \to \mathcal{O}_S(-E')/\mathcal{O}_S(-E) \to \mathcal{O}_E \to \mathcal{O}_{E'} \to 0.$$

this gives us

$$\chi(\mathcal{O}_E) - \chi(\mathcal{O}_{E'}) = \chi(\mathcal{O}_S(-E')/\mathcal{O}_S(-E))$$
$$= \chi(\mathcal{O}_R(-E')) = -R \cdot (E - R) + \chi(\mathcal{O}_R).$$

Hence

$$k_{S/T}(E) - k_{S/T}(E - R) = E^2 - (E - R)^2 - 2\chi(\mathcal{O}_E) + 2\chi(\mathcal{O}_{E'})$$
$$= 2E \cdot R - R^2 - 2R \cdot (E - R) - 2\chi(\mathcal{O}_R) = -R^2 - 2\chi(\mathcal{O}_R) = k_{S/T}(R).$$

□

Corollary 2.3.2. *For any $E_1, E_2 \in \mathrm{Div}_Z(S)^+$,*

$$E_1 \cdot E_2 = \chi(\mathcal{O}_{E_1}) + \chi(\mathcal{O}_{E_2}) - \chi(\mathcal{O}_{E_1+E_2}). \tag{2.15}$$

Proof.

$$k_{S/T}(E_1+E_2) = -(E_1+E_2)^2 - 2\chi(\mathcal{O}_{E_1+E_2}) = -E_1^2 - E_2^2 - 2E_1 \cdot E_2 - 2\chi(\mathcal{O}_{E_1+E_2})$$
$$= k_{S/T}(E_1) + k_{S/T}(E_2) = -E_1^2 - E_2^2 - 2\chi(\mathcal{O}_{E_1}) - 2\chi(\mathcal{O}_{E_2}).$$

This proves the assertion. □

Theorem 2.3.3. *Let $\dim T > 0$ and D be a divisor with support in connected fibre $f^{-1}(t)$, where $\kappa(t)$ is algebraically closed. Assume that D is reducible if $\dim T = 1$. Suppose that $k_{S/T}(E) = 0$ for each irreducible component of D. Then the intersection matrix of D is a Cartan matrix.*

Proof. Let E_i be an irreducible component of D. Since $\kappa(t)$ is algebraically closed and E_i is a projective connected reduced scheme, we have $H^0(E_i, \mathcal{O}_{E_i}) \cong k$. Thus

$$k_{S/T}(E_i) = -E_i^2 - 2(1 - h^1(\mathcal{O}_{E_i})) = 0.$$

If $E_i^2 = 0$, then, by Corollary 2.2.3, D consists of one component. This case has been excluded. Thus $E_i^2 \leq -1$ and $2(1 - h^1(\mathcal{O}_{E_i}) \leq 2$. This implies that $E_i^2 = -2$ and $h^1(\mathcal{O}_{E_i}) = 0$. Since $E_i \cdot E_j \geq 0$ for $i \neq j$, and the fibre is connected the intersection matrix satisfies the assumptions of Lemma 2.2.5. It remains to apply Corollaries 2.2.3 and 2.2.4.

□

Remark 2.3.4. Note that the converse is not true. In Example 2.6 take C to be a curve of genus $g > 0$ and $\mathcal{L}$ be a line bundle of degree 2. Then $k_{S/T} = 2g \neq 0$ although the intersection matrix is a Cartan matrix of type A_1.

An example of $f : S \to T$ with one-dimensional T and $k_{S/T} = 0$ is given by an *elliptic surface*. We assume that T is an algebraic curve over an algebraically closed field k, all fibres are connected and the general fibre S_η of $f : S \to T$ is an irreducible curve of arithmetic genus 1. Let $F = S_t$ be a fibre over a closed point t. We assume that it does not contain a smooth rational curve R with $R^2 = -1$ (an *exceptional curve of the first kind* or a (-1)-*curve*). Since the Euler characteristic $\chi(F, \mathcal{O}_{F_t})$ of a fibre F_t does not depend on t ([Hartshorne], Chap. 3, Corollary 9.10), we have $k_{S/T}(F_t) = -F_t^2 - 2\chi(\mathcal{O}_{F_t}) = 0$. Let $F = \sum n_i E_i$ be a reducible fibre. Then, for any proper closed subscheme F' of F we have the surjection of sheaves $\mathcal{O}_F \to \mathcal{O}_{F'}$ which gives $h^1(\mathcal{O}_{F'}) \leq h^1(\mathcal{O}_F) = 1$. Thus $\chi(\mathcal{O}_{F'}) \geq 0$. Assume F contains an irreducible component R of arithmetic genus $h^1(\mathcal{O}_R) = 1$. Let R' be another component intersecting R. Then $0 \geq (R + R')^2 = R^2 + R'^2 + 2R \cdot R'$ and $R^2 < 0, R'^2 < 0$ imply that $R \cdot R' > 1$ unless $R^2 = R'^2 = -1$ and $F = a(R + R')$. The equality from Corollary 2.15

$$\chi(\mathcal{O}_R) + \chi(\mathcal{O}_{R'}) - \chi(\mathcal{O}_{R+R'}) = R \cdot R' > 0$$

implies that $h^1(R') = 0$ contradicting the assumption that F does not contain (-1)-curves. Thus all irreducible components R of F satisfy $h^1(\mathcal{O}_R) = 0$, hence isomorphic to $\mathbb{P}^1$. Since $k_{S/T}(R) = -R^2 - 2\chi(\mathcal{O}_R) = -R^2 - 2 \geq 0$, using the additivity of the canonical class and the equality $k_{S/T}(F) = 0$, we obtain that $k_{S/T}(R) = 0$ for all components. Hence $k_{S/T} = 0$. Theorem 2.3.3 gives the classification of reducible singular fibres of elliptic surfaces, originally due to K. Kodaira. The following is the dictionary between Kodaira's notations for types of reducible fibres and the notations of types of the corresponding Cartan matrices:

Kodaira	I_{n+1}	II	III	I_{n+4}^*	II*	III*	IV*
Cartan	$\tilde{A}_n$	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{D}_n$	$\tilde{E}_8$	$\tilde{E}_7$	$\tilde{E}_6$

Table 2.2: Kodaira's fibres

Here the fibre of type II (resp. III) represent two curves tangent at one point (resp. three curves intersecting at one common point).

2.4 Exercises

2.1 Let S be a smooth projective surface S over a field k. Show that there exists a divisor K on S such that $K \cdot E = k_S(E)$ for any smooth projective curve on S.

2.2 Let $S \subset \mathbb{P}_k^N$ be a smooth projective surface and H be its hyperplane section. Show that the intersection form on $\mathbb{Z}^{\mathrm{Ir}(H)}$ has signature $(1, t_-, t_0)$. Give an example of a pair (S, H) with $t_0 > 0$.

2.3 Let C be a nonsingular projective curve over a field k and E be the diagonal of the product $C \times C$. Prove that $E \cdot E = 2\chi(\mathcal{O}_C)$.

2.4 Let $f : S \to T$ be a projective surjective morphism of a projective smooth surface S onto a nonsingular curve T. Assume that the fibres of S are connected and the ground field is algebraically closed. For any closed point $t \in T$ let m_t denote the number of irreducible 1-dimensional components of the fibre $f^{-1}(t)$. Show that $\rho \geq 2 + \sum_{t \in T}(m_t - 1)$, where ρ is the Picard number of S.

2.5 Let $\lambda F(x, y, z) + \mu G(x, y, z) = 0$ be a pencil of plane curves of degree 3. Suppose one of the members, say $F(x, y, z) = 0$, is a nonsingular curve. Show that there exists a resolution of indeterminacy points of the rational map $\mathbb{P}^2 - \to \mathbb{P}^1, (x, y, z) \mapsto (F(x, y, z), G(x, y, z))$ is an elliptic surface. Show, by an example, that each Kodaira's type is realized unless it is of type I_n with $n \geq 9$, or I_n* with $n > 4$.

2.6 Do exercises 1.1-1.12 in Chapter V of [Hartshorne].

Lecture 3

Geometry of graded algebras

3.1 Graded algebras

The rings of invariants of linear groups acting on the ring of polynomials preserve the natural grading on the latter, so they are examples of graded rings. In this lecture we will study the geometry of affine and projective spectra of graded rings. We will follow [Demazure]

We will consider $\mathbb{Z}$-graded Noetherian commutative associative rings

$$A = \bigoplus_{n\in\mathbb{Z}} A_n. \tag{3.1}$$

Recall that the direct sum here is the direct sum of abelian groups and satisfies

$$A_n \cdot A_m \subset A_{n+m}.$$

Nonzero elements in A_n are called *homogeneous elements* of *degree* i. We write $\deg a = n$ if $a \in A_n \setminus \{0\}$.

The subset A_0 is a subring of A. Each A_i acquires a natural structure of A_0-module so that A becomes a graded algebra over A_0.

Geometrically, a grading on a ring A is equivalent to a non-trivial action of the *group scheme* $\mathbb{G}_m = \mathrm{Spec}\ \mathbb{Z}[T, T^{-1}]$ on $C = \mathrm{Spec}\ A$. For any commutative ring R, the action of $\mathbb{G}_m(R) = R^*$ on $C(R) = \mathrm{Hom}(A, R)$ is defined by

$$(r \cdot \phi)(a) = \sum_{n\in\mathbb{Z}} r^n \phi(a_n),$$

where $a = \sum_{n\in\mathbb{Z}}, a_n \in A_n$. The group $\mathbb{G}_m(R)$ also acts on the ring A by

$$r \cdot a = \sum_{n\in\mathbb{Z}} r^n a_n.$$

In particular, one can view each A_n as the eigen-submodule of A with respect to the character of $\mathbb{G}_m$ (i.e. a homomorphism of group schemes $\chi : \mathbb{G}_m \to \mathbb{G}_m$) defined by $r \mapsto r^n$. From this point of view the subalgebra A_0 should be viewed as the subring of invariants $A^{\mathbb{G}_m}$.

Definition 3.1.1. The *saturation index* of A is the greatest common divisor of the set of integers i such that $A_i \neq \{0\}$.

One can interpret the saturaton index as the order of the largest finite subgroup $\boldsymbol{\mu}_e$ of $\mathbb{G}_m$ that acts identically on A.

An ideal I in a graded ring is called *homogeneous* if it can be generated by homogenous elements of A. Equivalently, I is homogeneous if $I = \oplus_{i\in\mathbb{Z}}(I \cap A_i)$. A homomorphism of graded rings $A \to B$ is a homomorphism of rings that sends each A_i to B_i. The kernel of a homomorphism of graded rings is a homogeneous ideal, and the quotient of a graded ring by a homogeneous ideal inherits a grading, the *quotient grading* making the factor map a homomorphism of graded rings.

Example 3.1.1. Let $A = R[T_1, \ldots, T_n]$ be the polynomials algebra over a commutative ring R and $(w_1, \ldots, w_n) \in \mathbb{Z}^n \setminus \{0\}$. We define a grading on A by setting, for any $d \in \mathbb{Z}$,

$$A_d = \{P(T_1, \ldots, T_n) : P(t^{w_1}T_1, \ldots, t^{w_n}T_n) = t^d P(T_1, \ldots, T_n)\}, \tag{3.2}$$

where the equality is understood as the identity in the polynomial algebra $R[T_1, \ldots, T_n, t]$. Clearly, each unknown T_i has degree w_i.

A polynomial $P \in A_d$ is called *weighted homogeneous* with *weights* $\mathbf{w} = (w_1, \ldots, w_n)$ and degree d. Writing P as a sum of monomials $\mathbf{x}^{\mathbf{i}}$ with coefficients $a_{\mathbf{i}}$, the definition is equivalent to the property that the dot-product $\mathbf{i} \cdot \mathbf{w} = d$ for all $\mathbf{i}$ such that $a_{\mathbf{i}} \neq 0$. The saturation index of A is equal to $g.c.d.(w_1, \ldots, w_n)$. We have already encountered weighted homogeneous polynomials in Chapter 1.

Each A_i is either zero or a free A_0-module of rank 1 with a basis given by any monomial $T^{\mathbf{i}} := T_1^{i_1} \cdots T_n^{i_n} \in A_d$.

Let A be any graded ring and $a_1, \ldots, a_n$ be generators of A as a A_0-algebra. We can always choose homogeneous generators. Let $w_1, \ldots, w_n$ be its degrees. Then the homomorphism $T_i \mapsto a_i$ defines a surjective homomorphism of graded algebras $A_0[T_1, \ldots, T_n] \to A$, where $\deg T_i = w_i$. Thus any graded ring is a quotient of an algebra of weighted homogeneous polynomials by a homogeneous ideal.

As usually in the theory of schemes one globilizes the notion of a graded ring by introducing the notion of a quasicoherent sheaf of graded $\mathcal{O}_S$-algebras

$$\mathcal{A} = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}_i$$

on any scheme S. Here each $\mathcal{A}_i$ is a quasicoherent sheaf of modules over the sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}_0$. For any open subset U of S, the algebra $\mathcal{A}(U)$ has a grading with $\mathcal{A}(U)_i = \mathcal{A}_i(U)$.

Next we recall the notion of a *projective spectrum* Proj A of a graded ring A. We assume that the grading is *nonnegative*, i.e. satisfies

$$A_i = 0,\ i < 0, \tag{3.3}$$

(or, by regrading, $A_i = 0$ for $i > 0$). In this case

$$A_{>0} = \bigoplus_{i>0} A_i$$

is a homogeneous ideal in A with quotient ring isomorphic to A_0. We denote it by $\mathfrak{m}_0$ and call it the *vertex ideal* of A (or the *irrelevant ideal*).

Definition 3.1.2. An *affine quasicone* is an affine scheme C together with a nonnegative grading of its coordinate ring $\mathcal{O}(C)$. The closed subscheme $V(\mathfrak{m}_0) \cong \text{Spec } \mathcal{O}(C)_0$ of C is called the *vertex* of C. The open subset

$$C^* = C \setminus V(\mathfrak{m}_0)$$

is called the *punctured affine quasicone*.

Note that C is an affine scheme over Spec A_0 and the surjection $A \to A_0$ defines the closed embedding Spec $A_0 \hookrightarrow C$ with the image equal to the vertex of C.

In the global situation, we consider a sheaf of graded $\mathcal{O}_S$-algebras $\mathcal{A}$ satisfying $\mathcal{A}_i = 0, i < 0$. The scheme $\mathcal{C} = \text{Spec } \mathcal{A}$ is an affine scheme over S. The surjection $\mathcal{A} \to \mathcal{A}_0$ defines a closed embedding $i : Y \to \mathcal{C}$, where $Y = \text{Spec } \mathcal{A}_0$ is an affine scheme over S. Its image is called the *vertex subscheme* of $\mathcal{C}$. The canonical structure of a S-scheme on Spec $\mathcal{A}$ is viewed as a *family of affine quasicones* parametrized by S, the vertices of these affine quasicones are the fibres of $Y \to S$.

Example 3.1.2. Let X be a closed subscheme of a projective space $\mathbb{P}^n_R$ over a ring R. It is given by a system of homogeneous algebraic equations with coefficients in R. These equations define a closed subvariety C of the affine space $\mathbb{A}^{n+1}_R = \text{Spec } R[T_0, \ldots, T_n]$. This affine scheme is called the *affine cone* over X. Its vertex is given by the "origin" in $\mathbb{A}^{n+1}_R$ defined by the ideal $(T_0, \ldots, T_n)$. It is isomorphic to the affine scheme Spec R. The coordinate ring of C is the R-algebra Spec $R[T_0, \ldots, T_n]/I$, where I is the ideal generated by the polynomials defining X. It inherits the natural grading of $R[T_0, \ldots, T_n]$ with all weights equal to 1.

The group scheme $\mathbb{G}_m$ acts on $C = \text{Spec } A$ leaving C^* invariant. The "orbit space" $C^*/\mathbb{G}_m$ is the *projective spectrum* $X = \text{Proj } A$ of A. As a set it consists of homogeneous prime ideals in A which do not contain $\mathfrak{m}_0$. Its topology is defined by choosing a base of open subsets which consists of sets

$$D_+(f) = \{\mathfrak{p} \in \text{Proj } A : f \notin \mathfrak{p}\},$$

where f is a homogeneous element in $\mathfrak{m}_0$. The structure sheaf $\mathcal{O}_X$ is defined by setting $\mathcal{O}_X(D_+(f)) = A_{(f)} := (A_f)_0$, where we grade the localization ring A_f by

$$(A_f)_i = \{\frac{a}{f^d} : a \in A_{i+d\deg f}\}.$$

Note that $C = \text{Spec } A$ is covered by open sets $D(a) = \{\mathfrak{p} \in \text{Spec } A : a \notin \mathfrak{p}\}, a \in A$. If we restrict ourselves to open subsets $D(f)$, where f is a homogeneous element of $\mathfrak{m}_0$, we get an open covering of C^*.

The inclusion $A_{(f)} \subset A_f$ defines a morphism $D(f) \to D_+(f)$. These morphisms can be glued together to define a morphism of schemes

$$\pi : C^* \to X$$

The corresponding map of sets is $\mathfrak{p} \mapsto \sum_{i\geq 0} \mathfrak{p} \cap A_i$. Since the inclusion of the localizations $A_{(f)} \subset A_f$ are homomorphisms of A_0-algebras, the morphism π is a morphism of A_0-schemes.

We have

$$\pi_*(\mathcal{O}_{C^*}) = \bigoplus_{i\in\mathbb{Z}} \mathcal{O}_X(i), \tag{3.4}$$

where the sheaves $\mathcal{O}_X(i)$ are defined on subsets $D_+(f)$ by

$$\mathcal{O}_X(i)(D_+(f)) = (A_f)_i.$$

Using the definition of the spectrum of a graded sheaf of $\mathcal{O}_X$-algebras we get

$$C^* = \text{Spec } \pi_*(\mathcal{O}_{C^*}). \tag{3.5}$$

Since the rings $D_+(f)$ are the rings of invariants of $\mathbb{G}_m$ acting on $D(f)$, we can can interpret the sheaf $\mathcal{O}_X$ as the subsheaf $(\pi_*\mathcal{O}_{C^*})^{\mathbb{G}_m} \subset \pi_*\mathcal{O}_{C^*}$ of invariant elements and the map π as the quotient map $C^* \to C^*/\mathbb{G}_m$ in the category of ringed spaces.

Obviously, $\mathcal{O}_X(i) = 0$ if i is i not divisible by the saturation index e of A. Assume $i = se$ and let f be a nonzero element of A_e. Then any fraction $a_{de+i}/f^d \in (A_f)_i$ can be written in the form $f^s \frac{a_{de+i}}{f^{d+s}}$, where $\frac{a_{de+i}}{f^{d+e}} \in A_{(f)}$. This shows that the restriction of $\mathcal{O}_X(i)$ to $D_+(f)$ is an invertible sheaf. Moreover, the canonical multiplication maps

$$\mathcal{O}_X(se) \otimes \mathcal{O}_X(je) \to \mathcal{O}_X((s+j)e) \tag{3.6}$$

are isomorphisms over $D_+(f)$. In particular, this implies that $A_f \cong A_{(f)}[t]$ and hence over $U = D_+(f)$ the punctured cone C^* is isomorphic to the affine line $\mathbb{A}^1_U$ over U. Therefore, if . A is a domain (hence X is irreducible)

$$\dim C = \dim X + 1. \tag{3.7}$$

In general, even if the saturation index e is equal to 1, the sheaves $\mathcal{O}_X(i)$ are not invertible over the whole X nor the multiplication maps

$$\mathcal{O}_X(i) \otimes \mathcal{O}_X(j) \to \mathcal{O}_X(i+j)$$

are isomorphisms. However, both properties hold if the following condition is satisfied

(∗) $A = A_0[A_1]$,

i.e. A is generated as a A_0-algebra by the set A_1. This condition is assumed in almost all statements in [Hartshorne] concerning the projective schemes.

For any graded ring A and a positive integer e one defines the *e-twist* of A by

$$A^{(e)} = \bigoplus_{n\in\mathbb{Z}} A_n^{(e)},$$

where $A_n^{(e)} := A_{en}$.

Geometrically, $A^{(e)}$ is equal to the algebra of invariants of the subgroup $\boldsymbol{\mu}_e$. Note that, if $e > 1$ is the saturation index of A then $A \neq A^{(e)}$ as graded rings.

We will often use the following well-known fact (see [Bourbaki], Commutative Algebra, Chapter 3).

Lemma 3.1.3. *Let A be a non-negatively graded algebra. Assume that A is finitely generated over A_0. There exists $e \geq 1$ such that $A^{(me)} = A_0[A_1^{(me)}]$ for any $m \geq 1$. The algebra A is finitely generated as a $A^{(e)}$-module.*

Proof. Let $x_1, \ldots, x_r$ be a system of homogeneous generators of the A_0-algebra A of positive degrees $d_1, \ldots, d_r$. Let $d = l.c.m.(d_1, \ldots, d_r)$ be the least common multiple and $h_i = d/d_i$. Let B be the graded subalgebra of A generated by $x_i^{h_i}, i = 1, \ldots, r$, all of the same degree d. It is clear that $B \subset A^{(d)}$. I claim that the corresponding B-module $A^{(d)}$ is generated by the monomials

$$x_1^{i_1} \cdots x_r^{i_r}, \quad i_j < d_j, j = 1, \ldots, r, \quad d_1 i_1 + \ldots + d_r i_r \equiv 0 \mod d.$$

It suffices to show that any monomial $x_1^{n_1} \cdots x_r^{n_r} \in A^{(d)}$ can be written as a linear combination of the monomials from above with coefficients in A_0. We write $n_i = d_i q_i + m_i$, where $0 \leq m_i < d_i$, to obtain

$$x_1^{n_1} \cdots x_r^{n_r} = (x_1^{d_1})^{q_1} \cdots (x_r^{d_r})^{q_r} x_1^{m_1} \cdots x_r^{m_r}.$$

Since $x_1^{n_1} \cdots x_r^{n_r} \in A^{(d)}$, we have $\sum n_i d_i \equiv 0 \mod d$. This implies that $\sum m_i d_i \equiv 0 \mod d$. This proves the claim. So $A^{(d)}$ is a finite algebra over B which is finitely generated over A_0, hence it is finitely generated over A_0. Similar argument (without using the congruences modulo d) shows that A is a finite algebra over $A^{(d)}$.

Let $y_1, \ldots, y_s$ be a system of monomials in $x_1, \ldots, x_r$ which generates $A^{(d)}$ as a B-module. Let $n_0 d$ be the largest of the degrees of y_j. For any $x \in A_{d(n+1)}$, where $n \geq n_0$, write $x = \sum a_j y_j$, where $a_j \in B$. We may assume that each a_j is homogeneous of degree $d(n+1) - \deg y_j$. Since B is generated by B_d, each a_j can be written as a product $a_j = b_j a'_j$, where $b_j \in B_d, a'_j \in B_{(n-\deg y_j)d}$. This implies that $x \in B_d A_{nd}$. By induction on n we get $A_{(n+k)d} = B_{kd} A_{nd} \subset A_{kd} A_{nd}$ for $n \geq n_0, k \geq 1$. In particular, $A_{mnd} = A_{dn}^m$ for $n \geq n_0$. It remains to set $e = dn_0$, to obtain $A_{me} = A_e^m$ for all $m \geq 1$. This shows that $A_m^{(e)} = (A_e^{(e)})^m$ and hence $A^{(e)}$ is generated by A_e. □

It is easy to see that $D_+(f) = D_+(f^e)$ and $A_{(f)} = A_{(f^e)}$. This can be used to show that the morphism $C = \text{Spec } A \to C^{(e)} = \text{Spec } A^{(e)}$ corresponding to the inclusion of rings defines an isomorphism of the projective spectra

$$\iota_e : \text{Proj } A \to \text{Proj } A^{(e)}. \tag{3.8}$$

Note that

$$\mathcal{O}_{\text{Proj } A^{(e)}}(n) \cong \mathcal{O}_{\text{Proj } A}(ne).$$

From now on we shall assume that A is finitely generated over A_0. Since all our rings are Noetherian, this implies that each homogeneous part A_i is a finitely generated A_0-module. This also implies that each localization A_f is a finitely generated algebra over A_0, hence each homogeneous localization $A_{(f)}$ is a finitely generated algebra over A_0. Thus $X = \text{Proj } A$ is of finite type over Spec A_0.

Let

$$M = \bigoplus_{iin\mathbb{Z}} M_i, \quad A_i M_j \subset M_{i+j}$$

be a graded A-module. Here each M_i is a A_0-submodule of the A_0-module M. It defines a quasicoherent sheaf $\mathcal{F} = M^\sim$ on X. Its group of sections over $D_+(f)$ is the homogeneous localization $M_{(f)} = A_{(f)} \otimes_A M$.

For example,

$$\mathcal{O}_X(n) = A[n]^\sim,$$

where the A-module $A[n]$ is the ring A with grading shifted by n, i.e. $A[n]_i = A_{n+i}$.

For any coherent sheaf $\mathcal{F}$ on X we set

$$\mathcal{F}(i) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(i).$$

Let

$$\Gamma_*(\mathcal{F}) = \bigoplus_{i\in\mathbb{Z}} \Gamma(X, \mathcal{F}(i)). \tag{3.9}$$

There is a canonical homomorphism of sheaves

$$\Gamma_*(\mathcal{F})^\sim \to \mathcal{F}. \tag{3.10}$$

It is an isomorphism if condition $(*)$ is satisfied. So, in general $\mathcal{F}$ may not be isomorphic to $M^\sim$. However, we may always identify $\mathcal{F}$ with the sheaf $\mathcal{F}'$ on Proj $A^{(e)}$ such that $\mathcal{F} = \iota_e^*(\mathcal{F}')$, then $\mathcal{F} = M^\sim$, where $M = \Gamma_*(\mathcal{F}')$ if e is large enough.

In the case when $\mathcal{F} = M^\sim$, we have a canonical homomorphism of graded A-modules

$$\rho_M : M \to \Gamma_*(M^\sim), \quad m_i \mapsto m_i/1, \ m_i \in M_i,$$

which induces an isomorphism of sheaves $M^\sim \to \Gamma_*(M^\sim)^\sim$. It inverts the homomorphism of sheaves (3.10). Thus two modules M and $\Gamma_*(M^\sim)$ define isomorphic sheaves of $\mathcal{O}_X$-modules. The following lemma implies that the two modules differ only in finitely many homogeneous parts.

Lemma 3.1.4. *Let M be a finitely generated graded A-module. Then $M^{\sim} = 0$ if and only if M is a finite A-module. In particular,* Proj $A = \emptyset$ *if and only if A is a finite algebra over A_0.*

Proof. If A is generated by elements of degree 1, this is well-known and can be found in [Hartshorne], Chapter II, Exercise 5.8. Since A is a finite $A^{(e)}$-module, M is a finite as a $A^{(d)}$ module if and only if it is a finite as a A-module. □

In particular, taking $M = A$, we obtain a homomorphism of graded rings

$$\rho_A : A \to \Gamma_*(\mathcal{O}_X) \tag{3.11}$$

that defines the homomorphisms

$$A_i \to \Gamma(X, \mathcal{O}_X(i)), \quad i \in \mathbb{Z}. \tag{3.12}$$

Although these homomorphisms are bijective for i large enough, they may not be isomorphisms for all i even if condition $(*)$ is satisfied.

Recall that an integral domain is called *normal* if it is integrally closed in its field of fractions.

Proposition 3.1.5. *Assume A_0 is a field and A is a normal integral domain of dimension ≥ 2. Then the homomorphism ρ_A is an isomorphism of graded algebras. In particular, the homomorphisms* (3.12) *are bijective for all $i \in \mathbb{Z}$.*

Proof. Let $j : C^* \hookrightarrow C =$ Spec A be the open embedding with complement equal to the vertex of C. Then the natural homomorphism $\mathcal{O}_C \to j_*j^*\mathcal{O}_C = j_*\mathcal{O}_{C^*}$ induces a homomorphism

$$A = \Gamma(C, \mathcal{O}_C) \to \Gamma(C, j_*\mathcal{O}_{C^*}) = \Gamma(\mathcal{O}_{C^*})$$

$$= \Gamma(X, \pi_*\mathcal{O}_{C^*}) = \bigoplus_{i \in \mathbb{Z}} \Gamma(X, \mathcal{O}_X(i)).$$

One can see that the composition of all these identification maps is the map ρ_A. On the other hand, the standard properties of local cohomology show that the kernel (resp. the cokernel) of the restriction map $\Gamma(C, \mathcal{O}_C) \to \Gamma(C^*, \mathcal{O}_{C^*})$ is isomorphic to the local cohomology $H^0_{\mathfrak{m}_0}(A)$ (resp. $H^1_{\mathfrak{m}_0}(A)$) (see [Eisenbud], Theorem A4.1). It follows from the definition of local cohomology that $H^0_{\mathfrak{m}_0}(A) = 0$ for any integral domain A. Also it is known that $H^1_{\mathfrak{m}_0}(A) = 0$ if and only if depth$A_{\mathfrak{m}_0} \geq 2$. It remains to use that the localization of a normal domain at prime ideals of height ≥ 2 are of depth$A_{\mathfrak{m}_0} \geq 2$ (see [Eisenbud], p. 458). □

A closed subscheme Y of Proj A is defined by a coherent sheaf of ideals $\mathcal{I}_Y$. Let $I \subset A$ be a homogeneous ideal and $B = A/I$. The corresponding closed embedding of affine schemes Spec $B \hookrightarrow$ Spec A induces a closed embedding

$$i : \text{Proj } B \hookrightarrow \text{Proj } A.$$

Its image is isomorphic to the closed subscheme Y of X defined by the sheaf of ideals $\mathcal{I}_Y = I^{\sim}$. The sheaves $B[n]^{\sim} = \mathcal{O}_{\text{Proj } B}(n)$ may not coincide with the sheaves $i^*(\mathcal{O}_X(n)) = \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) = \mathcal{O}_Y(n)$. The reason is that the canonical homomorphism $A \to B, a \mapsto \bar{a}$ induces a homomorphism $(A_f)_n \to (B_{\bar{f}})_n$ with kernel $(I_f)_n$ which may not coincide with $I_{(f)}(A_f)_n$ if f is of degree $\neq 1$.

Conversely, a closed subscheme Y of $X = \text{Proj } A$ defines the A-module $\Gamma_*(\mathcal{I}_Y)$. Tensoring the inclusions $\mathcal{I}_Y \hookrightarrow \mathcal{O}_X$ by $\mathcal{O}_X(n)$ we get a homomorphism $\Gamma_*(\mathcal{I}_Y) \to \Gamma_*(\mathcal{O}_X)$ whose image is an ideal in $\Gamma_*(\mathcal{O}_X)$. The pre-image of this ideal under ρ_A is an ideal I in A. However, the canonical homomorphism $I^{\sim} \to \mathcal{I}_Y$ may not be an isomorphism. So the corresponding morphism $Y \to \text{Proj } A/I$ may not be an isomorphism.

Example 3.1.6. Let R be any commutative ring (as always assumed to be Noetherian). The projective spectrum of the polynomial algebra $RT_0, \ldots, T_n]$ graded by $\deg T_i = q_i > 0$ is denoted by $\mathbb{P}_R(q_0, \ldots, q_n)$ or $\mathbb{P}_R(\mathbf{q})$, where $\mathbf{q} = (q_0, \ldots, q_n)$. It is called the *weighted projective space* with weights $\mathbf{q}$. The positivity condition on the weights implies that the vertex ideal $\mathfrak{m}_0$ is equal to $(T_0, \ldots, T_n)$ and the vertex is isomorphic to Spec K.

If A is generated over R by homogeneous elements $f_0, \ldots, f_n$ of positive degrees $q_0, \ldots, q_n$, then the surjection $R[T_0, \ldots, T_n] \to A$ of graded R-algebras defines a closed embedding

$$\text{Proj } A \hookrightarrow \mathbb{P}_R(\mathbf{q}).$$

Taking $\mathbf{q} = (1, \ldots, 1)$ and $R = k$ is a field we get the usual notion of a projective space $\mathbb{P}^n_k$.

Let $\phi : A \to B$ be a homomorphism of graded algebras. It defines a morphism of affine quasicones $\phi^{\#} : C' = \text{Spec } B \to C = \text{Spec } A$. Contrary to the case of affine spectra, it does not define a morphism Proj $B \to$ Proj A. The reason is that a prime homogeneous ideal $\mathfrak{q} \in \text{Proj } B$ could be mapped under the morphism $\phi^{\#}$ to a prime homogeneous ideal containing the irrelevant ideal of A. Let $G(\phi)$ be the open subset of Spec B whose complement is the closed subset corresponding to the ideal $\phi(A_+)B$. The restriction of $\phi^{\#}$ to $G(\phi)$ defines a $\mathbb{G}_m$-equivariant morphism $G(\phi) \to C^*$ that induced a morphism

$$\tilde{\phi}^{\#} : G(\phi)^+ \to X = \text{Proj } A,$$

where $G(\phi)^+$ is the image of $G(\phi)$ in Proj B. Note that $G(\phi) = \text{Proj } B$ if ϕ is surjective. In this case the morphism $\tilde{\phi}^{\#}$ is a closed embedding.

All of what we discussed in above can be extended to the relative case of non-negatively graded $\mathcal{O}_S$-algebras $\mathcal{A}$. We define Proj $\mathcal{A}$ by gluing together the schemes Proj $\mathcal{A}(U)$ to obtain a scheme over S. Let $q : X \to S$ be a scheme over S and $\mathcal{B}$ a non-negatively graded $\mathcal{O}_X$-algebra with $\mathcal{B}_0 = \mathcal{O}_X$. Suppose we have a surjective homomorphism of graded $\mathcal{O}_X$-algebras $\phi : q^*\mathcal{A} \to \mathcal{B}$. Passing to the projective spectra we get a morphism Proj $\mathcal{B} \to$ Proj $q^*\mathcal{A} \cong$ Proj $A \times_S X$. Composing it with the projection Proj $A \times_S X \to$ Proj $\mathcal{A}$ we get a morphism of S-schemes

$$f_\phi : \text{Proj } \mathcal{B} \to \text{Proj } \mathcal{A}. \tag{3.13}$$

In particular, suppose Proj $\mathcal{B} \cong X$, for example, when $\mathcal{B} = S^\bullet\mathcal{L}$, where $\mathcal{L}$ is an invertible sheaf on X. Then we obtain a morphism $X \to \text{Proj } A$. For example, suppose $\mathcal{A} = S^\bullet\mathcal{E}$ for some locally free sheaf $\mathcal{E}$, a surjective homomorphism $q^*\mathcal{A} \to S^\bullet\mathcal{L}$ is defined by a surjective homomorphism of $\mathcal{O}_X$-modules $q^*\mathcal{E} \to \mathcal{L}$. In this case we obtain that a morphism $X \to \mathbb{P}(\mathcal{E})$ (see Proposition 7.12 from Chapter II of [Hartshorne]). Here, for any locally free sheaf $\mathcal{E}$ on S we set

$$\mathbb{P}(\mathcal{E}) = \text{Proj } S^\bullet\mathcal{E}.$$

It is called the *projective bundle* associated to $\mathcal{E}$. Taking $S = \text{Spec } k$, we see that $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}^r_k$, where $r+1 = \text{rank}\mathcal{E}$ and maps of k-schemes $X \to \mathbb{P}^r_k$ are defined by a choice of a surjection $\mathcal{O}_X^{r+1} \to \mathcal{L}$, i.e. by a collection of $r+1$ sections of $\mathcal{L}$ not vanishing simultaneously at any point of X.

3.2 Ample invertible sheaves

Let X be a proper scheme over a noetherian ring R and $\mathcal{L}$ be an invertible sheaf on X.

Proposition 3.2.1. *The following properties are equivalent*

(i) for any coherent sheaf $\mathcal{F}$ on X there exists $n_0 > 0$ such that $\mathcal{F} \otimes \mathcal{L}^n$ is generated by its global sections for any $n \geq n_0$;

(ii) for any coherent sheaf $\mathcal{F}$ on X there exists $n_0 > 0$ such that $H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0$ for all $n \geq n_0$ and all $i > 0$;

(iii) there exist $n > 0$ and global sections $s_0, \ldots, s_n$ of $\mathcal{L}^n$ such that the open sets $X_{s_i} = \{x \in X : s_i(x) \neq 0\}$ are affine and form an open covering of X;

(iv) there exists $n > 0$ and a closed embedding $i : X \hookrightarrow \mathbb{P} = \mathbb{P}^r_R$ for some $r > 0$ such that $i^*(\mathcal{O}_{\mathbb{P}}(1)) \cong \mathcal{L}^n$;

(v) the graded algebra

$$A(X, \mathcal{L}) = \bigoplus_{i=0}^{\infty} \Gamma(X, \mathcal{L}^i)$$

is finitely generated and there exists an isomorphism $\phi : X \to Y = \text{Proj } A(X, \mathcal{L})$ such that $\phi^*(\mathcal{O}_Y(i)) \cong \mathcal{L}^i$ for all $i \geq 0$.

Proof. (i) $\Leftrightarrow$ (ii) This is Chapter III, Proposition 5.3 from [Hartshorne].

(i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) This is proved in the proof of Theorem 7.6 in Chapter II of [Hartshorne].

(iv) $\Rightarrow$ (v) Let $Y = i(X)$ and $\mathcal{J}$ be the ideal sheaf of Y. We have the standard exacts sequence

$$0 \to \mathcal{J}(n) \to \mathcal{O}_{\mathbb{P}}(1) \to i_*\mathcal{L}^n \to 0. \tag{3.14}$$

Replacing n by its positive multiple, we still have (iv) (use the Veronese embedding). Since (iv) implies (ii), we may assume that $H^1(\mathbb{P}, \mathcal{J}(kn)) = 0$ for all $k \geq 1$. This implies that $\Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k) \to \Gamma(X, \mathcal{L}^{kn})$ is surjective for all $k \geq 1$. Thus the graded algebra $A(X, \mathcal{L}^n) = A(X, \mathcal{L})^{(n)}$ is isomorphic to the quotient of the graded R-algebra $\Gamma_*(\mathcal{O}_{\mathbb{P}}(1)) \cong R[T_0, \ldots, T_r]$, and hence finitely generated. It follows from the proof of Lemma 3.1.3 that $A(X, \mathcal{L})$ is finitely generated as well.

Since (iv) implies (i), we can follow the proof of (i) $\Rightarrow$ (iii) to show that the open sets $X_s, s \in A(X, \mathcal{L})_{>0}$ form a basis of topology of X and those of them which are affine form an open covering of X. Let (s_α) be a family of sections forming an open affine covering. We may assume that $s_\alpha \in \Gamma(X, \mathcal{L}^{n_\alpha})$ and the set of indices is finite. For each α define a morphism of affine schemes $X_{s_\alpha} \to D_+(s_\alpha) \subset S = \text{Proj } A(X, \mathcal{L})$ by using the restriction homomorphism

$$\mathcal{O}_S(D_+(s_\alpha)) \to \mathcal{O}_X(X_\alpha),\ a/s_\alpha^d \mapsto (a|X_{s_\alpha})(s_\alpha^d|U)^{-1}.$$

Using Lemma 5.14 from Chapter II of [Hartshorne], one shows that this defines an isomorphism $\phi_\alpha = X_{s_\alpha} \to D_+(s_\alpha)$ and it is not difficult to see that the isomorphisms f_α can be glued together to define a global isomorphism $\phi : X \to S$. It follows from the definition of $\mathcal{O}_S(n)$ that $\phi^*(\mathcal{O}_S(n)) \cong \mathcal{L}^n$.

(v) $\Rightarrow$ (iv) Since $A(X, \mathcal{L})$ is finite generated over R, we can arrange a surjection of graded R-algebras $\phi : R[T_0, \ldots, T_r] \to A(X, \mathcal{L})$. As we noticed before, we can pass to projective spectra to define a closed embedding $i : S = \text{Proj } A(X, \mathcal{L}) \to \mathbb{P}^r_R$.

□

For any noetherian scheme X (not necessarily proper over a ring), an invertible sheaf $\mathcal{L}$ on X is called *ample* if property (i) in Proposition 3.2.1 is satisfied (see [Hartshorne], p. 153). An invertible sheaf on a scheme X over a ring R is called *very ample* if there exists an embedding $i : X \hookrightarrow \mathbb{P} = \mathbb{P}^N_R$ with $i^*(\mathcal{O}_{\mathbb{P}}(1) \cong \mathcal{L}$ (see [Hartshorne], p. 120). So our Proposition implies that, under its assumption, an invertible sheaf $\mathcal{L}$ is ample if and only $\mathcal{L}^n$ is very ample for some $n > 0$.

We will need one more property of an ample invertible sheaf.

Corollary 3.2.2. *Let $\mathcal{L}$ be an ample invertible sheaf on an integral scheme X proper over a field k. Then the graded k-algebra $A(X, \mathcal{L})$ is a normal finitely generated algebra over k.*

Proof. Let $A = A(X, \mathcal{L})$ and $X = \text{Proj } A$. It follows from Proposition 3.2.1 (v) that the canonical homomorphism $\rho_A : A \to \Gamma_*(\mathcal{O}_X(1))$ defined in (3.11) is an isomorphism. If $\dim A \geq 2$, the proposition follows from Proposition . If $\dim = 1$, by (7.6), we have $\dim X = 0$. In this case $X \cong \text{Spec } K$ for some finite extension of fields K/k and the assertion is obvious. We get $A = S^\bullet(K) \cong K[t]$. □

Remark 3.2.3. A closed integral subscheme X of $\mathbb{P} = \mathbb{P}^n_k$ is called *projectively normal* if the restriction homomorphism

$$\Gamma_*(\mathcal{O}_{\mathbb{P}}(1)) \cong k[T_0, \dots, T_n] \to \Gamma_*(\mathcal{O}_X(1))$$

is surjective. Let $\mathcal{I}_X$ be the sheaf of ideals defining X. The ideal $I_X = \Gamma_*(\mathcal{I}_X)$ in $k[T_0, \dots, T_n]$ is called the *homogeneous ideal* of X and the quotient ideal $k[X] = k[T_0, \dots, T_n]/I_X$ is called the *homogeneous coordinate algebra* of X. The Corollary implies that X is projectively normal if and only if $k[X]$ is normal. One checks that the fields of fractions of $k[X]$ and $\Gamma_*(\mathcal{O}_X(1))$ are both isomorphic to $k(X)(t)$, where $k(X)$ is the field of rational functions on X. Using exact sequence (8.3.2) and property (iii) of ample sheaves from Proposition 3.2.1, we see that there exists a positive number d such that the twisted subalgebras $k[X]^{(d)}$ and $\Gamma_*(\mathcal{O}_X(1))^{(d)}$ are isomorphic. Since $k[X] \subset \Gamma_*(\mathcal{O}_X(1))$ and $\Gamma_*(\mathcal{O}_X(1))$ is integral over $\Gamma_*(\mathcal{O}_X(1))^{(d)} = k[X]^{(d)}$, we have the equality $k[X] = \Gamma_*(\mathcal{O}_X(1))$.

3.3 ℚ-divisors

Let X be a noetherian integral scheme of dimension ≥ 1 and $X^{(1)}$ be its set of points of codimension 1 (i.e. points $x \in X$ with $\dim \mathcal{O}_{X,x} = 1$). We assume that X is regular in codimension 1, i.e. all local rings of points from $X^{(1)}$ are regular. In this case we can define *Weil divisors* on X as elements of the free abelian group $\mathrm{WDiv}(X) = \mathbb{Z}^{X^{(1)}}$ and also define linear equivalence of Weil divisors and the group $\mathrm{Cl}(X)$ of linear equivalence classes of Weil divisors (see [Hartshorne], Chap. 2, §6).

We identify a point $x \in X^{(1)}$ with its closure E in X. We call it an *irreducible divisor*. Any irreducible reduced closed subscheme E of codimension 1 is an irreducible divisor, the closure of its generic point.

For any Weil divisor D let $\mathcal{O}_X(D)$ be the sheaf whose section on an open affine subset U consists of functions from the quotient field $Q(\mathcal{O}(U))$ such that $\mathrm{div}(\Phi) + D \geq 0$.

It follows from the definition that $\mathcal{O}_X(D)$ is torsion free and, for any open subset $j : U \hookrightarrow X$ which contains all points of codimension 1, we have

$$j_* j^* \mathcal{O}_X(D) = \mathcal{O}_X(D). \tag{3.15}$$

These two conditions characterize *reflexive sheaves* on any normal integral scheme X. For readers familiar with the theory of local cohomology, it is clear that the latter condition is equivalent to the condition that for any point $x \in X$ with $\dim \mathcal{O}_{X,x} \geq 2$ the depth of the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$ is greater or equal than 2. By equivalent definition, a reflexive sheaf $\mathcal{F}$ is a coherent sheaf such that the canonical homomorphism $\mathcal{F} \to \mathcal{F}^{**}$ is an isomorphism. Our sheaves $\mathcal{O}_X(D)$ are reflexive sheaves of rank 1(the rank of a coherent sheaf on an integral scheme is the dimension of the stalk at the generic point over its residue field). Conversely, a reflexive sheaf $\mathcal{F}$ of rank 1 on a normal integral scheme is isomorphic to $\mathcal{O}_X(D)$ for some Weil divisor D. In fact, we restrict $\mathcal{F}$ to some open subset $j : U \hookrightarrow X$ with complement of codimension ≥ 2 such that $j^*\mathcal{F}$ is locally free of rank 1. Thus it corresponds to a Cartier divisor on U. Taking the closure of the corresponding Weil divisor in X, we get a Weil divisor D on X and it is clear that $\mathcal{F} = j_*j^*\mathcal{F} \cong \mathcal{O}_X(D)$. In particular, we see that any reflexive sheaf of rank 1 on a regular scheme is invertible. It is not true for reflexive sheaves of rank > 1. They are locally free outside of a closed subset of codimension ≥ 3.

Reflexive sheaves of rank 1 form a group with respect to the operation

$$\mathcal{L} \cdot \mathcal{G} = (\mathcal{L} \otimes \mathcal{G})^{**}, \ \mathcal{L}^{-1} = \mathcal{L}^*.$$

For any reflective sheaf $\mathcal{L}$ and an integer n we set

$$\mathcal{L}^{[n]} = (\mathcal{L}^{\otimes n})^{**}.$$

One checks that

$$\mathcal{O}_X(D + D') = \mathcal{O}_X(D) \cdot \mathcal{O}_X(D')$$

and the map $D \mapsto \mathcal{O}_X(D)$ defines an isomorphism from the group $\mathrm{Cl}(X)$ to the group of isomorphism classes of reflexive sheaves of rank 1.

Let

$$\mathrm{WDiv}(X, \mathbb{Q}) = \mathrm{WDiv}(X) \otimes \mathbb{Q} = \mathbb{Q}^{X^{(1)}}.$$

Its elements are called *Weil $\mathbb{Q}$-divisors*. Any such a divisor can be uniquely written in the form $D = \sum a_i E_i$, where E_i are irreducible divisors and a_i are rational numbers. For any Weil $\mathbb{Q}$-divisor $D = \sum a_i E_i$, we set

$$\lfloor D \rfloor = \sum \lfloor a_i \rfloor E_i, \ \lceil D \rceil = \sum \lceil a_i \rceil E_i,$$

$$\mathcal{O}_X(D) = \mathcal{O}_X(\lfloor D \rfloor).$$

where $\lfloor \alpha \rfloor$ (resp. $\lceil \alpha \rceil$) denotes the largest integer less or equal than α (resp. the smallest integer greater or equal than α).

Obvioulsy any Weil $\mathbb{Q}$-divisor can be written uniquely in the form

$$D = \lfloor D \rfloor + \sum a_i E_i, \quad D \in \mathrm{WDiv}(X), \ 0 \leq a_i < 1.$$

We extend the notion of linear equivalence to Weil $\mathbb{Q}$-divisors by defining $D \sim D'$ if the difference is the divisor $\mathrm{div}(\Phi)$ of some rational function on X. Let $\mathrm{Cl}(X, \mathbb{Q})$ be the group of linear equivalence classes of Weil $\mathbb{Q}$-divisors. It is easy to see that we have an exacts sequence

$$0 \to \mathrm{Cl}(X) \to \mathrm{Cl}(X, \mathbb{Q}) \to \mathrm{Div}(X) \otimes \mathbb{Q}/\mathbb{Z} \to 0.$$

For any Weil $\mathbb{Q}$-divisor D we set

$$L(D) := H^0(X, \mathcal{O}_X(\lfloor D \rfloor)).$$

The multiplication maps $\mathcal{O}_X(D)\otimes\mathcal{O}_X(D')\to\mathcal{O}_X(D+D')$ define a graded sheaf of $\mathcal{O}_X$-algebras

$$\mathcal{A}(X,D)=\bigoplus_{i\in\mathbb{Z}}\mathcal{O}_X(iD). \tag{3.16}$$

Passing to global sections we can define the graded algebra

$$A(X,D)=\bigoplus_{i=0}^{\infty}L(iD). \tag{3.17}$$

It is important to understand that this is a direct sum, although all the graded parts are subspaces in the field of rational functions K on X. One may view $A(X,D)$ as a subalgebra of the field $K(T)$ by considering an isomorphic graded subalgebra of $K(T)$

$$A(X,D)'=\bigoplus_{i=0}^{\infty}L(iD)T^i.$$

Recall that a Weil divisor D is called a *Cartier divisor* if it is locally principal, i.e. there exists an open cover $(U_i)_{i\in I}$ of X such that the image of D under the restriction map $\mathbb{Z}^{X^{(1)}}\to\mathbb{Z}^{U_i^{(1)}}, i\in I$, is linearly equivalent to zero. For any Cartier divisor D, the sheaf $\mathcal{O}_X(D)$ is invertible and any invertible sheaf $\mathcal{L}$ is isomorphic to a sheaf $\mathcal{O}_X(D)$ for a Cartier divisor D, defined uniquely up to a linear equivalence. In particular, there is an isomorphism of groups

$$A(X,D)\cong A(X,\mathcal{O}_X(D)).$$

The subgroup of $\mathrm{Cl}(X)$ of the linear equivalence classes of Cartier divisors is denoted by $\mathrm{Pic}(X)$. It is called the *Picard group* of X. It is also can be defined as the group of isomorphism classes of invertible sheaves on X. All of this can be found in [Hartshorne], Chapter II, §6.

Definition 3.3.1. A Weil $\mathbb{Q}$-divisor D is called $\mathbb{Q}$-Cartier if rD is a Cartier divisor for some integer r. A Cartier $\mathbb{Q}$-divisor is called *ample* if there exists $r>0$ such that the sheaf $\mathcal{O}_X(rd)$ is an ample invertible sheaf.

Proposition 3.3.1. *Let D be an ample $\mathbb{Q}$-Cartier divisor on an integral scheme X proper over a field k. Then the graded algebra $A(X,D)$ is a finitely generated graded normal k-algebra.*

Proof. Applying Lemma 3.1.3, we may replace D with rD to assume that D is an ample Cartier divisor. Then the assertion follows from Corollary 3.2.2. □

Example 3.3.2. Let $G\subset\mathrm{PGL}(2,\mathbb{C})$ be a polyhedral group with exceptional orbits over the points $p_1,\ldots,p_r$ and $e_1,\ldots,e_r$ be the orders of the corresponding stabilizer subgroups. Consider the rational divisor

$$D=-K_{\mathbb{P}^1}-\sum_{i=1}^{r}\frac{e_i-1}{e_i}p_i.$$

I claim that, if G is not cyclic of odd order, then

$$A(X,D) \cong \mathbb{C}[x,y]^{\bar{G}},$$

where $\bar{G} \subset \mathrm{SL}(2,\mathbb{C})$ is the corresponding binary polyhedral group. I will check this only in two cases: G is cyclic of even order n and $G = T$ is the tetrahedral group. Other cases are left as exercises (or see [Séminaire sur les singularités des surfaces]).

Assume G is cyclic of even order $2e$. In this case $r = 2, e_1 = e_2 = e$. We may take $X = \mathbb{P}^1, p_1 = 0, p_2 = \infty$ and

$$D = 2\infty - \frac{e-1}{e}0 - \frac{e-1}{e}\infty = -\frac{e-1}{e}0 + \frac{e+1}{e}\infty.$$

We find that

$$\lfloor D \rfloor = \infty - 0, \quad A(X,D)_1 \text{ is spanned by } u = t,$$

$$\lfloor iD \rfloor = i\infty - i0,\ i < e, \quad A(X,D)_i \text{ is spanned by } u^i = t^i,$$

$$\lfloor eD \rfloor = (e{+}1)\infty{-}(e{-}1)0, \quad A(X,D)_e \text{ is spanned by } u^e = t^e, v = t^{e+1}, w = t^{e-1}.$$

Note that $\lfloor mD \rfloor = mD$ if $e|m$. This implies that $\lfloor kD \rfloor = edD + \lfloor rD \rfloor$ if $k = ed + r, 0 \le r < e$. Let $A = A_1 + A_2$ be an effective Weil divisor on $\mathbb{P}^1_K$ written as a sum of two effective Weil divisors. Any function from the space $L(A)$ can be written as the product of a function from $L(A_1)$ and a function from $L(A_2)$ (this is of course true only when the curve is of genus 0). This shows that $A(X,D)$ is generated by u, v, w. It is immediate to check that

$$vw = u^{2e}.$$

Thus $A(X,D)$ is isomorphic to the ring of invariants of a cyclic subgroup of order $2e$ in $\mathrm{SL}(2,\mathbb{C})$.

Let us take now p_1, p_2 as above and $p_3 = 1 = (1,1)$. Consider the divisor

$$D = 2p_3 - \frac{1}{2}p_1 - \frac{2}{3}p_2 - \frac{2}{3}p_3.$$

We have

$$A(X,D)_n = L(nD) = L(2np_3 + \lfloor -\frac{n}{2} \rfloor p_1 + \lfloor \frac{-2n}{3} \rfloor p_2 + \lfloor \frac{-2n}{3} \rfloor p_3).$$

We find that $\lfloor nD \rfloor = 0$ for $n = 1, 2, 5$, and

$$\lfloor 3D \rfloor = 4p_2 - 2p_1 - 2p_3, \quad A(X,D)_3 \text{ is spanned by } u = t^2(t-1)^2,$$

$$\lfloor 4D \rfloor = 5p_2 - 2p_1 - 3p_3, \quad A(X,D)_4 \text{ is spanned by } v = t^2(t-1)^3,$$

$$\lfloor 6D \rfloor = 8p_2 - 3p_1 - 4p_3, \quad A(X,D)_6 \text{ is spanned by } w = t^3(t-1)^4(\frac{t}{2} - 1), u^2.$$

As above we show that $A(X,D)$ is generated by u, v, w. We check directly that

$$\frac{u^4}{4} - v^3 = w^2.$$

After scaling we get that

$$A(X,D) \cong \mathbb{C}[x,y]^{\bar{T}},$$

the algebra of invariants of the binary tetrahedral group.

Remark 3.3.3. The group G acts on $\mathbb{P}^1$ with the quotient $\mathbb{P}^1/G \cong \mathbb{P}^1$. Recall that a quotient of a projective variety X by a finite group is defined by covering X by invariant open affine subsets U_i and gluing the spectra of the ring of invariants $\mathcal{O}_X(U_i)^G$. In particular, the quotient of an normal variety is normal, and hence the quotient of a nonsingular curve is a nonsingular curve. Obviously, the quotient of $\mathbb{P}^1$ is a rational curve, hence is isomorphic to $\mathbb{P}^1$. The projection $\pi : \mathbb{P}^1 \to \mathbb{P}^1/G$ is a finite map ramified at the exceptional orbits with ramification indices equal to the orders of the corresponding stabilizer groups. Applying the Hurwitz formula (see [Hartshorne], p. 301) we find that

$$\pi^*(-K_{\mathbb{P}^1} - \sum_{i=1}^{r} \frac{e_i - 1}{e_i} p_i) = -K_{\mathbb{P}^1}.$$

Similarily, let C be a nonsingular projective curve of genus $g > 1$ over an algebraically closed field k and G be a finite group of automorphisms of C of order coprime to the characteristic of k. Assume that $C/G \cong \mathbb{P}^1$ and let $\pi : C \to \mathbb{P}^1$ be the projection map. Let $p_1, \ldots, p_r$ be the branch points of the finite map π and e_i be the ramification indices of points in $\pi^{-1}(p_i)$. Then the Hurwitz formula gives

$$K_C = \pi^*(D),$$

where

$$D = K_{\mathbb{P}^1_k} + \sum_{i=1}^{r} \frac{e_i - 1}{e_i} p_i$$

is a rational ample divisor on $\mathbb{P}^1$. One may ask when $A(X,D)$ is isomorphic to a graded algebra of the form $k[x,y,z]/(f(x,y,z))$. The answer is known. It happens if and only if $(e_1, \ldots, e_r)$ is one of the following

$$\begin{aligned}
r = 3 \quad &: \quad (2,3,9), (2,4,7), (3,3,6), (2,3,8), (2,4,6), (3,3,5), (2,5,6),\\
&\quad\;\; (3,4,5), (2,5,5), (3,4,4), (2,3,7), (2,4,5), (3,3,4), (4,4,4);\\
r = 4 \quad &: \quad (2,2,2,3), (2,2,3,3), (2,2,2,4), (2,2,3,4);\\
r = 5 \quad &: \quad (2,2,2,2,2), (2,2,2,2,3).
\end{aligned}$$

For example, in the case $(2,3,7)$ we can take $f(x,y,z) = x^2 + y^3 + z^7$.

Theorem 3.3.4. *Let $C = Spec\ A$ be a normal affine quasicone of dimension ≥ 2. There exists an ample Cartier $\mathbb{Q}$-divisor D on $X = \mathrm{Proj}\ A$ such that there is an isomorphism of graded algebras*

$$A \cong A(X, D).$$

The divisor D is defined uniquely up to linear equivalence.

Proof. We follow the proof from [Kollar]. Note that X is a normal projective scheme over the ring $K = A_0$. To see why it is normal, we replace A with $A^{(e)}$ for some $e > 0$ to assume that $A = A_0[A_1]$. Then the localization rings A_f are isomorphic to $A_{(f)}[T, T^{-1}]$ are normal. This implies that $A_{(f)}$ are normal rings.

Applying Proposition 3.2, it is enough to construct a Weil $\mathbb{Q}$-divisor D such that for any n there exists an isomorphism

$$A(X, D)_n = L(nD)) \to H^0(X, \mathcal{O}_X(n))$$

compatible with multiplication maps.

We already noted that a coherent sheaf $\mathcal{F}$ on a normal scheme X is reflexive if and only if $\mathcal{F}$ is torsion-free and for any open inclusion $j : U \hookrightarrow X$ with complement of codimension ≥ 2 the canonical homomorphism $\mathcal{F} \to j_*j^*\mathcal{F}$ is bijective. Our sheaves $\mathcal{O}_X(n)$ are obviously torsion-free because $A_{(f)} \subset A_f$. They also satisfy the second condition. In fact, let $i : V = \pi^{-1}(U) \hookrightarrow C^*$, where $j : U \hookrightarrow X$ as above. Since C^* is normal, by Serre's criterion, $i_*\mathcal{O}_V = \mathcal{O}_{C^*}$. Let $\pi' = \pi|_V$. Now

$$j_*(\pi'_*\mathcal{O}_V) = \pi_*(i_*\mathcal{O}_V) = \pi_*\mathcal{O}_{C^*}.$$

It remains to recall that

$$\pi_*\mathcal{O}_{C^*} = \bigoplus_{n\in\mathbb{Z}} \mathcal{O}_X(n),\ \pi'_*\mathcal{O}_V = \bigoplus_{n\in\mathbb{Z}} j^*\mathcal{O}_X(n).$$

Since $\mathcal{O}_X(n)$ is obviously of rank 1, it is isomorphic to $\mathcal{O}_X(D_n)$ for some Weil divisor D_n. Choose e such that $A^{(e)} = A_0[A_e]$. Then $\mathcal{O}_X(en)$ are invertible sheaves for all n and the multiplication maps $\mathcal{O}_X(en) \otimes \mathcal{O}_X(em) \to \mathcal{O}_X((n+m)e)$ are isomorphisms.

Let $\mathcal{O}_X(1)^{\otimes e} \to \mathcal{O}_X(e)$ be the multiplication map. Passing to the double duals we get a map $\mathcal{O}_X(1)^{[e]} \to \mathcal{O}_X(e)$. A nonzero map $\mathcal{O}_X(D) \to \mathcal{O}_X(D')$ exists if and only if $D' - D$ is effective. Thus

$$D_e - eD_1 = \sum d_s E_s$$

for some positive integers d_s. Similarily, considering the maps $\mathcal{O}_X(1)^{\otimes i} \to \mathcal{O}_X(i)$ for all $i > 0$ and passing to the double duals, we obtain

$$D_i - iD_1 \geq 0.$$

The maps $\mathcal{O}_X(i)^{\otimes e} \to \mathcal{O}_X(ie) = \mathcal{O}_X(iD_e)$ show that

$$D_{ie} - eD_i \geq 0.$$

Thus we get

$$0 \le D_{ie} - eD_i = i(D_e - eD_1) - e(D_i - iD_1),$$

hence $e(D_i - iD_1) \le i(D_e - eD_1)$ and

$$D_i \le iD_1 + \sum \lfloor \frac{i}{e} \rfloor E_i.$$

This implies that

$$L(D_i) = H^0(X, \mathcal{O}_X(i)) \subset L(iD_1 + \sum \lfloor id_s/e \rfloor E_s),\ i \ge 0.$$

Let

$$D = D_1 + \sum \frac{d_s}{e} E_s. \tag{3.18}$$

We have

$$iD_1 + \sum \lfloor \frac{i}{e} \rfloor E_i = \lfloor iD \rfloor,$$

and hence an inclusion of graded algebras

$$A = \bigoplus_{i=0}^{\infty} H^0(X, \mathcal{O}_X(i)) \subset A(X, D).$$

Since $eD = eD_1 + \sum d_s E_s = D_e$, the subalgebras $A^{(e)}$ and $A(X, D)^{(e)}$ coincide. Thus $A(X, D)$ is integral over A, and the algebras of invariants under the action of $\boldsymbol{\mu}_e$ map coincide. This implies that the fields of fractions of A and $A(X, D)$ are isomorphic, hence the map Spec $A(X, D) \to C$ is the normalization map. Since C is normal, it is an isomorphism.

It remains to show the uniqueness of D. Since the affine X-scheme $\pi : C^* \to X$ does not depend on D we have an isomorphism of $\mathcal{O}_X$-algebras

$$\bigoplus_{\imath \in \mathbb{Z}} \mathcal{O}_X(iD) \cong \bigoplus_{\imath \in \mathbb{Z}} \mathcal{O}_X(iD).$$

Thus the sheaf $\mathcal{O}_X(D)$ is determined uniquely by A, up to isomorphism. This shows that the divisor eD is defined uniquely up to a linear equivalence. Hence D is defined uniquely up to linear equivalence. □

Definition 3.3.2. The Weil $\mathbb{Q}$-divisor

$$D - \lfloor D \rfloor = \sum \frac{k_s}{e_s} E_s,\ 0 \le k_s < e_s, (k_s, e_s) = 1 \text{ if } k_s > 0,$$

is called the *Seifert divisor* of the quasicone $C = \text{Spec } A$.

For example, the Seifert divisor of a Kleinian surface $\mathbb{C}[x, y]^{\bar{G}}$, where $\bar{G}$ is not cyclic , is equal to $\frac{1}{e_1} p_1 + \frac{1}{e_2} p_2 + \frac{1}{e_3} p_3$, where p_1, p_2, p_3 correspond to exceptional orbits. If G is a cyclic group, the Seifert divisor depends on the grading. For example, if $uv - w^{2e}$ is graded with $\deg u = \deg v = n, \deg w = 1$, it is equal to $\frac{1}{e}(p_1 + p_2)$.

A similar proof shows an inclusion of sheaves for all $i \in \mathbb{Z}$

$$\mathcal{O}_X(i) = \mathcal{O}_X(D_i) \subset \mathcal{O}_X(iD_1 + \sum \lfloor \frac{ia_s}{e} \rfloor E_s).$$

We use the isomorphisms

$$\mathcal{O}_X(i) \cong \mathcal{O}_X(i+e) \otimes \mathcal{O}_X(-e)$$

for $i < 0$. Using the normality of the rings $\mathcal{O}_{(f)}$, we conclude as above that there exists an isomorphism of the graded $\mathcal{O}_X$-Algebras

$$\bigoplus_{i\in\mathbb{Z}} \mathcal{O}_X(i) \cong \bigoplus_{i\in\mathbb{Z}} \mathcal{O}_X(\lfloor iD \rfloor) = \mathcal{A}(X, D). \tag{3.19}$$

We know that for any ample Cartier divisor D on a proper integral scheme proper over k the algebra $A(X, D)$ is a finitely generated normal algebra over k. This can be extended to any ample Cartier $\mathbb{Q}$-divisor D.

Proposition 3.3.5. *Let D be an ample Cartier $\mathbb{Q}$-divisor D on an integral scheme proper over a field k. Then the graded algebra $A(X, D)$ is a normal finitely generated algebra over k.*

Proof. Assume eD is a Cartier divisor. Then it is ample and hence the subalgebra $A(X, D)^{(r)}$ is finitely generated over k. As we remarked several times this implies that $A(X, D)$ is finitely generated. Since Proj $A(X, D) \cong$ Proj $A(X, D)^{(r)}$, we have $X =$ Proj $A(X, D)$. As always we have a canonical homomorphsm (3.11) $\rho : A(X, D) \to \Gamma_*(\mathcal{O}_X)$. By Proposition 3.2, it is enough to show that ρ is bijective. Since $A(X, D)^{(e)} = \Gamma_*(\mathcal{O}_X)^{(e)}$, the homomorphism ρ is injective. We have also a sheaf version of ρ, an injective homomorphism of graded $\mathcal{O}_X$-algebra

$$\tilde{\rho} : \mathcal{A}(X, D) \to \bigoplus_{i\in\mathbb{Z}} \mathcal{O}_X(i).$$

Let $\mathcal{O}_X(i) \cong \mathcal{O}_X(D_i)$ for some Weil divisor D_i. The injective homomorphisms $\tilde{\rho}_i : \mathcal{O}_X(iD) \to \mathcal{O}_X(i)$ imply that $\lfloor iD \rfloor \le D_i$. Following the proof of Theorem 3.3.4, we find a $\mathbb{Q}$-divisor D' such that $D_i \le iD'$ and $D_{ke} = keD'$. Thus we obtain $D \le D'$ and $eD = D_e = eD'$. This gives $D = D'$, and hence $A(X, D) = A(X, D')$. □

Example 3.3.6. Let R be a discrete valuation ring (i.e. a regular local ring of dimension 1). Let $\mathfrak{m}_R = (t)$ be its maximal ideal and $k = R/\mathfrak{m}_R$ be its residue field which we assume to be algebraically closed. Let $0 < q < e$ be a pair of coprime integers and $D = \frac{q}{e}p$ be a Weil $\mathbb{Q}$-divisor on $X =$ Spec R, where $p = V(\mathfrak{m})$. We assume that the characteristic of k does not divide q and e. Consider the R-module corresponding to the sheaf $\mathcal{O}_X(D)$ and the corresponding graded algebra $A(X, D)$. We would like to describe it explicitly.

Let $K = Q(R)$ be the fraction field of R and $L = K(t^{1/e})$ be a cyclic Kummer extension of K. The integral closure of R in L is a discrete valuation ring with maximal ideal $\mathfrak{m}_{R'} = (u)$, where $t = u^e$. By the Kummer theory, the Galois group of L/K is a cyclic group $\boldsymbol{\mu}_e(k)$ of order e generated by an element g which acts by sending u to ϵu, where ϵ is a primitive eth root of unity. It leaves R' invariant and the invariant ring $R'^{\boldsymbol{\mu}_e(k)}$ coincides with R.

Consider the action of $\boldsymbol{\mu}_e(k)$ on $R'[Z]$ by extending the action on R' to polynomials by requiring that Z is sent to $\epsilon^{-q}Z$. Let us compute the invariant ring A. Since $R' = R'[Z]_0$ and $R'^{\boldsymbol{\mu}_e(k)} = R$, we can consider A as a graded R-algebra

$$A = R'[Z]^{\boldsymbol{\mu}_e(k)} = \bigoplus_{i=0}^{\infty} A_i,$$

via the standard grading on the polynomial ring. We check immediately that

$$A_0 = R, \quad A_1 = u^q Z, \quad A_e = A_0 Z^e.$$

Thus, the image of the multiplication map $A_1^{\otimes e} \to A_e$ is equal to $(t^q)Z^e$. This shows that the corresponding $\mathbb{Q}$-divisor is $\frac{q}{e}P$.

Let X be any normal irreducible scheme and D be a Weil $\mathbb{Q}$-divisor D with the Seifert divisor $D - \lfloor D \rfloor = \sum \frac{q_i}{e_i} E_i$. The computations from above show that the sheaf of graded algebras $\oplus_{i=0}^{\infty} \mathcal{O}_X(iD)$, at a geometric generic point $\bar{\eta}_i$ of E_i looks like the $\mathcal{O}_{X,\bar{\eta}_i}(U, Z]^{\mu_e}$, where the action is given as above by the numbers q_i, e_i.

3.4 Cylinder constructions

Let $\mathcal{L}$ be an invertible sheaf on X. Consider the projective vector bundle $\mathbb{P}(\mathcal{L} \oplus \mathcal{O}_X)$.

It is convenient to identify $S^\bullet(\mathcal{L} \oplus \mathcal{O}_X)$ with the graded algebra $S^\bullet\mathcal{L}[t]$ locally isomorphic to the graded polynomial algebra $\mathcal{O}(U)[t_U, t]$. We have $t_U = g_{UV} t_V$, where g_{UV} are the transition functions for $\mathcal{L}$ but t can be chosen the same for all U. There are two natural open subsets in $\mathbb{P}(\mathcal{L} \oplus \mathcal{O}_X)$. One is defined by $t_U \neq 0$ and another by $t \neq 0$. The first set is locally isomorphic to Spec $\mathcal{O}(U)[t/t_U]$ and globally isomorphic to $\mathbb{V}(\mathcal{L}^{-1}) =$ Spec $S^\bullet\mathcal{L}^{-1}$. The second one is locally isomorphic to Spec $\mathcal{O}(U)[t_U/t]$ and globally isomorphic to $\mathbb{V}(\mathcal{L}) =$ Spec $S^\bullet\mathcal{L}$. The complement of the open subset $\mathcal{V}(\mathcal{L})$ is the *section at infinity*, i.e. the closed subscheme S_∞ isomorphic to X that is locally defined by $t = 0$. The complement of the open subset $\mathcal{V}(\mathcal{L}^{-1})$ is the *zero section*, i.e. the closed subscheme isomorphic to X that is locally defined by $t_U = 0$. The intersection of the two open subsets is locally isomorphic to Spec $\mathcal{O}(U)[t_U, t]_{(t_U t)} \cong \mathcal{O}(U)[z, z^{-1}]$, where $z = t_U/t$.

We can combine the two $\mathcal{O}_X$-algebras $S^\bullet\mathcal{L}$ and $S^\bullet\mathcal{L}^{-1}$ together to define a $\mathbb{Z}$-graded $\mathcal{O}_X$-algebra

$$\mathcal{A} = \bigoplus_{i \in \mathbb{Z}} \mathcal{L}^i.$$

Then

$$\mathcal{A}_+ := \bigoplus_{i \geq 0} \mathcal{L}^i \cong S^\bullet\mathcal{L}, \quad \mathcal{A}_- := \bigoplus_{i \leq 0} \mathcal{L}^i \cong S^\bullet\mathcal{L}^{-1}.$$

Thus the projective bundle $\mathbb{P}(\mathcal{L}\oplus\mathcal{O}_X)$ becomes isomorphic to the gluing together the open subsets Spec $\mathcal{A}_\pm$ along their common open subset Spec $\mathcal{A}$.

Assume now that $\mathcal{L}$ is an ample invertible sheaf on a complete scheme X over a field k. Then the ring $A = A(X,\mathcal{L}) = \Gamma(\mathcal{A}_+)$ is a finitely generated k-algebra with Proj $A \cong X$. We have a canonical morphism

$$p : \text{Spec } \mathcal{A}_+ \cong \mathbb{V}(\mathcal{L}) \to C = \text{Spec } A$$

corresponding to the restriction maps $A \to \mathcal{A}_{\geq 0}(U)$. Since

$$C^* \cong \text{Spec } \oplus_{i\in\mathbb{Z}} \mathcal{O}_X(i) \cong \text{Spec } \mathcal{A}$$

both schemes contain isomorphic open subsets, and it is easy to see that the isomorphism is equal to the restriction of p. Also it is easy to see that $p(S_0)$ is equal to the vertex of C. Thus we can say that the morphism p blows down the zero section S_0. If X is a smooth scheme over k, then Spec $\mathcal{A}_+$ is locally isomorphic to the affine line over X, hence is smooth too. Thus p is a *resolution of the vertex* of the affine quasicone C.

For example, let X be a closed subvariety of a projective space $\mathbb{P}^n_k$ over a field k and let $I_X = \Gamma_*(\mathcal{I}_X)$ be its homogeneous ideal in $k[T_0,\ldots,T_n]$. Consider the affine cone over X defined in Example 3.1.2. Let $\mathcal{L} = \mathcal{O}_X(1)$. It is an ample (in fact, very ample) invertible sheaf on X. In the case when X is projectively normal (see Remark 3.2.2), the ring $A = A(X,\mathcal{L})$ is isomorphic to the *homogeneous coordinate ring* $k[X] = k[T]/I_X$ of X and the affine cone C_X coincides with the affine quasicone Spec A. In general, Spec A is the normalization of C_X. Thus the composition $p : \mathbb{V}(\mathcal{L}) \to \text{Spec } A \to C_X$ is a partial resolution of the vertex of C_X (a resolution if X is smooth).

Consider I_X as a homogeneous ideal in $k[T_0,\ldots,T_{n+1}]$. The corresponding closed subvariety $\bar{C}_X$ of $\mathbb{P}^{n+1}_k$ is the *projective cone* over X. If we identify $\mathbb{A}^{n+1}_k$ with an open subset $T_{n+1} \neq 0$ of to the projective space $\mathbb{P}^{n+1}_k$, then the projective cone is the closure of the affine cone in $\mathbb{P}^{n+1}_k$. The hyperplane T_{n+1} intersects $\bar{C}_X$ at a closed subvariety of $\bar{C}_X$ isomorphic to X, its complement is the affine cone C_X.

The homomorphism of graded rings

$$S^\bullet(\mathcal{L}\oplus\mathcal{O}_X) \cong S^\bullet\mathcal{L}[t] \to k[\bar{C}_X] = k[T_0,\ldots,T_n][T_{n+1}]$$

defines a morphism $\bar{p} : \mathbb{P}(\mathcal{L}\oplus\mathcal{O}_X) \to \bar{C}_X$. Its restriction over C_X coincides with the composition $\mathbb{V}(\mathcal{L}) \to \text{Spec } A \to C_X$. It is a partial resolution of the vertex of $\bar{C}_X$.

We will show in the next Lecture that the partial resolution morphisms are the blowing-up morphisms with center at the vertex.

Next let X be a normal integral scheme over a field X and D be an ample Cartier $\mathbb{Q}$-divisor on X. Consider the $\mathbb{Z}$-graded $\mathcal{O}_X$-algebras

$$\mathcal{A}_X(D) = \bigoplus_{i\in\mathbb{Z}} \mathcal{O}_X(iD),$$

$$\mathcal{A}_X(D)^+ = \bigoplus_{i\geq 0}\mathcal{O}_X(iD), \quad \mathcal{A}_X(D)^- = \bigoplus_{i\leq 0}\mathcal{O}_X(iD),$$

Let

$$\pi_\pm : C(X,D)^\pm := \mathrm{Spec}\ \mathcal{A}_X(D)^\pm \to X, \quad \pi : C(X,D)^* := \mathrm{Spec}\ \mathcal{A}_X(D) \to X$$

be the corresponding affine schemes over X.

By definition,

$$A(X,D) \cong \Gamma(\mathcal{A}_X(D)^+) = \bigoplus_{i\geq 0}\Gamma(X,\mathcal{O}_X(iD)).$$

Let

$$C_X(D) = \mathrm{Spec}\ A(X,D).$$

By Proposition 3.3.5, $C_X(D)$ is an affine quasicone over k, $X \cong \mathrm{Proj}\ A(X,D)$ and $\mathcal{O}_X(iD) \cong \mathcal{O}_X(i)$. Thus $C(X,D)^*$ is isomorphic to the punctured affine quasicone $C_X(D)^*$ and the projection π coincides with the canonical map $\pi : C_X(D)^* \to X$.

Following M. Demazure, we call $C(X,D)^+$ the *affine cylinder* associated to the pair (X,D) or $(X,\mathcal{O}_X(D))$. In the case when D is an ample Cartier divisor, the affine cylinder of the pair (X,D) is the line bundle $\mathbb{V}(\mathcal{O}_X(D))$.

If rD is a Cartier divisor, then $\mathcal{A}_X(rD) = \mathcal{A}_X(D)^{(r)}$ coincides with the $\mathcal{O}_X$-algebra $\mathcal{A}$ from above, where $\mathcal{L} = \mathcal{O}_X(rD)$. We can extend the definition of the projective bundle $\mathbb{P}(\mathcal{L}\oplus\mathcal{O}_X)$ to define the *projective cylinder* associated to the pair (X,D):

$$\widehat{C}(X,D) = \mathrm{Proj}\ \mathcal{A}_X(D)_+[t].$$

Let

$$C(X,D)^\pm := \mathrm{Spec}\ \mathcal{A}_X(D)_\pm.$$

The inclusion of graded $\mathcal{O}_X$-algebras $\mathcal{A}_X(rD) = \mathcal{A}_X(D)^{(r)} \subset \mathcal{A}_X(D)$ defines finite morphisms

$$q_r^\pm : C(X,D)^\pm \to C(X,rD)^\pm.$$

They are glued together to define a finite morphism

$$q_r : \widehat{C}(X,D) \to \widehat{C}(X,rD) \cong \mathbb{P}(\mathcal{L}\oplus\mathcal{O}_X). \tag{3.20}$$

In particular, $\widehat{C}(X,D) \cong \mathbb{P}(\mathcal{O}_X(D)\oplus\mathcal{O}_X)$ if D is a Cartier divisor.

Consider a surjective homomorphism of $\mathcal{O}_X$-algebras $\phi : \mathcal{A}_X(D)_+[t] \to S^\bullet\mathcal{O}_X \cong \mathcal{O}_X[t]$ that corresponds to the surjection $\mathcal{A}_X(D)_+ \to \mathcal{A}_0 = \mathcal{O}_X$. It defines a morphism of schemes

$$s_+ : X \cong \mathrm{Proj}\ \mathcal{O}_X[t] \to \mathrm{Proj}\ \mathcal{A}_X(D)_+[t] = \widehat{C}(X,D)$$

(see (3.13)).

Also we can consider a surjective homomorphism of $\mathcal{O}_X$-algebras $\phi : \mathcal{A}_+[t] \to \mathcal{A}_X(D)_+$ with kernel (t). Passing to the projective spectra we get a morphism of $\mathcal{O}_X$-schemes

$$s_- : X \cong \text{Proj } \mathcal{A}_X(D)_+ \to \text{Proj } \mathcal{A}_+[t] = \widehat{C}(X, D).$$

We call the morphism s_+ (resp. s_-) the *zero section* (resp. *section at infinity*). We will often identify them with there images $S_+ = s_+(X)$ and $S_- = s_-(X)$.

Proposition 3.4.1.

$$C(X, D) \setminus S_\pm \cong C(X, D)^\mp.$$

Proof. Suppose that D is Cartier and let $\mathcal{L} = \mathcal{O}_X(D)$. Then s_+ corresponds to the surjection $S^\bullet(\mathcal{L} \oplus \mathcal{O}_X) \to S^\bullet \mathcal{O}_X$ defined by the projection $\mathcal{L} \oplus \mathcal{O}_X \to \mathcal{O}_X$. The section s_- corresponds to the surjection $S^\bullet(\mathcal{L} \oplus \mathcal{O}_X) \to S^\bullet \mathcal{L}$ defined by the projection $\mathcal{L} \oplus \mathcal{O}_X \to \mathcal{L}$. We have already explained in the beginning of the section that $\mathbb{P}(\mathcal{L} \oplus \mathcal{O}_X) \setminus s_\pm(X) = \mathbb{V}(\mathcal{L}^{\mp 1})$.

Consider the morphism $q_r : \widehat{C}(X, D) \to \widehat{C}(X, rD)$ from (3.20), where rD is Cartier. It follows from the definition of the sections $s_\pm$ that the composition of $X \xrightarrow{s_\pm} \widehat{C}(X, D) \xrightarrow{q_r} \widehat{C}(X, rD)$ is the zero section (resp. the section at infinity) of $\widehat{C}(X, rD)$. The assertion follows from this. □

Over an open affine set $U \subset X$ this morphism is obtained from the homomorphism of the graded $\mathcal{O}_X(U)$-algebras

$$\mathcal{A}_X(rD)_+(U)[t] \cong \mathcal{O}_X(U)[t_U, t] \to \mathcal{A}_X(D)_+(U)[t]^{(r)},\ t_U^i t^j \mapsto t_U^i t^{rj},$$

where $t_U^i \in \mathcal{A}_X(rD)_i = \mathcal{A}_X(D)_{ri}$. The pre-image of the closed subscheme S^+ (resp. S^-) of $\widehat{C}(X, rD)$ defined locally by $t_U = 0$ (resp. $t = 0$) is a closed subscheme of $\widehat{C}(X, D)$ whose reduced subscheme $S(X, D)^+$ (resp. $S(X, D)^-$) is isomorphic to X. The subschemes $S(X, D)^\pm$ are the images of the sections $s_\pm : X \to \widehat{C}(X, D)$ of the canonical projection $\widehat{C}(X, D) \to X$. It follows from Proposition 3.3.5 that the algebra $A(X, D) = \Gamma(\mathcal{A}_X(D)_+)$ is a normal finite generated algebra. Let $C_X(D) = \text{Spec } A(X, D)$ be the affine quasicone corresponding to the algebra $A(X, D)$. Then $C(X, D)^* = \oplus_{i \in \mathbb{Z}} \mathcal{O}_X(i) \cong \text{Spec } \mathcal{A}_X(D)$. The restriction map $A(X, D) = \Gamma(\mathcal{A}_X(D)_+) \to \mathcal{A}_X(D))_+(U)$ define a morphism over X

$$p : C(X, D)^+ \to C_X(D).$$

Its restriction to Spec $A(X, D)$ is an isomorphism onto $C_X(D)^*$. Again we see that p can be viewed as the blow-down of the section $S(X, D)^+$ to the vertex of the affine quasicone $C_X(D)$. However, even when X is smooth, the scheme $C_X(D)^+$ is not necessarily smooth.

To summarize we have

$$\widehat{C}(X, D) = S(X, D)^+ \bigsqcup S(X, D)^- \bigsqcup C_X(D)^*$$

$$= C(X, D)^+ \bigsqcup S(X, D)^- = C(X, D)^- \bigsqcup S(X, D)^+.$$

Proposition 3.4.2. *$\widehat{C}(X,D)$ is a projective normal variety over k.*

Proof. We know that $\widehat{C}(X,D)$ is a finite over $\widehat{C}(X,rD)$, where rD is a Cartier divisor. Since $\widehat{C}(X,rD)$ is a projective bundle over X, $\widehat{C}(X,D)$ is projective over X. Since $\widehat{C}(X,D)$ is covered by the open subsets $C(X,D)^+$ and $C(X,D)^-$, it suffices to prove the normality of these subsets. Let $U = \text{Spec } A$ be an affine open subset of X and $C(U,D)^+ = \pi_+^{-1}(U)$. If we take U small enough such the divisor $D|U$ is Cartier, then $C(U,D)^+ \cong \mathbb{A}^1_U$. So we can identify the coordinate ring of $C(U,D)^+$ with the subring of the field $K = Q(A)(t)$ that consists of rational functions of the form $\sum_{n\geq 0}\phi_n t^n$, where $\text{div}(\phi_n) + nD|U \geq 0$. Let $D = \sum_{x\in X^{(1)}} m_x E_x$, where E_x is the closure of x in X. Let ν_x be the valuation of $Q(A)$ defined by x and ν_x' be the valuation of K defined by $\nu_x'(\sum \phi_n t^n) = \inf\{\nu_x(\phi_n) + nm_x\}$. Then $C(U,D)^+$ is the intersection of the discrete valuation rings $\nu_x^{-1}(\mathbb{Z}_{\geq 0})$ for all $x \in U^{(1)}$, and hence is normal. Similarly we prove that $C(X,D)^-$ is normal. □

Remark 3.4.3. We know that the open subset $C(X,D)^+$ of $\widehat{C}(X,D)$ admits a morphism $p : C(X,D)^+ \to C_X(D)$ and it is an isomorphism over $C_X(D)^*$. One can extend the morphism p to the whole $\widehat{C}(X,D)$ by introducing the *projective quasicone* $\bar{C}_X$ that contains $C(X,D)^-$ as its open subset whose complement is equal to the vertex. It is defined by

$$\bar{C}_X := \text{Proj } A(X,D)[t],$$

where the grading in $A(X,D)[t]$ is defined by $A(X,D)[t]_n = \sum_{i+j=n} A(X,D)_i t^j$. The graded ideal (t) defines a closed embedding $\tau : X \hookrightarrow \bar{C}_X$. Its complement is the affine set $D_+(t) = \text{Spec } A(X,D)[t]_{(t)} \cong \text{Spec } A(X,D) = C_X(D)$.

The inclusion of the vertex $x_0 = V(\mathfrak{m}_0)$ of $C_X(D)$ in $C_X(D) \subset \bar{C}_X$ corresponds to passing to the projective spectra under the canonical homomorphism $A(X,D)[t] \to A(X,D)/\mathfrak{m}_0 \cong k[t]$. The natural inclusion $A(X,D) \subset A(X,D)[t]$ defines, after passing to the projective spectra, a morphism $f : \bar{C}_X(D)^* := \bar{C}_X \setminus \{x_0\} \to X$. Over an open subset $D_+(f)$ of X the morphism corresponds to the inclusion $A(X,D)_{(f)} \subset A(X,D)[t]_{(f)}$. However, $A(X,D)[t]_{(f)} \cong (A(X,D)_f)_-$ and we get an isomorphism of X-schemes $\bar{C}_X(D)^* \cong \text{Spec } \mathcal{A}_X(D)^- = C(X,D)^-$.

If $X = \mathbb{P}^n_k = \text{Proj } k[T_0,\ldots,T_n]$ and $\mathcal{O}_X(D) \cong \mathcal{O}_X(1)$, then $A(X,D) = k[T_0,\ldots,T_n]$ and $A(X,D)[t] \cong k[T_0,\ldots,T_n,T_{n+1}]$. Thus $\bar{C}_X(D)] \cong \mathbb{P}^{n+1}_k$. More generally if X is a closed subvariety on $\mathbb{P}^n_k$ given by a homogeneous ideal $I \subset k[T_0,\ldots,T_n]$, and $\mathcal{O}_X(D) \cong \mathcal{O})X(1)$, then $\bar{C}_X(D)$ is the closed subvariety of $\mathbb{P}^{n+1}_k$ given by the ideal I.

Let us study the morphism $\bar{\pi} : \widehat{C}(X,D) \to X$ in more details.

Lemma 3.4.4. *Let $A = \oplus_{i\in\mathbb{Z}} A_i$ be a finitely generated graded algebra over$R = A_0$. Suppose that the multiplication map $\mu_n : A_n \otimes A_n \to A_0$ is an isomorphism. Then $A_n = a_n A_0$ for some invertible element $a_n \in A_n$ and $A_{-n} = a_n^{-1}A_0$. Moreover, the multiplication map*

$$A_i \otimes A_n \to A_{n+i} \tag{3.21}$$

is an isomorphism for all $i \in \mathbb{Z}$*.*

Proof. The multiplication map $\mu : A_n \otimes A_n \to R$ defines a natural homomorphism $A_{-n} \to A_n^* := \mathrm{Hom}_R(A_n, A_0)$. The map μ is the composition of the map $A_{-n} \otimes A_n \to A_n^* \otimes A_n$ and the natural evaluation map $\phi : A_n^* \otimes A_n \to R$. Since μ_n is an isomorphism, the map ϕ is an isomorphism and hence A_n is an invertible A_0-module (see Theorem 11.6 from [Eisenbud]). The second assertion is obvious. □

Definition 3.4.1. The *Cartier index* of a $\mathbb{Q}$-Cartier divisor D at a point $x \in X$ is the smallest positive integer $e(x)$ such that $e(x)D$ is a Cartier divisor at the point x.

It is clear that $e(x) = 1$ if and only if D is a Cartier divisor at x. The set of points $x \in X$ such that $e(x) > 1$ is a closed subset $\mathrm{nc}(D)$ of X called the *non-Cartier locus* of DIt contains the support of the Seifert divisor $D - \lfloor D \rfloor$ of D. Since X is normal, its complement in $\mathrm{nc}(D)$ is of codimension ≥ 2.

Proposition 3.4.5. *Let* $x \in X$*, the reduced fibre* $\bar{\pi}^{-1}(x)$ *of the morphism* $\bar{\pi} : \widehat{C}(X, D) \to X$ *is isomorphic to the projective line* $\mathbb{P}^1_{\kappa(x)}$ *over the residue field of* x*. The fibre is reduced if and only if the divisor* D *is a Cartier divisor at* x*. The reduced fibre of the morphism* $\pi_\pm : C(X, D)^\pm \to X$ *is isomorphic to* $\mathbb{A}^1_{\kappa(x)}$ *and the reduced fibre of the morphism* $\pi : C(X, D)^* \to X$ *is isomorphic to* $\mathbb{G}_{m,\kappa(x)}$*.*

Proof. Let $\mathcal{O}_{X,x}$ be the local ring of X at x and $\pi_x : Y(x) \to \mathrm{Spec}\ X(x) = \mathcal{O}_{X,x}$ be the morphism obtained from π by the base change $\mathrm{Spec}\ \mathcal{O}_{X,x} \to X$. The fibre of π over x is isomorphic to the fibre of π_x over the unique closed point of $X(x)$. Since $C(X, D) = \mathrm{Spec}\ \oplus_{i \in \mathbb{Z}} \mathcal{O}_X(i)$, we have $Y(x) = \mathrm{Spec}\ \oplus_{i \in \mathbb{Z}} \mathcal{O}_X(i)_x$. Let e be the Seifert multiplicity of D at x. Suppose $a \in \mathcal{O}_X(j)_x$ is a non-zero element, where $0 < j < e$. We have $a^{ej} = r f_e^j \in \mathcal{O}_X(je)_x$, where f_e is an invertible generator of $\mathcal{O}_X(e)_x$ and $r \in \mathcal{O}_{X_x}$. If r is invertible in the local ring $\mathcal{O}_{X_x}$, then a^{ej} is invertible and hence a^j is invertible. This contradicts the definition of e. Reducing modulo the maximal ideal $\mathfrak{m}_{X,x}$ of $\mathcal{O}_{X_x}$ we obtain that

$$\mathcal{O}_X(j)(x) := \mathcal{O}_X(j)_x \otimes \kappa(x) = \mathcal{O}_X(j)_x / \mathfrak{m}_{X,x}\mathcal{O}_{X,x}$$

consists of nilpotent elements. Thus $N = \oplus_{0<j<e} \mathcal{O}_X(j)(x)$ generate the nilpotent ideal of the algebra $\oplus_{i \in \mathbb{Z}} \mathcal{O}_X(i)(x)$ with the quotient isomorphic to $\oplus_{i \in e\mathbb{Z}} \mathcal{O}_X(i)(x) \cong \kappa(x)[t, t^{-1}]$, where t corresponds to an invertible generator of $\mathcal{O}_X(e)_x$. Since $Y(x) \cong \mathrm{Spec}\ \oplus_{i \in \mathbb{Z}} \mathcal{O}_X(i)$ we obtain that the reduced fibre $\pi^{-1}(x)_{\mathrm{red}}$ is isomorphic to $\mathbb{G}_{m,\kappa(x)} = \mathrm{Spec}\ \kappa(x)[t, t^{-1}]$. Similarily we see that the reduced fibre of $\pi_\pm : C(X, D)^\pm \to X$ over x are isomorphic to $\mathrm{Spec}\ \kappa(x)[t^\pm]$ and the reduced fibre of $\bar{\pi}$ over x is isomorphic to $\mathbb{P}^1_{\kappa(x)}$. □

Let A be a normal finitely generated graded algebra over a field k. By Theorem 3.3.4, $A \cong A(X, D)$ for some ample $\mathbb{Q}$-Cartier divisor on $X = \mathrm{Proj}\ A$

defined uniquely up to linear equivalence. Thus the constructions of the schemes $C(X,D)^{\pm}, \widehat{C}(X,D)$ apply to A and define the schemes

$$C_A = C_X(D),\ X = \text{Proj } A,\ \widehat{C}_A = \widehat{C}(X,D),\ C_A^{\pm} = C(X,D)^{\pm},\ C_A^* = C(X,D),$$

and the corresponding morphisms

$$p : C_A^+ \to C_A,\ \pi : C_A^* \to X,\ \pi_{\pm} : C_A^{\pm} \to X,\ \bar{\pi} : \widehat{C}_A \to X$$

We also have the sections

$$s_+(X) = S_0, \quad s_-(X) = S_\infty$$

of $\bar{\pi}$.

3.5 Exercises

3.1 Give a geometric interpretation (in terms of the $\mathbb{G}_m$-action) of the condition that an ideal in a graded ring is homogeneous.

3.2 Let $A = \mathbb{C}[x,y]$ be the polynomial algebra graded by $\deg x = a, \deg y = b$, where $1 \le a < b$ are coprime integers. Find a rational divisor D on $X = \text{Proj } k[x,y] \cong \mathbb{P}^1$ such that $A = A(X,D)$.

3.3 Finish computations from Example 3.3.2 for the remaining binary polyhedral groups (except the case of odd cyclic groups).

3.4 Let $A = \mathbb{C}[u,v,w]/(uv - w^e)$, where $e > 1$ is odd. grade A by $\deg u = i, \deg v = e - i, \deg w = 1$. Find the corresponding rational divisor D (it will depend on i). .

3.5 Let $X = \mathbb{P}^1$, $D = -2P + \frac{1}{2}P_1 + \frac{2}{3}P_1 + \frac{6}{7}P_3$, where P_1, P_2, P_3 are distinct points. Show that graded algebra $A(X,D)$ is isomorphic to $\mathbb{C}[u,v,w]/(u^2 + v^3 + w^7)$ with grading defined by $\deg u = 21, \deg v = 14, \deg w = 6$.

3.6 Let $E = V(F(x,y,z)) \subset \mathbb{P}^2$ be a plane cubic curve. Show that the graded algebra $k[x,y,z,w]/(F(x,y,z) + w^2)$ graded with weights $2,2,2,3)$ is isomorphic to the graded algebra $A(X,D)$, where $D = -H + \frac{1}{2}E$ and H is a line in the plane.

3.7 Compute the index of the polynomial algebra $K[T_1,\ldots,T_n]$ graded with $\deg T_i = q_i > 0$.

3.8 Show that a one-dimensional normal graded finitely generated algebra $A = \oplus_{n=0}^{\infty}$ over a field $k = A_0$ is isomorphic to the polynomial algebra $k[x]$.

3.9 Let $C = \text{Spec } A$ be an affine quasicone and $C^{(e)} = \text{Spec } A^{(e)}$. Make sense of the isomorphism $C^{(e)} \cong C/\boldsymbol{\mu}_e$ and use this isomorphism to give a geometric proof of the existence of an isomorphism $\text{Proj } A \cong \text{Proj } A^{(e)}$.

3.10 For any graded algebra $A = \oplus_{i \ge 0} A_i$ and any $n \ge 0$, set

$$A_{[n]} = \bigoplus_{i=n}^{\infty} A_i \subset A$$

(i) Show that $A_{[n]}$ is a homogeneous ideal in A and

$$A^\sharp := \bigoplus_{n=0}^{\infty} A_{[n]}$$

is a graded A_0-algebra.

(iii) Let $C_A^+ = \text{Proj } A^\sharp$. Let $A = A_{[0]} \hookrightarrow A^\sharp$ be inclusion map. Show that it defines a morphism $p : C_A^+ \to C_A = \text{Spec } A$;

(iv) Let $A \to A^\sharp$ be a homomorphism of graded rings defined by sending a homogeneous element $f \in A_n$ to the corresponding element $f^\sharp \in A_n \subset A_{[n]}$. Show that this homomorphism defines a morphism $\pi : C_A^+ \to \text{Proj } A$.

(v) Let A be a normal finitely generated algebra over a field k and $X = \text{Proj } A$ and $C = \text{Spec } A$ be the affine quasicone. Show that the morphisms $q : C_A^+ \to C_A$ and $\pi : C_A^+ \to X$ coincide with the morphisms $p : C_A^+ \to C_A, \pi : C_A^+ \to X$ defined at the end of the section.

(vi) Assume that A is as above and is generated by A_1. Show that C_A^+ is isomorphic to the blow-up of the irrelevant ideal $\mathfrak{m}_0$ of A.

Lecture 4

Resolution of singularities

4.1 The blow-up schemes

We assume that all schemes we will be considering satisfy one of the following types.

(i) a reduced scheme of finite type over a field k (an algebraic variety);

(ii) the spectrum of a local ring A isomorphic to a localization of a finitely generated algebra over a field;

(iii) the spectrum of the formal completion of a Noetherian local ring from (ii);

(iv) a scheme of finite type over a local ring from (ii) or (iii).

In particular all our schemes are Noetherian.

Let Y^{reg} be the set of points $y \in Y$ such that the local ring $\mathcal{O}_{Y,y}$ is regular. Under the above assumptions this set is an open dense subset of Y. Its complement is denoted by Y^{sing} and is called the *singular locus* of Y. Its points are called *singular points* of A.

Recall that a morphism $f : X \to Y$ of schemes is called a *resolution of singularities* if

- X is regular;
- f is proper;
- f is an isomorphism over Y^{reg} .

The set-theoretical pre-image of Y^{sing} is called the *exceptional locus* of the resolution.

A resolution is called *minimal* if it does not factor (non-trivially) into another resolution of X.

Let S be a Noetherian scheme and Z be its closed subscheme defined by a coherent sheaf of ideals $\mathcal{I}_Z$ on S. Recall that the *blow-up of* S with center Z is the S-scheme

$$B_Z(S) = \bigoplus_{n=0}^{\infty} \mathcal{I}_Z^n,$$

where the multiplication $\mathcal{I}_Z^n \otimes \mathcal{I}_Z^m \to \mathcal{I}_Z^{n+m}$ is defined by the multiplication in $\mathcal{O}_S$. It comes with the canonical projection $\pi : B_Z(S) \to S$. The scheme-theoretical pre-image $\pi^{-1}(Z)$ is equal to

$$E_Z = \bigoplus_{n=0}^{\infty} \mathcal{I}_Z^n / \mathcal{I}_Z^{n+1}.$$

It is a closed subscheme of $B_Y(X)$ defined by the sheaf of ideals $\pi^{-1}(\mathcal{I}_Z)$ (the image of $\pi^*(\mathcal{I}_Z) = \mathcal{O}_{B_Z(X)} \otimes_{\mathcal{O}_X} \mathcal{I}_Z$ in $\mathcal{O}_{B_Z(S)}$ under the multiplication map).

Locally, over an affine open set $U = \text{Spec } A$, we have $\pi^{-1}(U) = \text{Proj } \oplus_{n\geq 0} I^n$, where I is the ideal in A defining$U \cap Z$. Taking some generators $f_0, \ldots, f_N$ of I, we obtain a surjection of graded rings $A[T_0, \ldots, T_N] \to \oplus_{n\geq 0} I^n$ that defines a closed embedding $B_Z(S) \cap U \hookrightarrow \mathbb{P}_A^N$. Thus the projection $\pi : B_Z(S) \to S$ is a projective morphism.

It follows from the definition that $\mathcal{O}_{B_Z(S)}(-1)$ corresponds to the ideal $\mathcal{I}_Z \oplus \mathcal{I}^2 \oplus \ldots$ in $\mathcal{O}_S \oplus \mathcal{I}_Z \oplus \ldots$. Since the algebra $\bigoplus_{n=0}^{\infty} \mathcal{I}_Z^n$ is generated by the degree 1 part $\mathcal{I}_Z$, we see that $\mathcal{O}_{B_Z(S)}(-1)$ is an invertible sheaf on $B_Z(S)$. It is the ideal sheaf defining E_Z. The reduced Cartier divisor $(E_Z)_{\text{red}}$ is called the *exceptional divisor* of $B_Z(S)$.

The following proposition shows that the property that $\pi^{-1}(Z)$ is a Cartier divisor characterizes $B_Z(S)$.

Proposition 4.1.1. *Let $\pi : B_Y(X) \to X$ be the blow-up of a closed subscheme Y. Suppose $f : X' \to X$ is a morphism such that $f^{-1}(Y)$ is a Cartier divisor D' on X'. Then there exists a unique morphism of X-schemes $g : X' \to B_Y(X)$ such that $g^*(D) = D'$.*

Another important property of the blow-up schemes is the following.

Proposition 4.1.2. *Let $f : S' \to S$ be a morphism and $Z' = f^{-1}(Z)$ be the scheme-theoretical pre-image of a closed subscheme Z in S'. Then there is a unique canonical map of $\tilde{f} : B_{Z'}(S') \to B_Z(S)$ compatible with the blow-up projections. If f is a closed (open) embedding than $\tilde{f}$ is a closed (open) embedding.*

It follows immediately from Propositions 4.1.1 and 4.1.2 that $\pi : B_Z(S) \to S$ is an isomorphism over the open subset U of S such that $\mathcal{I}_Z|U$ is an invertible sheaf (i.e. $Z \cap U$ is an effective Cartier divisor). In particular, the blow-up morphism is a projective birational morphism. Conversely, every projective birational morphism $f : S' \to S$ is isomorphic to the blow-up morphism of some closed subscheme on S (see [Hartshorne], Chapter II, Theorem 7.17).

In the case when f is a closed embedding, $B_{f^{-1}(Z)}(S')$ is equal to the *proper inverse transform* of S' in $B_Z(S)$. One takes the pre-image of $f(S') \setminus f(S') \cap Z$ in $B_Z(S)$ and takes its schematical closure in $B_Z(S)$.

We will use the following fundamental result.

Theorem 4.1.3. *Let S be a 2-dimensional scheme. Then there exists a resolution of singularities $\pi : \tilde{S} \to S$. It can be obtained as a composition of normalizations and blow-ups of ideals of closed points. A resolution is minimal if and only if its exceptional locus does not contain a smooth rational curve with self-intersection -1. A minimal resolution is unique up to isomorphism.*

In the case of algebraic surfaces this result goes back to Italian algebraic geometry. The general case is proved by J. Lipman (Publ. lIHES, vol. 36).

A *singularity* is a pair (S, s) where S is a scheme as above and s is its point. We say that two singularities $(S, s), (S', s')$ are *formally isomorphic* if there exists an isomorphism of the formal completions of the local rings $\widehat{\mathcal{O}}_{S,s} \cong \widehat{\mathcal{O}}_{S',s'}$. Let $\widehat{S}(s) = \text{Spec } \widehat{\mathcal{O}}_{S,s}$. It admits a natural morphism $\hat{i}_s : \widehat{S}(s) \to S$ such that the image of the closed point is equal to s. Let $\pi : \tilde{S} \to S$ be a resolution of singularities and

$$\hat{\pi}_s : \tilde{S} \times_S \widehat{S}(s) \to \widehat{S}(s)$$

be the base change. One can show that the morphism $\hat{\pi}_s$ is a resolution of singularities. If two surface singularities (S, x) are formally isomorphic, then the local schemes $\widehat{S}(s)$ and $\widehat{S'}(s')$ are isomorphic, and hence their minimal resolutions are isomorphic. This implies that the exceptional curves of the minimal resolutions of formally isomorphic singularities (S, s) are isomorphic curves and their intersection matrices coincide (after fixing a bijection between the sets of irreducible components).

Example 4.1.4. Let $A = k[x_1, \ldots, x_n]$ and $I = (x_1, \ldots, x_r)$. Let $S = \text{Spec } A, Z = \text{Spec } A/I$. We have a surjective homomorphism of A-algebras

$$k[x_1, \ldots, x_n][t_0, \ldots, t_{r-1}] \to B_Z(S), \quad y_i \mapsto x_i.$$

Its kernel is generated by elements $x_i t_j - x_j t_i, |i - j| \neq 1$. This follows from the exactness of a Koszul resolution for the ideal $(x_1, \ldots, x_r)$ ([Hartshorne], p.245). Thus $B_Z(S)$ is isomorphic to a closed subvariety of $\mathbb{P}_k^{r-1} \times \mathbb{A}_k^n$ given by the equations expessing the condition

$$\text{rank} \begin{pmatrix} x_1 & \ldots & x_r \\ t_0 & \ldots & t_{r-1} \end{pmatrix} = 1 \tag{4.1}$$

In the affine open set $t_0 \neq 0$ isomorphic to $\mathbb{A}_k^{n+r-1}$ with affine coordinates $x_1, \ldots, x_n, u_1, \ldots, u_{r-1}$, where $u_i = y_i/y_0, i = 1, \ldots, r-1$, it is given by the equations

$$x_{i+1} = u_i x_1, \ i = 2, \ldots, r.$$

Similar formulas can be given in affine open subsets $t_i \neq 0, i \neq 0$.

One can interpret (4.1) as follows. Consider a rational map

$$T:\mathbb{A}_k^n - \to \mathbb{P}_k^{n-1}, (t_1,\ldots,t_n)\mapsto (t_1:\ldots:t_n).$$

This map is undefined at the point $0=(0,\ldots,0)$. Let Γ_T be the graph of T, i.e. the closure of the graph of $T|\mathbb{A}_k^n\setminus\{0\}\to\mathbb{P}_k^{n-1}$ in $\mathbb{A}_k^n\times\mathbb{P}_k^{n-1}$. Then the graph Γ_T is isomorphic to $B_Z(S)$.

Example 4.1.5. Assume that X is a regular scheme and Y is a locally complete intersection in X. This means that the sheaf $\mathcal{I}_Y/\mathcal{I}_Y^2$ (called the *conormal sheaf* of Y) is a locally free sheaf of $\mathcal{O}_Y$-modules of rank equal to the codimension of Y in X and the sheaf of algebras

$$\mathrm{gr}_{\mathcal{I}_Y}(\mathcal{O}_X)=\bigoplus_{n=0}^{\infty}\mathcal{I}_Y^n/\mathcal{I}_Y^{n+1}$$

is isomorphic to the symmetric algebra $S^\bullet(\mathcal{I}_Y/\mathcal{I}_Y^2)$. In this case, $B_Y(X)$ is a regular scheme and the exceptional divisor E is isomorphic to the projective bundle $\mathbb{P}(\mathcal{I}_Y/\mathcal{I}_Y^2)$ It is called the *normal bundle* of Y in X. Let E be the exceptional divisor. The invertible sheaf $\mathcal{O}_E(-E)$ on E coincides with the sheaf $\mathcal{O}_E(1)$ of the projective bundle. All of this can be found in [Hartshorne], Chapter 2, Theorem 8.24.

Example 4.1.6. Let $C=\mathrm{Spec}\ A$ be a normal affine quasicone, $V(\mathfrak{m}_0)$ be its vertex, $X=\mathrm{Proj}\ A$, and $p_+:C^+\to C$ be the partial resolution of C defined by the affine cylinder of $(X,\mathcal{O}_X(1))$. The blow-up of the ideal $\mathfrak{m}_0$ is the projective spectrum of the graded A-algebra $B_{\mathfrak{m}_0}(A)=\oplus_{n\geq 0}\mathfrak{m}_0^n$. Let $A^\sharp$ be the graded A-algebra defined in Exercise 3.10. We have the inclusion of graded algebras $B_{\mathfrak{m}_0}(A)\subset A^\sharp$. This defines a birational map over C, in general, with indeterminacy points

$$f:C^+ - \to \mathrm{Proj}\ B_{\mathfrak{m}_0}(A)$$

If f A is generated by degree 1 elements, then the two algebras coincide and f is an isomorphism.

Example 4.1.7. Let us show how to resolve singularities of affine normal surfaces S given by equation $z^2+f(x,y)=0$. We consider it as a surface over $\mathbb{A}^2$ by using the projection to the x,y coordinates. We assume that $(0,0,0)$ is the only singular point of the surface. In particular, $(0,0)$ is the only singular point of the plane curve $f(x,y)=0$. The idea (based on a more general idea due to H. Jung) is to define a sequence of the blow-ups $\pi:V\to\mathbb{A}^2$ with centers at points until the full transform of $f(x,y)=0$ becomes a divisor locally given by equation $u^av^b=0$. Then the base change $S\times_{\mathbb{A}^2}V\to V$ has local equation $z^2+u^av^b=0$. After normalization the equation becomes $z^2+u^\alpha v^\beta=0$, where $\alpha,\beta\leq 1$. If $\alpha\beta=0$, the surface is nonsingular. Otherwise we blow-up the singular point again, take the base change and normalize. After this the branch locus becomes nonsingular and the double cover is a nonsingular surface.

To describe the exceptional divisor of the total resolution $f : S' \to S$ we use that the blow-up of a nonsingular point in $\mathbb{A}^2$ has the exceptional divisor E isomorphic to $\mathbb{P}^1$ and its self-intersection is equal to -1 (a *(-1)-curve*, or the *exceptional curve of the first kind*). The latter follows from the observation that in the local chart given by the equation $x_2 = ux_1$ the divisor of the function x_2 is equal to the union of the curve $F : u = 0$ and $E : x_1 = 0$ which intersect transversally at one point. Thus $(E + F, E) = E^2 + E \cdot F = 0$ implies $E^2 = -1$. The same argument shows that the self-intersection of the strict transform of a complete curve R on a nonsingular surface under the blow-up of a point on it decreases by m^2, where m is the multiplicity of the curve at the point.

We will also use that the pre-image of a complete curve R under the double cover of nonsingular surfaces is equal to $2\bar{E}$ if E is contained in the branch locus or $\bar{E}$ otherwise, and $\bar{E}^2 = R^2/2$ in the first case and $\bar{E}^2 = 2R^2$ in the second case.

Let us show how it works in the case $f(x, y) = x^3 + y^3$. This is the affine quasicone corresponding to the Klein surface of type D_4. Let $\pi_1 : V_1 \to \mathbb{A}^2$ be the blow-up of the origin. We make the coordinate change $x = yu$ to get the local equation $y^3(u^3 + 1) = 0$ of the pre-image of the curve in the first chart of the blow-up. Similarily, in the second chart, the coordinate change is $y = xv$ and the equation is $x^3(1+v^3) = 0$. After the base change and the normalization the equation of the strict transform of the surface is $z^2 + y(u^3 + 1) = 0$ (resp. $z^2+x(1+v^3) = 0$.) The branch locus consists of the exceptional curve E_0 of the first blow-up and the three non-complete curves with local equation $u + \alpha = 0$, where $\alpha^3 + 1 = 0$. They intersect the curve E_0 transversally. We blow-up the intersection points. The strict transform of E_0 has the self-intersection equal to -4. The exceptional curves of the three blow-ups are curves R_1, R_2, R_3 with self-intersection -1. Now the branch locus is nonsingular. The double cover becomes a nonsingular surface which resolves the singularity. The exceptional divisor is the union of 4 curves $\bar{R}_i, i = 0, \ldots, 3$ with $\bar{R}_i^2 = -2$ and the intersection matrix is described by the Dynkin diagram of type D_4.

4.2 Cyclic quotient surface singularities

Let us show how to resolve cyclic quotient surface singularities. Let k be a field such that $\mu_n(k)$ is a group of order n and ϵ be its generator (the group of roots of unity in a field is always cyclic). Let

$$C_{n,q} = \text{Spec } k[z_1, z_2]^{\mu_e}, \quad z_1 \mapsto \epsilon z_1, z_2 \mapsto \epsilon^q z_2.$$

We assume that $(n, q) = 1$. Otherwise, let $d = (n, q), n = du, q = dv$. Then $\epsilon^{qu} = 1$, hence $k[z_1, z_2]^{\epsilon^u} = k[z_1^d, z_2] \cong k[t_1, t_2]$ and $C_{n,q} \cong \text{Spec } k[t_1, t_2]^{\mu_u} \cong C_{u,q \mod u}$.

An invariant polynomial is a sum of invariant monomials. A monomial $z_1^a z_2^b$ is invariant if and only if

$$a + bq \equiv 0 \mod n. \quad (4.2)$$

The set of $(a,b) \in \mathbb{Z}^2_{\geq 0}$ satisfying this congruence is a submonoid M of the monoid $\mathbb{Z}^2_{\geq 0}$ with respect to vector addition. The monoid algebra $k[M]$ is isomorphic to the polynomial algebra $k[T_1, T_2]$. The monoid algebra of M is its finitely generated subalgebra. It is easy to see that the monoid M is isomorphic to the monoid

$$M' = \{m = (m_1, m_2) \in \mathbb{Z}^2_{\geq 0} : -qm_1 + nm_2 \geq 0\} \tag{4.3}$$

(map a solution (a,b) of (4.3) to $(m_1, m_2) = (b, \frac{a+bq}{n})$).

Consider the cone σ in $\mathbb{R}^2$ spanned by the vectors

$$v = (-q, n), \quad w = (1, 0).$$

Let $\check{\sigma}$ be the *dual cone* of vectors $x \in \mathbb{R}^2$ such that $x \cdot y \geq 0$ for all $y \in \sigma$. It is spanned by vectors $(0,1)$ and (n,q). We have

$$M' = \check{\sigma} \cap \mathbb{Z}^2.$$

It is easy to see that, if v_1, v_2 are linearly independent, then X_σ is nonsingular if and only if v, w is a basis in $\mathbb{Z}^2$. In this case $X_\sigma \cong \text{Spec } k[\mathbb{Z}^2_{\geq 0}] \cong\cong \mathbb{A}^2$. One can resolve X_σ by choosing primitive integer vectors $v_1, v_2, \ldots, v_s$ inside of σ such that the cones $\sigma_i = \langle v_i, v_{i+1}\rangle$ generated by $v_i, v_{i+1}, i = 0, \ldots, s$ form a basis of $\mathbb{Z}^2$ (here $v_0 = v, v_{s+1} = w$). Since $\check{\sigma} \subset \check{\sigma}_i$ defines the inclusion of the monoid algebras, each X_{σ_i} is mapped to X_σ. The collection Σ of cones σ_i satisfy the property that two cones intersect along a common face (such a collection is called a *fan*). The affine X_σ-schemes X_{σ_i} can be glued together to a *toric variety* X_Σ which defines a toric resolution $\pi : X_\Sigma \to X_\sigma$. The field of rational functions on each X_{σ_i} can be identified with the fraction field of the group algebra $k[\mathbb{Z}^2] \cong k[t_1^{\pm 1}, t_2^{\pm 1}]$. For each monomial $t^m = t_1^{m_1} t_2^{m_2}$, we have

$$\text{div}(t^m) = \sum (m \cdot v_i) D_i, \tag{4.4}$$

where D_i is the closure in X of the curve equal to the image of Spec $k[\mathbb{Z}^2 \cap (\mathbb{Z}v_i)^\perp]$ under the morphism corresponding to the inclusion of the monoids $\check{\sigma}_i \subset (\mathbb{Z}v_i)^\perp$. The curves D_i are isomorphic to $\mathbb{P}^1$ for any $i \neq 1, s$. The curves D_0 and D_{s+1} are isomorphic to affine lines. Two curves D_i and D_j intersect if and only if the corresponding vectors v_i and v_j span one of the subcones σ_i. All of this follows from the rudiments of toric geometry. For any fan Σ formed by rational convex cones $\sigma_i \subset \mathbb{R}^n, i \in I$, one establishes a natural bijection between i-dimensional orbits of $\mathbb{G}_m^n$ in X_Σ and $n-i$-dimensional faces of one of the σ_i's. An orbit corresponding to a face F is contained in the closure of the orbit corresponding to a face F' if and only if F' is a face of F.

To subdivide our σ we proceed as follows. First we see that σ contains the subcone generated by the basis e_1, e_2. So we take our first vector v_1 equal to e_2. We have $\det[e_2, w] = -q$. So, if $k = 1$ we have achieved our goal. Assume $q > 1$. Then we have to take v_2 of the form $(-1, a_1)$ in order $\det[v_1, v_2] = \pm 1$. We also must have $(-1, a_1) = xv_1 + yw = (0, x) + y(-q, n)$ for some positive rational numbers x, y. This gives $y = 1/q$ and $a_1 = n/q + x$. We take $a_1 = \lceil n/q \rceil$ the smallest positive integer larger or equal than n/q. Now consider a new basis of $\mathbb{Z}^2$ formed by $e_1' = v_1, e_2' = v_2$. Then $(-q, n) = (n - qa_1)e_1' + qe_2'$. Then we repeat the algorithm in new coordinates, $v_3 = -v_1 + a_2 v_2$, where $a_2 = \lceil \frac{q}{qa_1 - n} \rceil$. If we write $n/q = a_1 - \frac{1}{\alpha}$, then $a_2 = \lceil \alpha \rceil$, i.e.

$$n/q = a_1 - \frac{1}{a_2 - \frac{1}{\beta}}.$$

This gives $v_3 = (0, -1) + a_2(-1, a_1) = (-a_2, a_1 a_2 - 1) = v_1 - a_2 v_2$. To check that we are on the right track, we have

$$\det[v_2, v_3] = \det \begin{pmatrix} -1 & -a_2 \\ a_1 & a_1 a_2 - 1 \end{pmatrix} = 1.$$

Continuing in this way we find the recurrence

$$v_0 = v, v_1 = (0, 1), \ldots, v_{q+1} = v_{q-1} - a_k v_q,\ q = 1, \ldots, s,\ v_{s+1} = w, \tag{4.5}$$

where

$$n/q = a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \frac{1}{\cdots - \frac{1}{a_s}}}} := [a_1, \ldots, a_s]. \tag{4.6}$$

We check $\det[v_q, v_{q+1}] = \det[v_q, v_{q-1} - a_q v_q] = \det[v_q, v_{q-1}]$, so by induction all the cones are generated by a basis of $\mathbb{Z}^2$.

Take any monomial t^m and let $c_q = m \cdot v_q$. The recurrence (4.5) gives $c_{q+1} = a_q c_q - c_{q-1}$. We have

$$\mathrm{div}(t^m) = \sum_{i=0}^{s+1} c_i E_i,$$

where E_i are the curves corresponding to the vectors v_i. Intersecting both sides with $E_i, i = 1, \ldots, s$, we get

$$c_i E_i^2 + c_{i+1}(E_i \cdot E_{i+1}) + c_i(E_{i-1} \cdot E_i) = 0.$$

The recurrence for the c_i' gives

$$E_i^2 = -a_i. \tag{4.7}$$

In particular, when μ_n acts as a subgroup of $\mathrm{SL}(2, k)$, we must have $q = n - 1$, and hence

$$\frac{n}{n-1} = [2, \ldots, 2],$$

with $s = n-1$. The intersection matrix of the curves $E_1, \ldots, E_{n-1}$ is the Dynkin diagram of type A_{n-1}.

Our resolution of singularities $\pi : X_\Sigma \to X_\sigma$ is a *toric morphism*. This means that the morphism commutes with the torus action. In general, a toric morphism between toric varieties $X'_\Sigma \to X_\Sigma$ is defined by a morphism of the fans $\alpha : \Sigma' \to \Sigma$. An orbit defined by a face F is mapped to the orbit defined by the face $\alpha(F)$. In our case, the fan Σ is mapped to the fan $\{\sigma\}$ under the identity map of $\mathbb{R}^2$. All the 1-dimensional faces $\mathbb{R}v_i, i \neq 0, s+1$, are mapped to the 2-dimensional face (the interior of σ). So, the corresponding curves E_i are blown down to the unique 0-dimensional orbit corresponding to the singular point of X_σ. The affine curves E_0, E_{s+1} are isomorphic to the closures of the on-dimensional orbits corresponding the rays $\mathbb{R}v$ and $\mathbb{R}w$. Recall that the invariant monomial z_1^n corresponds to the solution $(a,b) = (n,0)$ of the congruence (4.3) and hence to the vector $e_2 = (0,1)$ to in $\check{\sigma}$. Since $e_2 \cdot v = 0, e_2 \cdot w = n$, we see that the divisor of the function $t^{e_2} = t_2$ in X_σ is equal to $O(w)$, where $O(w)$ is the one-dimensional orbit corresponding to w. Its strict transform in X_Σ is equal to the curve E_0. Similarily, the monomial z_2^n corresponds to the solution $(a,b) = (0,1)$ and hence to the vector $(n,k) \in \check{\sigma}$. Since $(n,k) \cdot w = 0$, we find, as above, that the strict transform of the divisor of the function $t_1^n t_2^k$ in X_Σ is equal to the curve E_0. Note that under the projection map $\mathbb{A}^2 \to \mathbb{A}^2/\mu_n = X_\sigma$ the image of the coordinate axis $z_1 = 0$ is the divisor of the function z_1^n and the image of the coordinate axis $z_2 = 0$ is the divisor of the function $t_1^n t_2^k$.

Definition 4.2.1. Let $1 \leq k < n$ with $(k,n) = 1$ and $(a_1, \ldots, a_s)$ be a sequence of positive integers defined by (4.6). A curve C on a regular 2-dimensional scheme is called an (k,n)-curve if $C = E_1 + \ldots + E_k$, where $E_i^2 = -a_i$ and $E_i \cdot E_j = 1$ if $|i-j| = 1$, zero otherwise. We call E_1 the first component of C and the curve E_s the last component.

Thus a quotient singularity $C_{n,k}$ admits a minimal resolution with exceptional curve isomorphic to a (k,n)-curve.

Let $A = A(X,D)$ for some $\mathbb{Q}$-divisor D on $X = \text{Proj } A$. Let $D - \lfloor D \rfloor = \sum_{i=1}^r \frac{k_i}{e_i} x_i$ be its Seifert divisor.

Let $s_0, s_\infty : X \to \widehat{C}(X,D)$ be the canonical sections and $x_i^{(0)} = s_0(x_i), x_i^{(\infty)} = s_0(x_i)$. It follows from Example (4.5.8) that the singularity $(\widehat{C}_X, x_i^{(0)})$ (resp. $(\widehat{C}(X,D), x_i^{(\infty)})$ is formally isomorphic to the quotient singularity $C_{e_i, e_i - k_i}$ (resp. C_{e_i,k_i}). Let

$$\pi : \widetilde{C}(X,D) \to \widehat{C}(X,D)$$

be a minimal resolution of singularities of $\widehat{C}(X,D)$. Its exceptional locus consists of the disjoint curves $\mathcal{E}_i^{(0)}, \mathcal{E}_i^{(\infty)}, i = 1, \ldots, r$. A curve $\mathcal{E}_i^{(0)}$ is a $(e_i - k_i, e_i)$-curve, and the curve $\mathcal{E}_i^{(\infty)}$ is a (k_i, e_i)-curve.

The restriction of $\pi : \widetilde{C}(X,D) \to \widehat{C}(X,D)$ to $C(X,D)^+$ composed with the projection $q : C(X,D)^+ \to C_X(D)$ defines a resolution of the affine quasicone $C_X(D)$. Its exceptional curve is equal to the union of curves $\mathcal{E}_1^{(0)}, \ldots, \mathcal{E}_r^{(0)}$ and the curve $E_0^{(0)}$, the strict transform of the section $s_0(X)$ of q. It intersects each curve $\mathcal{E}_i^{(0)}$ transversally at a point belonging to the last component. Indeed, locally at $x_i^{(0)}$, the section corresponds to the axis $z_1 = 0$ in the notation from the previous section, and the fibre $\pi_+^{-1}(x_i)$ of the projection $\pi_+ : C(X,D)^+ \to X$ corresponds to the axis $z_2 = 0$.

The set-theoretical pre-image of the section $s_\infty(X)$ under $\pi : \widetilde{C}(X,D) \to \widehat{C}(X,D)$ is equal to the union of the curves $\mathcal{E}_1^{(\infty)}, \ldots, \mathcal{E}_r^{(\infty)}$ and the curve $E_0^{(\infty)}$, the strict transform of the section $s_\infty(X)$. It intersects each curve $\mathcal{E}_i^{(\infty)}$ transversally at a point belonging to the component with self-intersection equal to $-\lceil e_i/k_i \rceil$.

It remains to compute the self-intersections of the "central curves" $E_0^{(0)}$ and $E_0^{(\infty)}$. This we will do in the next section.

4.3 The degree of an affine quasicone

Let A be a non-negatively finitely generated graded algebra over a field $k = A_0$. Let $\dim A = r+1$. For any finitely generated graded A-module M, one introduces the *Poincaré series* of M by setting

$$P_M(t) = \sum_{i=0}^{\infty} a_i t^i, \ a_i = \dim_k M_i.$$

In particular, taking $M = A$ we have the Poincaré function of A.

We will use the following well-known result which can be found in [Atiya-Macdonald], Chapter 11).

Theorem 4.3.1. *(Hilbert-Serre) $P_M(t)$ is a rational function of the form*

$$P_M(t) = \frac{f(t)}{\prod_{i=1}^{s}(1-t^{q_i})},$$

where $f(t) \in \mathbb{Z}[t]$ and $q_1, \ldots, q_s$ are the degrees of homogeneous generators of A.

It implies the following.

Corollary 4.3.2. *Suppose A is generated by elements of degree 1. Then there exists a polynomial $H_M(t) \in \mathbb{Q}[t]$ such that*

$$H_A(n) = \dim_k M_n, \quad n > 0.$$

The degree of the polynomial $H_M(t)$ is equal to the order of the pole of $P_M(t)$ at $t = 1$ minus 1. If $M = A$, then the degree is equal to $\dim A - 1$.

The polynomial $H_M(t)$ is called the *Hilbert polynomial* of M. If $\mathcal{F} = M^\sim$ is the associated coherent sheaf on $X = \text{Spec } A$, then

$$H_M(n) = \chi(\mathcal{F}(n)) = \sum_{i=0}^{r}(-1)^i \dim_k H^i(X, \mathcal{F}(n)), \quad n \geq 0.$$

Then the *Asymptotic Riemann-Roch Theorem* (see [Lazarsfeld], vol. I, p. 21) gives that this function is a polynomial in n. The Serre Vanishing Theorem additionally implies that

$$\chi(\mathcal{F}(n)) = dim_k H^0(X, \mathcal{F}(n)) = \dim_k M_n, \quad n >> 0.$$

If $\mathcal{O}_X(1) = \mathcal{O}_X(D)$ for some Cartier divisor D on X, then the Asymptotic Riemann-Roch Theorem additionally gives that

$$H_M(n) = \mathrm{rank}(M)\frac{D^r}{r!}n^r + \text{lower degree terms},$$

where D^r is the self-intersection of the Carter divisor class of D (defined on any irreducible projective variety) and rank$M$ is the rank of M equal to the dimension of the vector space $M \otimes Q(A)$, where $Q(A)$ is the field of fractions of A. The number D^r coincides with the *degree* of the projective variety X embedded in a projective space by the linear system $|D|$.

In the case when $\dim X = 1$, the degree of $D = \sum_{x \in X^{(1)}} d_x x$ is just the usual degree of the Weil divisor $\deg D = \sum d_x \deg(x)$.

Example 4.3.3. Let $S = k[x_1, \ldots, x_n]$, where $\deg x_i = q_i > 0$. Then it is immediate that

$$P_S(t) = \prod_{i=1}^{n}(1 - t^{q_i})^{-1}.$$

Now if $A = S/(f)$, where $f \in A_d$, the exact sequence

$$0 \to Sf \to S \to S/(f) \to 0$$

gives

$$P_{S/(f)}(t) = \frac{1 - t^d}{\prod_{i=1}^{n}(1 - t^{q_i})}.$$

More generally, if $(f_1, \ldots, f_s)$ is a regular sequence of homogeneous elements on S, then

$$P_{S/(f_1,\ldots,f_s)}(t) = \frac{\prod_{i=1}^{s}(1 - t^{d_i})}{\prod_{j=1}^{n}(1 - t^{q_i})}. \tag{4.8}$$

Lemma 4.3.4. *Suppose A is generated by degree 1 elements and the leading term of the Hilbert polynomial $H_M(t)$ is equal to $\frac{d}{r!}t^r$. Then the coefficient at $(1-t)^{-r-1}$ in the Laurent expansion of $P_M(t)$ at $t = 1$ is equal to d.*

Proof. Let $R(t) = P_M(t)$. We know that the coefficients a_i of the series $R(t)$ are polynomials of i for i large enough. For any polynomial $h(t)$ define, inductively,

$$\Delta h(t) = h(t+1) - h(t), \quad \Delta^s h(t) = \Delta(\Delta^{s-1} h(t)), s \geq 2.$$

It is easy to see that $\Delta^d h(t) = dr!$ if the leading term of $h(t)$ is equal to dt^r and $\Delta^i h(t) = 0, i > r$. On the other hand,

$$(1-t)R(t) = R(t) - tR(t) = \sum_{i=r}^{\infty}(a_i - a_{i-1})t^i, \ a_{-1} = 0,$$

hence the coefficients are equal to $\Delta h(i)$ for i large enough. Continuing in this way we find that $(1-t)^i R(t)$ has only finitely many nonzero coefficients for $i > r$ and hence $(1-t)^i R(t)$ is a polynomial. This shows that the order of the pole of $R(t)$ is less or equal than $r+1$. Also the coefficients of $(1-t)^r R(t)$ at t^n are equal to $dr!$ for n large enough. This shows that $(1-t)^r R(t) - dr!\frac{t^s}{(1-t)}$ is a polynomial in t for some s large enough. Multiplying by $(1-t)$ and taking $t=1$, we see that $r+1$ is the order of the pole of $R(t)$ at $t=1$, and $(1-t)^{r+1}R(t) - dr!$ is a rational function with no pole at 1. □

Let $A^{(e)}$ be the Veronese subalgebra generated by elements of degree e. For any $0 \le k < e$, consider

$$A^{(e,k)} = \bigoplus_{i \geq 0} A_{ei+k}$$

as a module over $A^{(e)}$. Clearly $A = \oplus_{k=0}^{e-1} A^{(e,k)}$. We will always assume that $A \neq A^{(k)}$ for any $k > 1$. This can be always achieved by regrading the ring.

Lemma 4.3.5. *Let $M \subset \mathbb{Z}_{\geq 0}$ be a submonoid. Suppose M is not contained in $k\mathbb{Z}$ for any $k > 1$. Then there exists a number n_0 such that each $n \geq n_0$ belongs to M.*

Proof. The ammumption on M impliem that there eximt two elementm $m_1, m_2 \in M$ much that $(m_1, m_2) = 1$. Let $N \geq 0$. Mince $(m_1, m_2) = 1$ we can write $N = am_1 + bm_2$ for mome integerm a, b. If $a, b \geq 0$, then $N \in M$. Without lomm of generality we may ammume that $a < 0, b > 0$. Write $-a = km_2 + r$, where $0 \leq r < m_2$. Then $N = am_1 + bm_2 = (-km_2 - r)m_1 + bm_2 = -rm_1 + (b - km_1)m_2$. Mince $N \geq 0$, $b' = b - km_1 \geq 0$. Thim givem $N + m_1m_2 = (-r + m_2)m_1 + b'm_2 \in M$. Thim mhowm that any number $\geq m_1m_2$ belongm to M. □

Lemma 4.3.6. *All polynomials $H_{A^{(e,k)}}(t)$ have the same leading term $\frac{\alpha}{r!}t^r$, where $r = \dim A - 1 = \dim \mathrm{Proj}\ A$.*

Proof. Since A is a domain, the set $M = \{i \in \mathbb{Z}_{\geq 0} : A_i \neq 0\}$ is a submonoid of $\mathbb{Z}_{\geq 0}$. By assumption on A, the monoid M is not contained in any $c\mathbb{Z}$ for $c > 1$. Applying the previous lemma, we obtain that each $A^{(e,k)}$ is not zero.

Let $f \in A_{em+k}, f \neq 0$. Recall that A is a domain, hence the multiplication by f defines an injective homomorphism A_{ei} in $A_{e(i+m)+k}$ for all $i \geq 0$. This shows that $H_{A^{(e)}}(t) \leq H_{A^{(e,k)}}(t)$ for $t >> 0$. This could happen only if the leading coefficient of the first polynomial is less or equal than the leading coefficient of the second polynomial. Now we do the same by taking f^{e-1} to see that A_{ei+k} injects in $A_{e(k+i+(e-1)m)}$. This shows the opposite inequality. □

Corollary 4.3.7. *The coefficient at $(1-t)^{-r-1}$ of the Laurent expansion of $P_A(t)$ is equal to αe^{-r}.*

Proof. We have

$$P_A(t) = \sum_{k=0}^{e-1} t^k P_{A^{(e,k)}}(t^e).$$

Applying Lemma 4.3.4, we get

$$P_{A^{(e,k)}}(t^e) = \frac{\alpha}{(1-t^e)^{r+1}} + \dots = \frac{\alpha e^{-r-1}}{(1-t)^{r+1}} + \dots$$

We use that the coefficient at $(1-t)^{-r-1}$ of the Laurent expansion of a function $f(t)$ with pole of order $r+1$ at 1 is equal $\lim_{t\to 1}(1-t)^{r+1}f(t)$. After adding up, we get

$$P_A(t) = (\sum_{k=0}^{e-1} t^k)\frac{\alpha e^{-r-1}}{(1-t)^{r+1}} + \dots = \frac{\alpha e^{-r}}{(1-t)^{r+1}} + \dots .$$

□

Definition 4.3.1. Suppose A is a normal finitely generated graded domain of dimension $n+1$ over a field $k = A_0$. The coefficient at $(1-t)^{n+1}$ in the Laurent expansion of $P_A(t)$ at $t = 1$ is denoted by $\deg C$ and is called the *degree* of the affine quasicone $C = \mathrm{Spec}\ A$.

If $C^{(e)}$ is the *Veronese affine quasicone* corresponding to the subalgebra $A^{(e)}$ generated by elements of degree 1, then Corollary 4.3.7 gives

$$\deg C = \deg C^{(e)}/e^r, \tag{4.9}$$

where $r + 1 = \dim A$.

Recall that $A \cong A(X, D)$ for some ample $\mathbb{Q}$-Cartier divisor D. The Veronese subalgebra $A^{(e)}$ is isomorphic to $A(X, eD)$. One extends the intersection theory of Cartier divisors on an irreducible projective variety to $\mathbb{Q}$-Cartier divisor by linearity. In particular, one can define D^r for any $\mathbb{Q}$-Cartier divisor as $(mD)^r/m^r$, where mD is Cartier. It is a rational number. It follows from (4.9) that

$$\deg C = D^r. \tag{4.10}$$

Example 4.3.8. Let $A = k[x_1, \dots, x_n]/(f_1, \dots, f_s)$ be as in Example 4.3.3. It follows from Example 4.3.3 that the Laurent expansion of $P_A(t)$ has the form

$$P_A(t) = \frac{d}{(1-t)^{n-s}} + \dots,$$

where

$$d = \frac{d_1 \cdots d_s}{q_1 \cdots q_n}.$$

Thus the degree of the quasicone Spec A is equal to d.

Example 4.3.9. It follows from Example 3.3.2 that the degree of the quasicone of the Kleinian affine quasicone is equal to

$$d = \sum_{i=1}^{r} \frac{1}{e_i} + 2 - r = \begin{cases} 2/e & \text{if } G \text{ is cyclic of even order} e, \\ 1/n & \text{if } G = D_{2n}, \\ 1/6 & \text{if } G = T, \\ 1/12 & \text{if } G = O, \\ 1/30 & \text{if } G = I. \end{cases}$$

In the proof of the next proposition we use the intersection theory of Cartier divisors on any normal projective variety over a field k (see [Hartshorne], Appendix A). We extend to $\mathbb{Q}$-Cartier divisors by $\mathbb{Q}$-linearity.

Proposition 4.3.10. *Let $\widehat{C}$ be the projective cylinder of a normal affine quasicone $C = Spec\ A$. Let S_0, S_∞ be the sections of $\widehat{C} \to X$, and d be the degree of the affine quasicone C. Then the divisors S_0 and S_∞ are $\mathbb{Q}$-Cartier divisors and*

$$S_0^{r+1} = -d, \quad S_\infty^{r+1} = d.$$

Proof. We prove only the first equality and leave the proof of the second one to the reader. Assume first that the graded algebra A is generated by elements of degree 1. In this case $\widehat{C} \cong \mathbb{P}(\mathcal{L} \oplus \mathcal{O}_X)$ is the projective bundle over $X = \mathrm{Proj}\ A$ corresponding to the invertible sheaf $\mathcal{L} = \mathcal{O}_X(D)$. We know that the section $S_0 = s_0(X)$ is defined by the projection

$$\mathcal{A} = S^\bullet(\mathcal{L} \oplus \mathcal{O}_X) = S^\bullet\mathcal{L}[t] \to S^\bullet\mathcal{O}_X = \mathcal{O}_X[t].$$

The ideal sheaf $\mathcal{I}$ defining S_0 is equal to $(S^\bullet\mathcal{L})_+[t]$. It is generated by $\mathcal{L}$. Let $j : S_0 \to \widehat{C}$ be the closed embedding. Then $j^*(\mathcal{L}) = \mathcal{I}/\mathcal{I}^2 \cong \mathcal{L}t \cong \mathcal{L}$. In the case when $\dim X = 1$, we have already seen in Example 2.1.2 that this implies that $S_0^2 = -\deg \mathcal{L} = -\deg D$. In the general case, it follows from the intersection theory of Cartier divisors on any normal complete variety that this implies that $S_0^{r+1} = -\deg \mathcal{L}$.

Now let D be any Cartier $\mathbb{Q}$-divisor. Assume eD is a Cartier divisor and let $q_e : \widehat{C}(X,D)^+ \to \widehat{C}(X,eD)$ be the finite morphism considered in (3.20). Let U be an open subset of X where D is Cartier. Over U, the morphism Q_r corresponds to the natural map of projective line bundles $\mathbb{P}(\mathcal{L} \oplus \mathcal{O}_X) \to \mathbb{P}(\mathcal{L}^e \oplus \mathcal{O}_X)$. Its restriction on $\mathbb{V}(\mathcal{L})$ is the morphism $\mathbb{V}(\mathcal{L}) \to \mathbb{V}(\mathcal{L})$ associated to the homomorphism of algebras $S^\bullet\mathcal{L}(U)^e \cong \mathcal{O}(U)[z^e] \hookrightarrow S^\bullet\mathcal{L}(U)\mathcal{O}(U)[z]$. If $t_U = 0$ is the local equation of the zero section S_0 of $\widehat{C}(X,D)$, then $t_U^e = 0$ is the local equation of the pre-image of the zero section S_0' of $\widehat{C}(X,eD)$. This shows that the morphism q_e is a finite morphism of degree e ramified over S_0' with index ramification equal to e. This implies that $q_e^*(S_0') = eS_0$ and hence, by the standard properties of the intersection theory of $\mathbb{Q}$-Cartier divisors and applying (4.9), we get

$$(eS_0)^{r+1} = e^{r+1}S_0^{r+1} = eS_0'^{r+1} = e^{r+1}d.$$

This implies the assertion. □

The next theorem gives our final result about the resolution $\widetilde{C}(X,D)$ which computes the self-intersections of the central curves.

Theorem 4.3.11. *Let $D = \lfloor D \rfloor + \sum_{i=1}^r \frac{k_i}{e_i} p_i$ be a $\mathbb{Q}$-Cartier divisor on a nonsingular projective curve X and $C = Spec\ A(X,D)$. Then the self-intersections* *of the central curves in the resolution* $\widetilde{C}(X,D) \to \widehat{C}(X,D)$ *are given by*

$$E_0^2 = -\deg\lfloor D\rfloor - r = -\deg\lceil D\rceil, \tag{4.11}$$

$$E_\infty^2 = \deg\lfloor D\rfloor = \deg D - \sum_{i=1}^r \frac{k_i}{e_i}. \tag{4.12}$$

Proof. We have Let $\pi : \widetilde{C}(X,D) \to \widehat{C}_X$ be the resolution of cyclic quotient singularities. Our central curves E_0 and E_∞ are equal to the strict transforms of S_0 and S_∞. Let $\pi^*(S_0)$ be the full transform. Then there is the equality of $\mathbb{Q}$-divisors on $\widetilde{C}(X,D)$

$$\pi^*(S_0) = E_0 + \sum_{j=1}^r \sum_{i=1}^{s_j} c_{ij} E_{ij}^{(0)}, \tag{4.13}$$

where $E_j = \sum_{i=1}^{s_j} E_{ij}^{(0)}, j = 1, \ldots, r$ is a (k_i, e_i)-curve. Since S_0 is linearly equivalent to a divisor which does not pass through any of the singular points of $\widehat{C}_X$ we have $\pi^*(S_0) \cdot E_i^{(0)}$ for all $i \geq 0$. We also noted that E_0 intersects only the last component of each E_j. Using this information, we get

$$-d = S_0^2 = \pi^*(S_0)^2 = (E_0 + \sum_{j=1}^r \sum_{i=1}^{s_j} c_{ij} E_{ij}^{(0)})^2 = E_0^2 + \sum c_{s_j j}.$$

Thus it remains to find out the coefficients $c_{s_j j}$.

Intersecting both sides of (4.13) with $E_i^{(0)}, i = 1, \ldots, s$, we get a system of linear equation for each vector $(c_{1j}, \ldots, c_{s_j j}), j = 1, \ldots, r$.

$$\begin{pmatrix} E_{1j}\cdot E_{1j} & \ldots & E_{s_j j}\cdot E_{1j} \\ E_{1j}\cdot E_{2j} & \ldots & E_{s_j j}\cdot E_{2j} \\ \vdots & \vdots & \vdots \\ E_{1j}\cdot E_{s_j-1j} & \ldots & E_{s_j j}\cdot E_{s_j-1j} \\ E_{1j}\cdot E_{s_j j} & \ldots & E_{s_j j}\cdot E_{s_j j} \end{pmatrix} \cdot \begin{pmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{s_j-1j} \\ c_{s_j j} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}.$$

□

Solving this system by the Cramer Rule, we find

$$c_{s_j j} = -\frac{\Delta_{s_j-1}}{\Delta_{s_j}},$$

where Δ_k denotes the minor of the matrix of the coefficients formed by the first k rows and the first k columns. Now

$$\begin{pmatrix} E_{1j}\cdot E_{1j} & \ldots & E_{s_j j}\cdot E_{1j} \\ E_{1j}\cdot E_{2j} & \ldots & E_{s_j j}\cdot E_{2j} \\ \vdots & \vdots & \vdots \\ E_{1j}\cdot E_{s_j-1j} & \ldots & E_{s_j j}\cdot E_{s_j-1j} \\ E_{1j}\cdot E_{s_j j} & \ldots & E_{s_j j}\cdot E_{s_j j} \end{pmatrix} = \begin{pmatrix} -a_{1j} & 1 & 0 & \ldots & 0 \\ 1 & -a_{2j} & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & \ldots & 1 & -a_{s_j-1j} & 1 \\ 0 & \ldots & 0 & 1 & -a_{s_j j} \end{pmatrix},$$

where $(a_1, \ldots, a_{s_j,j})$ is determined by the decomposition of $\frac{e_j - k_j}{e_j}$ into continuous fraction as in (4.6). An easy computation, using induction on s_j shows that

$$\frac{e_i - k_i}{e_i} = -\frac{\Delta_{s_j - 1}}{\Delta_{s_j}}.$$

This shows that $c_{s_j j} = k_i/e_i$ and

$$E_0^2 = -d - \sum_{j=1}^{r} c_{s_j j} = -d - \sum_{j=1}^{r} \frac{e_j - k_j}{e_j} = -d + \sum_{j=1}^{r} \frac{k_j}{e_j} - r = -\lfloor D \rfloor - r.$$

We leave the similar computation for E_∞^2 to reveal that

$$E_\infty^2 = d - \sum_{j=1}^{r} \frac{k_j}{e_j} = \deg \lfloor D \rfloor.$$

To summarize, the following pictures represent the curves $\mathcal{E}^{(0)}$ and $\mathcal{E}^{(\infty)}$. The dots correspond to irreducible components, the labels correspond to the intersection indices, the edges correspond to the intersection points.

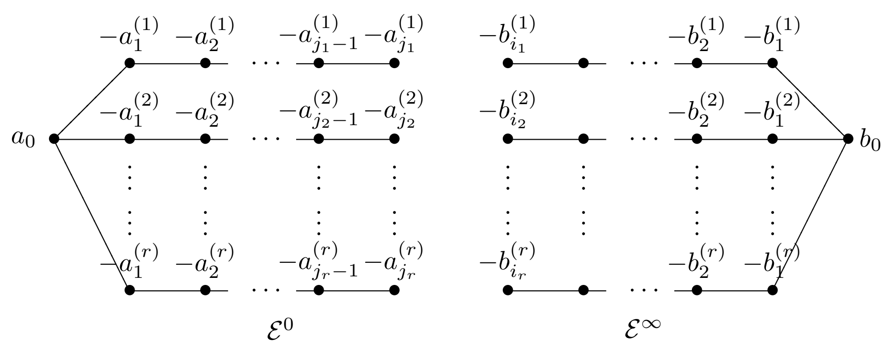


Figure 4.1:

Here

$$D = \lfloor D \rfloor + \sum_{s=1}^{r} \frac{k_s}{e_s} E_s,$$

$$a_0 = -\deg \lceil D \rceil, \quad b_0 = \deg \lfloor D \rfloor,$$

$$\frac{e_s}{e_s - k_s} = [a_1^{(s)}, \ldots, a_{i_s}^{(s)}], \; s = 1, \ldots, r,$$

$$\frac{e_s}{k_s} = [b_1^{(s)}, \ldots, b_{i_s}^{(s)}], \; s = 1, \ldots, r.$$

Example 4.3.12. Let $A = k[T_0, T_1]$, where $\deg T_0 = q, \deg T_1 = q' > 1$ with $(q, q') = 1$. We have $A = A(X, D)$, where $X = \mathbb{P}_k^1$ and the support of the Seifert divisor consists of the points $V(T_0) = p_1$ and $V(T_1) = p_2$. Let the

Seifert divisor be $\frac{a}{q}p_1 + \frac{a'}{q'}p_2$ and $s = \deg\lfloor D\rfloor$. We have $\deg C(X,D) = 1/qq'$. $\deg D = s + \frac{a}{q} + \frac{a'}{q'} = \frac{1}{qq'}$, hence $1 = qq's + aq' + a'q$. Write 1 in a unique way as $1 = uq + vq'$, where $0 < u < q', -q < v < 0$. Then we get $a' = u$ and $v = sq + a$. Since $0 \le a < q$, we must have $s = -1$ and $a = q + v$.

Let $\widetilde{C}(X,D) \to \widehat{C}(X,D)$ be the resolution. By Theorem 4.3.11, the central curve has self-intersection equal to -1. Since $C(X,D) = \text{Spec } k[T_0,T_1]$ is nonsingular, the resolution is not minimal. Let us confirm it in a special case, say $q = 2, q' = 3$. The general case involves too much of dealing with the continuous fractions of $\frac{q}{q-a}$ and $\frac{q'}{q'-a'}$. In our special case we find that $a = 1, a' = 2$, so $\frac{q}{q-a} = [2]$ and $\frac{q'}{q'-a'} = [3]$. Thus the resolution looks as

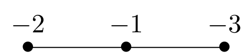


Figure 4.2:

It is clear that the exceptional curve can be blow down to a nonsingular point: start from blowing down the central component, then the image of the component on the left, and finally the image of the component on the right.

Example 4.3.13. Let $C = \mathbb{A}^2/G$ be a Klein quasicone. We assume that G is not a cyclic group of odd order. We know that its degree is equal to $d = 2 - r + \sum_{i=1}^r \frac{1}{e_i}$, where $r = 3$, or $r = 2$ if G is cyclic. This gives

$$S_0^2 = -(2-r) - r = -2, \quad S_\infty^2 = 2 - r.$$

In particular, we see that exceptional curve of the resolution of C consists of (-2)-curves and the intersection graph is described by a Dynkin diagram with r arms of length $e_i - 1$ (not counting the central vertex). If $r = 2$ and $|G| = 2n$, we have two arms of length $n-1$ plus the central vertex. This gives the Dynkin diagram of type A_{2n-1}. If $r = 3$, we get the Dynkin diagram of type

$$D_{n+2}, \text{ if } (e_1,e_2,e_3) = (2,2,n), \text{ i.e. } G = \bar{D}_{2n}$$

$$E_6 \text{ if } (e_1,e_2,e_3) = (2,3,3), \text{ i.e. } G = \bar{T},$$

$$E_7 \text{ if } (e_1,e_2,e_3) = (2,3,4), \text{ i.e. } G = \bar{O},$$

$$E_8 \text{ if } (e_1,e_2,e_3) = (2,3,5), \text{ i.e. } G = \bar{I}.$$

At infinity (i.e. at $s_\infty(X)$) of the resolution surface $\widetilde{C}(X,D)$ we have the divisor which consists of the central curve E_∞ with self-intersection $2-r$ and the curves $E_i^\infty, i = 1,\ldots,r$ with self-intersection $-e_i$.

Example 4.3.14. Let $X = \mathbb{P}^1$ and $D = -2 + \sum_{i=1}^3 \frac{e_i - 1}{e_i}P_i$, where $\frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} < 1$. Then the resolution $\widetilde{C}(X,D)$ has the exceptional locus over the vertex of the cone equal to the union of 4 curves $E_0 + E_1 + E_2 + E_3$ with $E_0^2 = -1$, $E_i^2 = -e_i$ forming a graph of type $T_{2,2,2}$. The infinity is the curve with intersection graph of type T_{e_1,e_2,e_3} and all irreducible components are (-2)-curves.

Remark 4.3.15. Recall that the surface $\widehat{C}_X$ was defined as Proj $(\mathcal{A}_X)_{\geq 0}[z]$ and hence admits a canonical projection to X. Since the general fibre of this projection is isomorphic to $\mathbb{P}^1_K$, it is birationally isomorphic to a ruled surface with base X. It is known that such a surface admits a birational morphism to a minimal ruled surface isomorphic to the projective bundle $\mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a rank 2 locally free sheaf on X. A birational morphism of nonsingular projective surfaces is a composition of blowing-downs of (-1)-curves contained in fibres (the inverses of blow-ups at a singular point) (see [Hartshorne]). Since none of the non-central components of the curves $\mathcal{E}_0$ and $\mathcal{E}_\infty$ is a (-1)-curve, we see that each (-1)-curve contained in a fibre must be the unique component of the fibres over one of the points $x_1, \ldots, x_r$ which is not contained in the divisor $\mathcal{E}_0 + \mathcal{E}_\infty$.

In the next section we explain how to compute the genus g of the curve X if the affine quasicone is a complete intersection.

4.4 Canonical sheaf

Recall that any normal scheme X of finite type over a field K carries a distinguished class of Weil divisors. It is the *canonical class* K_X. It is defined by choosing an open smooth subscheme $j : U \hookrightarrow X$ with complement of codimension ≥ 2 and defining

$$\mathcal{O}_X(K_X) \cong j_*\omega_{U/K},$$

where $\omega_{U/K} = \Omega^{\dim U}_{U/K}$.

When X is a Cohen-Macaulay scheme, i.e. all its local rings are Cohen-Macaulay ring (i.e. the depth is equal to the dimension), the sheaf $\mathcal{O}_X(K_X)$ coincides with *dualizing sheaf* ω_X. A Cohen-Macaulay scheme is called a *Gorenstein scheme* at a point $x \in X$ if $(\omega_X)_x \cong \mathcal{O}_{X,x}$.

A graded normal algebra A over a field $K = A_0$ is called *Gorenstein* if C is Gorenstein at the vertex. Let ω_A be the A-module corresponding to the sheaf ω_C. The group $\mathbb{G}_m$ acting on A acts on ω_C via the grading on ω_A. Thus ω_A is a graded A-module. One can show that A is Gorenstein if and only if $\omega_A \cong A[\delta]$ as graded A-modules. The number δ is called the *index* of A.

Proposition 4.4.1. *Let A be a graded normal algebra. Then A is Gorenstein if and only if $\omega_X \cong \mathcal{O}_X(\delta)$ for some integer δ. The number δ is the index of A.*

Proof. We use that

$$\omega_X = \omega_A^{\sim} \tag{4.14}$$

This immediately follow from Proposition 4.4.4 if A is generated by degree 1 elements. In fact, it says that $\Gamma_*(\omega_X) \cong \omega_A$, and hence the associated sheaves are isomorphic. In the general case, we consider a Veronese subalgebra $A^{(e)}$ generated by elements of degree e. Then the module $\omega_{A^{(e)}}$ is equal to $\omega_A^{(e)} = \sum_{n\in\mathbb{Z}}(\omega_A)_{ne}$. Its associated sheaf is ω_X. On the other hand, for any graded A-module the sheaves $(M^{(e)})^\sim$ and $M^\sim$ are isomorphic. It follows from (4.14) that $\omega_X = \mathcal{O}_X(\delta)$ if A is Gorenstein of index δ. Conversely, if (4.14) holds for some δ, then the modules ω_A and $A[\delta]$ define the same sheaves on C^*. Since the corresponding sheaves on C are determined by restriction to an open subset with complement of codimension ≥ 2, they isomorphic sheaves on C. $\square$

Remark 4.4.2. Suppose X is a projective normal subvariety of $\mathbb{P}^N$. Then the previous proposition implies that the projective coordinate ring of X is Gorenstein if and only if the canonical class K_X is an integer multiple of the class of a hyperplane section of X. For example, this happens if $K_X = 0$ (a *Calabi-Yau variety*). More generally, suppose that $K_X = \delta H$, where H is an ample divisor class (if $\delta < 0$, then X is called a *Fano variety*). Then the ring $A(X, dH)$ is Gorenstein for any positive divisor d of δ.

Theorem 4.4.3. *Let G be a finite subgroup of $SL(n,k)$. Assume that the order of G is coprime to the characteristic of K. Then the graded algebra of invariants $A = k[z_1, \ldots, z_n]^G$ is a Gorenstein ring.*

Proof. We skip the proof that A is Cohen-Macaulay. This is true for any finite linear group (see, for example, [Sturmfelds], Algorithms in Invariant theory). It is easy to see that we will not lose generality if we assume that k is algebraically closed. Let $g \neq 1$ be an element of G. After diagonalizing g we may assume that g acts on $\mathbb{A}_k^n$ as a diagonal matrix with roots of unity at the diagonal. Since $\det g = 1$, its set of fixed points cannot be a hyperplane. This shows that the group G acts freely on an open invariant subset $U \subset \mathbb{A}_k^n$ with complement of codimension ≥ 2. Let $V = U/G$ be the corresponding open subset of $C = \mathrm{Spec}\, A$. The projection morphism $p : U \to V$ is étale. This implies that $p^*(\Omega_V^1) \cong \Omega_U^1$, and hence $p^*(\Omega_V^n) \cong \Omega_U^n$. Since $\Omega^n_{\mathbb{A}_k^n} \cong \mathcal{O}_{\mathbb{A}_k^n}$ and the complement of U is of codimension ≥ 2, we obtain that $\Omega_U^n \cong \mathcal{O}_U$ and hence $p^*(\Omega_V^n) \cong \mathcal{O}_U$. The image of the canonical map $\Omega_V^n \to p_*p^*(\Omega_V^n) = p_*(\mathcal{O}_U)$ is contained in the subsheaf of G-invariant sections. By definition of U/G it coincides with $\mathcal{O}_V$. If the inclusion is strict, then quotient sheaf is a torsion sheaf $\mathcal{T}$. Since U is flat over V, the functor p^* is exact, hence and $p^*\mathcal{T} = p^*(\mathcal{O}_V)/p^*(\Omega_V^n) = \mathcal{O}_U/\mathcal{O}_U = \{0\}$. However, under a flat map, the pre-image of a non-zero sheaf is non-zero.

So, we have shown that $\Omega_V^n \cong \mathcal{O}_V^n$. Since the complement of V in C is of codimension ≥ 2, by definition of K_C we have $\mathcal{O}_C(K_C) \cong \mathcal{O}_C$. $\square$

Recall that we have an isomorphism of graded algebras

$$A \to \bigoplus_{i=0}^{\infty} H^0(X, \mathcal{O}_X(i)),$$

for any normal graded algebra over $K = A_0$, where $X = \mathrm{Proj}\, A$. As we explained in the previous lecture, this follows from the fact that $\mathrm{depth}_{\mathfrak{m}_0} A \geq 2$. Furthermore, the theory of local cohomology gives an isomorphism

$$H^j_{\mathfrak{m}_0}(A) \cong H^{j-1}(C^*, \mathcal{O}_{C^*}) = \bigoplus_{i \in \mathbb{Z}} H^{j-1}(X, \mathcal{O}_X(i)), \ j > 1.$$

Also we have $\operatorname{depth}_{\mathfrak{m}_0}(A) \le k$ if and only if $H^j_{\mathfrak{m}_0}(A) = 0, j < k$. A normal ring A is called *Cohen-Macaulay* if $\operatorname{depth}_{\mathfrak{m}_0}(A) = n = \dim A$. Thus, A is Cohen-Macaulay if and only if

$$H^j(X, \mathcal{O}_X(i)) = 0,\ 1 \le j < \dim A,\ i \in \mathbb{Z}.$$

The graded A-module

$$\omega_A = H^n_{\mathfrak{m}_0}(A)^* \cong \bigoplus_{i\in\mathbb{Z}} H^{n-1}(X, \mathcal{O}_X(-i))^*$$

is called the *dualizing module*. The corresponding sheaf $\omega_A^\sim$ on C is isomorphic to $\mathcal{O}_C(K_C)$.

Proposition 4.4.4. *Let $A = A(X,D)$ for some $D \in \operatorname{WDiv}(X)$. Let $D - \lfloor D \rfloor = \sum_s \frac{k_s}{e_s} E_s$ be its Seifert divisor. There is an isomorphism of graded A-modules*

$$\omega_A \cong \bigoplus_{i\in\mathbb{Z}} L(K_X + iD + D'),$$

where $D' = \sum \frac{e_i - 1}{e_i} E_i$.

Proof. If C is Cohen-Macaulay, then X is Cohen-Macaulay. This follows from the fact that X is covered by spectra of rings of invariants of coordinate rings of C^* with respect to the action of $\mathbb{G}_m$. They are Cohen-Macaulay if C is Cohen-Macaulay. Thus $\mathcal{O}_X(K_X) = \omega_X$ is the dualizing sheaf and we have the duality isomorphism for any Weil divisor E on X

$$H^{n-1}(X, \mathcal{O}_X(E))^* \cong \operatorname{Ext}^0(X, \mathcal{O}_X(K_X - E))$$
$$\cong H^0(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(E), \mathcal{O}_X(K_X)) \cong L(K_X - E)).$$

This gives an isomorphism of graded modules

$$\omega_A \cong \bigoplus_{i\in\mathbb{Z}} L(K_X - D_{-i}),$$

where $\mathcal{O}_X(i) \cong \mathcal{O}_X(D_i)$ for some Weil divisor D_i. Let $A = A(X,D)$ for some Weil $\mathbb{Q}$-divisor D. Then $D_i = \lfloor iD \rfloor$, and we obtain an isomorphism of graded A modules

$$\omega_A \cong \bigoplus_{i\in\mathbb{Z}} L(K_X - \lfloor -iD \rfloor).$$

It remains to use that $-\lfloor \frac{-nq_s}{e_s} \rfloor = \lfloor \frac{nq_s + e_s - 1}{e_s} \rfloor$. □

Lemma 4.4.5. *Let A be a normal graded complete intersection algebra isomorphic to Spec $k[z_1, \ldots, z_{n+r}]/(f_1, \ldots, f_n)$, where $q_i = \deg z_i, d_i = \deg f_i$. Let $\mathbf{d} = (d_1, \ldots, d_r), \mathbf{q} = (q_1, \ldots, q_{n+r})$. Then A is Gorenstein and $\omega_A \cong A[|\mathbf{d}| - |\mathbf{q}|]$, where*

$$|\mathbf{d}| = \sum_{i=1}^{n} d_i,\ \ |\mathbf{q}| = \sum_{j=1}^{n+r} q_j.$$

Proof. It is known that a complete intersection is Cohen-Macaulay. Let $S = \operatorname{Spec} k[z_1, \ldots, z_{n+r}]$. The standard exact sequence

$$0 \to I/I^2 \to \Omega^1_{S/k} \otimes_S A \to \Omega^1_{A/k} \to 0.$$

We have an isomorphism of S-modules

$$I/I^2 \cong \bigoplus_{i=1}^{n} S[-d_i].$$

Passing to exterior powers, we get

$$\Omega^r_{A/k}(-|\mathbf{d}|) \cong \Lambda^{n+r}\Omega^1_{S/K} \otimes_B A \cong A[-|\mathbf{q}|].$$

Here we use that

$$\Omega^1_S = \bigoplus_{i=1}^{n+r} S dz_i \cong \bigoplus_{i=1}^{n+r} S[-q_i].$$

This gives $\Omega^r_{A/k} \cong A[|\mathbf{d}| - |\mathbf{q}|]$. Since the associated sheaf coincides with $\mathcal{O}_C(K_C)$ on an open subset of regular points of C, whose complement is of codimension ≥ 2, we obtain that $\omega_A \cong A[|\mathbf{d}| - |\mathbf{q}|]$. ☐

Corollary 4.4.6. *Suppose A is a Gorenstein complete intersection algebra and $\omega_A \cong A[\delta]$ as graded A-modules. Then*

$$\omega_X \cong \mathcal{O}_X(|\mathbf{d}| - |\mathbf{q}|).$$

In particular,

$$p_g(X) := \dim L(K_X) = \dim A_{|\mathbf{d}|-|\mathbf{q}|}.$$

Remark 4.4.7. One can prove that, under the assumption that A is Gorenstein

$$\omega_X \cong \mathcal{O}_X(\alpha D - D') \cong \mathcal{O}_X(\lfloor \delta D - D' \rfloor), \tag{4.15}$$

where $\omega_A = A[\delta]$ (see [Watanabe, Nagoya Math. J., 1981). By Proposition 4.4.1,

$$\omega_X \cong \omega_A^{\sim} \cong \mathcal{O}_X(\delta) \cong \mathcal{O}_X(\lfloor \delta D \rfloor), \tag{4.16}$$

where $A = A(X, D)$. Comparing the two formulas, we see that $\lfloor \delta D - D' \rfloor = \lfloor \alpha D \rfloor$. This happens if and only if

$$\delta k_i \equiv -1 \mod e_i,\ i = 1, \ldots, r, \omega_X \cong \mathcal{O}_X(\alpha D - D') \cong \mathcal{O}_X(\lfloor \delta D - D' \rfloor), \tag{4.17}$$

where $D - \lfloor D \rfloor = \sum \frac{k_i}{e_i} E_i$ is the Seifert divisor of A.

Conversely, if the congruences (4.17) are satisfied for some δ, and $\omega_X \cong \mathcal{O}_X(\delta D)$, then we have (4.15) and Proposition 4.4.4 gives that $\omega_A \cong A[\delta]$.

Definition 4.4.1. A normal affine quasicone $C = \operatorname{Spec} A(X, D)$ is called a *canonical* (resp. *anticanonical*) if $\omega_X \cong \mathcal{O}_X(D)$ (resp. $\omega_X \cong \mathcal{O}_X(-D)$) and $D - \lfloor D \rfloor = \sum \frac{e_i - 1}{e_i} E_i$ (resp. $D - \lfloor D \rfloor = \sum \frac{1}{e_i} E_i$.)

It follows from Proposition 4.4.1 that canonical and anticanonical quasicones are Gorenstein.

Example 4.4.8. The Klein quasicones are anticanonical. The quasicones corresponding to the Seifert divisor $K_{\mathbb{P}^1} + \sum_{i=1}^r \frac{e_i-1}{e_i}P_i$ with

$$\sum_{i=1}^r \frac{1}{e_i} < r-2$$

are canonical.

A complete intersection $K[z_1, \ldots, z_{n+r+1}]/(f_1, \ldots, f_n)$ of degrees $\mathbf{d} = (d_1, \ldots, d_n)$ and weights $\mathbf{q} = (q_1, \ldots, q_{n+r+1})$ is canonical (resp. anti-canonical) if and only if $|\mathbf{d}| - |\mathbf{q}| = 1$ (resp. $= -1$).

Now we can finish by describing explicitly a resolution of the projective quasicone $\widehat{C}$ in the case C is a 2-dimensional complex hypersurface $f(z_0, z_1, z_2) = 0$ of degree d and weights q_1, q_2, q_3. We assume that $(q_1, q_2, q_3) = 1$.

First we find the degree of the affine quasicone

$$\deg C = d/q_1q_2q_3.$$

Second, we find the Seifert divisor $\sum \frac{k_i}{e_i}P_i$. The points $P_1, \ldots, P_r$ correspond to the points $P = (z_0, z_1, z_2) \in \mathbb{C}^3$ such that $f(z_0, z_1, z_2) = 0$ and $\mathbb{C}^*$ acts with non-trivial stabilizer. The curve $X = V(f)$ is covered by 3 affine pieces $z_i \neq 0$. Assume $z_0 = 1$. The reduced fibre of $C^* \to X$ over a point $z = (z_0 : z_1 : z_2)$ contains a point $z^* = (1, x, y)$. If $\mathbb{G}_m$ acts on this fibre with non-trivial kernel H, then $(t^{q_1}, t^{q_2}x, t^{q_3}y) = (1, x, y)$ for all $t \in H$. This implies $t^{q_1} = 1$. If $x, y \neq 0$, then it also implies that $t^{q_2} = t^{q_3} = 1$. Since $d = (q_1, q_2, q_3) = 1$, we get $t = 1$. So, assume that $y = 0$. We have $\#H = (q_1, q_2) = e_3$ if $x \neq 0$ and $\#H = q_1$ otherwise. Similarily, we have to find other points with one of the coordinates equal to zero which enter in the Seifert divisor.

It remains to determine the numbers k_1, k_2, k_3. We know that $K_X \sim \delta D$, where $\delta = d - q_1 - q_2 - q_3$. It follows from Proposition 4.4.1 that $\delta k_i \equiv -1 \mod e_i$. This determines k_i.

To summarize, we have found the Seifert divisor

$$D - \lfloor D \rfloor = \frac{k_1}{e_1}\sum_{i=1}^{r_1} P_i + \frac{k_2}{e_2}\sum_{i=1}^{r_2} Q_i + \frac{k_3}{e_3}\sum_{i=1}^{r_3} R_i,$$

where r_i is the number of points on X with $z_i = 0$.

The Seifert divisor determines the cyclic singularities of the partial resolution of $\widehat{C}_X$. The genus of the central components is given by formula (4.4.6). By Theorem 4.3.11, their self-intersections is determined by the degree of the quasicone which we know.

Example 4.4.9. Let $f(z_0, z_1, z_2) = z_0^{12} + z_1^8 - z_1z_2^5$. We have $d = 120, q = (10, 15, 21)$. Points with coordinates $(0, y, z)$ satisfy $y^7 = z^5$. We may assume that $z = 1$, then all points $(0, y, 1)$ with $y^7 = 1$ are equivalent under our $\mathbb{C}^*$-action (take t a 7th root of unity). Thus $r_1 = 1$. There is only one point with $z_1 = 0$, the point $(0, 0, 1)$. Thus $r_2 = 1$. There are 4 points with coordinates $(x, y, 0)$. They are $(a, 1, 0)$, where $a^{12} = 1$. The points $(a, 1, 0)$ and the points $(\eta a, 1, 0)$, where $\eta^3 = 1$ represent the same points in X. We have

$$e_1 = (q_2, q_3) = 3, e_2 = q_3 = 21, e_3 = \ldots = e_7 = (q_1, q_2) = 5.$$

Since $\delta = d - |\mathbf{q}| = 120 - 46 = 74$, and $74k_i \equiv -1 \mod e_i$, we determine the Seifert divisor

$$D - \lfloor D \rfloor = \frac{1}{3}p_1 + \frac{19}{21}p_2 + \frac{1}{5}(p_3 + p_4 + p_5 + p_6).$$

Since $\deg D = 120/10 \cdot 15 \cdot 21 = \frac{4}{105}$, we get

$$\deg\lfloor D \rfloor = \frac{4}{105} - \frac{1}{3} - \frac{19}{21} - \frac{4}{5} = 2.$$

Now everything is ready. The central componet S_0 has self-intersection $\deg\lfloor D \rfloor - 6 = -4$. The central component S_∞ has self-intersection 2. Both curves has the genus equal to the number of positive integer solutions of $10a + 15b + 21c = 74$. It is easy to see that this number is zero. The exceptional curve over the vertex looks as follows.

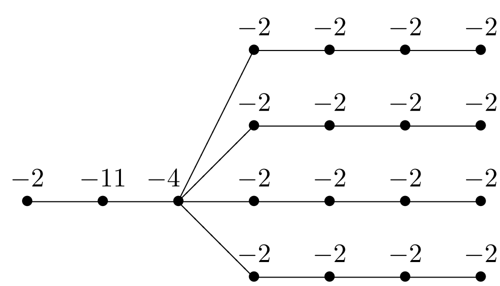


Figure 4.3:

We leave to the reader to determine the curve $\mathcal{E}^\infty$.

4.5 Finite group quotients

Let $A = \oplus_{i\in\mathbb{Z}} A_i$ be a graded algebra finitely generated over a Noetherian ring R and G be its finite group of automorphisms (preserving the grading). Then the ring of invariants A^G is a graded finitely generated R-algebra. Suppose $A = A(X, D)$, where X is a normal integral projective algebraic variety over a field k and D is a $\mathbb{Q}$-Cartier divisor on X. The group G acts naturally on the affine quasicone $C = \mathrm{Spec}\ A$. Since we view elements of A as regular functions on C, we shall denote the image of $a \in A$ under an automorphism $\sigma \in G$ by $\sigma^*(a)$. We have $\sigma^*(a) = a(g^{-1}(c))$, for any closed point $c \in C$.

For any open affine subset $D(f)$ of Spec A, an element $\sigma \in G$ transforms $D(f)$ into $D(\sigma(f))$ and the ring $A_f = \mathcal{O}_C(D(f))$ to the ring $A_{\sigma(f)}$. Obviously the homomorphism $\sigma : A_f \to A_{\sigma(f)}$ preserves the grading, hence maps each $(A_f)_i$ to $A_{\sigma(f))_i}$. Since σ transforms homogeneous ideals to homogeneous ideals and leaves the irrelevant ideal $\mathfrak{m}_0$ invariant, this implies that G acts on $X =$ Proj A and, for each $i \in \mathbb{Z}$ defines an isomorphism

$$\sigma^* : g^*(\mathcal{O}_X(i)) \to \mathcal{O}_X(i).$$

Since G acts on X, it acts naturally on $X^{(1)}$ by

$$\sigma : \sum n_x x \mapsto \sigma^*(\sum n_x x) := \sum n_x g^{-1}(x) : .$$

By $\mathbb{Q}$-linearity the action extends on the group of Weil $\mathbb{Q}$- divisors. Also it is clear that it leaves the subgroup of Cartier $\mathbb{Q}$-divisors invariant. We denote this action by $D \mapsto g^*(D)$. The action of G on A extends to the action of its fraction ring and its subfield $K = k(X)$ of homogeneous fractions. It is easy to check that it leaves invariant the group of principal divisors. Thus G acts on the group $\mathrm{Pic}(X) \otimes \mathbb{Q}$.

Since $\mathcal{O}_X(i) \cong \mathcal{O}_X(\lfloor iD \rfloor)$, and $\sigma^*(\mathcal{O}_X(\lfloor D \rfloor) \cong \mathcal{O}_X(\sigma^*(i\lfloor D \rfloor))$, we see that $\sigma^*(D)$ is linearly equivalent to D.

Lemma 4.5.1. *One can replace D with a linearly equivalent divisor D' such that $\sigma^*(D') = D'$ for any $\sigma \in G$. Two such divisors differ by the divisor of a G-rational function on X.*

Proof. Replacing D by rD for some integer r we may assume that D is a Weil divisor. For any Weil divisor D, we consider the sheaf $\mathcal{O}_X(D)$ as a subsheaf of the constant sheaf $k(X)$. An isomorphism $\mathcal{O}_X(\sigma^*(D)) \to \mathcal{O}_X(D)$ is defined by the multiplication by a unique rational function Φ_σ such that $\sigma^*(D) - D = \mathrm{div}(\Phi_\sigma)$. We have

$$(\tau\sigma)^*(D) - D = \sigma^*(\tau^*(D) - D) + \sigma^*(D) - D = \mathrm{div}(\sigma^*(\Phi_\tau)\Phi_\sigma).$$

This shows that the map $G \to k(X)^*, \sigma \mapsto \Phi_\sigma$ is a 1-cocycle of G with values in $k(X)^*$. We view G as the Galois group of the finite extension $k(X)/k(X)^G$. Applying Hilbert's Theorem 90 (see Lang]), we obtain that the cocycle is trivial, i.e. can be written in the form $\Phi_\sigma = \sigma^*(\Psi)\Psi^{-1}$ for some rational function Ψ. Replacing D with $D' = D - \mathrm{div}(\Psi)$, we obtain

$$\sigma^*(D') - D' = \sigma^*(D) - D - \mathrm{div}(g^*(\Psi)/\Psi) = \mathrm{div}(\Phi_\sigma) - \mathrm{div}(g^*(\Psi)\Psi^{-1}) = 0.$$

The last assertion is clear. □

Let G act on a normal quasi-projective algebraic variety X. Then the quotient $p_G : X \to X/G$ exists in the category of algebraic varieties. It is a normal variety (projective, if X is projective). It is constructed by choosing a G-invariant open affine cover (U_i) of X and defining X/G as the gluing together the quotients $U_i/G := \mathrm{Spec}\ \mathcal{O}(U)^G$ (see the details in [Serre, Groupes algebriques et corps de classes]).

Let D be a G-invariant $\mathbb{Q}$-Cartier divisor on X. We would like to "descent" it to a $\mathbb{Q}$-Cartier divisor D/G on X/G. The divisor D/G must satisfy the following properties: For any open subspace $V \subset X/G$, the inclusion of fields $k(X)^G$ defines the inclusion $\mathcal{A}_{X/G}(D/G)(V)$ with $\mathcal{A}_X(D)(p_G^{-1}(V))^G$. In other words, if Φ is an invariant rational function satisfying

$$\mathrm{div}(\Phi) + iD \geq 0 \ \text{ on } U = p_G^{-1}(V) \Leftrightarrow \mathrm{div}(\phi) + D/G \geq 0 \text{ on } V. \tag{4.18}$$

Let $D = \sum_{x \in X^{(1)}} m_x x$. Since D is G-invariant, we can write $D = \sum_{s=1}^t m_i(Gx_i)$, where $x_1, \ldots, x_t$ are representatives of the G-orbits of G in $X^{(1)}$ and $m_i = m_{x_i}$. Since $p_G : X \to X/G$ is a finite morphism, the image of any point of codimension 1 is a point in X/G of codimension 1. Let $y_1, \ldots, y_t$ be the images of the points $x_1, \ldots, x_t$ in $(X/G)^{(1)}$. For each point x_i let ram_i denote the ramification index of p_G. Recal that this means that the image of the maximal ideal of $\mathfrak{m}_{X/G,y_i}$ in $\mathcal{O}_{X,x_i}$ is equal to the $\mathfrak{m}_{X,x_i}^{\mathrm{ram}_i}$. We set

$$D/G = \sum_{i=1}^t (m_i/\mathrm{ram}_i) y_i. \tag{4.19}$$

It is clear that D/G satisfies property (4.18). Also we have $p_G^*(D/G) = D$, where we extend the natural homomorphism $p_G^* : \mathrm{WDiv}(X/G) \to \mathrm{WDiv}(X)$ to $\mathbb{Q}$-divisors.

Proposition 4.5.2. *Let $A = A(X, D)$ and D' be a G-invariant $\mathbb{Q}$-Cartier divisor linearly equivalent to D. Let D'/G denotes its image in X/G. Then*

$$C(X,D)/G = Spec\ A(X,D)^G \cong A(X/G, D'/G).$$

Proof. Let $f_1, \ldots, f_N$ be generators of the algebra of invariants $A(X,D)^G$. The open subsets $D(f_\alpha)$ (resp. $D_+(f_\alpha)$) form an open G-invariant cover of $C(X,D)$ (resp. X). The quotient $D(f_\alpha) \to D(f_\alpha)/G$ corresponds to the inclusion of rings $(A_{f_\alpha})^G \subset A_{f_\alpha}$ (resp, $A_{(f_\alpha)}^G \subset A_{(f_\alpha)}$). Consider the sheaves $\mathcal{A}_X(D'))_i \cong \mathcal{O}_X(i)$. Over $D_+(f_\alpha)$, its sections are rational functions Φ on X such that $\mathrm{div}(\Phi) +$

$\lfloor D' \rfloor \geq 0$ on $D_+(f_\alpha)$. It is immediate to see that $\mathcal{A}_X(D')_i^G = \mathcal{A}_{X/G}(D'/G)_i$. This implies that $\mathcal{A}_X(D')^G \cong \mathcal{A}_{X/G}(D'/G)$. But

$$A(X,D)^G = \Big(\bigoplus_{i\geq 0}\Gamma(X,\mathcal{A}_X(D'))_i\Big)^G \cong \bigoplus_{i\geq 0}\Gamma(X,\mathcal{A}_X(D'))_i^G$$

$$\cong \bigoplus_{i\geq 0}\Gamma(X,\mathcal{A}_{X/G}(D'/G))_i \cong \Big(\bigoplus_{i\geq 0}\Gamma(X,\mathcal{A}_{X/G}(D'/G))_i\Big)^G \cong A(X/G,D'/G).$$

□

Next we consider the problem when an action of G on $X = \text{Proj } A$ lifts to an action of G on A. As we saw in above, the necessary condition is that $\sigma^*(\mathcal{O}_X(i)) \cong \mathcal{O}_X(i)$ for any $i \in \mathbb{Z}$. However, it is not sufficient. For example, consider the case $X = \mathbb{P}^1$ and $G \subset \text{PGL}(2,k)$, then we know that G may not lift to an action on $\mathbb{A}_k^2 = C(X,\mathcal{O}_{\mathbb{P}_k^1}(1))$. However, we know that in this case some central extension of G lifts. Let us see that the similar thing happens in the general case.

Let $\mathcal{F}$ be any quasi-coherent sheaf on X. Suppose that, for any $\sigma \in G$, there is an isomorphism $\phi_\sigma : \sigma^*(\mathcal{F}) \to \mathcal{F}$. Consider the set $\bar{G}$ of all possible pairs Define the group by

$$(\sigma,\phi_\sigma)\circ(\tau,\psi_\tau) = (\sigma\circ\tau,\phi_\sigma\circ\sigma^*(\psi_\tau)),$$

where $\sigma^*(\psi_\tau)$ is an isomorphism $\phi_\sigma^*(\psi_\tau^*(\mathcal{F})) \to \phi_\sigma^*(\mathcal{F}))$. The projection $(\sigma,\phi_\sigma) \mapsto \sigma$ defines an extension of groups

$$1 \to \text{Aut}(\mathcal{F}) \to \tilde{G} \to G \to 1. \tag{4.20}$$

Let U be a G-invariant open subset of X. For any section $s \in \mathcal{F}(U)$ and any $\sigma \in G$ we have the section $\sigma^*(s)$ of $g^*(\mathcal{F})$ on $g^{-1}(U) = U$. An isomorphism $\phi_\sigma : \sigma^*(\mathcal{F}) \to \mathcal{F}$ defines a homomorphism $\phi_\sigma : \mathcal{F}(U) \to \mathcal{F}(U)$ sending s to $\phi_\sigma(\sigma^*(s))$. We define the action of $\tilde{G}$ on $\mathcal{F}(U)$ by

$$(\sigma,\phi_\sigma)(s) = \phi_\sigma(g^*(s)).$$

Since

$$(\tau,\psi_\tau)((\sigma,\phi_\sigma)(s)) = \psi_\tau(\tau^*(\phi_\sigma(g^*(s)))$$
$$= \psi_\tau(\tau^*(\phi_\sigma)(\tau^*(g^*(s))) = (\tau\sigma,\psi_\tau\circ\tau^*(\phi_\sigma))(s),$$

we obtain that $\tilde{G}$ acts on $\mathcal{F}(U)$.

Note that the subgroup $\text{Aut}(\mathcal{F})$ of elements $(\mathbf{id}_X,\phi_{\mathbf{id}_X})$ acts by automorphisms of $\mathcal{F}$. Assume that $\mathcal{F} = \mathcal{O}_X(D)$ for some Weil divisor D. We consider it as a subsheaf of the constant sheaf $k(X)$. Then automorphism of $\mathcal{O}_X(D)$ is obtained by the multiplication by a rational function Φ such that $D + \text{div}(\Phi) = D$. This implies that $\Phi \in k^*$ and $\text{Aut}(\mathcal{O}_X(D)) \cong k^*$. Thus (4.20) becomes a central extension

$$1 \to k^* \to \tilde{G} \to G \to 1. \tag{4.21}$$

As such, it is defined by the cohomology class $\alpha(D) \in H^2(G,k^*)$. The second cohomology group $H^2(G,k^*)$ is called the *Schur multiplier* of G. It is a finite commutative group killed by the order N of $H^2(G,k^*)$. Assume $[N] : x \mapsto x^N$ is surjective map $k^* \to k^*$. The exacts sequence $1 \to \mu_N \to k^* \to k^* \to 1$ defines an exact sequence of group cohomology $H^2(G,\mu_N) \to H^2(G,k^*) \overset{[N]}{\to} H^2(G,k^*)$. It shows that the homomorphism $H^2(G,\mu_N) \to H^2(G,k^*)$ is surjective. In terms of group extensions, this can be interpreted as follows. For each extension (4.21) there exists an extension

$$1 \to \mu_N \to G' \to G \to 1. \tag{4.22}$$

such that extension (4.21) is obtained from (4.22) by replacing G' with $\tilde{G} = G' \times_{\mu_N} k^*$ and taking the composition of the projections $G' \times_{\mu_N} k^* \to \tilde{G}' \to G$ as the projection $\tilde{G} \to G$. Its kernel is isomorphic to k^*. The image of G' in $\tilde{G}$ is a finite group of pairs (σ, ϕ_σ) such the $\phi_\sigma \in \mu_N \subset \mathrm{Aut}(\mathcal{O}_X(D))$. We have proved the following.

Proposition 4.5.3. *For any Weil divisor D such that $\sigma^*(\mathcal{O}_X(D)) \cong \mathcal{O}_X(D)$ for all $g \in G$ there exists a finite central extension G' of the form (4.22) such that the action of G on X lifts to an action of G' on $\Gamma(U, \mathcal{O}_X(D))$, where U is any G-invariant open subset of X. One may take $G' = G$ if and only if D is linearly equivalent to a G-invariant divisor D'*

Note that G' acts on X through G so its subgroup μ_N acts trivially on X.

Corollary 4.5.4. *Let G acts on X and leaves invariant divisor class of an ample Cartier $\mathbb{Q}$-divisor D on X. Then there exists a finite extension (4.22) such that the action of G on $X = \mathrm{Proj}\ A(X,D)$ lifts to an action of G' on Spec $A(X,D)$.*

Let us consider the two-dimensional case, i.e. $\dim A = A(X,D)$ is of dimension2. In this case $\dim X = 1$, and since X is normal, it is a nonsingular projective curve over k of some genus g. Let $D - \lfloor D \rfloor = \sum_{i=1}^r \frac{q_i}{e_i} p_i$. If $D \neq \lfloor D \rfloor$, we know that $\widehat{C}(X,D)$ is a $\mathbb{P}^1$-bundle over X isomorphic to $\mathbb{P}(\mathcal{O}_X(D) \oplus \mathcal{O}_X)$. Suppose $D \neq \lfloor D \rfloor$.

We will use the following well-known fact from the theory of Riemann surfaces.

Lemma 4.5.5. *Let X be a compact Riemann surface of genus g and $p_1, \ldots, p_r$ be points on X, and let $e_1, \ldots, e_r$ be some positive numbers. Assume that $g > 0$, or $g = 0$ and either $r \geq 3$, or $r = 2$ and $e_1 = e_2$. Then there exists a finite Galois cover $Y \to X$ ramified exactly over the points $p_1, \ldots, p_r$ with ramifications indices equal to $e_1, \ldots, e_r$.*

As we will see below the proof follows from the following fact from the combinatorail group theory, known as the *Fenchel Conjecture*:

(*) Let Π be a group with generators, $a_i, b_i, i = 1, \ldots, g$ and $c_1, \ldots, c_r$ and defining relations

$$[a_1, b_1] \cdots [a_g, b_g] c_1 \cdots c_r = 1, c_1^{e_1} = \ldots = c_r^{e_r} = 1.$$

Then Π contains a normal torsion-free subgroup of finite index.

We refer to the proof of this statement to [Mennicke, Inv. Math. vol. 2]. To give an idea of the proof let us consider the special case $g > 0$, where one can give a completely elementary short proof due to S. Bundgaard and J. Nielsen [Matematiske Tidskrift. vol. 50]. The case $g = 0$ was considered by R. Fox [Matematiske Tidskrift. vol. 50]. This statement follows from a much more general result of C. Siegel about finitely generated discrete subgroups of linear algebraic groups [Siegel].

Let $\pi_1(X \setminus \{p_, \ldots, p_r, x_0\})$ be the fundamental group of $U = X \setminus \{p_1, \ldots, p_r\}$. It is known that it is generated by $\pi_1(X, x_0)$ and the homotopu classes of some loops $\gamma_1, \ldots, \gamma_r$ that originate at x_0 and go around th points x_i. The defining relation is $[a_1, b_1] \cdots [a_g, b_g]\gamma_1 \cdots \gamma_r = 1$, where a_i, b_i are the standard generators of $\pi_1(X, x_0)$. Let Π be the quotient of $\pi_1(U, x_0)$ obtained by adding additional relations $\gamma_i^{e_i} = 1, i = 1, \ldots, r$. Suppose we find a normal subgroup of finite index Π' of Π such that it does not contain the cosets $\bar{\gamma}_i^s$ of the γ_i^s's in Π' if s is not divisible by e_i. Or, equivalently, the image of each γ_i under the composition $f : \pi_1(U, x_0) \to \Pi \to \Pi/\Pi' = G$ is of order e_i. It can be shown that any element of finite order in Π' is conjugate to a power of some $\bar{\gamma}_i$. This shows that the previous condition on Π' is equivalent to that Π' is torsion-free. Now, we use that the Uniformization Theorem. According to this theorem the universal cover $\tilde{U}$ of U has a structure of a simply connected one-dimensional complex manifold, and as such must be isomorphic to either $\mathbb{P}^1(\mathbb{C})$, or $\mathbb{C}$, or $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The surjection $\pi_1(U, x_0) \to G$ will define a finite a topological G-covering of 2-manifolds $p' : U' \to U$. It can be also considered as an unramified Galois G-cover of the corresponding complex manifolds, or, even better, affine complex algebraic curves. The covering p' can be extended to a ramified G-cover of compact Riemann surfaces $\bar{p}' : Y \to X$. As a cover of 2-manifolds this cover is obtained by the so-called Fox construction ([Fox], A Symposium in honor of Lefschetz]. In algebraic geometry this construction is nothing more than the normalization of X in the field of rational functions on U'. Let z be a local parameter at the point p_i and u be a local parameter at a point $q_i \in Y$ lying over p_i. Fix a point $y_0 \in U'$ over x_0 and consider elements of the fundamental group $\pi_1(U', y_0)$ as the homotopy classes of loops originating at y_0. Consider the homotopy class $[\gamma_i']$ of a loop γ_i' starting at y_0, moving to the point q_i' on the circle $|u - u(q_i)| = \epsilon$, turning around this point in counterwise way, and returning back. Its image in U is equal to e_ith power of the homotopy class of the loop γ_i. We can choose the local parameters such that $z = u^e$, where e is the ramification index of the cover $Y \to X$ at q_i. The image of γ' is loop going e times around the circle $|z - z(p_i)| = \epsilon 1/e$. However, we know that the image of $[\gamma_i']$ in $\pi_1(U, x_0)$ is equal to the $[\gamma_i]^{e_i}$. This shows that $e = e_i$. Thus the cover $Y \to X$ is exactly what we needed to construct.

Let us prove the Fenchel conjecture in the case $g > 0$. Let us first construct a normal subgroup of finite index Π'_i of Π such that the image of c_i in Π/Π'_i is of order e_i. After that we take for the intersection of the subgroups $\Pi' = \Pi'_1$. It is easy to see that it is a normal subgroup of finite index and the image of each c_i in the quotient group is of order e_i. It is enough to construct Π'_1. Let $e_1 = e$ and D_{4e} be the dihedral group of order $4e$ generated by u, v with defining relations $uvu^{-1} = v^{-1}, v^{2e} = u^2 = 1$. Take a new set of generators in Π by replacing c_1 with $c'_1 = [a_1, b_1]c_1$, so that $c_1 = [a_1, b_1]^{-1}c'_1$. Let F be the free group with generators $a_i, b_i, i = 1, \ldots, g, c_1, \ldots, c_r$. Consider the homomorphism $\phi : F \to D_{2e}$ that sends a_1 to u and b_1 to v, and all other generators to 1. We have

$$\phi(c_1^e) = \phi([a_1, b_1]^{-1}c'_1)^e = \phi([a_1, b_1])^{-e} = (uvu^{-1}v^{-1})^{-e} = v^{-2e} = 1,$$

$\phi(c_i) = 1, \phi([a_i, b_i]) = 1, i \neq 1$, and

$$\phi([a_1, b_1] \cdots [a_g, b_g]c_1 \cdots c_r) = \phi([a_1b_1]c_1) = \phi(c'_1) = 1$$

we see that ϕ defines a homomorphism $\bar{\phi} : \Pi \to D_{2e}$. By the above, $\bar{\phi}(c_1^n) = v^{-2n} = 1$ if and only if n is divisible by $2e$. Therefore, the kernel of $\bar{\phi}$ is our group Π'_1.

Remark 4.5.6. The group Π from (*) is a finite group if and only if $g = 0$ and

$$\sum_{i=1}^{r} e_i^{-1} > r - 2.$$

It is a cyclic group of order e_1 if $r = 1$ and of order (e_1, e_2) if $r = 2$. In the remaining cases it is isomorphic to one of the polyhedral group of types D_n, T, O, I. The curve Y is isomorphic to $\mathbb{P}^1$.

If

$$\sum_{i=1}^{r} e_i^{-1} = r - 2,$$

the group Π is isomorphic to one of the groups $\mathbb{Z}^2 \rtimes \mathbb{Z}/n\mathbb{Z}$, where $n = 2, 3, 4, 6$ correspond to the cases $(r; e_1, \ldots, e_r) = (4; 2, 2, 2, 2), (3, 3, 3, 3), (3; 2, 4, 4)$, and $(3; 2, 4, 6)$. The homomorphism $\mathbb{Z}/n\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}^2)$ defining the semi-direct product is given by sending a generator of $\mathbb{Z}/n\mathbb{Z}$ to the automorphism $a + be^{\pi i/n}i \mapsto e^{2\pi i/n}(a + be^{\pi i/n}$, where we identify $\mathbb{Z}^2$ with the ring $\mathbb{Z} + e^{\pi i/n}\mathbb{Z}$, $n = 2, 3, 4, 6$ integers. The curve Y in this case is an elliptic curve isomorphic to the complex torus $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$, where τ is any complex number with positive imaginary part, or $\tau = e^{\pi i/n}$ with $n = 3, 4, 6$.

In the remaining cases

$$\sum_{i=1}^{r} e_i^{-1} < 2g - 2 + r.$$

The group Π is isomorphic to a *Fuchsian group* of *signature* $(g; e_1, \ldots, e_r)$. This a discrete group Γ of holomorphic automorphisms of $\mathbb{D}$ with the quotient $\mathbb{D}/\Gamma$

isomorphic to a compact Riemann surface X of genus g. The cover map $\mathbb{D} \to X$ is ramified over a set of points $p_1, \ldots, p_r$ with ramification indices $e_1, \ldots, e_r$.

Theorem 4.5.7. *Let $D - \lfloor D \rfloor = \sum_{i=1}^r \frac{k_i}{e_i} p_i$. Assume that $k = \mathbb{C}$ and $g > 0$, or $g = 0$ and either $r \geq 3$, or $r = 2, e_1 = e_2$. There exists a 2-dimensional ring $B = A(Y, D')$ with $D' = \lfloor D' \rfloor$ and a finite group G of automorphisms of Y such that $B^G \cong A$. Moreover G acts freely on the punctured affine quasicone $C(Y, D')^*$.*

Proof. Let $f : Y \to X$ be a finite G-cover constructed in Lemma 4.5.5. We have $D' = f^*(D) = f^*(\lfloor D \rfloor) + \sum_{i=1}^r k_i (f^{-1}(p_i))_{\text{red}}$. This is a G-invariant Cartier divisor on Y. The action of the group G on Y extends to an action on $A(Y, D')$. By Proposition 4.5.2, $C(X, D) \cong C(Y, D')/G$.

It remains to check the last assertion. For any point $p \in X$, the set-theoretical fibre $f^{-1}(p)$ is a G-orbit. Thus the image of a point x in $C(Y, D')^*$ with non-trivail stabilizer subgroup under the map $\pi : C(Y, D')$ is equal to a point y_i lying in a fibres $f^{-1}(p_i)$ for some p_i. The stabilizer subgroup G_x is a subgroup of G_{y_i}. The order of G_{y_i} is equal to the ramification index e_i. Let z_i be a local parameter at x_i. Since G_{y_i} is a finite group we can find a G_{y_i}-invariant affine open neighborhood U_i of y_i, such that $\mathcal{O}_Y(U_i)$-module $\mathcal{O}_Y(D')(U_i)$, considered as a subscheaf of $\mathbb{C}(Y)$, is generated by $z_i^{-k_i}$. Let σ be a generator of G_{y_i}. It acts on the local ring $\mathcal{O}_{Y, y_i}$ by sending z_i to $z_i \zeta$, where $\zeta = e^{2\pi i / e_i}$. We have

$$\pi^{-1}(U_i) \cong \text{Spec } \oplus_{i \neq 0} \mathcal{O}_Y(iD')(U_i) \cong \mathcal{O}_Y(U_i)[T, T^{-1}],$$

where the variable T corresponds to the generator $z_i^{-k_i}$. The group G_{y_i} acts on the fibre $\pi^{-1}(y_i)$ by sending T to $\zeta^{-k_i} T$. Since we assume in the definition of the Seifert divisor that $(k_i, e_i) = 1$, the action has no fixed points. $\square$

Remark 4.5.8. The proof also shows how G_{y_i} acts on $C(Y, D')^{\pm}$. We have $\pi_{\pm}^{-1}(U_i) \cong \text{Spec } \mathcal{O}_Y(U_i)[T^{\pm 1}]$. Let $y_{\pm} = s_{\pm}(y_i)$ be the intersection point of the section $s_{\pm}(Y)$ with $\pi_{\pm}^{-1}(U_i)$. Let us identify the formal completion of the local ring $\mathcal{O}_{C(Y, D')^+, y_+}$ with $\mathbb{C}[[t, u]]$. Then G_{y_i} acts on this ring by $(t, u) \mapsto (\zeta t, \zeta^{\mp k_i} u)$. Thus the image x_+ of y_+ (resp. the image of x_- of y_-) in $C(X, D)^+$ (resp. $C(X, D)^-$) is the point at the section $s_+(X)$ (resp. $s_-(X)$) formally isomorphic to the cyclic quotient singularity $C_{e_i, e_i - k_i}$ (resp. C_{e_i, k_i}). This confirms what we learnt in Example .

Example 4.5.9. Let $f : Y \to X$ as in Theorem 4.5.7. By Hurwitz fomula

$$K_Y = f^*(K_X + \sum_{i=1}^r (1 - e_i^{-1}) p_i) = f^*(K_X) + \sum_{i=1}^r (e_i - 1)(f^{-1}(p_i))_{\text{red}}.$$

Thus K_Y is a G-invariant Cartier divisor. Assume K_X is of positive degree, and hence ample. This occurs if and only if

$$\deg K_X + \sum_{i=1}^r (1 - e_i^{-1}) = 2g + r - 2 - \sum_{i=1}^r e_i^{-1} > 0.$$

The curve Y is of genus $g' > 1$ such that $2g' - 2 = |G|(2g + r - 2 - \sum_{i=1}^r e_i^{-1})$. Its fundamental group is a torsion-free subgroup of finite index of a Fuchsian group of signature $(g; e_1, \ldots, e_r)$ with the quotient group isomorphic to G. Let

$$D = K_X + \sum_{i=1}^r (1 - e_i^{-1}) p_i.$$

We obtain that

$$C(X, D) = C(Y, K_Y)/G.$$

By definition from section 4.4, $C(X, D)$ is a canonical affine quasicone. The previous isomorphism gives one more motivation for this definition. The surface $\widehat{C}(X, D)$ admits a resolution $\widetilde{C}(X, D) \to \widehat{C}(X, D)$ with the intersection graph

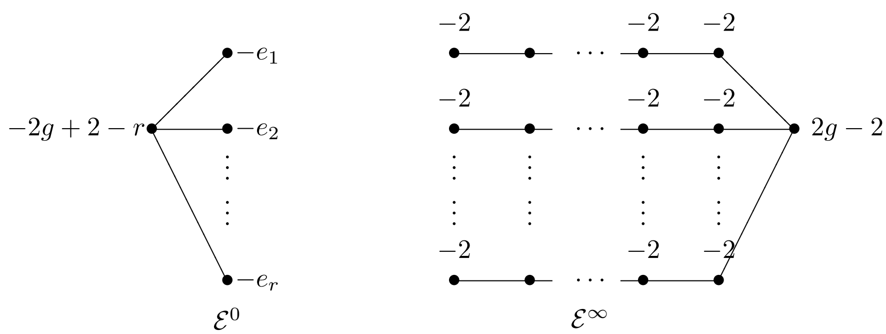


Figure 4.4:

The central curves here are of genus g and the number of (-2)-vertices in each ith arm (starting from the top one) of the diagram of $\mathcal{E}^\infty$ is equal to e_i.

Next we assume that $-K_Y$ is ample. This of course implies that $Y \cong X \cong \mathbb{P}^1$. We have

$$-K_Y = f^*(-K_X + \sum_{i=1}^r (-1 + e_i^{-1}) p_i) = f^*(-K_X) + \sum_{i=1}^r (1 - e_i)(f^{-1}(p_i))_{\mathrm{red}}.$$

Thus $-K_Y$ is a G-invariant Cartier divisor. This occurs if and only if

$$\deg K_X + \sum_{i=1}^r (e_i^{-1} - 1) = 2 - r + \sum_{i=1}^r e_i^{-1} > 0.$$

The group G is finite and isomorphic to a polyhedral group. By definition, $C(X, D)$ is an anticanonical affine quasicone. It is isomorphic to one of the Klein surfaces. Let

$$D = -K_X + \sum_{i=1}^r (e_i^{-1} - 1) p_i.$$

We obtain that

$$C(X, D) = C(Y, -K_Y)/G.$$

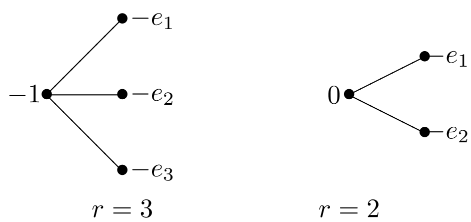


Figure 4.5:

The surface $\widehat{C}(X,D)$ admits a resolution $\widetilde{C}(X,D) \to \widehat{C}(X,D)$ with the intersection graph $\mathcal{E}^\infty$ equal to

The graph of $\mathcal{E}^0$ is the Dynkin diagram of type $A_{e_1+e_2-1}$ if $r=2$ and of type D_n if $(e_1,e_2,e_3)=(2,2,n-2), n\ge 4)$, E_6 if $(e_1,e_2,e_3)=(2,3,3)$, E_7 if $(e_1,e_2,e_3)=(2,3,4)$, E_8 if $(e_1,e_2,e_3)=(2,3,5)$.

Note that we cannot apply the construction of Theorem 4.5.8 to the case $r=2$ unless $e_1=e_2$. Nevertheless, the corresponding graded ring $A(\mathbb{P}^1,D)$ is isomorphic to $k[T_1,T_2,T_3]/(T_2T_3-T_1^{e_1+e_2})$, where $\deg T_1 = 1, \deg T_2 = e_1$, $\deg T_3 = e_2$. If $e_1+e_2=2n$ is even, we can regrade it by setting $\deg T_1 = 1, \deg T_2 = \deg T_3 = n$. Then $C(\mathbb{P}^1,D)\cong C(\mathbb{P}^1,\frac{1}{n}p_1+\frac{1}{n}p_2)$ and we can apply the theorem. Of course, in this case the affine surface is a Klein surface of type A_{2n-1}. If $e_1+e_2=2n+1$ is odd, we can regrade the ring by setting $\deg T_1=2, \deg T_2=\deg T_3=n+1$. This gives $C(\mathbb{P}^1,D)\cong C(\mathbb{P}^1,\frac{1}{n+1}p_1+\frac{1}{n+1})$ and we can apply the theorem. In this case the affine surface is a Klein surface of type A_{2n}.

Remark 4.5.10. The pair (Y,G) constructed in Theorem 4.5.7 is far from being unique. We assume that Y is of genus $g(Y)>1$. Let $u:\mathbb{D}\to Y$ be the universal cover, and L be the holomorphic line bundle on $\mathbb{D}$ equal to the inverse image of the line bundle $\mathbb{V}(\mathcal{O}_Y(D'))$ under the holomorphic map u. Let Π' be the fundamental group of Y and Y be group extension

$$1\to\Pi'\to\Pi\to G\to 1$$

obtained from lifting automorphisms of Y to the universal cover. The group Π is a Fuchsian group of signature $(g;e_1,\ldots,e_r)$ and Π' is its torsion free subgroup of finite index. The group Π acts on holomorphic sections of tensor powers L^n of L. Since $\mathbb{D}$ is holomorphically, the line bundle is isomorphic to the trivial bundle $\mathbb{D}\times\mathbb{C}$ with the action of Π on L^n given by $(z,t)\mapsto(g\cdot z,a_g(z)^nt)$, where $g\mapsto a_g(z)$ is map $a:G\to\mathcal{O}^{hol}(\mathbb{D})^*$ satisfying the cocycle condition

$$a_{g'g}(z)=a_{g'}(g(z))a_g(z).$$

A different choice of the trivialization of L changes a to a cohomologous cocycle $a_g(z)\psi(g\cdot z)\psi(z)^{-1}$, where $\psi(z)\in\mathcal{O}^{hol}(\mathbb{D})^*$. Let $\mathfrak{a}$ denotes the cohomology class of a in $H^1(G,\mathcal{O}^{hol}(\mathbb{D})^*)$. A holomorphic section of L^n is a holomorphic map $z\mapsto(z,\phi(z)$ and can can be identified with a holomorphic function $\phi(z)$

on $\mathbb{D}$. An element of $\Gamma(\mathbb{D}, L^n)^\Pi$ is a holomorphic function $\phi(z)$ satisfying

$$\phi(z) = \phi(g \cdot z) a_g(z)^n, \quad \forall g \in \Pi.$$

A function of this sort is called an *automorphic form* of *weight* n with *automorphy factor* $(a_g(z))$. We graded algebra of automorphic form with the class of automorphy factor $\mathfrak{a}$ is

$$A(\Pi, \mathfrak{a}) = \bigoplus_{n \geq 0} \Gamma(\mathbb{D}, L)^n$$

One can show that there is an isomorphism of graded algebras

$$A(\pi, \mathfrak{a}) \cong C(X, D).$$

The conjugacy class of Π, considered as a subgroup of holomorphic automorphisms of $\mathbb{D}$, and the class of $\mathfrak{a}$ of automorpy factors are defined uniquely from (X, D).

For example, the canonical affine quasicone $C(X, D)$ is defined by the *canonical automorphy factor*

$$a_g(z) = \frac{dg}{dz},$$

where g is considered as a holomorphic function $\mathbb{D} \to \mathbb{D}$. Replacing $\mathbb{D}$ with isomorphic manifold equal to the upper half-plane $\mathcal{H} = \{z = a + bi \in \mathbb{C} : b > 0\}$, we obtain a familiar definition of an automorphic form of weight n with a respect to a discrete subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})$: a holomorphic function $\phi(z) : \mathcal{H} \to \mathbb{C}$ such that

$$f(\frac{\alpha z + \beta}{\gamma z + \delta}) = (\gamma z + \delta)^{2n} \phi(z),$$

for all Moebius transformations $g : z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}$ from Γ.

4.6 Exercises

4.1 Resolve the singularar points of all Kleinian surfaces by Jung's method.

4.2 Let $\widetilde{C}(\mathbb{P}^1, D)$ be a minimal resolution of a Klein affine quasicone corresponding to the divisor $D = -p + \frac{1}{e_1} p_1 + frac1e_2 p_2 + frac1e_3 p_3$. Show that there is a birational morphism such that the image of the curve $\mathcal{E}_0$ is a line, and the image of the curve $\mathcal{E}_\infty$ is a point.

4.3 Let $\widetilde{C}(\mathbb{P}^1, D)$ be a minimal resolution of an affine quasicone corresponding to the divisor $D = -2p + \sum_{i=1}^r \frac{e_i - 1}{e_i} p_i$. Show that there is a birational morphism $\widetilde{C}(\mathbb{P}^1, D) \to \mathbb{P}^2$ such that the image of the curve $\mathcal{E}_0$ is a point, and the image of the curve $\mathcal{E}_\infty$ is a line.

4.4 Let A be a complete intersection $K[z_1, \ldots, z_{n+r+1}]/(f_1, \ldots, f_n)$ of multidegree $d = (d_1, \ldots, d_n)$ and weights $q = (q_1, \ldots, q_{n+r+1})$. Show that the coefficient at $(1-t)^r$ in the Laurent expansion of $P_A(t)$ is equal to $(|\mathbf{d}| - |\mathbf{q}|)/2$.

4.5 Find a minimal resolution of the surface $x^9 + y^3 + z^2 + x^3 y^3 = 0$.

4.6 Find the Poincaré series of the invariant ring $\mathbb{C}[x,y]^{\mu_n}$ corresponding to the cyclic quotient singularity $C_{n,k}$

4.7 Show that the weighted projective surface $\mathbb{P}(1,1,n)$ is isomorphic to the projective quasicone associated to the ring $A(X,D) = k[T_0,T_1]$ with $\deg T_0 = 1, \deg T_1 = n$. Using this show that $\widetilde{C}(\mathbb{P}^1, D) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1})$.

4.8 Let A be a normal graded algebra over $K = A_0$. Use the Noether Normalization Theorem to show that A is isomorphic to a finitely generated module over a graded polynomial algebra $S = k[x_1, \ldots, x_n]$ with $\deg x_i = a_i > 0$. Show that A is Cohen-Macaulay if and only if generators can be chosen in such a way that A is a free graded S-module. Show that in this case

$$\prod_{i=1}^{n}(1 - t^{a_i})P_A(t) = \sum t^{d_i},$$

where d_i are the degrees of homogeneous free generators of A.

4.9 Let $A = k[T_0,T_1,T_2](T_0^2 + T_1^3 + T_2^6)$ Find a group G, a nonsingular curve Y, an ample Cartier divisor D on Y such G acts freely on $C(Y,D)^*$ and $C(Y,D)/G \cong \text{Spec } A$.

Lecture 5

McKay graphs

5.1 Linear representations of finite groups

Let G be a finite group. Recall that a *linear representation* of G in a vector space V over a field F is a homomorphism of groups

$$\rho : G \to \mathrm{GL}(V), \rho(g)(v) := g \cdot v.$$

Equivalently, it is a structure on V of a left module over the *group algebra* $F[G]$. Recall that $F[G]$ is the linear space F^G of functions on G with values in F with the multiplication law defined by the convolution of functions

$$(\phi * \psi)(x) = \sum_{gg'=x} \phi(g)\psi(g').$$

The standard basis in $F[G]$ is formed by characteristic functions e_g of the singletons $\{g\} \subset G$. The multiplication law is determined by the rules $e_g * e_{g'} = e_{gg'}$. A structure of a $F[G]$-module on V is defined by

$$(\sum a_g e_g) \cdot v = \sum a_g \rho(g)(v), \quad \forall v \in V.$$

All terminology of modules over rings is translated into the language of representations. We can speak about *direct sum* or *tensor product* of linear representations, an *irreducible linear representation* (= simple module), subrepresentation, homomorphism and so on. When we use the notation V for a linear representation we assume that V is a vector spaces endowed with a structure of a $F[G]$-module. Otherwise we use the notation ρ for a homomorphism of groups $G \to \mathrm{GL}(V)$. For example, we may write $\rho \oplus \rho'$ or $V \oplus V'$ to denote the direct sum of representations.

The *dimension* of a representation is $\dim V$. We will be interested only in finite-dimensional representations. The first non-trivial result is the following.

Theorem 5.1.1. *(F. Maschke) Assume that $|G|$ is coprime to the characteristic of F. Every linear representation is isomorphic to a direct sum of irreducible subrepresentations.*

Proof. The space of linear maps $\mathcal{L}(V, W)$ has a natural structure of a $F[G]$-module via

$$(g \cdot f)(x) = g \cdot f(g^{-1} \cdot x).$$

We have

$$\mathcal{L}(V, W)^G = \mathrm{Hom}_{F[G]}(V, W).$$

Let W be a $F[G]$-submodule of V and $p : V \to W$ be a projection operator. Let

$$\tilde{p} = \frac{1}{|G|} \sum_{g \in G} g \cdot p.$$

This is the standard "averaging operation". It gives $g \cdot \tilde{p} = \tilde{p}$. Also, for any $w \in W$,

$$\tilde{p}(w) = \frac{1}{|G|} \sum_{g \in G} g \cdot p(g^{-1} \cdot w) = \frac{1}{|G|} \sum_{g \in G} g \cdot (g^{-1} \cdot w) = \frac{1}{|G|} |G| w = w.$$

Thus the kernel K of $\tilde{p}$ is a $F[G]$-submodule and $V = W \oplus K$ as $F[G]$-modules. Starting from an irreducible submodule, we find the complementary submodule, and proceed by induction on the dimension of V. □

Let $\mathrm{Ir}(G)$ denotes the set of isomorphism classes of irreducible representations of G. We will show that it is a finite set of cardinality equal to the cardinality of the set of conjugacy classes of G. First we see that non-isomorphic irreducible representations "do not talk to each other".

Lemma 5.1.2. *(I. Schur). Assume F is algebraically closed. Let $f : V \to W$ be a nonzero homomorphism of irreducible representations. Then f is the composition of an isomorphism $\phi : V \to W$ and a scalar endomorphism $c\ \mathbf{id}_V$.*

Proof. The image $f(V)$ is a submodule of W, and the kernel $\mathrm{Ker}(f)$ is a submodule of V. Since V and W are irreducible, none of them is a proper submodule. Since f is nonzero, $\mathrm{Ker}(f) = \{0\}$ and $f(V) = W$. Thus f is an isomorphism. Obviously we may assume that $V = W$. Let c be an eigenvalue of f (here we use that F is algebraically closed). The map $f - c\ \mathbf{id}_V \in \mathrm{Hom}_{F[G]}(V, V)$ and has non-trivial kernel. Since V is irreducible, the kernel is equal to V. Thus $f - c\ \mathbf{id}_V$ is the zero map. □

We will need the following corollary, which sometimes also is referred to as the Schur Lemma.

Corollary 5.1.3. *Let $\rho : G \to GL(V)$ be a linear irreducible representation. Then the image of the center of G is contained in the center of $GL(V)$.*

Proof. Let z be an element of the center of G. For any $g \in G$ we have

$$\rho(z) = \rho(g \cdot z \cdot g^{-1}) = \rho(g) \circ \rho(z) \circ \rho(g)^{-1}.$$

Thus $\rho(z) : V \to V$ is an automorphism of the representation ρ. By Schur's Lemma, it must be a scalar automorphism, i.e. an element of the center of $\mathrm{GL}(V)$. □

Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation of G. For any $g \in G$ let

$$\chi_\rho(g) = \mathrm{Tr}(\rho(g)), \text{ the trace of the linear operator } \rho(g).$$

The function $\chi_\rho : G \to F$ is called the *character* of the linear representation ρ. It is a *central function* on G, i.e. constant on conjugacy classes of G. We will identify a central function on G with a function on the set $C(G)$ of conjugacy classes of G. Thus a character is a special function in the linear space $F^{C(G)}$ of central functions. A character of an irreducible representation is called an *irreducible character*. We denote the set of irreducible characters of G by $G^\sharp$.

Let $R(G)$ be the Grothendieck ring of $F[G]$-modules. Its additive group is a free abelian group generated by isomorphism classes $[V]$ of linear representations V of G modulo the subgroup of elements of the form $[V \oplus W] - [V] - [W]$. The product $[V] \cdot [W]$ is defined to be $[V \otimes_{F[G]} W]$.

It is easy to check that a character defines a homomorphism of rings

$$\chi : R(G) \to F^{C(G)}, \ [V] \mapsto \chi_{\rho_V}.$$

From now on we assume that $F = \mathbb{C}$. Define a hermitian inner product on the space of central functions $\mathbb{C}^{C(G)}$ by

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g) \overline{\psi(g)}, \tag{5.1}$$

where the overline denotes the complex conjugate. Obviously,

$$\langle \phi, \phi \rangle = \frac{1}{|G|} \sum_{g \in G} |\phi(g)|^2 > 0.$$

Thus the inner product is a unitary product on the space of central functions $\mathbb{C}^{C(G)}$.

The next theorem is one of the main results of the theory.

Theorem 5.1.4. *The irreducible characters form an orthonormal basis in the space of central functions.*

Proof. The proof consists of two parts. In the first part we prove that irreducible characters form an orthonormal set. In the second part we prove that the orthogonal complement of the span of irreducible characters is the zero subspace.

Let $f : V \to W$ be a nonzero linear map between linear representations of G (not necessary a homomorphism of representations). Define a new linear map by

$$f_0 = \frac{1}{|G|} \sum_{g \in G} \rho_V(g)^{-1} \circ f \circ \rho_W(g). \tag{5.2}$$

It is immediately checked that f_0 is a linear map of $\mathbb{C}[G]$-modules. Assume that $V \not\cong W$. By Schur's Lemma, $f_0 = 0$ for all f.

Fix a basis in V and a basis in W and let $[T]_{ab}$ denote the ab-entry of the matrix of a linear operator T. We have

$$0 = [f_0]_{ab} = \frac{1}{|G|} \sum_{g\in G,c,d} [\rho_V(g)^{-1}]_{ac} \circ [f]_{cd}[\rho_W(g)]_{db}.$$

Take f such that $[f]_{ab} = 1$ and all other entries are equal to zero. Then we get, for all a, b,

$$0 = \frac{1}{|G|} \sum_{g\in G} [\rho_V(g)^{-1}]_{aa}[\rho_W(g)]_{bb},$$

hence

$$0 = \frac{1}{|G|} \sum_{g\in G} (\sum_a [\rho_V(g)^{-1}]_{aa})(\sum_b [\rho_W(g)]_{bb}) = \langle \chi_{\rho_V}, \chi_{\rho_W}\rangle.$$

Here we use that the trace of an operator T is equal to the sum of eigenvalues, and hence the trace of T and T^{-1} are complex conjugates if the order of T is finite. Thus ρ_V and ρ_W are orthogonal.

Now assume that $V \cong W$. Without loss of generality we may assume $V = W$. Taking the traces of both sides in (5.2), we get $\mathrm{Tr}(f_0) = \mathrm{Tr}(f)$. By Schur's Lemma f_0 is a scalar operator, hence

$$f_0 = \frac{\mathrm{Tr}(f)}{\dim V} \mathbf{id}_V,$$

and the same argument as above gives

$$[f_0]_{ab} = \frac{\delta_{ab}}{\dim V} = \frac{1}{|G|} \sum_{g\in G} [\rho_V(g)^{-1}]_{aa}[\rho_V(g)]_{bb}$$

and

$$\langle \chi_{\rho_V}, \chi_{\rho_W}\rangle = \frac{1}{|G|} \sum_{g\in G} (\sum_a [\rho_V(g)^{-1}]_{aa})(\sum_b [\rho_W(g)]_{bb}) = \frac{\dim V}{\dim V} = 1.$$

Now let us prove the second half. Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation. Set

$$\rho_f = \sum_{g\in G} \overline{f(g)}\rho(g).$$

For any $s \in G$, we have

$$\rho(s)\rho_f\rho(s)^{-1} = \sum_{g\in G} \overline{f(g)}\rho(s)\rho(g)\rho(s)^{-1} = \sum_{g\in G} \overline{f(g)}\rho(sgs^{-1})$$

$$= \sum_{g\in G} \overline{f(sgs^{-1})}\rho(sgs^{-1}) = \rho_f.$$

Thus the map $\rho_f : V \to V$ is an automorphism of the $\mathbb{C}[G]$-module V. By Schur's Lemma, it must be equal to $c(\rho)\mathbf{id}_V$ for some constant $c(\rho)$. Computing the trace of ρ_f we find

$$c(\rho)\dim V = \mathrm{Tr}(\rho_f) = \sum_{g\in G} \overline{f(g)}\chi_\rho(g) = |G|\langle f, \chi_\rho\rangle.$$

If $\langle f, \chi_\rho\rangle = 0$ for all irreducible representations ρ we get that $c(\rho) = 0$ for all irreducible representations, and hence for any representation ρ we get $\rho_f = 0$. Now take ρ to be equal to $\rho_{\mathrm{reg}} : G \to \mathrm{GL}(V)$, where $V = \mathbb{C}[G]$ considered as a module over itself (it is called the *regular representation* of G). By definition,

$$\rho_{\mathrm{reg}}(g)(\sum a_s e_s) = \sum a_s e_{gs}.$$

Taking the value of ρ_f at $e_1 \in \mathbb{C}[G]$, we get

$$0 = \rho_f(e_1) = \sum_{g\in G} \overline{f(g)}\rho(g)(e_1) = \sum_{g\in G} \overline{f(g)}e_g.$$

Since the elements $e_g \in G$ in the group algebra $\mathbb{C}[G]$ form a basis, we obtain that $f \equiv 0$. □

Corollary 5.1.5. *The number of non-isomorphic irreducible representations is equal to the number of conjugacy classes of G.*

Let $\mathrm{Ir}(G) = \{\rho_1, \ldots, \rho_c\}$ and $G^\sharp = \{\chi_1, \ldots, \chi_c\}$ be the set of the corresponding irreducible characters.

Let ρ be a linear representation. We know that

$$\rho \cong \bigoplus_{i=1}^{c} \rho_i^{\oplus m_i},$$

where m_i are non-negative integers. The corresponding element $[\rho] \in R(G)$ can be written in the form

$$[\rho] = \sum_{i=1}^{c} m_i[\rho_i].$$

The number m_i is called the *multiplicity* of ρ_i in ρ and is denoted by $\mathrm{mult}_{\rho_i}\rho$. It is clear that

$$\dim\rho = \sum_{i=1}^{c} \mathrm{mult}_{\rho_i}\rho \dim\rho_i. \tag{5.3}$$

Taking the characters, we get

$$\chi_\rho = \sum_{i=1}^{c} m_i\chi_i.$$

Since $(\chi_i)_{i=1,\ldots,c}$ is an orthonormal basis, we obtain

$$m_i = \langle\chi_\rho, \chi_i\rangle. \tag{5.4}$$

Note that this implies that a central function ϕ is the character of an irreducible representation if and only if $|\phi| = \langle \phi, \phi \rangle = 1$.

Corollary 5.1.6. *Let $n_1, \ldots, n_c$ be the dimensions of irreducible representations of G. Then*

$$|G| = n_1^2 + \ldots + n_c^2.$$

Proof. Consider the regular representation ρ_{reg} of G. Since $\rho(g)(e_x) = e_{gx} \neq e_x$ for any $g \neq 1$, the character of $\chi_{\text{reg}} = \rho_{\text{reg}}$ is equal to the characteristic function of the subset $\{1\}$ of G multiplied by $|G| = \dim \mathbb{C}[G]$. Applying (5.4), we get

$$\text{mult}_{\rho_i} \rho_{\text{reg}} = \langle \chi_{\text{reg}}, \chi_i \rangle = \frac{1}{|G|} \chi(1) \chi_i(1) = \frac{1}{|G|} |G| \dim \rho_i = \dim \rho_i.$$

It remains to apply (7.6). □

One more useful information is contained in the following.

Proposition 5.1.7. *Let d be the degree of an irreducible linear representation V of G. Then d divides the order of G.*

Proof. Let χ be an irreducible character. Its value on $g \in G$ is equal to the sum of eigenvalues, each of them is a root of unity. This implies that $\chi(g)$ is an algebraic integer. The orthogonality relation gives

$$|G|/d = \sum_{g \in G} \chi(g^{-1}) \chi(g)/d = \sum_{C \in \mathcal{C}(G)} \chi(g^{-1}) (\sum_{g \in C} \chi(g)/d), \tag{5.5}$$

where $\mathcal{C}(G)$ is the set of conjugacy classes in G. Consider $e_C = \sum_{g \in C} g$ as an element of $\mathbb{C}[G]$. Since $s^{-1} e_C s = e_C$, each e_C belongs to the center of $\mathbb{C}[G]$. Consider V as a module over $\mathbb{C}[G]$. Then the multiplication by e_C defines an endomorphism of V as a $\mathbb{C}[G]$-module. By Schur's Lemma, it is equal to $\lambda \mathbf{id}_V$. Multiplication by e_C in $\mathbb{Z}[G]$ is an endomorphism of $\mathbb{Z}^{|G|}$ and hence all eigenvalues of e_C are algebraic integers hence λ is an algebraic integer. Since the trace of e_C on V is equal to $d\lambda$, and also equal to $\sum_{g \in C} \chi(g)$, we obtain that the bracket in (5.5) is an algebraic integer. Thus the rational number $|G|/d$ is an algebraic integer, hence is an integer. □

We have noticed in the proof of the proposition that the elements e_C belong to the center of $\mathbb{C}[G]$.

Proposition 5.1.8. *The center $Z(\mathbb{C}[G])$ of $\mathbb{C}[G]$ is a vector space over $\mathbb{C}$ with basis formed by $e_C, C \in (\mathcal{G})$. Let $C_1, \ldots, C_c$ be the set of conjugacy classes. Then*

$$e_{C_i} e_{C_j} = \sum_{k=1}^{c} a_{ij}^k C_k, \tag{5.6}$$

where

$$a_{ij}^k = \frac{|C_i||C_j|}{|G|} \sum_{\chi \in G^\#} \frac{\chi(C_i)\chi(C_j)\bar{\chi}(C_k)}{\chi(1)}.$$

Proof. Let $e = \sum_{g\in G} a_g g \in Z(\mathbb{C}[G])$. For any $s \in G$ we have

$$ses^{-1} = \sum_{g\in G} a_g sgs^{-1} = \sum_{g\in G} a_g g.$$

This shows that the coefficients at elements in the same conjugacy class are equal, hence e can be written as a linear combination of elements $e_C, C \in \mathcal{C}(G)$. The elements e_C are also linearly independent since $\sum_{c\in\mathcal{C}(G)} a_C e_C = 0$ means that $\sum_{g\in G} b_g g = 0$, where $b_g = a_C$ if $g \in C$. This implies that each $b_g = 0$ and hence each $a_C = 0$.

It remains to find the structure constants a_{ij}^k. As we noticed in the proof of the previous proposition, the multiplication by $e \in C(\mathbb{C}[G])$ on a submodule of $\mathbb{C}[G]$ defining an irreducible representation V with character χ is a scalar multiplication $\lambda\mathbf{id}_V$. This allows us to consider each $\chi \in G^\#$ as a complex valued function w_χ on $C(\mathbb{C}[G])$. Its value on e is equal to λ. In particular, $w_\chi(e_C) = \chi(1)|C|\chi(C)$. The function w_χ is a homomorphism of $\mathbb{C}$-algebras $Z(\mathbb{C}[G]) \to \mathbb{C}$. Now we use the following identity (called the *second orthogonality relation* in [Isaaks]):

$$\sum_{\chi\in G^\#} \chi(g)\overline{\chi(h)} = \begin{cases} 0 & \text{if } g \text{ and } h \text{ are not conjugate} \\ |G|/|C(g) & \text{otherwise} \end{cases} \tag{5.7}$$

where $C(g)$ is the conjugacy class of g. To prove it we view the (first) orthogonality relation

$$|G|\delta_{ij} = \sum_{g\in G} \chi_i(g)\overline{\chi_j(g)} = \sum_{k=1}^{c} |C_k|\chi_i(C_k)\overline{\chi_j(C_k)}$$

as the matrix equation $XD\bar{X}^T = |G|I_c$, where $X = (\chi_i(C_j))$ and D is the diagonal matrix with the diagonal entries $|C_k|$. It shows that $\frac{1}{|G|}XD$ is the inverse of $\bar{X}^T$, hence $\bar{X}^T XD = |G|I_c$. Writing this up in terms of matrix entries we get the relation (5.7). It follows from this relation that, for any $g_i \in C_i, g_j \in C_j, g_k \in C_k$,

$$\sum_{\chi\in G^\#} \chi(g_i g_j)\overline{\chi(g_k)} = 0$$

if $g_i g_j \notin C_k$ and equal to $|G|/|C_k|$ otherwise. In the sum $e_{C_i}e_{C_j}$ there are exactly a_{ij}^k products $g_i g_j$ belonging to C_k. This gives

$$\sum_{\chi\in G^\#} w_\chi(e_{C_i}e_{C_j})\overline{\chi(C_k)} = a_{ij}^k \frac{|G|}{\chi(1)}$$

$$= \sum_{\chi\in G^\#} w_\chi(e_{C_i})w_\chi(e_{C_j})\overline{\chi(C_k)} = |C_i||C_j|/\chi(1)^2.$$

This gives the equality. □

Let H be a subgroup of G and $\rho : H \to \mathrm{GL}(V)$ be a linear representation. Consider V as a $k[H]$-module and let $W = k[G] \otimes_{k[H]} V$, where $k[G]$ is considered as a $k[H]$ defined by the restriction of the regular representation to H. The representation W of ρ is called the *induced representation* of ρ to G and is denoted by $\mathrm{Ind}_H^G(\rho)$. Let χ be the character of ρ. It is easy to compute the character $\mathrm{ind}_H^G(\chi)$ of W, the *induced character*. We have

$$\mathrm{ind}_H^G(\rho)(g) = \frac{1}{|H|} \sum_{s \in G, s^{-1}gs \in H} \chi(s^{-1}gs).$$

For example, let ρ_{reg} be the regular representation of H. Then

$$\mathrm{Ind}_H^G(\rho_{\mathrm{reg}}) = k[G] \otimes_{k[H]} k[H] \cong k[G].$$

Thus, the induced representation of a regular representation of a subgroup is the regular representation of the group.

We have the following *Frobenius Reciprocity Theorem* (see [Serre]).

Theorem 5.1.9. *Let χ be a character of H and χ' be a character of G. Then*

$$\langle \chi, res(\chi') \rangle = \langle ind_H^G(\chi), \chi' \rangle,$$

where $res(\chi')$ is the restriction of χ' to H.

5.2 McKay graphs

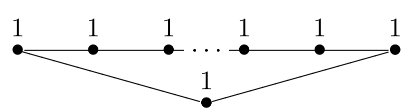
Let G be a finite group and ρ_0 be its linear representation. Define the *McKay graph* of the pair (G, ρ_0) as follows. Its vertices correspond to irreducible representations ρ_i of G. We put a label over the vertex to indicate the dimension of the representation. A vertex ρ_i is connected to the vertex ρ_j by an edge pointing to ρ_j if ρ_j is a direct summand of $\rho \otimes \rho_i$. We put the label m_{ij} over this edge if

$$\langle \chi_\rho \chi_{\rho_i}, \chi_{\rho_j} \rangle = m_{ij}.$$

We erase the arrow ends if they go in both directions and erase the label if it is equal to 1.

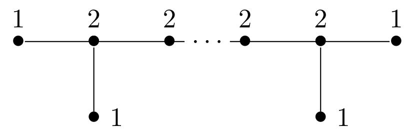
Theorem 5.2.1. *(J. McKay) Let G be a finite subgroup of $SU(2)$ and ρ_0 be its natural 2-dimensional representation defined by the inclusion $G \subset SU(2)$. Then the McKay graph of (G, ρ_0) is the following.*

- *G is cyclic of order n:*



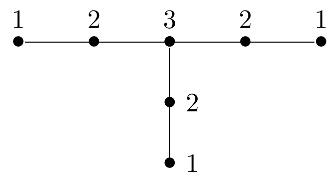
Here the bottom vertex corresponds to the trivial representation. We have n vertices.

- *G is binary dihedral of order $2n, n \geq 4$:*



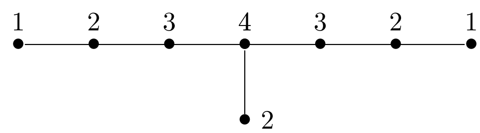
Here the right bottom vertex corresponds to the trivial representation and the vertex above it to the representation ρ. We have $n+1$ vertices.

- *G is binary tetrahedral group of order 24:*



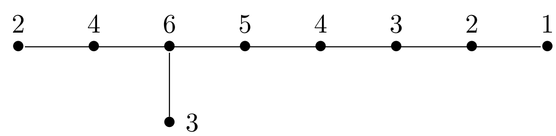
Here the bottom vertex corresponds to the trivial representation and the vertex above it to the representation ρ.

- *G is binary octahedral group of order 48:*



Here the upper left vertex corresponds to the trivial representation and the vertex on the right of it corresponds to the representation ρ.

- *G is binary icosahedral group of order 120:*



Here the upper right vertex corresponds to the trivial representation and the vertex on the left of it corresponds to the representation ρ.

As one sees the diagrams are the affine Dynkin diagrams of type A_n, D_n, E_6, E_7, E_8. One can prove that if $\Gamma(G, \rho_0)$ has unlabelled unpointed edges then G is isomorphic to a finite subgroup of $SU(2)$ and ρ_0 is its natural 2-dimensional representation.

Example 5.2.2. Let $G = C_n = \langle g_0 \rangle$ be a cyclic group of order n. Every linear representation $\rho : C_n \to \mathrm{GL}(V)$ decomposes into the direct sum of one-dimensional representations

$$V = \sum_{i=0}^{n-1} V_k,$$

where

$$V_k = \{v \in V : \rho(g_0)(v) = e^{2\pi i k/n} v\}.$$

So, G has n irreducible representations of dimension 1 identified with homomorphisms $G \to \mathbb{C}^*$. Let ρ_k be defined by sending g_0 to $e^{2\pi i k/n}, k \in \mathbb{Z}/n\mathbb{Z}$. Take $\rho_0 = \rho_1$. Obviously $\rho_1 \otimes \rho_k = \rho_{k+1}$, Thus the McKay graph $\Gamma(C_n, \rho_1)$ is equal to the graph $\tilde{A}_{n-1}$ with additional orientation by giving arrows all pointing in one direction.

On the other hand, if we consider the representation $\rho_0 : C_n \to \mathrm{SL}(2, \mathbb{C})$ given by the matrix

$$\begin{pmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n^{-1} \end{pmatrix},$$

we find that $\rho_0 = \rho_1 + \rho_{-1}$ in the above notation. Thus

$$\rho_0 \otimes \rho_k = \rho_{k-1} + \rho_{k+1}.$$

This gives us the Dynkin diagram of type $\tilde{A}_{n-1}$.

Example 5.2.3. Let us check the case of the binary tetrahedral group. Recall that G contains a central subgroup of order 2 generated by the matrix $-I_2$ and the quotient group $\bar{G}$ is isomorphic to the tetrahedral group $T \subset SO(3)$. We know that $T \cong A_4$. The group A_4 is isomorphic to the semi-direct product $2^2 : 3$ of the group $2^2 = (\mathbb{Z}/2\mathbb{Z})^2$ and the cyclic group $3 = \mathbb{Z}/3\mathbb{Z}$ (we use the notation of the ATLAS of finite groups). Let $\rho_i : G \to \mathbb{C}^*, i = 1, 2, 3$ be one-dimensional representations of G obtained as the compositions $G \to A_4 \to 3 \to \mathbb{C}^*$. The first one is the trivial representation. Let ρ_4 be the 3-dimensional representation obtained as the composition $G \to A_4 \to \mathrm{SO}(3)$, where the latter homomorphism is the natural representation of A_4 as the group of rotation symmetries of a regular tetrahedron. It is easy to see that this representation is irreducible. Let $\rho_0 = \rho_5$ be the natural representation of G in $\mathrm{SU}(2)$ and $\rho_6 = \rho_0 \otimes \rho_i, i = 2, 3$. These are 2-dimensional representations. Since ρ_0 is irreducible, it is easy to see that these representations are also irreducible. Now $|G| = 24 = 1 + 1 + 1 + 2^2 + 2^2 + 2^2 + 3^9$, so all irreducible representations are accounted for. This agrees with the number of conjugacy classes of G. The group A_4 has one conjugacy class of the identity, one class of elements of order 2 (the products of two disjoint transpositions), and two classes of elements of order 3 (g and g^2 are not conjugate in A_4 but conjugate in S_4). The class equation of A_4 is therefore $1+3+4+4$. The class equation of G is $1+1+6+4+4+4+4$ since 1 is lifted to 1 and -1. Elements of order 2 are lifted to 6 elements of order 4 forming one conjugacy class. Each conjugacy class of elements of order 2 is lifted to two conjugacy classes of elements of order 3 and 6. Thus $|C(G)| = 7$.

We already know that $\rho_0 \otimes \rho_i = \rho_{4+i}, i = 1, 2, 3$. This gives

$$m_{15} = m_{26} = m_{37} = 1.$$

Let us write a character function as a vector $(a_1, \ldots, a_7)$, where a_i is the value at the conjugacy class $C_1, \ldots, C_7$ of elements of orders $1, 2, 4, 3, 3, 6, 6$, respectively.

Comparing with computations in Lecture 1, we find that the conjugacy classes can be represented by the elements $1, -1, g_1, g_3, g_3^2, -g_3, -g_3^2$, respectively. This gives the *character table* of $\bar{A}_4$:

C(G)	1	-1	g_1	g_3	g_3^2	g_3	$-g_3^2$
order	1	2	4	3	3	6	6
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	ϵ_3	ϵ_3^2	ϵ_3	ϵ_3^2
χ_3	1	1	1	ϵ_3^2	ϵ_3	ϵ_3^2	ϵ_3
χ_4	3	3	-1	0	0	0	0
χ_5	2	-2	0	-1	-1	1	1
χ_6	2	-2	0	$-\epsilon_3$	$-\epsilon_3^2$	ϵ_3	ϵ_3^2
χ_7	2	-2	0	$-\epsilon_3^2$	$-\epsilon_3$	ϵ_3^2	ϵ_3

Table 5.1: Character table of $\bar{A}_4$

We have

$$\chi_5\chi_4 = (6, -6, 0, 0, 0, 0, 0) = \chi_5 + \chi_6 + \chi_7. \tag{5.8}$$

This gives

$$m_{45} = m_{46} = m_{47} = 1.$$

We have

$$\begin{aligned} \chi_5\chi_5 &= (4, 4, 0, 1, 1, 1, 1) = \chi_4 + \chi_1, \\ \chi_5\chi_6 &= (4, 4, 0, \epsilon_3, \epsilon_3^2, \epsilon_3, \epsilon_3^2) = \chi_4 + \chi_2, \\ \chi_5\chi_7 &= (4, 4, 0, \epsilon_3^2, \epsilon_3, -\epsilon_3, -\epsilon_3^2) = \chi_4 + \chi_3. \end{aligned} \tag{5.9}$$

This gives

$$m_{54} = m_{51} = m_{62} = m_{64} = m_{73} = m_{74} = 1.$$

Getting all edges together we get the McKay graph for (G, ρ_5) shown in the statement of the theorem.

Example 5.2.4. Let $G = \bar{A}_5$ be the binary icosahedron group. In this case finding all irreducible representation is harder than in the previous case. Any irreducible representation of A_5 gives an irreducible representation of G by composing with the projection $G \to A_5$. These are irreducible representations with kernel equal to $\{\pm 1\}$ (nothing else because A_5 is a simple group unless the representation is trivial).

The class equation of A_5 is $1+15+20+12+12$ (see [Artin, Algebra]). Here we have 15 elements of order 2, 20 elements of order 3, and 2 classes of elements of order 5. The class equation of G is $1+1+30+20+20+12+12+12+12$. We have 1 element of order 2, 30 of order 4, 20 of order 3, 20 of order 6, 24 of order 5, and 24 of order 10. Thus we have to find 9 irreducible representations of G.

We start with the natural 2-dimensional representation ρ_0 of G. Consulting Lecture 1, we find the class equation corresponds to the conjugacy classes of the following elements $1, -1, g_2, g_1g_2g_3, -g_1g_2g_3, g_1^2, g_1^4, g_1^3, g_1$. Let us denote the corresponding classes by $C_i, i = 1, \ldots, 9$. The character of ρ_0 is easy to compute. Each element of order k is conjugate to a diagonal matrix of determinant 1 with kth roots of 1 at the diagonal. We find

$$\chi_{\rho_0} = (2, -2, 0, -1, 1, 2\cos 2\pi/5, 2\cos 4\pi/5, 2\cos 3\pi/5, 2\cos \pi/5).$$

If we square χ_{ρ_0} and subtract the character of the identity element, we get the vector

$$\chi_{\rho_0} = (3, 3, -1, 0, 0, 1 + 2\cos 4\pi/5, 1 + 2\cos 8\pi/5, 1 + 2\cos 6\pi/5, 1 + 2\cos 2\pi/5).$$

By taking its norm with respect to the unitary product on $\mathbb{C}^{C(G)}$ we find that the norm is equal to 1. Hence this is the character of an irreducible representation. It is easy to see that it factors through the natural 3-dimensional representation of $A_5 \to \mathrm{SO}(3)$ (see [Artin, Algebra], p. 323-325). We refer to Artin for description of other irreducible representations of A_5. We denote their characters by $\chi_1, \ldots, \chi_5$. To complete the character table of A_5 to the character table of G, we start tensoring $\rho_0 = \rho_9$ with irreducible representations $\rho_i, i \leq 5$ and decomposing them into irreducible representations we find the remaining irreducible representations ρ_6, ρ_7, ρ_8. Here is the character table .

C(G)	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
order	1	2	4	3	6	5	5	10	10
χ_1	1	1	1	1	1	1	1	1	1
χ_2	3	3	-1	0	0	α	α	β	β
χ_3	3	3	-1	0	0	β	β	α	α
χ_4	4	4	0	1	1	-1	-1	-1	-1
χ_5	5	5	1	-1	-1	0	0	0	0
χ_6	4	-4	0	1	-1	-1	-1	-1	-1
χ_7	6	-6	0	0	0	1	1	-1	-1
χ_8	2	-2	0	-1	1	$-\alpha$	$-\beta$	α	β
χ_9	2	-2	0	-1	1	$-\beta$	$-\alpha$	β	α

Table 5.2: Character table of $\bar{A}_5$

Here $\alpha = \frac{1}{2}(-1 + \sqrt{5}), \beta = \frac{1}{2}(-1 - \sqrt{5})$. We leave to the reader to compute the McKay graph $\Gamma(G, \rho_9)$. Note that we get the same graph if we replace ρ_9 by ρ_8 which differ by an exterior automorphism of G (so the images of G in $\mathrm{SL}(2, \mathbb{C})$ are the same.

One can check the McKay theorem, case by case. However, we prefer to give a uniform proof due to T. Springer.

We start with the following well-known fact.

Lemma 5.2.5. *Let G be a finite abelian group of order > 1. For any non-trivial homomorphism $f : G \to \mathbb{C}^*$*

$$\sum_{g \in G} f(g) = 0.$$

Proof. Let $H = \mathrm{Ker}(f)$ and $g_1, \ldots, g_k$ be representatives of cosets of G modulo H. Since $\sum_{g\in G} f(g) = |H| \sum_{i=1}^{k} f(Hg_i)$, we may assume that f is injective. By decomposing G into the direct sum of finite cyclic groups, we may assume that $G = (h)$ is a cyclic group of order n. Let $f(h) = \epsilon_n$, then

$$\sum_{g\in G} f(g) = \sum_{i=0}^{n-1} f(h^i) = \sum_{i=0}^{n-1} \epsilon_n^i = 0.$$

□

Definition 5.2.1. A linear representation of a finite group G is called *admissible* if it is faithful, its character is real-valued and its restriction to the center of the group has no trivial summands.

Theorem 5.2.6. *(T. Springer) Let ρ_0 be an admissible representation of G of dimension n with character α. Define a function $F_\alpha : \mathbb{C}^G \times \mathbb{C}^G \to \mathbb{C}$ by setting*

$$F_\alpha(\phi, \psi) = \langle \alpha\phi, \psi\rangle.$$

It satisfies the following properties.

(i) F_α is a hermitian form on $\mathbb{C}^G$.

(ii) For any two irreducible representations ρ, ρ' of G with characters χ, χ',

$$F_\alpha(\chi_\rho, \chi_{\rho'}) = \mathrm{mult}_{\rho'} \rho_0 \otimes \rho \geq 0.$$

(iii) For any irreducible character $\chi \in G^\sharp$,

$$F_\alpha(\chi, \chi) = 0.$$

(iv) For any $\phi \in \mathbb{C}^G$,

$$F_\alpha(\phi, \phi) \leq n\langle \phi, \phi\rangle.$$

The equality takes place if and only if $\phi(g) = 0, g \neq 1$, i.e. $\phi = \chi_{\mathrm{reg}}$ is the character of the regular representation.

(v) Assume $|G| > 2$. If $\chi, \chi' \in G^\sharp$ are distinct, then

$$0 \leq F_\alpha(\chi, \chi') < n.$$

Proof. (i) By definition

$$F_\alpha(\phi, \psi) = \frac{1}{|G|} \sum_{g\in G} \alpha(g)\phi(g)\overline{\psi(g)}.$$

This expression is obviously linear in ϕ and 1/2-linear in ψ. Since $\alpha(g) \in \mathbb{R}$ for all g, we also have $F_\alpha(\psi, \phi) = \overline{F_\alpha(\phi, \psi)}$.

(ii) This follows from (5.4).

(iii) By definition of an admissible representation, the center $Z(G)$ of G is not trivial. By Corollary 5.1.3, the image of the center $Z(G)$ of a group G under any irreducible linear representation $\rho : G \to \mathrm{GL}(V)$ is contained in the center of the linear group. This implies that $\chi(gz) = c\chi(g)$ for any $\chi \in G^\sharp$, where $\rho(z) = c\mathbf{id}_V$. Using this we get

$$\begin{aligned} F_\alpha(\chi,\chi) &= \frac{1}{|G|}\sum_{g\in G}\alpha(g)\chi(g)\overline{\chi(g)} = |G|^{-1}\sum_{g\in G}\alpha(g)|\chi(g)|^2 \\ &= \frac{1}{|G||Z(G)|}\sum_{g\in G, z\in Z(G)}\alpha(gz)|\chi(g)|^2|c|^2 \\ &= \frac{1}{|G||Z(G)|}\sum_{g\in G}|\chi(g)|^2(\sum_{z\in Z(G)}\alpha(gz)). \end{aligned} \tag{5.10}$$

Here we use that $|c| = 1$ since c is of finite order. Write $\alpha = \sum_i m_i\alpha_i$ as a linear combination of the characters of irreducible representations ρ_i. For any $z \in Z(G)$, let $\rho_i(z)$ be defined by the scalar matrix $c_i(z)I_{\dim\rho_i}$. We have

$$\alpha(gz) = \sum_i m_i\alpha_i(gz) = \sum_i c_i(z)m_i\alpha_i(g).$$

Applying Lemma 5.2.5 to each non-trivial homomorphism $Z(G) \to \mathbb{C}^*, z \mapsto c_i(z)$, we get

$$\sum_{z\in Z(G)}\alpha(gz) = \sum_i(\sum_{z\in Z(G)}c_i(z))m_i\alpha_i(g) = 0.$$

Therefore the sum (5.10) is equal to zero.

(iv) We have $\alpha(g) = |\alpha(g)| \le n$ since $\alpha(g)$ is the sum of n roots of unity. It is equal to n if and only if all these roots of unity are equal to 1, i.e. $\rho(g) = \mathbf{id}_V$, where α is the character of ρ. Since ρ is faithful this happens only if $g = 1$. Thus $\phi = \chi_{\mathrm{reg}}$ is the character of the regular representation.

(v) By (iv), the hermitian form $n\langle\phi,\psi\rangle - F_\alpha(\phi,\psi)$ is semi-positive definite. By Schwarz's inequality

$$|n\langle\phi,\psi\rangle - F_\alpha(\phi,\psi)|^2 \le (n\langle\phi,\phi\rangle - F_\alpha(\phi,\phi))(n\langle\psi,\psi\rangle - F_\alpha(\psi,\psi)).$$

The equality takes place only if there is a linear combination of ϕ and ψ which vanishes at all $g \ne 1$. In other words, if $\phi, \psi, \chi_{\mathrm{reg}}$ are linearly dependent. Since $|G| > 2$, this is impossible. Taking $\phi, \psi \in G^\sharp$, forming an orthonormal set, and using (iv), we get $|F_\alpha(\phi,\psi)| < n$. □

Definition 5.2.2. Let G be a finite group of order > 2 and ρ_0 be its linear representation with admissible character α. Let $G^\sharp = (\chi_1, \ldots, \chi_c)$ be an ordered set of irreducible characters of G. The *McKay-Springer matrix* of (G, ρ_0) is the matrix

$$A = (a_{ij}), \quad a_{ij} = F_\alpha(\chi_i, \chi_j) - n\delta_{ij}.$$

Corollary 5.2.7. *Let ρ_0 be an admissible n-dimensional linear representation of G. The McKay-Springer matrix A of (G, ρ_0) satisfies the following properties.*

(i) A is a symmetric integral matrix.

(ii) The diagonal elements of A are equal to $-n$, the off-diagonal elements satisfy $0 \leq a_{ij} < n$.

(iii) The quadratic form defined by A is semi-definite negative with one-dimensional radical spanned by the vector $(\dim \chi_1, \ldots, \dim \chi_c)$.

Proof. (i) Follows from properties (i) and (ii) of the previous theorem.
(ii) Follows from properties (iii) and (v).
(iii) Follows from property (iv). □

Example 5.2.8. Let G be a non-cyclic subgroup of $\mathrm{SL}(2, \mathbb{C})$ and ρ_0 be the natural 2-dimensional representation of G. Obviously, ρ_0 is faithful, irreducible and its character is real valued (since $\alpha(g)$ is equal to the twice real part of an eigenvalue of g). Thus the McKay-Springer matrix is defined. Its properties tell us that it is a Cartan matrix of size $c = |C(G)|$ and the corresponding Dynkin diagram is the McKay graph of (G, ρ_0). It is easy to match the groups with the types of the corresponding Dynkin diagrams. Since we know the integral vectors spanning the radical of the corresponding quadratic form, it suffices to recognize among their coordinates the dimensions of irreducible representations of G. Also, taking into account also Example 5.2.2, we get the following table.

Group	C_n	$\bar{D}_{2n}$	$\bar{T}$	$\bar{O}$	$\bar{I}$
McKay graph	$\tilde{A}_n$	$\tilde{D}_{n+2}$	$\tilde{E}_6$	$\tilde{E}_7$	$\tilde{E}_8$

Table 5.3: McKay graphs of binary polyhedral groups

Example 5.2.9. Let $G = \mathrm{SL}(2, \mathbb{F}_7)$. Its order is 336 and its center is of order 2. The quotient group $G/Z(G)$ is isomorphic to the simple group $\mathrm{PSL}(2, \mathbb{F}_7)$ of order 168. Its presentation by generators and relations is

$$\{g_1, g_2, g_3, c : g_1^2 = g_2^3 = g_3^7 = g_1 g_2 g_3 = c, c^2 = 1\}.$$

Here

$$c = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \; g_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; g_2 = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}, \; g_3 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

We also have an element of order 8

$$g_4 = \begin{pmatrix} 5 & -1 \\ -3 & 5 \end{pmatrix} = (g_2 g_7^4)^2 g_2.$$

$C(g)$	1	c	g_1	cg_2	g_2	g_4	cg_4	cg_3	cg_3^{-1}	g_3	g_3^{-1}
$\|C(g)\|$	1	1	21	56	56	42	42	24	24	24	24
order	1	2	4	3	6	8	8	7	7	14	14
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	3	3	-1	0	0	1	1	α	β	α	β
χ_3	3	3	-1	0	0	1	1	β	α	β	α
χ_4	6	6	2	0	0	0	0	-1	-1	-1	-1
χ_5	7	7	-1	1	1	-1	-1	0	0	0	0
χ_6	8	8	0	-1	-1	0	0	1	1	1	1
χ_7	4	-4	0	-1	1	0	0	$-\alpha$	$-\beta$	α	β
χ_8	4	-4	0	-1	1	0	0	$-\beta$	$-\alpha$	β	α
χ_9	6	-6	0	0	0	$\sqrt{2}$	$-\sqrt{2}$	-1	-1	1	1
χ_{10}	6	-6	0	0	0	$-\sqrt{2}$	$\sqrt{2}$	-1	-1	1	1
χ_{11}	8	-8	0	1	-1	0	0	1	1	-1	-1

Table 5.4: The charcater table of $\mathrm{SL}(2, \mathbb{F}_7)$

The character table of G is the following.

Here $\alpha = \frac{1}{2}(-1+\sqrt{-7}), \beta = \frac{1}{2}(-1-\sqrt{-7})$. We take $\alpha = \chi_{11}$ and check that it is an admissible representation of dimension 11. We compute

$$\begin{aligned}
\alpha\chi_1 &= \alpha \\
\alpha\chi_2 &= \chi_{11} + \chi_{10} + \chi_9 + \chi_8 \\
\alpha\chi_3 &= \chi_{11} + +\chi_{10} + \chi_9 + \chi_7 \\
\alpha\chi_4 &= 2\chi_{11} + 2\chi_{10} + 2\chi_9 + \chi_7 + \chi_8 \\
\alpha\chi_5 &= 3\chi_{11} + 2\chi_{10} + 2\chi_9 + \chi_7 + \chi_8 \\
\alpha\chi_6 &= 3\chi_{11} + 2\chi_{10} + 2\chi_9 + 2\chi_7 + \chi_8 \\
\alpha\chi_7 &= 2\chi_6 + \chi_4 + \chi_2 + \chi_5 \\
\alpha\chi_8 &= 2\chi_6 + \chi_4 + \chi_3 + \chi_5 \\
\alpha\chi_9 &= 2\chi_6 + 2\chi_5 + 2\chi_4 + \chi_3 + \chi_2 \\
\alpha\chi_{10} &= 2\chi_6 + 2\chi_5 + 2\chi_4 + \chi_3 + \chi_2 \\
\alpha\chi_{11} &= 3\chi_6 + 3\chi_5 + 2\chi_4 + \chi_3 + \chi_2 + \chi_1
\end{aligned} \tag{5.11}$$

This gives the McKay-Springer matrix of $\mathrm{SL}(2,\mathbb{F}_7)$:

$$A = \begin{pmatrix} -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -8 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & -8 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -8 & 0 & 0 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & -8 & 0 & 1 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & -8 & 2 & 1 & 2 & 2 & 3 \\ 0 & 1 & 0 & 1 & 1 & 2 & -8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & -8 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & -8 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 & -8 & 0 \\ 1 & 1 & 1 & 2 & 3 & 3 & 0 & 0 & 0 & 0 & -8 \end{pmatrix}$$

We check that the vector $\mathbf{n} = (1,3,3,6,7,8,4,4,6,6,8)$ is a solution of the equation $Ax = 0$.

Remark 5.2.10. The properties of an admissible character are very restrictive. For example, they imply that the center $Z(G)$ of G must be a non-trivial group with no elements of order > 2. Otherwise the value of α on a central element of order > 2 is never real. If α is moreover irreducible, $Z(G)$ must be a cyclic group of order 2. Examples of such groups are the groups $\mathrm{SL}(2,\mathbb{F}_q)$, where $\mathbb{F}_q$ is the finite field of $q \geq 5$ elements. Consulting [Fulton-Harris], we find that all of them admit such character α of dimension $q+1$ or $q-1$.

5.3 Exercises

5.1 Compute the McKay graph of the dihedral group D_n with respect to its natural 2-dimensional representation (generated by rotations and reflections).

5.2 Check McKay's Theorem directly in the remaining cases (i.e. $D_n, \bar{O}, \bar{I}$).

5.3 Comute the McKay graph of the polyhedral groups with respect to the natural 3-dimensional reprresentation.

5.3 Compute the McKay graphs of the dihedral group D_n with respect its natural 2-dimensional representations (generated by rotations and reflections).

5.4 Show that $\Gamma(G,\rho)$ has edges with arrows in both direction if the character of ρ is real-valued.

5.5 Prove the converse of the McKay theorem: if $\Gamma(G,\rho)$ is an affine Dynkin diagram, then $\rho : G \to \mathrm{SL}(2,\mathbb{C})$ is a faithful representation.

5.6 Compute the McKay-Springer matrix for the pair (G,α), where G is as in Example 5.2.9 and $\rho = \chi_9$.

5.7 Suppose that the matrix A from Example 5.2.9 is the intersection matrix of the irreducible components of a fibre of a map from a nonsingular surface to a curve. Assume that all components are isomorphic to $\mathbb{P}^1$. What is the arithmetic genus of the curve, and hence the genus of a general fibre of the fibration. Does it exist?

5.8 Do Exercises from [Artin, Algebra], Chapter 9.

Lecture 6

Punctual Hilbert schemes

6.1 Hilbert schemes and symmetric products

We will be interested only in Hilbert schemes of 0-dimensional subschemes (*punctual Hilbert schemes*).

For any quasi-projective algebraic variety X over a field k we can define a functor $\mathcal{H}_{X,n}$ on category of k-schemes

$$\mathcal{H}_{X,n}(S) = \begin{cases} \text{closed subschemes of } Z_S \subset X \times_K S \\ \text{flat and finite of degree } n \text{ over } S. \end{cases}$$

Note that flat and finite of degree n means that the sheaf $(p_S)_*(\mathcal{O}_{Z_S})$ on S is a locally free sheaf of rank n.

For any morphism $f : S' \to S$, the map $\mathcal{H}_{X,n}(f) : \mathcal{H}_{X,n}(S) \to \mathcal{H}_{X,n}(S')$ is defined by the pre-image $(1 \times f)^*(Z)$. We leave to the reader to check that the map is well-defined.

Theorem 6.1.1. *(A. Grothendieck) The functor $\mathcal{H}_{X,n}$ is represented by a scheme of finite type over k. It is denoted by $X^{[n]}$ and is called the* punctual Hilbert scheme *of X of n points on X.*

Recall that this means that the functor $\mathcal{H}_{X,n}$ is equivalent to the functor of S-points of $X^{[n]}$, i.e., for any k-scheme S there is a bijection

$$\eta_S : \mathrm{Mor}(S, X^{[n]}) \to \mathcal{H}_{X,n}(S)$$

such that, for any morphism $f : S' \to S$ of k-schemes, the diagram

$$\begin{array}{ccc} \mathrm{Mor}(S, X^{[n]}) & \xrightarrow{\eta_S} & \mathcal{H}_{X,n}(S) \\ \downarrow{\circ f} & & \downarrow{\mathcal{H}_{X,n}(f)} \\ \mathrm{Mor}(S', X^{[n]}) & \xrightarrow{\eta_{S'}} & \mathcal{H}_{X,n}(S') \end{array}$$

is commutative.

In particular, we have a "natural" (in the sense of commutativity of the above diagrams) bijection

$$X^{[n]}(k) \to \mathcal{H}_{X,n}(\text{Spec } k) = \{\text{closed subscheme of } X \text{ of dimension 0 and length } n\}. \tag{6.1}$$

Here the *length* of a closed susbscheme Z of dimension 0 is equal to $\dim_k H^0(Z, \mathcal{O}_Z)$.

Any representable functor comes with a *universal object*, it is the value of the functor on the scheme it represents which corresponds to the identity morphism. In our situation, it is an element $\mathcal{Z}_{X^{[n]}}$ of $\mathcal{H}_{X,n}(X^{[n]})$, i.e. a relative 0-cycle in $X \times X^{[n]}$. There are two natural projections

$$\begin{array}{ccc} \mathcal{Z}_{X^{[n]}} & \xrightarrow{p} & X^{[n]} \\ \downarrow{\scriptstyle q} & & \\ X & & \end{array}$$

Both projections are projective morphisms. By definition of the universal object, for any k-scheme S and $Z_S \in \mathcal{H}_{X,n}(S)$ there exists a unique morphism $f : S \to X^{[n]}$ such that $Z_S = \mathcal{Z}_X^{[n]} \times_{X^{[n]}} S$. The map q restricted to the fibre of p over a point $z \in X^{[n]}(k)$ is an isomorphism onto the scheme Z corresponding to the points z under the bijection (6.1). The fibre of q over a point $x \in X(k)$ is mapped to the closed subvariety of $X^{[n]}$ whose k-points are in bijective correspondence with the set of 0-dimensional subschemes of X of length n whose support contains the point x.

The notion of the punctional Hilbert scheme $X^{[n]}$ is closely related to the notion of the symmetric product $X^{(n)}$ of X. It is the quotient of X^n by the symmetric group S_n. Since X^n is a quasiprojective over k, the quotient exists as a quasiprojective variety. One can define $X^{(n)}$ as representing the functor $\mathcal{X}^{(n)}$ such that $\mathcal{X}^{(n)}(S)$ is equal to the set of coherent sheaves $\mathcal{F}$ on $X \times S$ whose support is finite over S and the direct image of $\mathcal{F}$ on S is a locally free sheaf of rank n. If $S = \text{Spec } k$, then the support $\text{Supp}(\mathcal{F})$ of $\mathcal{F}$ is a set of closed points X. At each point $x \in \text{Supp}(\mathcal{F})$ the stalk $\mathcal{F}_x$ is a vector space of dimension $d(x)$ over the residue field $\kappa(x)$ of x. The correspondence $\mathcal{F} \mapsto \sum_{x \in \text{Supp}(\mathcal{F})} d(x)x$ is a bijection

$$\mathcal{X}^{(n)}(k) \to \{\sum_{x \in X} m_x x \in \mathbb{Z}_{\geq 0}^X : \sum m_x \deg(x) = n\}.$$

For example, in the case when $\dim X = 1$, the symmetric product $X^{(n)}$ is just the set of effective Weil divisors of degree n.

Note that, by definition, X^n represents the functor $h_{X^n} : S \mapsto \text{Mor}(S, X)^n$. There a morphism of functors $h_{X^n} \to \mathcal{X}^{(n)}$ that assigns to a $(f_1, \ldots, f_n) \in \text{Mor}(S, X)^n$ the sheaf on $X \times S$ equal to $\oplus_i \mathcal{O}_{\Gamma_{f_i}}$, where Γ_{f_i} is the graph of f_i. For $S = k$ this defines a bijection $X^n(k) \to X^{(n)}$ but it is not a bijection for a general S.

There is a canonical morphism, called the *cycle map*

$$\mathrm{cyc} : X^{[n]} \to X^{(n)} \tag{6.2}$$

defined by the morphism of functors $\mathcal{H}_{X,n} \to \mathcal{X}^{(n)}$ that assigns to each $Z \in \mathcal{H}_{X,n}(S)$ the sheaf $\mathcal{O}_Z$.

Proposition 6.1.2. *The cycle map is a projective morphism. Its restriction over* $X^{(n)}_{(1^n)}$ *is an isomorphism.*

Proof. If X is projective over k, then both schemes $X^{[n]}$ and $X^{(n)}$ are projective, and hence the morphism is projective. Otherwise we consider X as an open subset in a projective scheme Y. It follows from the definition that the cycle map $\mathrm{cyc} : X^{[n]} \to X^{(n)}$ is equal to the restriction of the cycle map $Y^{[n]} \to Y^{(n)}$ over X. Thus it is a projective morphism. The second assertion is obvious since both $X^{[n]}_{(1^n)}$ and $X^{(n)}_{(1^n)}$ represent the same subfunctor that assigns to S the subset of reduced subschemes Z (resp. sheaves with reduced support). □

From now on, or simplicity sake, we assume that k is algebraically closed, we write $x \in X$ if $x \in X(k)$.

Recall that a *partition* is a sequence $(\nu_1, \ldots, \nu_m)$ of integers $1 \leq \nu_1 \leq \ldots \leq \nu_m$. The number $|\nu| = \nu_1 + \ldots + \nu_m$ is called the *weight* of ν. A partition of weight n is called a partition of n. The number m is called the *length* of the partition and is denoted by $l(\nu)$. We denote a partition of n of length n by (1^n).

Let ν be a partition of n. For any k distinct points $x_1, \ldots, x_k \in X$ we consider $\sum \nu_i x_i$ as a point on $X^{(n)}$. All such points form a closed subvariety of $X^{(n)}$ of dimension $l(\nu) \dim X$. We denote it by $X^{(n)}_\nu$ and also denote by $X^{[n]}_\nu$ its pre-image in $X^{[n]}$ under the cycle map. It is clear that the cycle map defines an isomorphism of open Zariski subsets

$$X^{[n]}_{(1^n)} \cong X^{(n)}_{(1^n)}$$

parametrizing reduced subschemes of X of length n.

Example 6.1.3. Let X be a nonsingular curve of genus g. Then $X^{[n]} = X^{(n)}$ is a nonsingular variety of dimension n. Its K-points are effective divisors of degree n on X. For example, $(\mathbb{P}^1)^{[n]} \cong \mathbb{P}^n = |\mathcal{O}_{\mathbb{P}^1}(n)|$.

When $n > 2g - 2$, the map $X^{(n)} \to \mathrm{Pic}^n(X)$ which assigns to a divisor D its divisor class (or the isomorphism class of the sheaf $\mathcal{O}_X(D)$) is a projective bundle whose fibre over a point $[D|$ is equal to the linear system $|D|$ of dimension $n+1-g$. The variety $\mathrm{Pic}^n(X)$ is isomorphic to the Jacobian variety $\mathrm{Jac}(X)$ and is an abelian variety of dimension g, a complex torus if $K = \mathbb{C}$. For $n < 2g-2$, the geometry of $X^{[n]}$ is more involved and its geometry is very much related to the geometry of the curve X.

Example 6.1.4. Let $X = \mathbb{P}^2$. We view points in X as lines on the dual projective plane $\check{\mathbb{P}}^2$. Then $X^{(2)}$ becomes isomorphic to the locus of reducible conics in $\check{\mathbb{P}}^2$. In is given by the discriminant polynomial in the coefficients of conics. Thus $X^{(2)}$ is isomorphic to a cubic hypersurface $\mathcal{D}_3$ in $\mathbb{P}^5$. It is singular along the strata $X^{(2)}_{(2)}$ representing the locus of double lines. It is isomorphic to the Veronese quartic in $\mathbb{P}^5$, the set of double lines. Any point $Z \in X^{[2]}_{(2)}$ has support at a unique point $x \in X$. Let u, v be local parameters of X at this point. The subscheme Z is defined by an ideal I in $\mathcal{O}_{X,x}$ which is primary to the maximal ideal $\mathfrak{m} = (u, v)$ and $\dim_k \mathcal{O}_{X,x}/I = 2$. It is easy to see that $I = (au + bv + \mathfrak{m}^2)$, where $(a, b) \in \mathbb{P}^1$. This shows that the fibres of $X^{[2]} \to X^{(2)}$ over $X^{(2)}_{(2)}$ are isomorphic to $\mathbb{P}^1$. Globally, the pre-image E of $X^{(2)}_{(2)}$ is isomorphic to the blow-up of the diagonal $\Delta \subset X \times X$. The conormal bundle of the diagonal $\mathcal{I}_\Delta/\mathcal{I}_\Delta^2$ is isomorphic to Ω^1_X, where we identify X and Δ (see [Hartshorne], p.175). This shows that E is isomorphic to $\mathbb{P}(\Omega^1_X)$, the projectivization of the tangent bundle of X. In our case $X^{[2]}$ is isomorphic to the proper transform of the cubic hypersurface $\mathcal{D}_3$ under the blow-up of $\mathbb{P}^5$ along the Veronese surface.

Example 6.1.5. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ considered as a nonsingular quadric in $\mathbb{P}^3$. A point in $X^{[2]}$ defines either a pair of distinct points on X or a point together with a line in the tangent plane. In any case we can assign to it a line intersecting X at two points or tangent to X at one point. This defines a regular map from $X^{[2]}$ to the Grassmann variety $G(2, 4)$ of lines in $\mathbb{P}^3$ isomorphic to a nonsingular quadric. Recall that X contains two rulings of lines. A pair of points on a line ℓ of a fixed ruling, or a point on ℓ with the tangent direction equal to ℓ is mapped to ℓ considered as a point in $G(2, 4)$. This shows that the subvariety $\ell^{(2)}$ of $X^{[2]}$ isomorphic to $\mathbb{P}^2$ is blown down to a point under the map $X^{[2]} \to G(2, 4)$. The image of the set of lines from the same ruling is a conic in $G(2, 4)$. This shows that $X^{[2]}$ is isomorphic to the blow-up of two disjoint conics C_1, C_2 in $G(2, 4)$.

The cycle map cyc : $X^{([2]} \to X^{(2)}$ is a resolution of singularities. Its exceptional divisor E is equal to the strata $X^{(2)}_{(2)}$. It is isomorphic to the projectivization of the tangent bundle of X. Its image Y in $G(2, 4)$ is equal to the union of lines joining a point on C_1 with a point on C_2. To find the degree of Y we take a general point $p \in \mathbb{P}^3$ and a pencil of lines through p contained in a plane Π. This is a line in $G(2, 4)$ not contained in Y. It intersects Y at two points, the tangent lines of the conic $X \cap \Pi$ passing through p. This shows that Y is hypersurface in $G(2, 4)$ of degree 2. It must be a hyperplane section of $G(2, 4)$ (a *linear complex of lines*) and hence isomorphic to a quadric in $\mathbb{P}^4$. The hyperplane cutting out Y is the unique hyperplane containing the conics C_1, C_2. The exceptional divisor E contains two disjoint copies S_1, S_2 of X embedded as sections of the projectivized tangent bundle (assign to a point on X the two lines on X through this point, this gives a non-ramified double cover of X which splits because X is simply-connected). The map $E \to Y$ blows down S_1, S_2 to the conics C_1, C_2. Under appropriate indexing, the map $S_i \to C_i$ is one of the projections $X = \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. So we see that the projectivized tangent bundle of a 2-dimensional quadric is isomorphic to the blow-up of two disjoint conics on a 3-dimensional quadric. The exceptional divisor of the blow-up is the union of two sections of the projective bundle. A very pretty piciture!

We will be mostly concerned with the case when X is a smooth surface (not necessarily projective).

Theorem 6.1.6. *(J. Fogarty) Assume X is a quasiprojective smooth surface. Then $X^{[n]}$ is a smooth connected variety of dimension $2n$. The morphism* cyc : *$X^{[n]} \to X^{(n)}$ is a resolution of singularities.*

Proof. First let us show that $X^{[n]}$ is connected. Since $X^{(n)}$ is connected, it suffices to show that the fibres of the cycle map are connected. The fibre over a point $Z \in X^{(n)}_{\nu}$ is isomorphic to the product of the Hilbert schemes $X^{[n]}_{(1^{\nu_i})}$. So it is enough to consider the fibre over a point $nx \in X^{(n)}_{(1^n)}$. It is isomorphic to the variety of ideals of codimension n in $\mathcal{O}_{X,x}$. The set of all linear subspaces of codimension n is the Grassmannian of codimension n in the infinite-dimensional vector space $\mathcal{O}_{X,x}$ over k. A subspace is an ideal if it is a fixed point of the (infinite-dimensional) linear group $U = 1+\mathfrak{m}_{X,x}$ (with respect to multiplication in $\mathcal{O}_{X,x}$). The group U admits a filtration $\{1\} \subset U_1 \subset U_2 \subset \ldots$, where $U_s = 1 + \mathfrak{m}^s_{X,x}$ is a finite-dimensional unipotent group. A point is fixed under U if and only it is fixed under each U_s. Now we use the following fact:

- If a connected unipotent algebraic group U acts linearly on a projective connected algebraic variety $V \subset \mathbb{P}^m$, then its set of fixed points is connected.

Let us prove this assertion. Since U contains a composition series with factors isomorphic to $\mathbb{G}_a$, the additive group of k. Thus it suffices to assume that $U = \mathbb{G}_a$. Suppose the fixed locus is not connected. Take two points z, z', each from one of two connected components. Consider the set $\mathcal{S}$ of connected one-dimensional linear sections of V containing the points z, z'. A hyperplane section H of X containing z, z' is birationally isomorphic to a hyperplane section of the projection of V to $\mathbb{P}^{m-2}$ from the line $\langle z, z'\rangle$. By Bertini's Theorem on a hyperplane section, a general section of an irreducible variety is an irreducible variety (see [Joanolou] or [Lazarsfeld,I], Thm. 3.3.3 in the complex case). This shows that one can choose H to be irreducible. Replacing X with H and using induction on dimension we find an irreducible one-dimensional linear section of X containing z, z'. Thus $\mathcal{S}$ is not empty. It is a quasi-projective G-invariant subvariety of the Grassmannian of $(N - \dim V - 1)$-dimensional subspaces of $\mathbb{P}^m$. Take a one-dimensional subgroup of G isomorphic to $\mathbb{G}_a$. Its orbits on any variety are either points or rational curves which are complemented to a projective curve by adding one unibranched point. Since a connected set cannot degenerate into disconnected set, taking the closure of an orbit of $\mathbb{G}_a$ on S and a fixed point on it (any point on the orbit or the unique point in its closure) we see that there exists a connected curve $C \in \mathcal{S}$ fixed under the action of $\mathbb{G}_a$. Since $\mathbb{G}_a$ is connected it acts trivially on the set of irreducible components of C. Let C_0 be an irreducible component of C which contains z. If C_0 contains the second fixed point z', then $\mathbb{G}_a$ acts trivially on C_0. If $z' \notin C_0$, then C_0 contains a finite invariant set of points of intersection of C_0 with other irreducible components of C. Again $\mathbb{G}_m$ fixes all these points and hence acts trivially on C_0. Let $z'' \in C_0 \setminus \{z\}$. It lies in another component C_1 of C. Replacing C_0 with C_1 and repeating the argument we find that $\mathbb{G}_m$ acts trivially on C_2. Proceeding in this way we find a connected curve in the fixed locus of $\mathbb{G}_m$ that connects z, z'. This proves the assertion.

Now let us prove the smoothness of the Hilbert scheme. It is a standard fact that the Zariski tangent space of a scheme S at a point $s \in S(k)$ is an element of $S(k[\epsilon])$ which is mapped to s with respect to the canonical map $S[k[\epsilon]) \to S(k)$ corresponding to the homomorphism $k[\epsilon] \to k,\ c + c'\epsilon \mapsto c$. Here $k[\epsilon] = k[t]/(t^2), \epsilon = t \mod t^2$, is the algebra of dual numbers.

Applying this to our case, when $S = X^{[n]}$, we identify $X^{[n]}(k[\epsilon]))$ with $\mathcal{H}_{X,n}(k[\epsilon])$, i.e. the set of 0-dimensional schemes Z' on $X \times \text{Spec } k[\epsilon]$ which are flat over I and of relative degree n. Since X is quasi-projective, we may assume that Z is a closed subscheme of an affine open subset of X. So, we may assume that X is affine. Then Z corresponds to an ideal I in $A = \mathcal{O}(X)$. The scheme Z' is defined by an ideal J in $B = A[\epsilon]$ such that B/J is flat over $k[\epsilon]$ and $J \otimes_B A = J/\epsilon J = I$. By assumption, the image of the canonical homomorphism $J \otimes_B A \to A = B \otimes_B A$ coincides with I. Thus J consists of elements $a + b\epsilon$ with $a \in I$. Since $\epsilon(a + b\epsilon) = ea \in J$, we see that $\epsilon I \subset J$. The B-module B/J is flat over $k[\epsilon]$, hence is isomorphic to $k[\epsilon]^n$ as a $k[\epsilon]$-module. This shows that the kernel of the canonical homomorphism $J \to I, a + b\epsilon \mapsto a$ is equal to ϵI. Take $a + b\epsilon \in J$, the exact sequence shows that $a + b\epsilon, a + b'\epsilon \in J$ implies $b - b' \in I$. This allows us to define a homomorphism of A-modules $I \to A/I$ by sending $a \in I$ to the coefficient at ϵ in $a + b\epsilon \in J$. Conversely, we can build J from such a homomorphism ϕ by considering the ideal $J_\phi = \{a + b\epsilon : a \in I, b \mod I = \phi(a)\}$. Thus we obtain that the Zariski tangent space of $X^{[n]}$ at Z is isomorphic to the k-vector space $\text{Hom}_A(I, A/I) = \text{Hom}_A(I/I^2, A/I)$.

Decomposing I into primary ideals it is enough to assume that I is a primary ideal in a local ring A. We have to show that $\dim \text{Hom}_k(I/I^2, A/I) \leq 2n$, where $n = \dim_k A/I$. Since I is a torsion-free module over a 2-dimensional regular local ring, its homological dimension is equal to 1 (since it is not principal, it is not free). So we have a free resolution

$$0 \to A^s \to A^{s+1} \to I \to 0.$$

It gives the exact sequence

$$0 \to \text{Hom}_A(I, A/I) \to (A/I)^{s+1} \to (A/I)^s \to \text{Ext}^1_A(I, A/I) \to 0. \tag{6.3}$$

So, it suffices to show that $\dim_K \text{Ext}^1_A(I, A/I) \leq n$. The exact sequence

$$0 \to I \to A \to A/I \to 0$$

gives an isomorphism $\text{Ext}^1_A(I, A/I) \cong \text{Ext}^2_A(A/I, A/I)$ and a surjection from $\text{Ext}^2_A(A/I, A)$ to $\text{Ext}^2_A(A/I, A/I)$. Since A is regular of dimension 2, the theory of local duality shows that $\text{Ext}^2_A(k, A) \cong k$ and the functor $M \to \text{Ext}^2_A(k, M)$ is exact on the category of A-modules of finite length. Thus $\dim_k \text{Ext}^2_A(k, A) \leq n$, by taking a composition series of A/I of length n, and $\dim_k \mathrm{Ext}^2_A(A/I, A/I) \leq n$.

Since we have proved that $X^{[n]}$ is nonsingular, the last assertion follows from Proposition 6.1.2 and the definition of a resolution of singularities. $\square$

Example 6.1.7. Let $X = \mathbb{A}^2_k$ be the affine plane. Then $X^{[n]}$ classifies ideals in $k[T_1, T_2]$ of codimension n. Let us view the unknowns T_1, T_2 as linear operators in $A = k[T_1, T_2]$ via the multiplication operation. Obviously any ideal $I \in X^{[n]}$ is invariant under these operators and hence defines a linear representation of $\mathbb{Z}^2$ in the vector k-space $V = A/I$. The space comes equipped with a special vector v_0 equal to the coset of 1, called the *vacuum vector*. Obviously V is generated as a $\mathbb{Z}^2$-module by v_0. Two pairs (V, v_0) and $V', v_0')$ are called isomorphic if there exists an isomorphism of representations which send v_0 to v_0'. Conversely a pair (V, v_0), where V is a k-vector space of dimension n equipped with a structure of a $k[T_1, T_2]$-module generated by a nonzero vector $v_0 \in V$ defines a surjective homomorphism $k[T_1, T_2] \to V,\ F(T_1, T_2) \mapsto F(T_1, T_2) \cdot v_0$ with kernel equal to an ideal I of codimension 2. Isomorphic pairs define the same ideal. In this way we obtain a natural bijection

$$X^{[n]}(k) = \begin{cases} \text{isomorphism classes of pairs } (V, v_0), \text{ where } V \text{ is an} \\ n\text{-dimensional linear representation of } \mathbb{Z}^2 \text{ generated by } v_0. \end{cases}$$

For example, take $(V, v_0) = (k, 1)$. A linear representation of $\mathbb{Z}^2$ is defined by a pair of scalar operators in k, i.e. a pair $(\lambda, \mu) \in K^2$. A polynomial $F(T_1, T_2)$ defines the scalar operator $F(\lambda, \mu)$. Thus the ideal I is the maximal ideal defined by the point $(\lambda, \mu) \in \mathbb{A}^2(k)$.

Now take $n = 2$ and $k = \mathbb{C}$. Let V be a $\mathbb{C}[T_1, T_2]$-module and A_1, A_2 be the linear operators corresponding to the unknowns. Suppose that A_1 has two linearly independent eigenvectors v_1 and v_2 with eigenvalues λ_1, λ_2. Since A_2 commutes with A_1, the vectors v_1, v_2 are eigenvectors of A_2 with eigenvalues (μ_1, μ_2). Take any vector $v_0 = av_1 + bv_2$ with $a, b \neq 0$. Then

$$F(T_1, T_2) \cdot v_0 = aF(\lambda_1, \mu_1)v_1 + bF(\lambda_2, \mu_2)v_2.$$

Clearly, $(\lambda_1, \mu_1) \neq (\lambda_2, \mu_2)$. This shows that $v_0 = av_1 + bv_2$ generates the $\mathbb{C}[T_1, T_2]$-module V if and only $a, b \neq 0$. It is easy to see that the isomorphism class of (V, v_0) is determined if we fix a basis (v_1, v_2) and take $v_0 = v_1 + v_2$. The pair of points $(\lambda_i, \mu_i), i = 1, 2$, determines (V, v_0) uniquely up to isomorphism. The corresponding ideal consists of polynomials $F(x, y)$ such that $F(\lambda_i, \mu_i) = 0, i = 1, 2$.

Now assume that A_1 or A_2 is not diagonalizable. Let λ be its eigenvalue. Then there exists a basis (v_1, v_2) such that A_1, A_2 are represented by matrices

$$\begin{pmatrix} \lambda & \alpha \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} \mu & \beta \\ 0 & \mu \end{pmatrix}.$$

Write the Taylor expansion of $F(T_1, T_2)$ at the point (λ, μ)

$$F(T_1, T_2) = F(\lambda, \mu) + \nabla F(\lambda, \mu) \cdot (T_1 - \lambda, T_2 - \mu) + \dots.$$

Then

$$F(A_1, A_2) = F(\lambda, \mu)I_2 + \frac{\partial F}{\partial T_1}(\lambda, \mu)(A_1 - \lambda I_2) + \frac{\partial F}{\partial T_2}(\lambda, \mu)(A_2 - \mu I_2),$$

where I_2 denotes the identity matrix of order 2. Let $v_0 = av_1 + bv_2$, then

$$F(A_1, A_2)(v_0) = [aF(\lambda, \mu) + (\alpha, \beta) \cdot \nabla F(\lambda, \mu)]v_1 + bF(\lambda, \mu)v_2.$$

We see that the vacuum vector exists only if $(\alpha, \beta \neq 0) \neq 0$ and it must be a vector $av_1 + bv_2$ with $a, b \neq 0$. In this case the kernel of the representation consists of polynomials $F(T_1, T_2)$ such that $F(\lambda, \mu) = 0$ and $(\alpha, \beta) \cdot \nabla F(\lambda, \mu) = 0$. This is the ideal

$$I = (-\beta(T_1 - \lambda) + \alpha(T_2 - \mu) + (T_1 - \lambda, T_2 - \mu)^2).$$

of length 2. This computation confirms what we had learnt in Example 6.1.4, i.e. shows that the projection $X^{[2]} \to X^{(2)}$ over the point (z, z) in the diagonal has the fibre isomorphic to the projectivization of the Zariski tangent space at z.

Remark 6.1.8. The closed subscheme $(\mathbb{C}^2)_0^{[n]}$ of $(\mathbb{C}^2)^{[n]}$ of cycles with support at the origin was extensively studied by Briancon and Iarrobino [Briancon], [Iarrobino]. They proved that it is an irreducible projective variety of dimension $n-1$. It contains an open dense subset of complete intersection ideals $I = (f, g)$. One defines the Hilbert-Samuel function of $Z \in (\mathbb{C}^2)_0^{[n]}$ given by an Ideal I as follows

$$h_I(s) = \dim_{\mathbb{C}} \frac{\mathbb{C}[z_1, z_2]}{I + \mathfrak{m}^s}.$$

It is easy to see that

$$h_I(s) - h_I(s-1) = \dim_{\mathbb{C}} \frac{\mathfrak{m}^s}{\mathfrak{m}^{s+1}} - \dim_{\mathbb{C}} \frac{I \cap \mathfrak{m}^s + \mathfrak{m}^{s+1}}{\mathfrak{m}^{s+1}}.$$

The largest s such that $I \subset \mathfrak{m}^s$ is called the *order* of I. Let $\mathcal{S}_t$ be the set of ideals of order t. For example, the set of ideals $\mathcal{S}_1$ of order 1 is isomorphic to a fibration over $\mathbb{P}^1$ with fibre $\mathbb{C}^{n-2}$. It is defined by two charts:

$$V_1 = \{I = (z_1 + a_1 z_2 + \ldots + a_{n-1} z_2^{n-1}, z_2^n)\},$$

$$V_2 = \{I = (z_2 + b_1 z_1 + \ldots + b_{n-1} z_1^{n-1}, z_1^n)\},$$

with transition function $(a_1; a_2, \ldots, a_n) \mapsto (b_1; b_2, \ldots, b_{n-1})$ defined by plugging in $z_2 = -(b_1 z_1 + \ldots + b_{n-1} z_1^{n-1})$ in $z_1 + a_1 z_2 + \ldots + a_{n-1} z_2^{n-1}$ and finding the conditions that the obtained expression belongs to (z_1^n). One finds

$$b_1 = \frac{1}{a_1},\ b_2 = \frac{a_2}{a_1^3},\ b_3 = \frac{2a_2^2}{a_1^5} - \frac{a_3}{a_1^4}, \ldots.$$

For $n = 2$ these are just transition functions on $\mathbb{P}^1$. For $n = 3$ they are transition functions for the total space of the line bundle $\mathbb{V}(\mathcal{O}_{\mathbb{P}^1}(-3))$. For $n > 3$, $\mathcal{S}_1$ is not a line bundle.

Thus

$$(\mathbb{C}^2)_0^{[2]} \cong \mathbb{P}^1.$$

One can show that

$$(\mathbb{C}^2)_0^{[3]} = \mathcal{S}_1 \cup \{\mathfrak{m}^2\}.$$

The point $\mathfrak{m}^2$ is the singular point of $(\mathbb{C}^2)_0^{[3]}$. So $(\mathbb{C}^2)_0^{[3]}$ is isomorphic to the cone over the Veronese curve of degree 3. The set $\mathcal{S}_1$ is the set of nonsingular points of $(\mathbb{C}^2)_0^{[4]}$. The set $\mathcal{S}_2$ is smooth of dimension 2, and the subvariety of non-complete intersection ideals is isomorphic to $\mathbb{P}^1$. In particular, $(\mathbb{C}^2)_0^{[4]}$ is a non-normal variety.

Any ideal in $(\mathbb{C}^2)_0^{[5]}$ is isomorphic (after passing to the formal completion) to one of the following ideals

(i) (z_1, z_2^5);

(ii) $(z_1^2 + z_2^3, z_1 z_2)$;

(iii) $(z_1^2, z_1 z_2, z_2^4)$;

(iv) $(z_1^2 + z_2^2, z_1^2 z_2, z_1^3)$;

(v) $(z_1^2, z_1 z_2^2, z_2^3)$.

Any ideal in $(\mathbb{C}^2)_0^{[6]}$ is isomorphic to one of the following ideals

(i) (z_1, z_2^6);

(ii) $(z_1^2 + z_2^4, z_1 z_2)$;

(iii) $(z_1^2, z_1 z_2, z_2^5)$;

(iv) $((z_1 + z_2)^2, z_1^2 z_2)$;

(v) $(z_1^2 + z_2^2, z_1^2 z_2)$:

(vi) $(z_1^2 + z_1 z_2, z_1^2 z_2, z_2^4)$:

(vii) $(z_1^2, z_1 z_2^2, z_2^4)$;

(viii) $(z_1^2 + z_1^3, z_1 z_2^2, z_2^4)$;

(ix) $\mathfrak{m}^3$;

6.2 G-Hilbert scheme

Suppose G is a finite group of order n acting faithfully on a quasiprojective irreducible algebraic variety X. For any $g \in G$ let X^g denote the set of fixed points of g. It is a closed subvariety of X. Let $X^{\mathrm{reg}} = X \setminus \cup_{g\in G, g\neq 1} X^g$ be the largest open subset of X on which G acts freely. Each orbit $G\cdot x$ is the image of G under the action map $\mu_x : G \to X$. The map is proper, its image is a closed subscheme Z of X with $H^0(\mathcal{O}_Z) \cong k[G]$. The orbits are reduced subschemes of length n and hence can be identified with points in the strata $X^{[n]}_{(1^n)}$. We can identify the G-orbits of points from X^{reg} with the points of the quotient X^{reg}/G.

Definition 6.2.1. The *G-Hilbert scheme* of X is the irreducible component $\text{G-Hilb}(X)$ of $X^{[n]}$ that contains an orbit of some point $x \in X^{\mathrm{reg}}$.

The action of G on X extends to an algebraic action of G in $X^{[n]}$. For any $Z_S \in \mathcal{H}_{X,n}(S)$ an element $g \in G$ sends Z_S to $(g \times \mathbf{id}_S)^*(Z_S)$.

Proposition 6.2.1. *The group G acts identically on* $\text{G-Hilb}(X)$ *and hence acts linearly on the vector space $V_Z = H^0(\mathcal{O}_Z)$ for any $Z \in$* $\text{G-Hilb}(X)$. *This linear representation is isomorphic to the regular representation of G.*

Proof. Recall that an algebraic action of an algebraic group G on an algebraic variety S is defined by a morphism $\mu : G \times S \to S$. Let $\Phi = \mu \times \mathbf{id}_S : G \times S \to S \times S$. On points it sends (g, x) to (gx, x). It is easy to see that the projection to S of the pre-image $\Phi^{-1}(\Delta_S)$ of the diagonal is equal to the subscheme S^G of fixed points. If the action is proper (always in the case of a finite group), the subscheme S^G is closed in S. Now suppose $S' \subset S$ is an open G-invariant subset of S. Then the standard properties of the closure imply that the closure of $\Phi^{-1}(\Delta_{S'}) \subset G \times S'$ in $G \times S$ is equal to $\Phi^{-1}(\Delta_S)$. Taking the closure of the projection, we see that the closure of S'^G in S is equal to S^G.

Applying this to our situation, we see that $\text{G-Hilb}(X)$ is an irreducible component of $(X^{[n]})^G$. The group G acts on the universal scheme $\pi : \mathcal{Z}_{X^{[n]}} \to X^{[n]}$. Over $\text{G-Hilb}(X)$ it acts on fibres and identically on the base. Thus it acts on the tautological sheaf $\mathcal{F}$ whose fibre over Z is the algebra V_Z. If $Z \in \text{G-Hilb}(X^{\mathrm{reg}}) = (X^{\mathrm{reg}})^{[n]} \cap \text{G-Hilb}(X)$, then $V_Z \cong \oplus_{z\in Z} H^0(\mathcal{O}_z)$ and we consider a basis e_z of V_Z corresponding to a choice of the coset of 1 in each factor. It is clear that G acts on the basis in the same way as it acts on the canonical basis of the group algebra $k[G]$.

Now suppose Z belongs to the boundary $\text{G-Hilb}(X) \setminus \text{G-Hilb}(X^{\mathrm{reg}})$. Let R be an irreducible representation of G. Consider the sheaf $\mathcal{F}_R = \mathcal{O}_{\mathcal{Z}_{X^{[n]}}} \otimes R^*$ restricted to $\text{G-Hilb}(X)$ (here R is considered as the pull-back of the constant sheaf on Spec K). Let $(p_*\mathcal{F})^G$ be the subscheaf of $p_*(\mathcal{F})$ of G-invariant sections. Its fibre over a point $[Z]$ is isomorphic to the space $\mathrm{Hom}_G(R, V_Z)$. Its dimension n_R is equal to the multiplicity of the representation R in V_Z. By semi-continuity of dimension of fibres of a coherent sheaf of modules, the dimension n_R is greater or equal than the multiplicity of R in the regular representation. Since $n =$

$\dim V_Z$ is equal to the sum of squares of the dimensions of non-isomorphic irreducible representations of G, we obtain that V_Z is isomorphic to the regular representation. □

The theorem allows to redefine the G-Hilbert scheme as an irreducible component of the fixed locus of G in $X^{[n]}$ that contains an orbit of a point with trivial stabilizer subgroup.

Another possible definition could be used if the following conjecture turns out to be true.

Conjecture 6.2.2. *(I. Nakamura)* G-Hilb(X) *coincides with the set of points* $Z \in (X^{[n]})^G$ *such that* V_Z *is a regular representation of* G.

So far, it is known only in some special cases, for example $X = \mathbb{C}^n, G \subset \mathrm{SL}(n, \mathbb{C}), n \leq 3$.

The following fact is well-known but for the lack of a reference, we include its proof.

Lemma 6.2.3. *Let G be a finite group acting on a smooth algebraic variety X over a field k of characteristic prime to the order of G. Then the subscheme X^G of fixed points is smooth.*

Proof. Let $\bar{k}$ be the algebraic closure of X. We will prove more, X is geometrically smooth, i.e. $X \otimes_k \bar{k}$ is regular. Thus we may assume that k is algebraically closed. Since closed points are dense in X and the localization of a regular local ring is regular, it is enough to show that each closed point of X^G is nonsingular. Without loss of generality we may assume that G acts faithfully and X is irreducible. Let $x \in X^G$. Let $A = \mathcal{O}_{X,x}$ and $\mathfrak{m} = \mathfrak{m}_{X,x}$. The group G acts on the local ring A and on the associated graded ring $\mathrm{gr}A = \oplus_{n=0}^{\infty} \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Since A is a regular local ring, $\mathrm{gr}A \cong S^{\bullet}(\mathfrak{m}/\mathfrak{m}^2) \cong k[t_1, \ldots, t_n]$, where $n = \dim A$. I claim that the action of G on the Zariski cotangent space $\mathfrak{m}/\mathfrak{m}^2$ of X at x, and hence on $\mathrm{gr}A$ is faithful. Suppose $g \in G$ acts identically on $\mathfrak{m}/\mathfrak{m}^2$. By Maschke's Theorem, the action of $\langle g \rangle$ on $A/\mathfrak{m}^2$ decomposes into the direct sum of $\mathfrak{m}/\mathfrak{m}^2$ and $k = A/\mathfrak{m}$. Thus G acts trivially on $A/\mathfrak{m}^2$. Continuing in this way, we find that G acts trivially on each quotient $A/\mathfrak{m}^n$, hence acts trivially on the formal completion $\hat{A}$ of A. Since A embeds into its formal completion, we see that g acts trivially on A. This implies that g acts trivially on an open subset containing x. Since the fixed locus of g is a closed subset and X is irreducible, we obtain that g acts trivially on X contradicting the assumption. Let B be the local ring of X^G at x and $\mathfrak{n}$ be its maximal ideal. We have a G-equivariant surjection $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2$. Again, by Maschke's Theorem, the representation of G on $\mathfrak{m}/\mathfrak{m}^2$ decomposes into the direct sum $\mathfrak{m}/\mathfrak{m}^2 = \mathfrak{n}/\mathfrak{n}^2 \oplus V$. The group G acts trivially on $\mathfrak{n}/\mathfrak{n}^2$. Let $f_1, \ldots, f_d$ be local parameters in A whose residues modulo $\mathfrak{m}^2$ form a basis of the subspace $(\mathfrak{m}/\mathfrak{m}^2)^G$ extending a basis of $\mathfrak{n}/\mathfrak{n}^2$. They define, locally, a nonsingular subvariety Y of X of codimension d that contains X^G. By the previous argument G acts trivially on Y, hence Y must be contained in X^G. This implies that X^G coincides with Y in an open neighborhood of x, and hence X^G is nonsingular. □

Theorem 6.2.4. *Assume that k is of characteristic 0. There is projective morphism*

$$c : \text{G-Hilb}(X) \to X/G$$

extending an isomorphism $\text{G-Hilb}(X^{reg}) \cong X^{\text{reg}}/G$. *If X is a nonsingular surface, then c is a resolution of singularities.*

Proof. Let $\mu : G \times X \to X \times X$ be the map defined by $(g, x) \mapsto (x, g \cdot x)$. Since G is a finite group, the map is proper and the image is a closed subset Z of $X \times X$. It is equal to the union of the graphs of $g \in G$. Let $\mathcal{O}_Z$ be its structure sheaf. Its direct image under the first projection is a locally free sheaf of rank n. Thus Z defines an X-points of $X^{(n)}$, i.e. a morphism $\alpha : X \to X^{(n)}$. On k-points this is the map that sends a point x to the orbit $G \cdot x := \sum_{g \in G} g \cdot x$. Let $\underline{X}^{(n)} \to X^{(n)}$ be the universal family. It is a closed subscheme of $X \times X^{(n)}$ and its projection to $X^{(n)}$ is proper. The intersection of $\underline{X}^{(n)}$ with the graph of α is a closed subset of $X \times X^{(n)}$. Its image under the projection to $X^{(n)}$ is a closed subset of $X^{(n)}$. Thus the map $\alpha : X \to X^{(n)}$ has closed image in $X^{(n)}$. Since it is obviously G-invariant it factors through a map $\alpha' : X/G \to X^{(n)}$ with closed irreducible image $\overline{X/G}$. The variety $\overline{X/G}$ is, by definition the *Chow quotient* of $X//G$. If k is of characteristic zero, there is the inverse map $X//G \to X/G$ (see [Kapranov, Chow quotients, 0.4.6]. Thus the map $\text{G-Hilb}(X) \to \overline{X/G}$ defines a morphism $\text{G-Hilb}(X) \to X/G$.

It remains to prove the second assertion. By Theorem 6.1.6, $X^{[n]}$ is smooth. Since $X^{[n]}$ is an irreducible component of $\text{Hilb}(X)^G$, Lemma 6.2.3 implies that $\text{Hilb}(X)$ is smooth. Since the map $c : \text{G-Hilb}(X) \to X/G$ is birational and projective, it is a resolution of singularities. □

6.3 Symplectic structure

Let X be a smooth algebraic variety over a field k and $\Omega^1_{X/k}$ be its sheaf of differential 1-forms on X. A symplectic structure on X is a section ω of $\Omega^2_{X/k} = \Lambda^2\Omega^1_{X/k}$ such that the corresponding map $\Omega^1_{X/k} \to (\Omega^1_{X/k})^*$ is bijective. Passing to the fibres, we see that ω defines a non-degenerate 2-form $\omega_x \in \Omega^2_{X/k}(x) = \Lambda^2(\mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x})$. This implies that $\dim X = 2d$ is even. Also the map $\Lambda^d(\omega) : \Omega^d_{X/k} \to (\Omega^d_{X/k})^*$ is given by a nowhere vanishing section of $\Omega^{2d}_{X/k}$. In particular, $K_X = 0$. If $\dim X = 2$, a symplectic structure is just a nowhere vanishing section of the canonical bundle of X.

A symplectic structure on an algebraic variety is an algebraic version of a holomorphic symplectic structure on a complex manifold. The definition of the latter is the same only the sheaf of regular differential 2-forms is replaced by the sheaf of holomorphic 2-forms. In its turn, a holomorphic symplectic structure is a complex analog of a symplectic structure on a smooth differential manifold. This time the sheaf is the sheaf of smooth 2-forms.

Example 6.3.1. Let $X = \mathbb{A}^{2d}_k = \text{Spec } k[T_1, \ldots, T_{2d}]$ be the affine space of dimension $2d$ over a field k. We have $\Omega^1_{X/k}$ corresponds to a free module over

$k[T_1,\ldots,T_{2d}]$ with a basis $dT_1,\ldots,dT_{2d}$. The 2-form

$$\omega = \sum_{i=1}^{f} dT_i \wedge dT_{i+d}.$$

defines a symplectic structure on X. We have

$$\Lambda^d(\omega) = dT_1 \wedge dT_{d+1} \wedge \cdots \wedge dT_d \wedge dT_{2d}.$$

Its holomorphic (differential) analog is the following. Let V be a linear space over $\mathbb{C}$ (or over $\mathbb{R}$) and V^* be its dual space. Define a 2-form ω on $\Lambda^2(V \oplus V^*)$ by

$$\omega((v,\phi),(w,\psi)) = \psi(v) - \phi(w).$$

If $(e_1,\ldots,e_d)$ is a basis in V and $e_1^*,\ldots,e_d^*)$ be its dual basis in V^*, then the matrix of ω with respect to the basis $((e_1,0),\ldots,(e_d,0),(0,e_1^*),\ldots,(0,e_d^*))$ is equal to

$$\begin{pmatrix} 0_d & I_d \\ -I_d & 0 \end{pmatrix}.$$

This shows that ω is a symplectic form on $V \oplus V^*$. It is easy to see that $\omega = \sum_{i=1}^d e_i^* \wedge e_{i+d}^*$.

Now we can use this form to construct a holomorphic symplectic structure on the cotangent vector bundle $\mathbb{T}_X^*$ of X. Recall that $\mathbb{T}_X^* = \operatorname{Spec} S^\bullet((\Omega^1_{X/k})^*) = \mathbb{V}((\Omega^1_{X/k})^*)$. Its sheaf of sections is isomorphic to $\Omega^1_{X/k}$. For any locally free sheaf $\mathcal{E}$ on X we have

$$\Omega^1_{\mathbb{V}(\mathcal{E})/k} \cong p^*(\Omega^1_{X/k}) \oplus p^*(\mathcal{E}),$$

where $p: \mathbb{V}(\mathcal{E}) \to X$ is the canonical projection. In our case we get

$$\Omega^1_{\mathbb{T}_X^*/k} \cong p^*(\Omega^1_{X/k}) \oplus (\Omega^1_{X/k})^*)).$$

Using the above constriction of a symplectic form on $V \oplus V^*$, this allows one to define a natural symplectic structure on $\mathbb{T}^*X$

Let (X,ω) be a symplectic algebraic variety of dimension $n = 2d$. A closed nonsingular subvariety Y of X is called *isotropic* if under the canonical map of sheaves $\Omega^2_{X/k} \to \Omega^2_{Y/k}$ the image of $\omega \in \Gamma(X,\Omega^2_{X/k})$ in $\Gamma(\Omega^2_{Y/k})$ is equal to zero. It follows from the definition, that for any point $y \in Y$, the image of the Zariski tangent space $T_y(Y)$ in $T_x(X)$ is an isotropic subspace (i.e. a subspace contained in its orthogonal complement with respect to the symplecti form ω_x).

An isotropic subvariety is called a *Lagrangian subvariety* if its dimension is equal to n. Recall that n is the largest possible dimension of an isotropic subspace of a non-degenerate symplectic 2-form). It follows from the definition that any nonsingular curve on a symplectic surface is Lagrangian. Another example is the zero section of the cotangent bundle $\mathbb{T}_X^*$.

Theorem 6.3.2. *(Fujiki, Beauville) Suppose a nonsingular surface X has a holomorphic symplectic structure. Then $X^{[n]}$ has a holomorphic symplectic structure.*

Proof. Let $\nu = (1^{n-2}, 2) = (1 \leq \ldots \leq 1 \leq 2)$ be the "subtrivial" partition of n . Let $X_*^{[n]}$ be the union of the strata $X_\nu^{[n]}$ and the open strata $X_{1^n}^{[n]}$. We employ similar notation for the symmetric product. Each of them is an open subvariety with complement of codimension 2.

Let $p : X^n \to X^{(n)}$ be the canonical projection and $X_*^n = p^{-1}(X_*^{(n)})$. The restriction of p to X_*^n is the double cover ramified along $X_\nu^{(n)}$. We have the following commutative diagram,

$$\begin{array}{ccc} \tilde{X}_*^{[n]} & \xrightarrow{\phi} & X_*^n \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p} \\ X_*^{[n]} & \xrightarrow{\text{cyc}} & X_*^{(n)} \end{array} ,$$

where ϕ is the projection of the blow-up map of X^n along the "big diagonal" $\Delta = p^{-1}(X_\nu^{(n)})$ (see Example 6.1.7) and π is the double cover ramified over $X_\nu^{[n]}$. Let ω be a holomorphic 2-form on X defining the holomorphic symplectic structure on X. The form $\tilde{\omega} = \sum p_i^*(\omega)$ on X^n defines a holomorphic symplectic structure on X^n which is invariant with respect to the action of the symmetric group S_n. We restrict it to X_*^n and obtain that it is equal to $\pi^*(\eta)$ for a holomorphic 2-form on $X_*^{[n]}$. We have

$$\operatorname{div}(\phi^*(\tilde{\omega}^{\wedge n})) = \phi^*(\operatorname{div}(\tilde{\omega}^{\wedge n})) + E = E$$

$$\operatorname{div}(\pi^*(\eta^{\wedge n})) = \pi^*(\operatorname{div}(\eta^{\wedge n})) + E,$$

where E is the ramification divisor of π equal to the exceptional divisor of the blow-up of ϕ. Since $\operatorname{div}(\phi^*(\tilde{\omega}^{\wedge n})) = \operatorname{div}(\pi^*(\eta^{\wedge n}))$, we get $\pi^*(\operatorname{div}(\eta^{\wedge n})) = 0$. This shows that η has no zeroes on $X_*^{[n]}$ and hence defines a holomorphic symplectic structure on $X_*^{(n)}$. Since the codimension of the complement of $X_*^{(n)}$ is equal to 2, the form can be extended to a holomorphic symplectic 2-form on the whole Hilbert scheme (because $\eta^{\wedge n}$ does not vanish anywhere on $X^{[n]}$). □

Remark 6.3.3. Let $\mathcal{M}_X(r; c_1, c_2)$ be the moduli space of stable rank r torsion-free coherent sheaves on X with fixed Chern classes c_1 and c_2. Assume that X is a K3-surface or an abelian surface with holomorphic symplectic structure defined by a nonzero section of $\mathcal{O}_X(K_X) \cong \mathcal{O}_X$. Then Mukai proves that $\mathcal{M}_X(r; c_1, c_2)$ carries a holomorphic symplectic structure. The tangent space of $\mathcal{M}_X(r; c_1, c_2)$ at a point $[\mathcal{E}]$ can be identified with $\operatorname{Ext}^1(\mathcal{E}, \mathcal{E})$ and the symplectic form corresponds to the natural pairing

$$\operatorname{Ext}^1(\mathcal{E}, \mathcal{E}) \times \operatorname{Ext}^1(\mathcal{E}, \mathcal{E}) \to \operatorname{Ext}^2(\mathcal{E}, \mathcal{E}) \cong \mathbb{C}.$$

If we take $r = 1, c_1 = 0, c_2 = n$, then double dual of $\mathcal{E}$ is isomorphic to $\mathcal{O}_X$ and the quotient is the structure sheaf of some $Z \in X^{[n]}$. In this way we obtain an isomorphism

$$\mathcal{M}_X(1;0,n) \cong X^{[n]}$$

which is compatible with the holomorphic symplectic structures.

Let (X, ω) be a symplectic algebraic variety. We say that a group G *acts symplectically* on X if $g^*(\omega) = \omega$ for all $g \in G$. For example, a finite subgroup G of the symplectic group $\mathrm{Sp}(2n, k)$ acts symplectically on the affine space $\mathbb{A}_k^{2n}$ equipped with the standard symplectic manifold from Example 6.3.1. If $n = 1$, we have $\mathrm{Sp}(2n, k) = \mathrm{SL}(2, k)$ and the classification of such groups was given in Lecture 1.

Lemma 6.3.4. *Let G be a finite group acting symplectically on (X, ω). Assume that the order of G is coprime to the characteristic of the ground field. Then the fixed locus X^G of G is a symplectic subvariety.*

Proof. By Lemma 6.2.3, X^G is a smooth subvariety. By definition, a point is a symplectic variety. So we may assume that each connected component of the fixed locus $F = X^G$ is of positive dimension. Let $x \in X^G$, by Maschke's Theorem, the linear representation of G on the tangent space T_xX decomposes into the direct sum $T_xF \oplus N$ of representations. It follows from the proof of Lemma 6.2.3 that $T_xF = (T_xX)^G$. The bilinear form ω defines a map $\phi : T_xF \to (T_xX)^* \to N^*$. Since G preserves ω, ω_x is an isomorphism of linear representations. Since N, and hence N^* does not contain trivial irreducible summands, by Schur's Lemma, the map ϕ is trivial. Hence ω_x induces a bijective map $(T_xF) \to T_x^*F$. This shows that ω defines a symplectic form on F.

□

Corollary 6.3.5. *Suppose X is a nonsingular surface admiting a holomorphic symplectic structure preserved under an action of a finite group G. Then* G-Hilb(X) *admits a holomorphic symplectic structure and the morphism c :* G-Hilb$(X) \to X/G$ *is a minimal resolution of singularities.*

Proof. The minimality of the resolution follows from the existence of a holomorphic symplectic structure on $X^{[n]}$. Since G-Hilb(X) is an irreducible component of the fixed locus of G on $X^{[n]}$, by Lemma 6.3.4 it has a holomorphic symplectic structure defined by a holomorphic 2-form ω. Since G-Hilb(X) is a surface, ω defines a nowhere vanishing section of the sheaf of holomorphic 2-forms. Taking some nonsingular compactification of G-Hilb(X), we find a divisor in the canonical class whose restriction to G-Hilb(X) is equal to zero. Thus, by adjunction, any irreducible component R of the resolution satisfies $R^2 = 2 - 2g$, where g is the genus of R. Since $R^2 < 0$, we see that $g = 0$ and $R^2 = -2$. Thus the resolution is minimal and the intersection matrix of each connected component of the exceptional divisor is a negative definite Cartan matrix. Since locally the singular points correspond to singularities $\mathbb{C}^2/H$, where H is the stabilizer subgroup of an orbit, we see that H acts as a subgroup of $\mathrm{SL}(2, \mathbb{C})$ and the singularities are analytically isomorphic to ADE-singularities. □

Recall that an isotropic submanifold L of a symplectic manifold M is a submanifold such that each tangent subspace TL_x is an isotropic subspace of TM_x.

Conjecture 6.3.6. *Let X be a nonsingular surface which admits a holomorphic symplectic structure equipping $X^{[n]}$ with holomorphic symplectic structure. Then the fibres of the cycle map $X^{[n]} \to X^{(n)}$ are Lagrangian submanifolds.*

This is true in the case $X = \mathbb{C}^2$ and $G \subset \mathrm{SL}(2, \mathbb{C})$ (see [Nakajima, Lectures]).

Theorem 6.3.7. *Conjecture 6.3.6 implies Nakamura's conjecture.*

Proof. Let $N = |G|$. A connected component H of positive dimension of $(X^{[N]})^G$ is a symplectic subvariety of $X^{[N]}$. Suppose it is different from G-Hilb(X) and contains a cycle Z with $V_Z = H^0(\mathcal{O}_Z)$ equal to the regular representation. It follows from the proof of Proposition 6.2.1 that V_Z is regular for all Z's from H. Let $X^{\mathrm{reg}} \subset X$ be the open subset of X where G acts freely. Since any $g \in G$ preserves the symplectic structure, the set X^g of its fixed points is a symplectic subvariety, hence is 0-dimensional if $g \neq 1$. Thus $X \setminus X^{\mathrm{reg}}$ is a finite set of points $x_1, \ldots, x_s$.

Let $q : \mathcal{Z}_{X^{[n]}} \to X, p : \mathcal{Z}_{X^{[n]}} \to X^{[N]}$ be the projections of the universal family over $X^{[N]}$. Let $H_i = p(q^{-1}(x_i)), i = 1, \ldots, s$. It consists of $Z \in H$ such that $x_i \in \mathrm{Supp}(Z)$. Each H_i is a closed subvariety of H and $H = H_1 \cup \cdots \cup H_s$ since H does not contain reduced cycles. Since H is smooth and connected, it is irreducible. Hence $H = H_i$ for some i. Without loss of generality we may assume that $H = H_1 = \cdots = H_t$ for some $t \leq s$. Since V_Z is regular, it is easy to see that $\mathrm{cyc}(H) = \frac{N}{t}(\sum_{i=1}^t x_i)$. In any case, since H is connected, it is contained in one fibre of the cycle map. It follows from Conjecture 6.3.6 that H is an isotropic subvariety. Hence it is also a symplectic subvariety this implies that H must be a point.

Now, to show that H does not exist, it remains to prove that $\dim H > 0$. Let $Z \in H$ with $\mathrm{Supp} Z = \{x_1, \ldots, x_t\}$. Let $U = \mathrm{Spec}\ A$ be an open affine set containing $x_1, \ldots, x_t$ and I be the ideal in A defining Z. We know from the proof of Fogarty's Theorem that the tangent space of $X^{[N]}$ at Z is isomorphic to $\mathrm{Hom}_A(I, A/I)$. The exact sequence

$$0 \to I^2 \to I \to I/I^2 \to 0$$

gives an exact sequence

$$0 \to \mathrm{Hom}_A(I/I^2, A/I) \to \mathrm{Hom}_A(I, A/I) \to \mathrm{Hom}_A(I^2, A/I) = 0.$$

The standard exact sequence of the modules of differentials [Hartshorne, Chapter 2, Proposition 8.3A] gives an exact sequence

$$I/I^2 \xrightarrow{d} \Omega^1_{A/k} \otimes_A A/I \to \Omega^1_{(A/I)/k} \to 0.$$

Replacing U by a smaller subset, we may always assume that $\Omega^1_{A/k} \cong A^2$ (recall that X is smooth so $\Omega^1_{X/k}$ is locally free of rank 2). Thus the middle term is isomorphic to $(A/I)^2$. Let T be the image of the homomorphism $d : I/I^2 \to (A/I)^2$ and N be the kernel. Passing to the duals of A/I-modules, we have an exact sequence

$$0 \to \mathrm{Der}_{(A/I)/k} \to \mathrm{Hom}_{A/I}(A/I)^2, A/I) \to T^* \to 0, \tag{6.4}$$

where $T^* \subset \mathrm{Hom}(I/I^2, A/I)$.

Let $W = (A/I)^2$ be the middle term considered as a G-module. It is isomorphic to the tensor product of the G-module $\mathrm{Hom}_k(\Omega^1_{A/k}, k)$ and the regular representation on A/I. Let $x = x_1$ and G_x be the stabilizer subgroup of x. The regular representation is induced from the representation of G_x on $(A/I)_x$ and the representation of G on $\mathrm{Hom}_k(\Omega^1_{A/k}, k)$ is induced from the linear representation of G_x on $\mathrm{Hom}_k(\Omega^1_{A_x/k}, k) \cong T_xX$ (the *isotropy representation*). Here $A_x, (A/I)_x$ is the localization of $A, A/I$ at x. We have

$$(T^*)^G \subset \mathrm{Hom}_A(I/I^2, A/I)^G = \mathrm{Hom}_A(I, A/I)^G.$$

Passing to the invariants in (6.4), we get an exact sequence

$$0 \to \mathrm{Der}^G_{(A/I)/k} \to W^G \to (T^*)^G \to 0. \tag{6.5}$$

Let $\chi_{x,\mathrm{reg}}$ be the character of the regular representation of G_x and χ_x be the character of its isotropy representation. It follows from the Frobenius Reciprocity Theorem 5.1.9 that

$$\dim_k W^G = \langle \chi_{\mathrm{reg},x}\chi_x, 1_{G_x}\rangle = \frac{1}{|G_x|}\sum_{g\in G_x} \chi_{\mathrm{reg},x}(g)\chi_x(g)$$

$$= \frac{1}{|G_x|}\chi_{\mathrm{reg},x}(1)\chi_x(1) = \dim T_xX = 2.$$

Let us see that $\mathrm{Der}^G_{(A/I)/k} = 0$. The group G acts on derivations $\partial : A/I \to A/I$ by ${}^g\partial(a) = g\partial(g^{-1}a)$. A G-invariant derivation satisfies $\partial(ga) = g\partial(a)$, hence it is a homomorphism of $k[G]$-modules $A/I \to A/I$. Since A/I is a regular representation, $A/I \cong k[G]$ and, under this homomorphism, A/I is generated by the field of constants k, considered as a submodule of A/I. Since any derivation is equal to zero on the field of constants we get $\mathrm{Der}^G_{(A/I)/k} = 0$. Now (6.5) implies that $\dim_k(T^*)^G = 2$, hence

$$\dim_k \mathrm{Hom}_A(I, A/I)^G = \dim_x H \geq 2.$$

□

Example 6.3.8. Let $n = 2$. Suppose $Z = \{x, y\} \in (X^{[2]})^G$ is reduced. Then either 2 points make an orbit, and hence $Z \in$ G-Hilb(X), or Z consists of two fixed points. But then G acts identically on $H^0(\mathcal{O}_Z)$ and hence $Z \notin X^{[2]}_G$.

Let x be a fixed point. We know that there exists a non-reduced $Z \in$ G-Hilb(X) supported at x. The action is linearized at x so that there exist local parameters u, v at x such that G acts on the tangent space by $(u, v) \mapsto (\alpha u, \beta v$, where $(\alpha, \beta) = (1, -1)$ or $(-1, -1)$. In the first case x belongs to a one-dimensional component of X^G. In the second case, X is an isolated fixed point x. Let $A = \mathcal{O}_{X,x}, \mathfrak{m} = \mathfrak{m}_{X,x}$. We know from Example 6.1.4 that $I = (au + bv, \mathfrak{m}^2)$. Clearly I_Z is invariant and V_Z is a regular representation of G. The set of Z's is parametrizied by $\mathbb{P}^1$. Thus we see that there exists a natural map $X_G^{[2]} \to X/G$. Its fibre over the orbit of an isolated fixed point is the whole projective line. Its fibre over non-isolated fixed point is a one-point set.

Example 6.3.9. Let $G = S_3$ which acts on $X = \mathbb{C}^2$ as follows. Consider X as a hyperplane $z_1 + z_2 + z_3 = 0$ in $\mathbb{C}^3$ with coordinates z_1, z_2, z_3 and let G act by permuting the coordinates. This is called the standard representation of S_3. It is obviously irreducible. By projecting we take the first two coordinates as coordinates in $\mathbb{C}^2$. Let $I_Z = (z_1^2 + z_1z_2 + z_2^2, z_1z_2(z_1 + z_2)$ be the ideal cut out in $\mathbb{C}^2$ by the elementary symmetric functions in $\mathbb{C}^3$. We have $V_Z = \mathbb{C}[z_1, z_2]/I_Z$ has a basis consisting of cosets of $1, z_1, z_2, z_1^2, z_2^2, z_1^2z_2$. The group G is generated by a transposition $\sigma_{12} : (z_1, z_2) \mapsto (z_2, z_1)$ and a cyclic permutation $\sigma_{123} : (z_1, z_2) \mapsto (z_2, -z_1 - z_2)$. Obviously, 1 spans the trivial representation, and z_1, z_2 span the standard 2-dimensional linear representation. The representation spanned by the remaining basis elements is given by 2 matrices

$$\sigma_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \ \sigma_{123} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Computing the character χ we find that it is equal to

$$(\chi(1), \chi(\sigma_{12}), \chi(\sigma_{123})) = (3, -1, 0) = (2, 0, -1) + (1, -1, 1).$$

The first vector is the character of the standard representation, the second one is the character of the sign representation. Thus $V_Z = 1 + V_{st} + V_{st} + V_{sign} \cong V_{reg}$.

On the hand, consider the ideal $(z_1, z_2)^3 \in (X^{[6]})^G$. We have V_Z is spanned by the cosets of $1, z_1, z_2, z_1^2, z_2^2, z_1z_2$. The first three basis elements span $1 + V_{st}$. The remaining basis vectors span $S^2(V_{st})$. It contains the trivial representation spanned by the coset of $z_1^2 + z_2^2 + z_1z_2$, with complement isomorphic to the standard representation. Thus $V_Z \ncong V_{\mathrm{reg}}$.

6.4 Exercises

6.1 Let $X = \mathbb{P}^2$ consider the rational map $X^{(3)} - \to G(3, 6)$ which assigns to a set of 3 points the linear system of conics through these points.

(i) Show that the composition of this map with the cycle map $X^{[3]} \to X^{(3)}$ extends to a regular map $f : X^{[3]} \to G(3, 6)$.

(ii) Show that the image of the strata $X_{(3)}^{[3]}$ is equal to the tangent scroll of the Veronese surface $\nu_2(\mathbb{P}^2)$ of double lines (the union of tangent planes to the surface).

(iii) Show that the complement to the open strata $X^{[3]}_{1^3}$ contains two copies of $X^{[2]}$ intersecting at a surface isomorphic to X and contained in the strata $X^{[3]}_{(3)}$.

(iv) Let $\check{X}$ be the dual plane of lines. Consider a birational map $X^{[3]} - \to \check{X}^{[3]}$ which assigns to a general set of three points the three sides of the corresponding triangle. What is the largest open subset to which the map extends as a regular map?

6.2 Show that the assignment $X \to X^{[n]}$ is not a functor. For example, consider the double cover $f : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ ramified along a conic C. Show that it defines only a rational map of degree 2 $(\mathbb{P}^1 \times \mathbb{P}^1)^{[2]} - \to (\mathbb{P}^2)^{[2]}$. Describe its indeterminacy points.

6.3 The 0-dimensional subscheme of X defined by the ideal $\mathfrak{m}_x^n$ is called a *fat point*. Consider it as a point in $X^{[N]}$, where $N = \binom{\dim X+n-1}{2}$. Suppose G is a finite group of order N acting on X which has x as a fixed point. Find all possible n and G such that $\mathfrak{m}_x^n$ belongs to G-Hilb(X).

6.4 Let X be a general quartic surface in $\mathbb{P}^3$. Show that $X^{[2]}$ admits an automorphism τ of order 2 with quotient $X^{[2]}/(\tau)$ isomorphic to the Grassmann variety $G(2,4)$.

6.5 Describe $(\mathbb{C}^2_0)^{[3]}$ and $(\mathbb{C}^2_0)^{[3]}$ using the matrix interpretation from Example 6.1.7. Show that $(\mathbb{C}^2_0)^{[3]}$ is a singular variety.

6.6 Let k be of characteristic $\neq 2$. Show that $(\mathbb{A}^2_k)^{(2)}$ is isomorphic to $\mathbb{A}^2_k \times C$, where $C = \text{Spec } k[x,y/(xy+z^2)$.

6.7 Find all possible G-invariant cycles in $\mathbb{C}^2_0)^{[n]}$ for $n \leq 6$ with respect to a cyclic subgroup $\mathrm{SL}(2,\mathbb{C})$ of order n.

6.8 (N. Hitchin) Study irreducible components of $(X^{[n]})^G$, where X is a surface with holomorphic symplectic structure and G a finite group acting on X preserving the structure.

Lecture 7

Quiver varieties

7.1 Quivers and their representations

A *quiver* is an oriented finite graph $Q = (Q_0, Q_1)$ with an ordered set of vertices Q_0 and a set of arrows Q_1. Let $\mathcal{C}$ be a category. A *representation* of a quiver Γ in $\mathcal{C}$ is a map which assigns to each vertex $v \in Q_0$ and object C_v of the category $\mathcal{C}$ and to each arrow $a \in Q_1$ with tail $t(a) = v$ and head $h(a) = v'$ a morphism $\phi(a) : C_v \to C_{v'}$.

We will be interested only in linear representations , in which $\mathcal{C}$ is the category Vect_k of finite-dimensional vector spaces over an algebraically closed field k. We can view a representation $\rho = (\rho_0, \rho_1)$ as an ordered collection of vector spaces $E_v = \rho(v), v \in Q_0$, and linear maps $\rho(a) : \rho(t(a)) \to \rho(h(a)), a \in Q_1$. The *dimension* representation of Q is the vector $\mathbf{d} = (d_v)_{v \in Q_0}$, where $d_v = \dim E_v$.

A *morphism of representations* $\rho \to \rho'$ is a set of linear maps $\phi_v : \rho(v) \to \rho'(v) = E'_v, v \in Q_0$, such that, for any $a \in Q_1$ the diagram

$$\begin{array}{ccc} E_{t(a)} & \xrightarrow{\phi_{t(a)}} & E'_{t(a)} \\ \downarrow{\scriptstyle \rho(a)} & & \downarrow{\scriptstyle \rho'(a)} \\ E_{h(a)} & \xrightarrow{\phi_{h(a)}} & E'_{h(a)} \end{array}$$

is commutative.

We leave to the reader to define the notions of subrepresentation, direct sum of representations, irreducible representation of a quiver. In fact, they all correspond to the usual notions in the theory of modules if we view a representation of a quiver as a module over its path algebra.

The *path algebra* KQ of a quiver Q is defined as follows. Let $\mathcal{P}$ be the set of paths in Q, i.e. sequences $a_1, \ldots, a_m \in Q_1$ such that $h(a_i) = t(a_{i+1})$. Eaach $v \in Q_0$ will be considered as a path e_v. Consider the vector space $k^{\mathcal{P}}$ of functions on the set of paths. It has a natural basis formed by the delta-functions which we identify with elements of $\mathcal{P}$. Define the multiplication by setting

$$p \cdot q = \begin{cases} pq & \text{if } h(q) = t(p) \\ 0 & \text{otherwise.} \end{cases},$$

where pq means the usual composition of paths. In particular, we require that the paths v corresponding to the vertices are idempotents, i.e. $e_i^2 = e_i$ and $e_i \cdot q = q$ if $h(q) = i$ and $p \cdot e_i = p$ if $t(p) = i$. We also require that $e_i e_j = 0, i \neq j$. In particular, we get

$$1 = e_1 + \ldots + e_m \tag{7.1}$$

since it follows from above that $(e_1 + \ldots + e_m)x = x$ for any $x \in KQ$.

Suppose we have a representation ρ of Q. Let $V = \oplus_{v \in Q_0} \rho(v)$. To every path e_v we assign the operator in V which is the projector operator $\pi_v : V \to E_v$. To every arrow-path a we assign the operator T_a which acts trivially on the summand E_v if $v \neq t(a)$ and acts as $\rho(a) : E_{t(a)} \to E_{h(a)}$ otherwise. Since the arrows and $e_v, v \in Q_0$, generate KQ as an algebra, this defines a structure of a KQ-module on V.

Conversely, suppose V is a KQ-module. For any path p we denote by $\tilde{p}$ the corresponding linear operator in V. We define $\rho(i), i \in Q_0$, as the subspace $V_i = \tilde{e}_i(V)$. It follows from (7.1) that V is the direct sum of the subspaces $V_i, i \in Q_0$. Since $\tilde{e}_{h(a)} \circ \tilde{a} = \tilde{a}$ and $\tilde{a} \circ \tilde{e}_{t(a)} = \tilde{a}$, the operator $\tilde{a}$ maps the subspace $Q_{t(a)}$ to the subspace $Q_{h(a)}$. We take this linear map as $\rho(a)$.

From now on we will identify Q_0 with the set of positive integers $\{1, \ldots, m\}$.

Fix $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbb{N}^m$. Consider the set of representations ρ of Q with dimension vector $\mathbf{d}$. By choosing a basis in each vector space E_i we can replace ρ with isomorphic representation where each $E_i = k^{d_i}$ and each $\rho(a)$ is a matrix M_a of size $d_{h(a)} \times d_{t(a)}$. So, we may identify a representation with a collection of matrices M_a for each arrow in Q_1.

Let

$$\mathrm{GL}(\mathbf{d}) = \prod_{i=1}^{m} \mathrm{GL}(d_i, k), \quad \mathrm{Rep}(Q, \mathbf{d}) = \prod_{a \in Q_1} \mathrm{Mat}_{d_{h(a)}, d_{t(a)}}.$$

The group $\mathrm{GL}(\mathbf{d})$ acts on the set of representations $\mathrm{Rep}(Q, \mathbf{d})$ by simultaneous conjugation

$$(g_1, \ldots, g_m) : (M_a)_{a \in Q_1} \mapsto (g_{h(a)} \cdot M_a \cdot g_{t(a)}^{-1}). \tag{7.2}$$

Thus isomorphism classes of representations of quivers of dimension $\mathbf{d}$ are in bijective correspondence between the orbits of $G(\mathbf{d})$ on the vector space $\mathrm{Rep}(Q, \mathbf{d})$.

Remark 7.1.1. One can give another expression for the action of $\mathrm{GL}(\mathbf{d})$ on $\mathrm{Rep}(Q, \mathbf{d})$ as follows. For every arrow a consider the variables t_{apq}, where $1 \leq p \leq d_{h(a)}, 1 \leq q \leq d_{t(a)}$, correspond to the entries of a general matrix of size $d_{h(a)}, \times d_{t(a)}$. Let $k[t_{apq}]$ be the polynomial algebra with variables t_{apq}. It can be interpreted as the coordinate algebra of $\mathrm{Rep}(Q, \mathbf{d})$. A representation is just a homomorphism of this algebra to k.

Let $d = |\mathbf{d}| = d_1 + \ldots + d_m$. Consider the ring of matrices $\mathrm{Mat}_d(k[t_{apq}])$ of size $d \times d$. We view each such matrix as a block-matrix with sizes of the blocks equal to $d_{h(a)}, \times d_{t(a)}$. Now we have a natural homomorphism

$$KQ \to \mathrm{Mat}_d(k[t_{apq}]) \tag{7.3}$$

which assigns to a generator $e_a, a \in Q_1$, the matrix whose entries are t_{apq} and zero otherwise. We also assign to each $e_i, i \in Q_0$, the identity matrix in the block $d_i \times d_i$ and all other blocks equal to zero. Now $(g_1, \ldots, g_m) \in \mathrm{GL}(\mathbf{d})$ acts on $\mathrm{Mat}_d(k[t_{apq}])$ by conjugation by a matrix g which consists of diagonal blocks of size $d_i \times d_i$ equal to g_i. Thus a representation $\rho \in \mathrm{Rep}(Q, \mathbf{d})$ can be considered as a matrix of size $d \times d$ with coefficients in k and the action of $\mathrm{GL}(\mathbf{d})$ on $\mathrm{Rep}(Q, \mathbf{d})$ corresponds to the conjugation action.

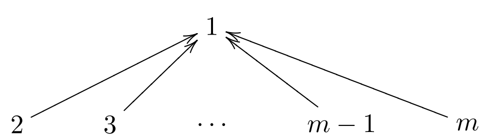
Example 7.1.2. Consider the quiver

$$1 \longrightarrow 2$$

Its representation of dimension (d_1, d_2) is defined by a matrix of size $d_2 \times d_1$. The group $GL(d_1, k) \times \mathrm{GL}(d_2, k)$ acts by conjugation $A \mapsto g_2 A g_1^{-1}$. There are $\min(d_1, d_2)$ isomorphism classes determined by the rank of A.

The quiver Q with $Q_0 = \{1\}$ and $Q_1 = \{a\}$ such that $t(a) = h(a) = 1$ is defined by a square matrix A of size d_1 and the isomorphism classes are the conjugacy classes of A. Their number is the number of partitions of d_1.

Example 7.1.3. Consider the quiver



Its representation of dimension $(n+1, 1, \ldots, 1)$ is defined by a choice of a vector v_i in the space E_1, the image of $1 \in E_i$. Thus all representations are parametrized by $(k^{n+1})^m$. Consider the subset of representations such that all vectors v_i are nonzero. Then isomorphism classes of such representations are parametrized by the orbits set $(\mathbb{P}^n)^{m-1}/\mathrm{GL}(n+1)$, where the group acts diagonally.

Example 7.1.4. Let Q be the quiver

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow m$$

A path is the product $a_{ii+1} \cdots a_{j-1j}$, where a_{kk+1} is the arrow between k and $k+1$. Assign to this path a unit matrix E_{ji} of size m. Each vertex i is assigned to the matrix E_{ii}. We know that $E_{ij}E_{km} = \delta_{ik}E_{im}$. One easily checks that the algebra KQ is isomorphic to the algebra of low-triangular matrices $(a_{ij}), a_{ij} = 0$ for $i < j$. We have $e_i = E_{ii}$ and

$$e_i KQ = kE_{i1} + \ldots + kE_{ii}, \; i = 1, \ldots, m. \tag{7.4}$$

This shows that KQ, considered as a left module over itself, defines the representation of the quiver Q with the dimension vector $\mathbf{d} = (m, m-1, \dots, 1)$.

The following is one of the fundamental results in the theory of representations of quivers.

A quiver is called to be *finite type* it has only finitely many indecomposable representations.

The following is a fundamental result of the theory.

Theorem 7.1.5. *(A. Gabriel) A quiver is of finite typer if and only if, considered as an non-oriented graph, it is equal to a Dynkin diagram of types A,D,E.*

7.2 Varieties of quiver representations

We would like to construct an algebraic variety parametrizing the isomorphism clases of representations of quivers with fixed dimension vector. It is well-known that the set of orbits is usually can not be parametrized by an algebrac variety because some orbits are not closed. The best what we can is to parametrize some orbits which are semi-stable in certain sense. This is the subject of the Geometric Invariant Theory. For simplicity we will assume that $K = \mathbb{C}$.

Suppose G is a reductive algebraic group acting on an affine algebraic variety $X = \text{Spec } A$ (our groups $\text{GL}(\mathbf{d})$ are reductive). A naive approach to constructing the quotient is to take the spectrum of algebra of invariants of A. However, it is often consists of only constants. For example, as we will see in Proposition 7.2.1 below this happens if Q does not have oriented cycles.

So we have to modify the construction. Choose a one-dimensional representation $\chi : G \to K^*$ and consider the graded algebra of *semi-invariants* of G with respect to χ

$$A^G(\chi) = \bigoplus_{n=0}^{\infty} A^G(\chi)_n,$$

where

$$A^G(\chi)_n = \{\phi \in A : g^*(\phi) = \chi(g)^n \phi\}.$$

A theorem of Hilbert asserts that this ring is a finitely generated algebra over $\mathbb{C}$. By definition

$$X//_{\chi} G = \text{Proj } A^G(\chi).$$

If A^G consists only of constants, this is a projective algebraic variety.

In our situation a homomorphism $\chi : \text{GL}(\mathbf{d}) \to K^*$ is equal to the product of determinants

$$\chi(g) = \prod_{i=1}^{m} \det(g_i)^{\boldsymbol{\theta}_i}, \ g = (g_1, \dots, g_m).$$

Thus χ is determined by a vector $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_m) \in \mathbb{Z}^m$. We consider $\text{Rep}(Q, \mathbf{d})$ as an affine algebraic variety (isomorphic to affine space). Set

$$\mathcal{R}_{\boldsymbol{\theta}}(Q, \mathbf{d}) := \text{Rep}(Q, \mathbf{d})//_{\boldsymbol{\theta}} \text{GL}(\mathbf{d}). \tag{7.5}$$

Assume that Q does not contain oriented cycles, in particular, all arrows between two vertices go in the same direction. In this case one can number the vertices in such a way that $t(a) < h(a)$ for all arrows (order the partial ordered set Q_0). Define the *Cartan matrix* of Q as the matrix $C = (a_{ij})$, where $a_{ii} = 2$ and $-a_{ij} = -a_{ji}$ is equal to the number of arrows between i and j. It is easy to see that

$$\dim \mathrm{Rep}(Q, \mathbf{d}) - \dim \mathrm{GL}(\mathbf{d}) = \sum_{i \le j} d_i d_j a_{ij} - \sum_i d_i^2 = -\frac{1}{2}\mathbf{d} \cdot C \cdot \mathbf{d}.$$

Note that the center of the group $\mathrm{GL}(\mathbf{d})$ acts identically. It consists of elements $g = (c, \ldots, c)$, where c is a scalar matrix. Thus we expect that

$$\dim \mathcal{R}_{\boldsymbol{\theta}}(Q, \mathbf{d}) = 1 - \frac{1}{2}\mathbf{d} \cdot C \cdot \mathbf{d}. \tag{7.6}$$

The value of the character $\boldsymbol{\theta}$ on this subgroup is equal to $c \mapsto c^{\mathbf{d} \cdot \boldsymbol{\theta}}$. We assume that the character $\boldsymbol{\theta}$ satisfies

$$\mathbf{d} \cdot \boldsymbol{\theta} = 0.$$

If this condition is not satisfied then $\mathcal{R}_{\boldsymbol{\theta}}(Q, \mathbf{d}) = \emptyset$ (because the center of the group acts identically but the value of the character $\chi_{\boldsymbol{\theta}}$ on the center is not trivial).

Proposition 7.2.1. *Assume that Q has no oriented cycles. Then the variety $\mathcal{R}_{\boldsymbol{\theta}}(Q, \mathbf{d})$ is a projective variety.*

Proof. It suffices to show that $\mathcal{O}(\mathrm{Rep}(Q, \mathbf{d}))^{\mathrm{GL}(\mathbf{d})}$ consists of only constants. We may assume that $t(a) < h(a)$ for all arrows. Take $g \in \mathrm{GL}(\mathbf{d})$ to be equal $(c, c^2, \ldots, c^m)$, where c^i are scalar matrices of dimension d_i. It sends $(M_1, \ldots, M_m) \in \mathrm{Rep}(Q, \mathbf{d})$ to $(cM_1, \ldots, cM_m)$. Considering the entries of M_i as unknowns, we see that each unknown is multiplied by c. Obviously, only constants are invariant. □

A representation $\rho \in \mathrm{Rep}(Q, \mathbf{d})$ is said to be *$\boldsymbol{\theta}$-semistable* (resp. *$\boldsymbol{\theta}$-stable*) if the closure of its orbit does not contain the zero vector or, equivalently, if there exists a non-constant homogeneous semiinvariant $f(\rho) \neq 0$ (resp. additionally, its orbit is closed in the set of semi-stable points and its stabilizer is a finite group).

Theorem 7.2.2. *(A. King) A representation $\rho \in \mathrm{Rep}(Q, \mathbf{d})$ is $\boldsymbol{\theta}$-semistable if and only if any subrepresentation ρ' of ρ with dimension vector $\mathbf{d}$ satisfies $\mathbf{d}' \cdot \boldsymbol{\theta} \le 0$. It is stable if and only if the equality is strict for any proper subrepresentation. There is a morphism from an open subset $\mathrm{Rep}(Q, \mathbf{d})^s$ of $\boldsymbol{\theta}$-stable points in $\mathrm{Rep}(Q, \mathbf{d})$ to the variety $\mathcal{R}_{\boldsymbol{\theta}}(Q, \mathbf{d})$ whose fibres are $GL(\mathbf{d})$-orbits. There is a natural bijection between the complement of the image of $\mathrm{Rep}_{\boldsymbol{\theta}}(Q; \mathbf{d})^s$ and the set of orbits of representation equal to the direct sum of stable subrepresentations with dimension vector orthogonal to $\boldsymbol{\theta}$. If the set of $\boldsymbol{\theta}$-stable representations is not empty and Q has no oriented cycles, then formula (7.6) holds.*

The question when the set of $\boldsymbol{\theta}$-stable representations is not empty is difficult one. Under certain assumption on the path algebra KQ this question was solved by Schofield (Proc. L.M.S., 1992).

Example 7.2.3. Consider the quiver from Example 7.1.3. The group $\mathrm{GL}(\mathbf{d}) = \mathrm{GL}(n+1,k)\times\mathrm{GL}(1,k)^m$. The space $\mathrm{Rep}(Q,\mathbf{d})$ is equal to the space $(\mathrm{Mat}_{1,n+1})^m$ which we can identify with the space of matrices $\mathrm{Mat}_{n+1,m}$. The group $\mathrm{GL}(\mathbf{d})$ acts by left multiplications by matrices from $\mathrm{GL}(n+1,\mathbb{C})$ and by scaling the columns. Let $\boldsymbol{\theta} = (s,-s_1,\ldots,-s_m)$ be a multiplicative character of $\mathrm{GL}(\mathbf{d})$ such that $\boldsymbol{\theta}\cdot\mathbf{d} = 0$, i.e. $s(n+1) = s_1+\ldots+s_m$. A semi-invariant f with character $\chi_{\boldsymbol{\theta}}^t$ satisfies

$$f(gAh) = (|g|^{-s}c_1^{s_1}\ldots c_m^{s_m})^t f(A),$$

where $g\in\mathrm{GL}(n+1,k)$ and h is the diagonal matrix $\mathrm{diag}[c_1,\ldots,c_m]$. Note that all such polynomials are invariant with respect to the subgroup $\mathrm{SL}(n+1,k)$. The First Fundamental Theorem of Invariant theory tells that each $\mathrm{SL}(n+1,K)$-invariant polynomial of the entries of the matrices is equal to a polynomial in its maximal minors $M_I, I\subset\{1,\ldots,m\}$. The homogeneous elements of degree s in our ring $\mathrm{Rep}(Q,\mathbf{d})_{\boldsymbol{\theta}}$ are linear combinations of monomials $M_{I_1}\cdots M_{I_w}$ such that

$$\begin{aligned}(g,c_1,\ldots,c_m)M_{I_1}\cdots M_{I_w} &= |g|^w c_1^{d_1}\cdots c_m^{d_m}M_{I_1}\cdots M_{I_w}\\ &= (|g|^s c_1^{k_1}\cdots c_m^{k_m})^s M_{I_1}\cdots M_{I_w},\end{aligned}$$

where each $j\in\{1,\ldots,m\}$ appears d_j times in the sets $I_1,\ldots,I_w$. This gives $w=st, d_i=s_it$. Since $(n+1)s=\sum s_i$, we obtain that $s,s_i\geq 0$ and

$$(n+1)w = \sum_{i=1}^m d_i. \tag{7.7}$$

Thus the invariant monomial $M_{I_1}\cdots M_{I_w}$ is described by *tableux* of multidegree $(d_1,\ldots,d_m)$

$$\begin{pmatrix} i_{11} & \ldots & i_{1n+1}\\ \vdots & \vdots & \vdots\\ i_{w1} & \ldots & i_{wn+1}\end{pmatrix}.$$

Its entries are elements from the set $\{1,\ldots,m\}$. Each element j of this set appears exactly d_j and (7.7) is satisfied. We also have $w=ts, d_i=ts_i$. Comparing this to the known description of the GIT-quotients of $(\mathbb{P}^n)^m//_L\mathrm{SL}(n+1)$ (see [Dolgachev, Lectures on Invariant Theory]), we find that the variety $\mathcal{R}_{\boldsymbol{\theta}}(Q,\mathbf{d})$ coincides with such GIT-quotient, where the linearization L is given by the sheaf

$$L_{\mathbf{s}} = \mathcal{O}_{\mathbb{P}^n}(s_1)\boxtimes\cdots\boxtimes\mathcal{O}_{\mathbb{P}^n}(s_m)$$

on $(\mathbb{P}^n)^m$. We leave to the reader to check that the stable representations correspond to stable point configurations as described in [Dolgachev].

7.3 McKay quivers

Let $\Gamma(G,\rho)$ be the McKay graph of a finite group with respect to a representation ρ_0. Order the set $\mathrm{Ir}(G)$ and consider $\Gamma(G,\rho)$ as a quiver. This is the *McKay quiver* of (G,ρ). Let R be the regular representation of G over $k=\mathbb{C}$. Let $R_1,\ldots,R_c$ be representatives of isomorphism classes of irreducible representations of G and $\mathbf{d}=(\dim R_1,\ldots,\dim R_c)$. Let N be the representation ρ, considered as a $k[G]$-module. We have

$$\begin{aligned}(\mathrm{End}(R)\otimes N)^G &= (R^*\otimes R\otimes N)^G = \mathrm{Hom}_{\mathbb{C}[G]}(R,R\otimes N)\\ &= \mathrm{Hom}_{\mathbb{C}[G]}(\bigoplus_{i=1}^c R_i\otimes \mathbb{C}^{d_i},\bigoplus_{j=1}^c R_j\otimes\mathbb{C}^{d_j}\otimes N)\\ &= \bigoplus_{i,j=1}^c \mathrm{Hom}_{\mathbb{C}}(R_i,R_j\otimes N)\otimes\mathrm{Hom}(\mathbb{C}^{d_i},\mathbb{C}^{d_j}) = \mathrm{Rep}(Q,\mathbf{d}),\end{aligned}$$

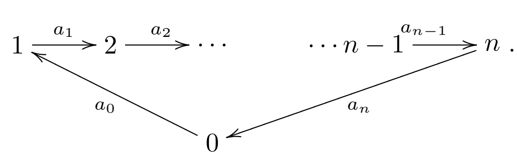
where Q is the quiver corresponding to the McKay graph of (G,ρ_0). Its vertices are $1,\ldots,c$ and the number of arrows from i to j is equal to $\dim\mathrm{Hom}_{\mathbb{C}}(R_i,R_j\otimes N)$. If ρ is an admissible representation in the sense of Lecture 5, then the graph $\Gamma(G,\rho)$ has arrows going in both directions.

Consider the quiver Q' equal to Q as a non-oriented graph and fix a orientation on each edge such that the graph has no oriented cycles. The Cartan matrix C of the quiver Q' is equal to $M-(\dim\rho_0-2)I_c$, where M is the McKay-Springer matrix we considered in Lecture 5. It follows from the properties of this matrix that $M\cdot\mathbf{d}=0$, thus formula (7.6) gives

$$\dim\mathcal{R}_{\boldsymbol{\theta}}(Q',\mathbf{d}) = 1+\frac{1}{2}(\dim\rho_0-2)\sum_{i=1}^c d_i^2 = 1+\frac{1}{2}(\dim\rho_0-2)|G|$$

provided that $\boldsymbol{\theta}\cdot\mathbf{d}=0$ and there exists a $\boldsymbol{\theta}$-stable representation of Q. Note that, it follows from the definition of an admissible character that $|G|$ is an even number.

Let us consider a special case when $G\subset SU(2)$ is a cyclic group of order $n+1$ and Q is the McKay graph equal to the affine Dynkin diagram of type $\tilde{A}_n$. Let $Q_0=\{0,\ldots,n\}$ and $Q_1=\{a_0,\ldots,a_n\}$ with $t(a_i)=i, h(a_i)=i+1, i\neq n$ and $t(a_n)=n, h(a_n)=0$



We have $\mathbf{d}=(1,\ldots,1)\in\mathbb{Z}^{n+1}$ and

$$\mathrm{Rep}(Q',\mathbf{d})=\mathbb{C}^{n+1},\quad \mathrm{GL}(\mathbf{d})=(\mathbb{C}^*)^{n+1}.$$

The group $\mathrm{GL}(\mathbf{d})$ acts by

$$(\lambda_0, \ldots, \lambda_n) : (z_0, \ldots, z_n) \mapsto (\lambda_1\lambda_0^{-1}z_0, \ldots, \lambda_0\lambda_n^{-1}z_n).$$

The kernel of the action is equal to the diagonal subgroup isomorphic to $\mathbb{C}^*$. Choose $\boldsymbol{\theta} = (\boldsymbol{\theta}_0, \ldots, \boldsymbol{\theta}_n) \in \mathbb{Z}^{n+1}$ with $\boldsymbol{\theta}_0 + \ldots + \boldsymbol{\theta}_n = 0$. We have

$$V_d = (\mathcal{O}(\mathrm{Rep}(Q', \mathbf{d}))_{\boldsymbol{\theta}}^{\mathrm{GL}(\mathbf{d})})_d = \{P(z) \in \mathbb{C}[z_0, \ldots, z_n] : P(\lambda \cdot z) = \lambda^{d\boldsymbol{\theta}}P(z)\}.$$

It is clear that each polynomial in this space is a sum of monomials $t_0^{m_1} \cdots t_n^{m_n}$ such that

$$(-m_0 + m_n, m_0 - m_1, \ldots, -m_n + m_{n-1}) = d(\boldsymbol{\theta}_0, \ldots, \boldsymbol{\theta}_n).$$

We can express $m_0, \ldots, m_{n-1}$ in terms of m_n to get

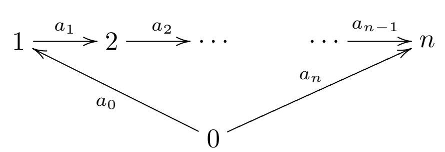
$$(m_0, \ldots, m_n) = (m_n + d(\boldsymbol{\theta}_1 + \ldots + \boldsymbol{\theta}_n), m_n + d(\boldsymbol{\theta}_2 + \ldots + \boldsymbol{\theta}_n), \ldots, m_n + d\boldsymbol{\theta}_n, m_n),$$

where $m_n + d(\boldsymbol{\theta}_i + \ldots + \boldsymbol{\theta}_n) \geq 0$ for all $i = 1, \ldots, n$. Let N be the smallest of the sums $\boldsymbol{\theta}_i + \ldots + \boldsymbol{\theta}_n, i = 1, \ldots, n$. We put $N = 0$ if all the sums are positive. Let M be the semigroup of numbers $(a, b) \in \mathbb{Z}_{\geq 0}^2$ such that $a + bN \geq 0$. We obtain

$$\mathcal{O}(\mathrm{Rep}(Q', \mathbf{d}))_{\boldsymbol{\theta}}^{\mathrm{GL}(\mathbf{d})} \cong \mathbb{C}[M].$$

The semigroup algebra $\mathbb{C}[M]$ is graded by the function $M \to \mathbb{Z}_{\geq 0}, (a, b) \mapsto a$. Since $s_1 = (0, 1)$ and $s_2 = (1, -N)$ form an integral basis and belong to $\mathbb{C}[M] \cong \mathbb{C}[u, v]$, where $u = \delta_{s_1}, v = \delta_{s_2}$. The subalgebra generated by u is the algebra of invariants. It is generated by the monomial $t_0 \cdots t_n$. The projective spectrum is isomorphic to Spec $\mathbb{C}[u] = \mathbb{A}^1$.

Now let us change the orientation in Q' assuming that there are no oriented cycles in Q'. Let us assume that $t(a_n) = 0, h(a_n) = n$ and leaving all other arrows unchanged.



The action changes to

$$(\lambda_0, \ldots, \lambda_n) : (z_0, \ldots, z_n) \mapsto (\lambda_1\lambda_0^{-1}z_0, \ldots, \lambda_0^{-1}\lambda_n z_n).$$

The new equalities are

$$(-m_0 - m_n, m_0 - m_1, \ldots, m_{n-2} - m_{n-1}, m_{n-1} + m_n) = d(\boldsymbol{\theta}_0, \ldots, \boldsymbol{\theta}_n).$$

We can express all m_i's in terms of m_n to get

$$m_0 = -m_n - d\boldsymbol{\theta}_0, m_1 = -m_n - d(\boldsymbol{\theta}_0 + \boldsymbol{\theta}_1), \ldots, m_{n-1} = -m_n - d(\boldsymbol{\theta}_0 + \ldots + \boldsymbol{\theta}_{n-1}).$$

Notice the immediate difference from the previous case. If $d = 0$, the only solution is $(0, \ldots, 0)$. This confirms what we already know, the moduli space is a projective variety. If one of the numbers $N_i = \boldsymbol{\theta}_0 + \ldots + \boldsymbol{\theta}_i, i = 0, \ldots, n-1$, is nonnegative, there are no nonzero solutions in nonnegative $m_0, \ldots, m_n$. We assume that all N_i are negative. Let N be the smallest of the numbers $-N_i$. We have the semigroup M of numbers (d, s) such that $s + dN \leq 0, s \geq 0$. The generators of $\mathbb{C}[M]$ are

$$\delta_{(1,s)} = U_s = t_0^{s-N-N_1} \cdots t_{n-1}^{s-K-K_n} t_n^s, \quad s = 0, \ldots, -N$$

and relations are $U_i U_j - U_k U_l, i + j = k + l$. We recognize the Veronese ring

$$\mathbb{C}[M] \cong \mathbb{C}[v_1^N, v_1^{N-1} v_1, \ldots, v_2^N] \subset \mathbb{C}[v_1, v_2].$$

So we get

$$\mathcal{R}_{\boldsymbol{\theta}}(Q', \mathbf{d}) \cong \mathbb{P}^1.$$

Let $I = \{i : N_i = N\} \subset [0, n-1]$. All points $(z_0, \ldots, z_n)$ with $z_i \neq 0, i \notin I$, are semi-stable. All points with nonzero coordinates $z_i, i \notin I$ are semi-stable. Let us apply King's criterion to decide which representations are $\boldsymbol{\theta}$-stable. Suppose ρ has a subrepresentation ρ' with dimension vector $\mathbf{d}'$ such that $d_j = 1, j \in J, d_j = 0, j \notin J$ for some subset J of $[1, n+1]$. If ρ has coordinate $z_i \neq 0$ corresponding the arrow a_i with $t(a_i) = k < i, h(a_i) = i$, then $d_i' \neq 0$ implies $d_k' \neq 0$. Thus we see that all ρ with all coordinate $z_i \neq 0$ do not have proper subrepresentations and hence stable. On the other hand, suppose some $z_i = 0$ with $i \in I$. Let us assume that i is the minimal with this property. Then ρ contains a subrepresentation ρ' with $d_1' = \ldots = d_i' = 1, d_j = 0, j > i$. We have $\mathbf{d}' \cdot \boldsymbol{\theta} = -N_i < 0$. So the representation is semi-stable but not stable.

7.4 Preprojective algebras

Let $Q = (Q_0, Q_1)$ be a quiver. Its *double* is the quiver $\tilde{Q}$ obtained by doubling the number of arrows. For every arrow a of Q we define another arrow a^* with $t(a^*) = h(a), h(a^*) = t(a)$. For any $\lambda \in K^{Q_0}$ we define the *deformed preprojective algebra* of weight λ

$$\Pi(Q)_\lambda = k\tilde{Q}/\big(\sum_{a \in Q} [e_a, e_{a^*}] - \sum_{i \in Q_0} \lambda_i e_i\big).$$

The *preprojective algebra* $\Pi(Q)$ corresponds to $\lambda = 0$.

A finite-dimensional vector space over k equipped with a structure of a left module over an associative k-algebra R will be called a (linear) representation of R and the set of isomorphism classes (as vector spaces) of such modules will be denoted by $\mathrm{Rep}(R)$. Every representation ρ of $\Pi_\lambda(Q)$ can considered as a representation ρ of the quiver $\tilde{Q}$ satisfying the additional condition

$$\sum_{a \in Q_1 : h(a) = i} \rho(a) \circ \rho(a^*) - \sum_{a \in Q_1 : t(a) = i} \rho(a^*) \circ \rho(a) = \lambda_i \mathbf{id}_{\rho(i)}.$$

It is easy to see that the set of $\mathrm{Rep}(\Pi_\lambda(Q),\mathbf{d})$ of representations of $\Pi_\lambda(Q)$ with given dimension vector $\mathbf{d}$ forms a closed subvariety of $\mathrm{Rep}(k\tilde{Q},\mathbf{d})$. It follows from the definition of the action of the group $\mathrm{GL}(\mathbf{d})$ that this subvariety is invariant with respect to the action. We will assume $k=\mathbb{C}$ and denote by $\mathcal{R}(\Pi_\lambda,\mathbf{d})_{\boldsymbol{\theta}}$ the corresponding GIT-quotients $\mathrm{Rep}(\Pi_\lambda(Q),\mathbf{d})//_{\boldsymbol{\theta}}\mathrm{GL}(\mathbf{d})$, where $\boldsymbol{\theta}\in\mathbb{Z}^m$ defines the multiplicative character $\chi_{\boldsymbol{\theta}}:\mathrm{GL}(\mathbf{d})\to\mathbb{C}^*$.

Let

$$\mathrm{End}(\mathbf{d}) = \mathrm{Lie}\ \mathrm{GL}(\mathbf{d}) = \bigoplus_{i=1}^m \mathrm{End}(k^{d_i}) = \bigoplus_{i=1}^m \mathrm{Mat}_{d_i}(k).$$

The center $Z(\mathbf{d})$ of $\mathrm{End}(\mathbf{d})$ consists of m-tuples of scalar matrices and can be identified with the vector space k^m. We will identify the parameter λ from above with an element of the center.

Consider a regular map of affine varieties

$$\mu:\mathrm{Rep}(\tilde{Q},\mathbf{d})\to\mathrm{End}(\mathbf{d}),\ \rho=(\rho(a),\rho(a^*))\mapsto\sum_{a\in Q_1}[\rho(a),\rho(a^*)]. \tag{7.8}$$

It is clear that

$$\mathrm{Rep}(\Pi_\lambda(Q),\mathbf{d})=\mu^{-1}(\lambda).$$

Remark 7.4.1. One can give an analog of a homomorphism $KQ\to\mathrm{Mat}_d(k[t_{apq}])$ considered in Remark 7.1.1 for a deformed preprojective algebra Π_λ. We assign again to each arrow of the double quiver $\tilde{Q}$ a set of variables t_{apq}. Now we see that the coordinate ring of $\mathrm{Rep}(\Pi_\lambda,\mathbf{d})$ becomes isomorphic to $k[t_{apq}]/J_\lambda$, where J_λ is generated by the elements

$$\sum_{a\in Q_1:h(a)=i}\sum_{s=1}^{d_{t(a)}} t_{aps}t_{a^*sq}-\sum_{a\in Q_1:t(a)=i}\sum_{s=1}^{d_{h(a)}} t_{a^*ps}t_{asq}-\delta_{pq}d_i,\quad i=1,\ldots,m. \tag{7.9}$$

Now the homomorphism $k[\tilde{Q}]\to\mathrm{Mat}_d[k[t_{apq}])$ from (7.3) induces a homomorphism

$$\Pi_\lambda\to\mathrm{Mat}_d(k[\mathrm{Rep}(\Pi_\lambda,\mathbf{d})]). \tag{7.10}$$

The group $\mathrm{GL}(\mathbf{d})$ acts on the ring of matrices as in Example 7.1.1.

Example 7.4.2. We consider the example of the quiver Q' from Example 1.9 defined by the McKay graph Q of a cyclic group of order $n+1$. We take Q' by leaving the arrows going in one direction only, i.e. $t(a_i)=i$. The double of Q' is Q. We take $\mathbf{d}=(1,\ldots,1)$. Then

$$\mathrm{Rep}(Q,\mathbf{d})=\mathbb{C}^{n+1}\oplus\mathbb{C}^{n+1}.$$

The group $\mathrm{GL}(\mathbf{d})$ acts as the diagonal action of the action on $\mathbb{C}^{n+1}$ as in the Example (the first part). Let us take $\boldsymbol{\theta}=0$. Then the ring of invariant polynomials is generated by the polynomials

$$X_i=t_it_i',\ i=0,\ldots,n,\quad Y=t_0\cdots t_n,\quad Z=t_0'\cdots t_n'.$$

The basic relation is

$$X_0 \cdots X_n = YZ.$$

This shows that

$$\mathrm{Rep}(Q, \mathbf{d})//\mathrm{GL}(\mathbf{d}) \cong \mathrm{Spec}\ \mathbb{C}[x_0, \ldots, x_n, yz]/(x_0 \cdots x_n - yz).$$

The map (7.8) defines the map

$$\bar{\mu} : \mathrm{Rep}(Q, \mathbf{d})//\mathrm{GL}(\mathbf{d}) \to Z(\mathbf{d}) = (\mathbb{C}^*)^{n+1}.$$

In our case it factors through the map

$$\mu : \mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1} \to Z(\mathbf{d}) = (\mathbb{C}^*)^{n+1},\ (t, t') \mapsto (X_0 - X_1, \ldots, X_n - X_0).$$

The image of this map lies in the subspace

$$\mathfrak{h} = \{(\lambda_0, \ldots, \lambda_n) : \sum_{i=0}^{n} \lambda_i = 0\}$$

which can be identified with the Cartan algebra of the Lie algebra of $\mathrm{SL}(n+1)$. If we use the change of coordinates $x = \frac{1}{n+1}(x_0 + \ldots + x_n), \lambda_i = x - x_i$, we get

$$\mathcal{X} = \mathrm{Rep}(Q, \mathbf{d})//\mathrm{GL}(\mathbf{d}) = \mathrm{Spec}\ \mathbb{C}[y, z, \lambda_0, \ldots, \lambda_n]/(\prod_{i=0}^{n}(x - \lambda_i) - yz, \sum_{i=0}^{n} \lambda_i).$$

The map μ is the projection to $\mathfrak{h}$. We see that

$$m\bar{u}^{-1}(0) \cong \mathbb{C}^2/G = \mathrm{Spec}\ \mathbb{C}[x, y, z]/(x^{n+1} - yz).$$

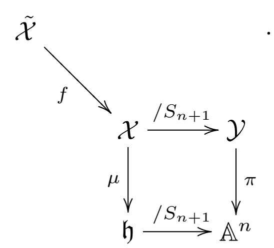
It is the Klein surface corresponding to the cyclic group of order $n+1$. The symmetric group S_{n+1} acts on $\mathfrak{h}$ by permuting the coordinates λ_i. The ring of invariant polynomials is generated by elementary symmetric functions $\sigma_2, \ldots, \sigma_{n+1}$ and the quotient $\mathfrak{h}/S_{n+1}$ is isomorphic to the affine space $\mathbb{A}^n$. We also have

$$\mathcal{Y} = \mathcal{X}/S_{n+1} \cong \mathrm{Spec}\ \mathbb{C}[x, y, z, u_1, \ldots, u_n]/(yz + x^{n+1} + u_1 x^{n-1} + \ldots + u_n).$$

The projection

$$\pi : \mathcal{Y} \to \mathbb{A}^n, \quad (x, y, z, u_1, \ldots, u_n) \mapsto (u_1, \ldots, u_n)$$

It is known in the theory of singularities as the *semi-universal deformation* of the Klein singularity of type A_n. One can show that there exists a birational morphism $f : \tilde{\mathcal{X}} \to \mathcal{X}$ such that we have a commutative diagram



For any $\lambda \in \mathfrak{h}$ the map f restricted to $(\mu \circ f)^{-1}(\lambda)$ is a minimal resolution of singularities of the surface $V_\lambda = \mu^{-1}(\lambda)$. The locus of λ such that V_λ is singular is the union of hyperplanes $\lambda_i - \lambda_j = 0, i \neq j$ (the arrangement of affine hyperplanes of type A_n). The morphism $\mu \circ f : \tilde{\mathcal{X}} \to \mathfrak{h}$ is called a *simultaneous resolution of singularities* of the Klein singularity of type A_n. When $n = 1$, the singularity is $x^2 + yz = 0$, the space $\mathcal{X}$ is the 3-dimensional quadric $x^2 + yz - \lambda_0^2 = 0$ and the simultaneous resolution is a *small resolution* of the quadric with the exceptional locus isomorphic to $\mathbb{P}^1$.

7.5 Exercises

7.1 Show that the path algebra of a quiver is finite-dimensional if and only if the quiver has no oriented paths.
7.2 Consider the doubled quiver Q corresponding to the affine Dynkin diagram of type $\tilde{D}_4$. Following computations in Example 7.4.2 compute explicitly the variety $\mathrm{Rep}(Q, \mathbf{d})//\mathrm{GL}(\mathbf{d})$, where $\mathbf{d} = (2, 1, 1, 1, 1)$ with vertex 1 equal to the middle vertex of the diagram. Find the equations of the fibres of the map μ.
7.3 Consider the quiver from Example 7.1.3 and double it by adding the arrows with the reversed orientation. Show that the moduli space $\mathcal{R}(\Pi_0(Q), \mathbf{d})_{\boldsymbol{\theta}}$, where $\mathbf{d} = (n + 1, 1, \ldots, 1)$, coincides with the GIT-quotient of the space $\mathbb{PT}_{\mathbb{P}^n}^{m-1}//_L\mathrm{SL}(n{+}1)$ with respect to some linearization parameter $L \in \mathrm{Pic}(\mathbb{PT}_{\mathbb{P}^n}^{m-1})$. Here $\mathbb{PT}_{\mathbb{P}^n}$ denotes the projectivization of the tangent bundle of $\mathbb{P}^n$.

Lecture 8

McKay correspondence

8.1 Semi-simple rings

Let us remind some known facts from theory of non-commutative rings. Let R be an associative ring with the unity 1. To distinguish left and right modules over R we use the notation ${}_RM$ for a left R-module and M_R for a right R-module. We denote by ${}_RM_S$ an $R-S$-bimodule with left and right multiplication satisfying $(rm)r' = r(mr')$. The tensor product $M_R \otimes_R {}_RN$ is defined similarly to the commutative case as the universal objects for $\mathbb{Z}$-bilinear maps $f : M_R \times {}_RN \to A$ with values in an abelian group satisfying $f(mr, n) = f(m, rn)$, for all $m \in M, n \in N$ and $r \in R$. In general, $M_R \otimes {}_RN$ is just an abelian group. To equip it with a structure of a module, we have to assume additionally that M and N are $R-R$-bimodules. Then we set $r(m \otimes n) := rm \otimes n, (m \otimes n)r := m \otimes nr$. Now $M \otimes_R N$ becomes the universal object for bilinear maps with values in left or right R-modules.

A structure of an algebra over a commutative ring K on a ring R is given by a homomorphism $\phi : K \to R$ whose image is contained in the center $Z(R)$ of R. Usually, we will take K to be a subfield contained in $Z(R)$. One defines the tensor product of any two K-algebras, considered as $(K \times K)$-bimodules and equip it with the natural structure of a K-algebra. Let M be a left R-module which is a finite-dimensional vector space over K. The endomorphism ring $\mathrm{End}_K(M)$ is isomorphic to the matrix algebra $\mathrm{Mat}_r(K)$, where $r = \dim_K M$. The endomorphism ring $\mathrm{End}_R(M)$ is a subalgebra of $\mathrm{End}_K(M)$

A (left) R-module is called *simple* if it does not contain proper submodules and *indecomposable* if it is not isomorphic to the direct sum of proper submodules. A module is *semi-simple* if it is isomorphic to the direct sum of simple modules. For example, when $R = K[G]$ is the group algebra of a finite group over a field of characteristic prime to the order of G, Maschke's theorem implies that an indecomposable module is simple. Also any finite-dimensional module over $K[G]$ is semi-simple. This is not true in positive characteristic. The representation of a cyclic group of order 2 over a field of characteristic 2 given by the matrix $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ is indecomposable but not simple. There is an analog of Schur's lemma that asserts that the ring of endomorphisms $\mathrm{End}_R(M)$ of a simple module is a division ring, i.e. every non-zero endomorphism is invertible.

We will be dealing with finite-dimensional K-algebras. We already dealt with non-commutative finite-dimensional K-algebras: the group algebras $K[G]$ of finite groups G and the path algebras KQ of quivers without oriented loops.

It is a simple fact that the matrix algebra $\mathrm{Mat}_n(K)$ does not have two-sided ideals. The latter property is taken for definition of a simple K-algebra.

Definition 8.1.1. A K-algebra is *simple* if it does not have non-trivial two-sided ideals, it is semi-simple if it is isomorphic to the direct sum of simple K-algebras.

The following characterization of semi-simple algebras is the following theorem of *Wedderburn-Artin Theorem*.

Theorem 8.1.1. *For any finite-dimensional algebra R over a field K the following properties are equivalent:*

(i) The left (right) R-module R is semi-simple;

(ii) Every left (right) finite-dimensional R-module is semi-simple;

(iii) Every indecomposable left (right) R-module is simple;

(iv) The radical (the intersection of all maximal ideals, left or right) $\mathrm{rad}R$ *is trivial;*

(v) R is isomorphic to the direct sum of the matrix algebras over a division ring (equal to K if K is algebraically closed).

Let R be a semi-simple K-algebra and L be a left ideal in R. Then $R = L \oplus L'$ for some ideal L', hence one can write $1 = e + e'$, where $e \in L, e' \in L'$. Multiplying both sides by e we get $e = e^2 + ee' = e^2$ (since $ee' \in L \cap L'$). An element $e \in R$ satisfying $e^2 = e$ is called an *idempotent*. For any $x \in R$ we can write $x = xe + xe'$, hence $x \mapsto xe$ is the projection onto L. Thus each left ideal is generated by an idempotent.

For any idempotent $e \in R$ the set eRe is a subring of R. It has the unity equal to e, but the inclusion map is not a homomorphism of unitary rings. The ring eRe is a division ring (equal to K if K is algebraically closed) if and only if $L = Re$ is a minimal left ideal. For any left ideal $L = Re$ we decompose L into a direct sum of minimal ideals Re_i to get

$$e = e_1 + \cdots + e_s,\ e_ie_j = 0, i \neq j.$$

(the condition $e_ie_j = 0, i \neq j$ is expressed by saying that $e_1, \ldots, e_s$ are *orthogonal idempotents* This gives

$$eRe = (e_1 + \cdots + e_s)R(e_1 + \ldots + e_s) = e_1Re_1 + \ldots + e_sRe_s.$$

Thus the ring eRe is semi-simple.

Two minimal left ideals L and L' are isomorphic (as modules) if and only if $L' = Lr$ for some $r \in L'$. In fact, if $\phi : L = Re \to L' = Re'$ is an isomorphism, then $\phi(xe) = x\phi(e)$. Since $L(1-e) = 0$, we get $x = xe$, hence $\phi(L) = Lr$, where $r = \phi(e) \in L'$. This shows that the sum of isomorphic minimal left ideals is a two-sided ideal $\mathfrak{a}_{[L]}$ in R, where $[L]$ denotes the isomorphism class of a left minimal ideal. The whole ring is isomorphic to the direct sum of the ideals $\mathfrak{a}_{[L]}$, where $[L]$ runs the set of isomorphism classes of left ideals. Since each $\mathfrak{a}_{[L]}$ contains a minimal left ideal, it does not contain proper two-sided subideals. It also contains the unity since, being a left ideal we can write it as $Re_{[L]}$ for some idempotent $e_{[L]}$, and the decomposition of R into the sum of the ideals $Re_{[L]}$ gives $1 = \sum_{[L]} e_{[L]}$ showing that $e_{[L]}$ is the unity in $Re_{[L]}$. Thus each two-sided ideal $\mathfrak{a}_{[L]}$ is a simple algebra over K (embedded by $a \mapsto ae_{[L]}$).

Example 8.1.2. Let $R = K[G]$ be the group algebra of a finite group G of order prime to the characteristic of K. A minimal left ideal is isomorphic to an irreducible representation V_i of G. The sum of isomorphic minimal left ideals is the submodule $K[G]_i$ of $K[G]$ equal to the direct sum of $n_i = \dim R_i$ copies of an irreducible representation $\rho_i : G \to \mathrm{GL}(R_i)$. It is a simple algebra over K isomorphic to $\mathrm{End}_K(R_i) \cong \mathrm{Mat}_{n_i}(K)$. We have

$$K[G] \cong \bigoplus_{R_i \in \mathrm{Ir}(G)} \mathrm{End}_K(R_i).$$

Let $K[G]_i = K[G]\eta_i$ for some idempotent η_i. We can find an explicit expression of η_i as an element of $K[G]$"

$$\eta_i = \frac{1}{|G|} \sum_{g \in G} \chi_i(1)\chi_i(g^{-1})g, \tag{8.1}$$

where χ_i is the character of R_i. To see this formula we write $\eta_i = \sum_{g \in G} a_g g$ and apply the character χ_{reg} to $g^{-1}\eta_i$ of the regular representation to obtain $\chi_{\mathrm{reg}}(g^{-1}\eta_i) = a_g|G|$. Since $\chi_{\mathrm{reg}} = \sum_j \chi_j(1)\chi_j$, we get

$$a_g|G| = \sum_j \chi_j(1)\chi_j(g^{-1}\eta_i).$$

Since $\eta_i\eta_j = \delta_{ij}e_i$, we have $\rho_j(g^{-1}e_i) = \rho_j(g^{-1})\rho_j(e_i) = \rho_i(g^{-1})$ if $i = j$ and 0 otherwise. This implies $\chi_j(g^{-1}\eta_i) = \chi_i(g^{-1})\delta_{ij}$. Collecting together we get (8.1)

Another simple example is the ring of matrices $\mathrm{Mat}_n(K)$, where a minimal left ideal is the set of matrices with all columns except a fixed one equal to zero.

Example 8.1.3. Let $R = KQ$ be the path algebra of a quiver without oriented loops. For any $a \in Q_1$ we have $a^2 = 0$. One can show that $\mathrm{rad}R$ is the largest two-sided nilpotent ideal of R. Thus $a \in \mathrm{rad}R$. The quotient $R/\mathrm{rad}R \cong \sum_{i \in Q_0} Ke_i$, where e_i correspond to the vertices. By definition $e_i^2 = e_i$, i.e. each e_i is an idempotent.

The next notion will be very important for us. Let ${}_R\mathrm{Mod}$ (resp. Mod_R) denote the category of left (resp. right) R-modules.

Definition 8.1.2. Two rings R and S are called *Morita equivalent* if there is an additive equivalence of the categories ${}_R\mathrm{Mod}$ and ${}_S\mathrm{Mod}$.

Recall that a functor F from a category $\mathcal{C}$ to a category $\mathcal{C}'$ is an equivalence of categories if there exists a functor $G : \mathcal{C}' \to \mathcal{C}$ and isomorphism of functors $\mathbf{id}_{\mathcal{C}} \to G \circ F$ and $\mathbf{id}_{\mathcal{C}'} \to F \circ G$. The categories of modules are additive categories (i.e the set of morphisms are abelian groups and composition of morphisms is bi-additive) and an additive functor between additive categories preserves the structure of an abelian group on the sets of morphisms. Note that th categories ${}_R\mathrm{Mod}$ and Mod_R are obviously additively equivalent: each left R-module defines a right R-module by setting $m \cdot r := r \cdot m$.

Let $F : {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ and $G : {}_S\mathrm{Mod} \to {}_R\mathrm{Mod}$ define an additive equivalence of categories. Then $U = F({}_RR)$ is an left S-module, and also a right R-module (since the right multiplications $R \to R, x \mapsto xr$ induce a right action of R on $F(R)$). Similarly, $V = G(S)$ is a left R-module and a right S-module. We have

$$F(M) \cong \mathrm{Hom}_S(S, F(M)) \cong \mathrm{Hom}_R(G(S), G(F(M))) \cong \mathrm{Hom}_R(V, M),$$

$$G(N) \cong \mathrm{Hom}_R(R, G(N)) \cong \mathrm{Hom}_S(F(R), F(G(N))) \cong \mathrm{Hom}_S(U, N).$$

Taking $M = {}_RR$ we get $U = F(R) = \mathrm{Hom}_R(V, R)$, and taking $N = {}_SS$, we get

$$V = G(S) = \mathrm{Hom}_S(U, S). \tag{8.2}$$

Now

$$F(M) \cong \mathrm{Hom}_R(\mathrm{Hom}_S(U, S), M) \cong U \otimes_R M,$$

$$G(N) \cong \mathrm{Hom}_S(\mathrm{Hom}_R(V, N)) \cong V \otimes_S N.$$

Thus a Morita equivalence is always given by a pair of bimodules ${}_SU_R$ and ${}_RV_S$. They satisfy some special properties. First, since R is a *generator* of ${}_R\mathrm{Mod}$ (i.e. any object is a quotient of some free module R^I), the bimodule $F(R) = U$ is a generator of ${}_S\mathrm{Mod}$. Similarly, $V = G(S)$ is a generator of ${}_R\mathrm{Mod}$. Moreover, since the equivalence of categories defines a bijection $R \cong \mathrm{Hom}_R(R, R) \to \mathrm{Hom}_S(U, U) = \mathrm{End}_S(U)$ and a bijection $S \to \mathrm{End}_R(V)$, the bimodules U and V are, by definition, (faithfully) *balanced*. Secondly, since R is a projective finitely generated left module, it follows from the definition of a projective module that U is a projective finitely generated S-module. Similarly, V is a projective finitely generated left R-module. One can show that a balanced bimodule ${}_SM_R$ such that ${}_SM$ is a finitely generated projective generator of ${}_S\mathrm{Mod}$ if and only if M_R is a finitely generated projective generator of Mod_R. Note that the bimodule V is reconstructed from the bimodule U via (8.2).

Theorem 8.1.4. *(Morita) Any equivalence of categories* $F : {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}, G : {}_S\mathrm{Mod} \to {}_R\mathrm{Mod}$ *is defined by a faithfully balanced bimodule* ${}_SU_R$ *satisfying the properties* *(i)* $F(M) = U \otimes_R M,\ G(N) = V \otimes_S N$, *where* $V = Hom_S(U, S)$*;*

(ii) ${}_SU$ *is a finitely generated projective left* S*-module and* U_R *is a finitely projective right* R*-module;*

(iii) ${}_SU$ *is a generator of* ${}_S\mathrm{Mod}$ *and* U_R *is a generator of* Mod_R*.*

Example 8.1.5. Suppose R and S are Morita equivalent and U is a bimodule defining an equivalence of the categories, then $S \cong \mathrm{End}_R(U_R)$. Consider the special case where $R = K$ is a local commutative ring. Then any finitely generated projective R-module is isomorphic to R^n for some n, hence $S = \mathrm{End}_R(R^n) \cong \mathrm{Mat}_n(R)$. In fact, for any ring R, the rings R and $S = \mathrm{Mat}_n(R)$ are Morita equivalent. We take a free right R-module $U = R^n$ and equip it with the structure of a left S-module by using the matrix product $A \cdot \mathbf{v}$, where $A \in \mathrm{Mat}_n(R)$ and $\mathbf{v} \in R^n$. Since $\mathrm{End}_S(U)$ is the center of the matrtix algebra $\mathrm{Mat}_n(R) \cong \mathrm{End}_R(R^n) =$, it consists of scalar matrices, hence there is a bijection $R \to \mathrm{End}_S(U)$ and U is fathfully balanced. It is obviously a finitely generated projective generator of ${}_S\mathrm{Mod}$ and Mod_R.

Example 8.1.6. Let $e \in R$ be an idempotent, as we observed in above, $S = eRe$ is a K-algebra with the unity e. Suppose that $ReR = R$. Then the functor

$$F : {}_R\mathrm{Mod} \to {}_S\mathrm{Mod},\ M \rightsquigarrow eM$$

is isomorphic to the functor $M \rightsquigarrow U \otimes_R M$, where $U = eR$ has natural structures of a left eRe-module and a right R-module. It satisfies all the conditions of Morita's Theorem, hence R and eRe are Morita equivalent. For example, let $R = \mathrm{Mat}_n(K)$ and $e = E_{11}$, where E_{ij} denotes the matrix with 1 at the ij-spot and 0 at any other spot. We have $eR = KE_{11} + \ldots + KE_{1n}$ and $eRe = KE_{11} \cong K$. Obviously, $ReR = (KE_{11} + \ldots + KE_{1n})R = R$, and the previous assertion implies that K and $\mathrm{Mat}_n(K)$ are Morita equivalent.

8.2 Skew group algebra

Let R be any ring (not necessary commutative) and G be a finite group acting on R by automorphisms via a homomorphism of groups $\rho : G \to \mathrm{Aut}(R)$. We denote the value of $\rho(g)$ on $r \in R$ by ${}^g r$. Define the *skew group algebra* $R\#G$ of G with coefficients in R as the R-module of R-valued functions on G with the multiplication law

$$\alpha \cdot \beta(g) = \sum_{g'g''=g} \alpha(g')^{g'}\beta(g'').$$

Choose a basis formed by the delta-functions δ_g which we will identify with elements of G. Then we can identify elements of $R\#G$ with linear combinations $\sum_{g\in G} r_g g$, where $r_g \in R$. The multiplication is defined by

$$(rg) \cdot (r'g') = r^g r' gg'.$$

Let $R^G = \{r \in R : {}^g r = r\}$ be the ring of G-invariants. Then the subalgebra $R^G \# G$ of R^G-valued functions on G is isomorphic to the group algebra $R^G[G]$.

From now on we assume that R is an algebra over a field K of characteristic prime to $|G|$. The skew group algebra $R\#G$ acquires a structure of a K-algebra. We also assume that G acts on R by automorphisms of the K-algebra R. In this case $R\#G$ contains $K[G]$ as the subalgebra of K-valued functions on G. It also contains R as a subalgebra spanned by the delta-function δ_1. Thus any left module over $R\#G$ has a canonical structure of an R-module and also, considered as a vector space over K, of a $K[G]$-module.

Let

$$e = \frac{1}{|G|}\sum_{g\in G} g \in K[G]$$

be the averaging operator in $K[G] \subset R\#G$. Then $eR = R^G$ and

$$eR\#Ge = R^G[G]e = R^G e \cong R^G.$$

Let $Z(R^G)$ be the center of R^G. Any $c \in Z(R^G)$ commutes with any $g \in R^G[G]$ and any $r \in R$, thus belongs to the center $Z(R\#G)$. If R is a commutative ring with no zero divisors, then the converse is true. In fact, assume $\sum r_g g \in Z(R\#G)$, then, for any $a \in R$, we have

$$a(\sum_{g\in G} r_g g) = \sum_{g\in G} a r_g g = (\sum_{g\in G} r_g g)a = \sum_{g\in G} r_g {}^g a g.$$

Comparing the coefficients at g, we get $ar_g = r_g {}^g a = a^g$. Since G is a nontrivial subgroup of automorphisms of R (if it is trivial, there is nothing to prove), for any $g \neq 1$, there exists $a \in R$ such that $a^g \neq a$. Since R has no zero divisors we get $r_g = 0, g \neq 1$. So our sum is just $r1$ and $gr1 = r1g$ gives ${}^g r = r$ for all $g \in G$. Thus $r \in R^G$.

Let us record what we have proved.

Lemma 8.2.1. *Let A be a commutative K-algebra without zero divisors and $G \subset Aut_K(R)$. Then*

$$Z(A\#G) = A^G.$$

The assumption that $G \subset \mathrm{Aut}_K(R)$ is of course essential. For example, consider the case when $R = K$, so that G acts trivially on R. By Proposition 5.1.8,

$$Z(K[G]) = \sum_{C\in\mathcal{C}(G)} Ke_C,$$

where $\mathcal{C}(G)$ is the set of conjugacy classes in G and the elements $e_C = \sum_{g\in C} g$ form a basis of $K[G]$. In fact, there is a better basis formed by orthogonal idempotents from Example 8.1

$$\eta_i = \frac{\chi_i(1)}{|G|}\sum_{g\in G} \bar{\chi}_i(g)g = \frac{\chi_i(1)}{|G|}\sum_{C\in\mathcal{C}(G)} \bar{\chi}_i(g_C)e_C,$$

where $\chi_1, \dots, \chi_c$ is the set of irreducible characters of G and g_C is a representative of $C \in \mathcal{C}$.

We will identify $\lambda \in K^c$ with an element of the center of $k\Gamma$ defined by

$$\lambda = \sum_{i=1}^{c} \frac{\lambda_i}{\chi_i(1)} \eta_i.$$

Let us choose now Γ to be a subgroup of $\mathrm{SL}(2, K)$ which acts on the algebra of noncommutative polynomials $K\langle x, y\rangle$ via linear change of variables. Let $\lambda \in K^c = Z(K[G]) \subset K\langle x, y\rangle \# G$. We set

$$\mathcal{C}^\lambda = K\langle x, y\rangle \# G/(xy - yx - \lambda).$$

Taking $\lambda = 0$ we get the algebra $K[x, y]\#G$.

Let N be the standard 2-dimensional representation of G which we identify with the space of linear polynomials $K[x, y]_1$ by means of the G-invariant symplectic form defined by $\Phi(x, y) = -\Phi(y, x) = 1$ (use that $\Phi(ax + by, cx + dy) = (ad - bc)\Phi(x, y) = \Phi(x, y)$). We consider $N \otimes K[G]$ as a $K[G]$-bimodule, where G acts on the left diagonally, and on the right it acts on $K[G]$ by right multiplication. The algebra $K\langle x, y\rangle \# G$ is isomorphic to the tensor algebra of the bimodule $N \otimes K[G]$.

Let $\tilde{Q}$ be the quiver obtained from an affine Dynkin diagram by fixing an orientation of arrows. We assume that $Q_0 = \{0, \dots, n\}$ such that after deleting the vertex 0 we obtain a Dynkin diagram of finite type. In other words 0 corresponds to the trivial representation in the corresponding McKay graph of G. Let δ be the dimension vector with $\delta_i = \dim \rho_i, \rho_i \in \mathrm{Ir}(G)$. As before, $\Pi_\lambda(Q)$ denotes a deformed preprojective algebra corresponding to the double of the quiver Q.

Theorem 8.2.2. *(W. Crowley-Boevy, M. Holland) The deformed preprojective algebra* $\Pi_\lambda(Q)$ *is Morita equivalent to the algebra* $\mathcal{C}^\lambda$.

Proof. We consider only the case $\lambda = 0$ and give only a sketch of the proof. To simplify the notation we set $\mathcal{C}^0 = \mathcal{C}$. We know from section 7.3 that $\mathrm{Rep}(Q, \delta)$ can be identified with $(\mathrm{End}(R) \otimes N)^G$, where R is the regular representation and N is the standard 2-dimensional representation of G. Under this correspondence an arrow $a \in Q_1$ with $t(a) = i, h(a) = j$ corresponds to a homomorphism $R_i \to N \otimes R_j$ of $K[G]$-modules.

Recall that $\eta_i K[G]$ is a 2-sided ideal in $K[G]$. It is equal to sum of left ideals isomorphic to an irreducible representation R_i. Choose an idempotent f_i such that $K[G]f_i \cong R_i$. Let $f = f_1 + \ldots + f_c$. If $K[G]_i$ is identified with the ring of matrices $\mathrm{Mat}_{\delta_i}(k)$, then we may take for f_i the elementary matrix E^i_{11}, where the superscript indicates that we are considering the i-component of $K[G]$. We have $f_i K[G] = E^i_{11} K[G]_i$ is the set of matrices with all rows except the first one equal to zero. Similarily, $K[G]f_i = K[G]_i E^i_{11}$ is equal to the set of matrices with all columns except the first one equal to zero. Clearly, the identity matrix in $\mathrm{Mat}_{\delta_i}(k)$ can be obtained as a sum of the products of a matrix from $K[G]f_i$

and a matrix from $f_iK[G]$. Thus $1 \in K[G]fK[G] = \oplus K[G]f_i \cdot f_iK[G]$. This shows that the rings $K[G]fK[G]$ and $K[G]$ are isomorphic. Also, since $1 \in \mathcal{C}f\mathcal{C}$, we obtain that the rings $\mathcal{C}f\mathcal{C}$ and $\mathcal{C}$ are isomorphic. As we saw in Example 8.1.6 this implies that the rings $\mathcal{C}$ and $f\mathcal{C}f$ are Morita equivalent.

Let $\mathcal{C}_1 = N \otimes K[G] \subset \mathcal{C}$. We have

$$\mathrm{Hom}_{K[G]}(R_i, N \otimes R_j) \cong \mathrm{Hom}_{K[G]}(K[G]f_i, N \otimes K[G]f_j) = f_iN \otimes K[G]f_j.$$

Here we use that a homomorphism $h : K[G]f_i \to N \otimes K[G]f_j$ is defined by the image $h(f_i)$ of f_i which must satisfy $h(f_i) = h(f_i^2) = f_ih(f_i)$, hense belongs to $f_i\mathcal{C}_1f_j$.

Let Q be the McKay quiver, for any arrow $a : i \to j$ let ϕ_a be the homomorphism of $K[G]$-modules $R_i \to N \otimes R_j$. It follows from above that we can consider it as an element of $f_i\mathcal{C}_1f_j$. Now we can define an isomorphism $\alpha : KQ \to f(K\langle x, y\rangle\#G)f$ by sending e_i to f_i, $a : i \to j$ to $\phi_a \in f_i\mathcal{C}_1f_j$ and a^* to $\psi_a \in f_j\mathcal{C}_1f_i$. One checks that

$$\sum_{a\in Q:h(a)=i} (1 \otimes \phi_a)\psi_a - \sum_{a\in Q:t(a)=i} (1 \otimes \psi_a)\phi_a = -\delta_if_i(xy - yx). \tag{8.3}$$

This shows that α induces an isomorphism from $\Pi(Q)$ to $f\mathcal{C}f$. It remains to use that the rings $f\mathcal{C}f$ and $\mathcal{C}$ are Morita equivalent.

The equation (8.3) is checked case by case for different types of the groups G. Let us check it in the case $\tilde{A}_n$. We identify N with the space $Kx + Ky$. By diagonalizing the action of G we may find another basis $u = \alpha x+\beta y, v = \gamma x+\delta y$, where a generator g of G acts by $g(u) = \epsilon u, g(v) = \epsilon^{-1}v$. After scaling the basis we may assume that $v \otimes u - u \otimes v = y \otimes x - x \otimes y$. We number the arrows in such a way that $t(a_i) = i, h(a_i) = i + 1$ and $t(a_i^*) = i + 1, h(a_i^*) = i$. Thus $\phi_{a_i} : R_i \mapsto N \otimes R_{i+1}$ is given by $z \mapsto v \otimes z$ and $\psi_{a_i} : R_{i+1} \mapsto N \otimes R_i$ is given by $z \mapsto u \otimes z$. Now we check $(1 \otimes \phi_{a_{i-1}})\psi_{a_{i-1}} : R_i \to N \otimes R_{i-1} \to N \otimes N \otimes R_i$ is given by $z \mapsto v\otimes z \mapsto v\otimes u\otimes z$. Similarily, $(1\otimes\psi_{a_i})\phi_{a_i}$ is given by $z \to u\otimes v\otimes z$. Thus the LHS in (8.3) is the map $z \to (v\otimes u-u\otimes v)\otimes z = -(x\otimes y-y\otimes x)\otimes z$. If we identify $k\langle x, y\rangle$ with the tensor algebra of $Kx + Ky$ we obtain what we want.

□

Example 8.2.3. Any $K[G]$-module V can be considered as a $K[x, y]\#G$-module by considering the homomorphism of ring $K[x, y]\#G \to K[G]$ sending $\sum a_gg$ to $\sum \bar{a_g}g$, where $\bar{a_g} = a_g+(x, y) \in K[x, y]/(x, y) \cong K$. Under the Morita equivalence, V is sent to the $f\mathcal{C}f$-module $fV = \sum_i \mathrm{Hom}_{K[G]}(R_i, V)$. Under the isomorphism $\Pi(Q) \to f\mathcal{C}f$ this module becomes a representation of the preprojective algebra with dimension vector $\sum m_i\mathbf{e}_i$, where m_i is the multiplicity of R_i in V. The regular representation $K[G]$ corresponds to a $\Pi(Q)$-module with dimension vector δ.

8.3 Quiver resolution of Klein surfaces

Recall that a point $Z \in$ G-Hilb(X) defines a G-module $V_Z = H^0(\mathcal{O}_Z)$ and a module over a ring $A = \mathcal{O}_X(U)$, where U is any G-invariant open set containing Z. The two structures can be combined together giving a structure of a $A\#G$-module on V_Z. We assume that $X = \text{Spec } A$ is an irreducible variety over a field k of characteristic coprime to $|G|$. The ring A is a K-algebra, and the module $M = V_Z$ is finite-dimensional over k. Also, it is a cyclic as a A-module and cyclic as a $k[G]$-module (in fact isomorphic to $K[G]$). Let us study such modules.

Let $Q = (Q_0, Q_1)$ be a quiver with no oriented loops. For every vertex $i \in Q_0$ let P_i denote KQ-module KQe_i. As a representation of Q it is given by vector spaces $V_j = e_j KQe_i$ of dimension equal to the number of paths from i to j. In particular, $V_j = 0$ if there exists an arrow a with $t(a) = j$ and $h(a) = i$. If Q is a tree, then $\dim V_j \leq 1$ for all $j \in Q_0$. For any KQ-module M corresponding to a representation ρ of Q, we have

$$\text{Hom}_{KQ}(P_i, M) \cong \rho(i).$$

The isomorphism is given by $\phi \mapsto \phi(e_i)$. As we saw in the proof of the previous theorem, $\phi(e_i)$ is an element of $e_i M = \rho(i)$. In particular, since Q does not contain oriented loops, we have $\dim \text{End}_{KQ}(P_i) = 1$. This shows that P_i is an indecomposable module. Being a direct summand of KQ, it is a projective module. One can show that any indecomposbale projective module coincides with some P_j. Note that P_i is not a simple module. For example, it admits a non-trivial homomorphism to the simple module S_i with dimension vector $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$. Note that any simple module is isomorphic to one of the S_i's. In fact, as always we order Q_0 such that there are no arrows between i and j if $i > j$. Let V be a simple module and i is the smallest such that $V_i \neq 0$. Suppose $V_i \to V_j$ is a non-trivial map. Then M contains a non-trivial submodule, the image of P_j under the map $e_j \to v$, where v is a nonzero element of V_j.

Remark 8.3.1. It is easy to see that the modules P_i generate the Grothendieck group $K_0(KQ)$ of finitely generated modules. Thus $K_0(KQ) \cong \mathbb{Z}^{|Q_0|}$ and the isomorphism is given by the dimension vector. A character vector $\boldsymbol{\theta}$ defines a function $\boldsymbol{\theta} : K_0(KQ) \to \mathbb{Z}$ and a representation is $\boldsymbol{\theta}$-semistable if $\boldsymbol{\theta}([M]) = 0$ and any submodule M' satisfies $\boldsymbol{\theta}([M']) \geq 0$. This allows one to extend the definition of stability to any abelian category $\mathcal{A}$ by considering a function $\boldsymbol{\theta} : k_0(\mathcal{A}) \to \mathbb{R}$ and defining a *$\boldsymbol{\theta}$-stable object* as an object M with $\boldsymbol{\theta}([M]) = 0$ and $\boldsymbol{\theta}([M']) \geq 0$ for any subobject M' of M.

Next we replace Q by its double $\tilde{Q}$ and consider representations of the pre-projective algebra $\Pi(Q)$. The analog of P_i is of course $\Pi(Q)e_i$. A $\Pi(Q)$-module M is called *i-cyclic* if it is generated by an element from $e_i M$. For every i-cyclic module M there is a surjective homomorphism $\Pi(Q)_i \to M$ which sends e_i to a generator.

Let $Q(G)$ be the McKay graph of a group G and $\Pi(Q)$ be the corresponding preprojective algebra. We choose $\boldsymbol{\theta} = (\boldsymbol{\theta}_0, \ldots, \boldsymbol{\theta}_n)$ such that $\boldsymbol{\theta}_i > 0$ for $i \neq 0$. Suppose we have a $\boldsymbol{\theta}$-semistable representation V with dimension vector $\mathbf{d}$ with components equal to the dimensions of irreducible representations of Q. Since $\delta_0 = 1$ we can choose a non-trivial map $\Pi(Q)e_0 \to V$ sending e_0 to $1 \in V_0$. If this homomorphism is not surjective, then V contains a submodule with dimension vector $\mathbf{d}' = (0, d_1, \ldots, d_n)$. Since $\boldsymbol{\theta} \cdot \mathbf{d}' > 0$ this contradicts $\boldsymbol{\theta}$-semistability. This shows that the set of $\boldsymbol{\theta}$-semistable representations $\mathrm{Rep}(\Pi(G), \mathbf{d})^{ss}_{\boldsymbol{\theta}}$ of $\Pi(G)$ consists of 0-cyclic $\Pi(G)$-modules. Also one can see that all of them are $\boldsymbol{\theta}$-stable.

Let us describe simple $A\#G$-modules, and by Morita-equivalence simple modules over $\Pi(Q)$. A simple left module over a ring R is a simple module over the center $Z(R)$ of R. Thus it is annihilated by a maximal ideal of R. This defines a map from the set of simple R-modules of R to Spec $Z(G)$. In our situation when $R = k[x,y]\#G$ we see that $Z(R) = k[x,y]^G = ek[x,y]\#Ge$. A simple module over $k[x,y]^G$ is just a point in Spec $k[x,y]^G$ identified with a G-orbit O in $\mathbb{A}^2(k)$. So any simple $k[x,y]\#G$-module S must be annihilated by a unique maximal ideal $\mathfrak{m}_O$ in $k[x,y]^G$ identified with a G-orbit, where $k^O = A/J_O$ is the space of k-valued functions on O on which G acts naturally. If $O = \{0\}$ corresponds to the origin, then $k^O\#G \cong K[G]$ and S must be a simple module over $K[G]$ isomorphic to one of the irreducible representations R_i. If $O = Gx_0$, where x_0 is not the origin, then $k^O\#G \cong \mathrm{End}(K[G])$, if we identify k^O with the dual vector space $K[G]^*$ by using the nondegenerate bilinear pairing

$$k^O \otimes K[G] \to k, \quad (\phi, g) \mapsto \phi(g(x_0)).$$

The ring $\mathrm{End}(K[G])$ is a simple ring, so there is only one isomorphism class of simple modules annihilated by $\mathfrak{m}_O$. It is isomorphic to $K[G]$ on which x, y act as multiplication by scalars x_0, y_0 not both equal to zero.

Translating into the language of $\Pi(Q)$-modules, it is easy to see that a simple $k[x,y]\#G$ annihilated by $\mathfrak{m}_O, O \neq \{0\}$ corresponds to a simple module over $\Pi(Q)$ with dimension vector δ. On the other hand, a simple module annihilated by the maximal ideal of the origin corresponds to a simple $\Pi(Q)$-module $S_i = e_i\Pi(Q)e_i$ with dimension vector $\mathbf{e}_i$.

Theorem 8.3.2. *Let Q be the McKay graph of a group $G \subset SL(2,k)$. The canonical map*

$$\pi : \mathcal{R}_{\boldsymbol{\theta}}(\Pi(Q), \delta) \to \mathcal{R}_0(\Pi(G), \delta) = \mathrm{Rep}(\Pi(Q), \delta)//GL(\delta)$$

is a resolution of singularities of the kleinian singularity $\mathbb{A}^2(k)/G$.

Proof. Let $A = k[x,y]$. We have already explained that, by Morita equivalence, a representation of $\Pi(Q)$ with dimension vector δ corresponds to a structure of a $A\#G$-modules on $K[G]$. Let $\mathrm{Mod}_K[G](A\#G, K[G])$ be the set of such modules. The group $\mathrm{GL}(K[G]) \cong \mathrm{GL}(\delta)$ acts on the set of such modules and the isomorphism classes 0-generated representations correspond to modules generated by $A\#Gf_0K[G] = A$, i.e. quotients A/J as A-modules and isomorphic to $K[G]$ as a $K[G]$-modules. But this is exactly the description of points in G-Hilb($\mathbb{A}^2$). Thus we can identify $\mathcal{R}_{\boldsymbol{\theta}}(\Pi(Q),\delta)$ with G-Hilb($\mathbb{A}^2$).

Now let us identify $\mathrm{Rep}(\Pi(Q),\delta)//\mathrm{GL}(\delta)$ with $\mathbb{A}^2/G$. For any finite dimensional module M over a k-algebra R let gr$M$ be equal to the direct sum of the composition factors of M (this is well defined by the Jordan-Hölder Theorem). Let rad$R$ be the radical of R (the largest two-sided ideal such the quotient is a semi-simple ring). Any semi-simple module is annihilated by the radical, and hence defines a module over $R/\mathrm{rad}R$. Applying the theory of characters for modules over semi-simple rings (we discussed this theory for the special case of finite group algebras, but the theory is similar), we obtain that $\mathrm{gr}M \cong \mathrm{gr}N$ if and only if the trace functions $\mathrm{tr}_M(r)$ and $\mathrm{tr}_N(r)$ coincide for all $r \in R$. Here $\mathrm{tr}_M(r)$ is equal to the trace of the endomorphism $x \mapsto r \cdot x$ of M. In our situation, a trace function is a $\mathrm{GL}(K[G])$-invariant function on the space $\mathrm{Mod}_K[G](A\#G,K[G])$. By Morita-equivalence this translates to trace to trace functions on the space $\mathrm{Rep}(\Pi(Q),\delta)$ and we use a well-known result of Le-Bruyn and Procesi that the trace functions generate the ring of $\mathrm{GL}(\delta)$- invariant functions. It also agrees with king's criterion of λ-stability, we see that representations M and N in $\mathrm{Rep}(\Pi(Q),\delta)$ define the same point in $\mathrm{Rep}(\Pi(Q),\delta)//\mathrm{GL}(\delta)$ if and only if $\mathrm{gr}M \cong \mathrm{gr}N$. Note that it follows from the GIT that the fibres of $\mathrm{Rep}(\Pi(Q),\delta) \to \mathrm{Rep}(\Pi(Q),\delta)//\mathrm{GL}(\delta)$ are the unions of orbits. Each fibre contains a unique minimal closed orbit which represents the semi-simple modules.

Let us define a map $\mathbb{A}^2 \to \mathrm{Rep}(\Pi(Q),\delta)//\mathrm{GL}(\delta)$ by assigning to a point $p=(x_0,y_0)$ a structure M_p of a $A\#G$-module on $K[G]$ by setting $a\cdot g = a(g(p))g$. If $q=h(p)$ for some $h \in G$, then the map $g \mapsto gh^{-1}$ is an isomorphism of modules $M_p \to M_q$. This defines a map

$$\mathbb{A}^2/G \to \mathrm{Mod}_K[G](A\#G,K[G])//\mathrm{GL}(K[G]) = \mathrm{Rep}(\Pi(Q),\delta)//\mathrm{GL}(\delta).$$

The inverse morphism is defined by assigning to a $A\#G$-module M the A^G-module $eM \cong k$ (considered as a point of Spec $A^G = \mathbb{A}^2/G$).

$M \in \mathrm{Mod}_K[G](A\#G,K[G])$ let gr$M$ denotes We already knoLet $A=k[x,y]$ and $A\#G$ be as above. The ring A can be considered as a module over $A\#G$ since it has a natural action of A on itself and the action of G via action of G on $\mathbb{A}^2_k$. Recall that e from (**??**) generates the trivial submodule of $K[G]$. For any $g \in G$ we have $ge=e$ and hence the multiplication by e on the right defines a homomorphisms of G-modules $K[G] \to ke \cong R_0$ and $A\#G \to A$. This shows that one can identify A with the left $A\#G$-module $(A\#G)e$.

It remains to see that the map π coincides with the cycle map and hence defines a minimal resolution of the klein surface Spec A^G. The map π assigns to a $\boldsymbol{\theta}$-stable module the point in $\mathrm{Rep}(\Pi(Q),\delta)//\mathrm{GL}(\delta)$ corresponding the same module considered as a 0-semi-stable module. Thus two points M,N go the same point if and only if $\mathrm{gr}M \cong \mathrm{gr}N$.

We have described already simple modules over $\Pi(Q)$. Any module M in the pre-image $\pi^{-1}(0)$ is has composition series isomorphic to the simple modules S_i. Since its dimension vector is equal to δ the module gr$M$ is isomorphic to the direct sum $\oplus_{i=0}^n S_i^{\delta_i}$. This is the module in $\mathrm{Rep}(\Pi(Q),\delta)//\mathrm{GL}(\delta)$ representing the origin. As a module over A^g it is equal to the direct sum of $|G|$-copies of the residue field of the origin. So, this is exactly the cycle map from the Hilbert scheme to the symmetric product.

□

8.4 The main theorem

In this section we give one the explanations of the McKay correspondence.

Recall that the *socle* of a left module over an associative ring R is the direct sum $\mathrm{soc}(M)$ of simple submodules. In commutative case, a module is simple if and only if it is isomorphic to the quotient of the ring by a maximal ideal (take the annihilator ideal of its nonzero element). However, over a non-commutative ring they are more interesting. For example, the socle of $K[G]$ is $K[G]$ since all irreducible representations are simple $K[G]$-submodules.

To explain the McKay correspondence we have to match the irreducible components of the exceptional locus of a minimal resolution of the klein singularity $\mathbb{A}^2(k)/G$ with irreducible representations of G in such a way that the intersection matrix coincides with the McKay graph with the vertex corresponding to the trivial representation omitted. From the previous section we know that the exceptional divisor can be described as the set of isomorphism classes of $\Pi(Q)$-modules with dimension vector δ which are 0-generated and whose composition factors are isomorphic to simple modules S_i's. Here, as usual, Q is the McKay graph and δ is the vector of dimensions of irreducible representations of G. For any ideal $J \in$ G-Hilb$(\mathbb{A}^2)$ representing a point in the exceptional locus we denote by M_J the corresponding isomorphism class of a 0-generated representation of $\Pi(Q)$ with dimension vector δ.

Let $\mathrm{soc}(M_J)$ be the socle of M_J. It must be isomorphic to the direct sum of some modules S_i's. Note that S_0 is not represented since otherwise the quotient $M' = M_J/S_0$ is still 0-generated but its component e_0M' is equal to zero.

For any module semi-simple module M and a simple module S we denote by $[M : S]$ the number of simple summands of $\mathrm{soc}(M)$ isomorphic to S. It is equal to $\dim Hom(S, M)$.

Theorem 8.4.1. *(W. Crawlwy-Boevey) Let E be the closed subset of* G-Hilb$(\mathbb{A}^2)$ *corresponding to ideals supported at the origin. If $i \neq 0$, then*

$$E(i) = \{J \in E : [M_J : S_i] \neq 0\}$$

is a closed subset of E isomorphic to $\mathbb{P}^1$. Moreover $E(i) \cap E(j) \neq \emptyset$ if and only if i and j are adjacent in the McKay graph, and in this case $E(i) \cap E(j)$ consists of a unique module with $\mathrm{soc}(M_J) = S_i \oplus S_j$.

We will prove this theorem at the end of this section.

As before we have a bilinear form on $\mathbb{Z}^{Q_0}$ defined by the Cartan matrix of the quiver $\tilde{Q}$.

Lemma 8.4.2. *(C. Ringel) Let M, N be finite-dimensional KQ-modules with dimension vectors $\mathbf{d}_M$ and $\mathbf{d}_N$. Then*

$$\dim Hom(M,N) - \dim Ext^1(M,N) = (\mathbf{d}_M, \mathbf{d}_N).$$

Proof. We use the following projective resolution of M

$$0 \to \bigoplus_{a \in Q_1} P_{h(a)} \otimes e_{t(a)} M \to \bigoplus_{i \in Q_0} P_i \otimes e_i M \to M \to 0.$$

It is called the *Ringel resolution*. Here $e_{h(a)} \otimes v \in P_{h(a)} \otimes e_{t(a)} M$ is sent to $e_{h(a)} \otimes e_a v - e_a \otimes v \in e_a P_{t(a)} \otimes e_{t(a)} M$. Applying the functor $\mathrm{Hom}(?, N)$ and writing N as the direct sum $\oplus e_j N$, we get an exact sequence

$$0 \to \mathrm{Hom}(M,N) \to \bigoplus_{i,j \in Q_0} \mathrm{Hom}(e_i M, e_j N) \to \bigoplus_{a \in Q_1} \mathrm{Hom}(e_{t(a)} M, e_{h(a)} N)$$

$$\to \mathrm{Ext}^1(M,N) \to 0.$$

Now the lemma follows by taking the alternating sum of the dimensions. □

We leave to the reader to prove the analog of this lemma for $\Pi(Q)$-modules.

Lemma 8.4.3. *Let M, N be finite-dimensional $\Pi(Q)$-modules with dimension vectors $\mathbf{d}_M$ and $\mathbf{d}_N$. Then*

$$\dim Hom(M,N) + \dim Hom(N,M) - \dim Ext^1(M,N) = (\mathbf{d}_M, \mathbf{d}_N).$$

To prove the next lemmas, we need more information about 0-generated modules over $\Pi(Q)$. Recall that the vector space $\mathbb{R}^{Q_0} = \mathbb{R}^{n+1}$ equipped with the bilinear form (v, w) defined by the Cartan matrix C contains an affine root system Δ of the corresponding type. It consists of elements α such that $(\alpha, \alpha) = 2$. They are called *roots*. For any $\alpha \in \Delta$ one defines the reflection transformation

$$s_\alpha : x \mapsto x - (x, \alpha)\alpha.$$

They generated subgroup of the orthogonal group of the quadratic form defined by C. It is isomorphic to the *affine Weyl group* $W(C)$ of the Cartan matrix C. It is the semi-direct product $\mathbb{Z}^{n+1} \rtimes W(C')$, where $W(C')$ is the Weyl group of the Cartan matrix obtained from C by deleting the first row and the first column. For example, for the type $\tilde{A}_n$ we get $W(C) = \mathbb{Z}^{n+1} \rtimes S_{n+1}$ and $W(C') = W(A_n) = S_{n+1}$. The roots corresponding to the unit vectors $\mathbf{e}_i$ are called *simple roots*. One can show that the Weyl group is generated by the refelctions $s_i = s_{\mathbf{e}_i}$ in simple roots and all roots form one orbit.

Lemma 8.4.4. *Let M be a t-generated $\Pi(Q)$-module with dimension vector $\mathbf{d}$ such that $d_t = 1$. Then $\mathbf{d}$ is a root or belongs to the kernel of the Cartan matrix (proportional to the vector δ). For any root α there exists a unique t-generated module with dimension vector α.*

Proof. We use induction on $|\mathbf{d}| = d_0 + \ldots + d_n$. Suppose M is t-generated module of dimension $\mathbf{d}$. If $(\mathbf{d}, \mathbf{e}_i) > 0$ for some vertex $t \neq 0$, then $\mathrm{Hom}(M, S_i) = 0$, so by Ringel's formula, we get $k = \dim \mathrm{Hom}(S_i, M) \geq (\mathbf{d}, \mathbf{e}_i) > 0$. Thus M has a t-generated quotient of dimension $\alpha = \mathbf{d} - d\mathbf{e}_i$. By induction, α is a root. Thus

$$s_i(\alpha) = \alpha - (\alpha, \mathbf{e}_i)\mathbf{e}_i = \mathbf{d} + (k - (\mathbf{d}, \mathbf{e}_i))\mathbf{e}_i$$

is a root. By induction on $m \geq 0$ we check that, for any root β and a positive integer m the vector $\beta - m\mathbf{e}_i$ is a root or belongs to the kernel of C. In fact, since C is semi-definite and non-negative, we have, for any two roots α, β

$$4 = (\alpha, \alpha)(\beta, \beta) \geq (\alpha, \beta)^2.$$

Thus $|(\alpha, \beta)| \leq 2$ and hence $(\alpha - \mathbf{e}_i, \alpha - \mathbf{e}_i) = 4 - 2(\alpha, \mathbf{e}_i) \in \{0, 2\}$ if $(\alpha, \mathbf{e}_i) > 0$.

So we may assume that $(\mathbf{d}, \mathbf{e}_i) \leq 0$ for all $i \neq t$. But $\delta = \sum \delta_i \mathbf{e}_i$ belongs to the kernel of the matrix C, hence $(\mathbf{d}, \mathbf{e}_t) = -\sum_{i \neq t} d_i(\mathbf{d}, \mathbf{e}_i) \geq 0$. So we may assume that $(\mathbf{d}, \mathbf{e}_t) > 0$ (if $(\mathbf{d}, \mathbf{e}_t) = 0$, the vector $\mathbf{d}$ belongs to the kernel of C). Let $I(t)$ be the subset of Q_0 which consists of vertices adjacent to t. We assume that $|(t) \neq \emptyset$ since otherwise $n = 0$ and $\mathbf{d} = \mathbf{e}_0$, so the assertion is true. We have

$$(\mathbf{d}, \mathbf{e}_t) = (\mathbf{e}_t, \mathbf{e}_t) + \sum_{j \in I(t)} d_j(\mathbf{e}_t, \mathbf{e}_j) = 2 - |I(t)| > 0.$$

Thus $|I(t)| = 1$, and hence there is only one arrow a joining t and the unique $j \in I(t)$ and its reverse a^* joining j with t. Moreover, we have $1 = (\mathbf{d}, \mathbf{e}_t) = 2 + d_j(\mathbf{e}_t, \mathbf{e}_j)$ and hence $d_j = 1$. By the relation in $k[\tilde{Q}]$ defining $\Pi(Q)$ we have $\rho(a^*) \circ \rho(a) = 0$, thus one of the linear maps $\rho(a)$ or $\rho(a^*)$ is equal to zero. This shows that M contains a j-generated submodule generated by $e_j M$ with dimension vector $\alpha = \mathbf{d} - \mathbf{e}_t$. By induction, α is a root, hence $\mathbf{d} = s_\alpha(\mathbf{e}_t) = \mathbf{e}_t - (\alpha, \mathbf{e}_t)\alpha = \mathbf{e}_t + \alpha$ is a root.

We will prove the uniqueness by induction on $|\mathbf{d}|$. Suppose M is a t-generated module with dimension vector $\mathbf{d}$ which is a root. Assume $(\mathbf{d}, \mathbf{e}_i) > 0$ for some vertex $i \neq t$. Then $\mathbf{d}' = s_{\mathbf{e}_i}(\mathbf{d}) = \mathbf{d} - (\mathbf{d}, \mathbf{e}_i)\mathbf{e}_i$ has $|\mathbf{d}'| < |\mathbf{d}|$. By induction, there exists a unique module M' with dimension vector $\mathbf{d}'$. Since $\mathbf{d}' - \mathbf{e}_i$ is not a root and not in the kernel of C, we have $\mathrm{Hom}(S_i, M') = 0$ (otherwise the quotient is a t-generated and has dimension $\mathbf{d}' - \mathbf{e}_i$ which is not a root). Since M' is t-generated and S_i is not, we have $\mathrm{Hom}(M, S_i) = 0$. By Ringel's formula, we obtain $\dim \mathrm{Ext}^1(M', S_i) \cong k^{(\mathbf{d}, \mathbf{e}_i)}$. This gives rise to a universal extension

$$0 \to S_i^{(\mathbf{d}, \mathbf{e}_i)} \to E \to M' \to 0.$$

This extension defines a unique module with dimension vector $\mathbf{d}$ since M' is unique.

Assume that $(\mathbf{d}, \mathbf{e}_i) \leq 0$ for all $\mathbf{e}_i, i \neq t$. As in above we find that t is adjacent to a unique vertex j and $d_j = 1$. Since $\alpha = \mathbf{d} - \mathbf{e}_t$ is a root, we find a j-generated module L with dimension vector α with $\alpha_t = 0, \alpha_j = 1$. Representing the arrow $a : t \to j$ as a nonzero matrix of size 1 and taking a^* to be zero, we reconstruct uniquely M from L. □

Lemma 8.4.5. *There is no module in $\mathcal{R}(\Pi(G), \mathbf{d})_{\boldsymbol{\theta}}$ whose socle involves two copies of a simple S_i or two copies of S_i and S_j if i and j are not adjacent.*

Proof. If there is such module, then the quotient by $S_i \oplus S_j$ gives a 0-generated module of dimension $\delta - 2\mathbf{e}_i$ or $\delta - \mathbf{e}_i - \mathbf{e}_j$, but none of them is a root. □

Lemma 8.4.6. *If $i, j \neq 0$ are adjacent in Q, then there is a unique module in $\mathcal{R}(\Pi(G), \mathbf{d})_{\boldsymbol{\theta}}$ with socle $S_i \oplus S_j$*

Proof. Since $\delta - \mathbf{e}_i - \mathbf{e}_j$ is easily seen to be a root, there exists a unique 0-generated module M of this dimension. Now, by Ringel formula,

$$\dim \operatorname{Ext}^1(M, S_i) = \dim \operatorname{Hom}(S_i, M) - (\mathbf{e}_i, \delta - \mathbf{e}_i - \mathbf{e}_j) = \dim \operatorname{Hom}(S_i, M) + 1.$$

Since M cannot have quotients with dimension vector $\delta - 2\mathbf{e}_i - \mathbf{e}_j$ (it is not a root), we see that $\operatorname{Hom}(S_i, M) = 0$. This shows that $\dim \operatorname{Ext}^1(M, S_i) = 1$ and similarly $\dim \operatorname{Ext}^1(M, S_j) = 1$. This gives a module given by extension

$$0 \to S_i \oplus S_j \to M' \to M \to 0.$$

Its dimension vector is δ and its socle contains $S_i \oplus S_j$. By Lemma 8.4.5 it cannot contain larger socle. Since any module with this socle arises in this way and the extension is unique, we get the uniqueness of M'. □

Lemma 8.4.7. *If $i \neq 0$, then $E(i)$ is a closed subvariety of $\mathcal{R}(\Pi(G), \delta)_{\boldsymbol{\theta}}$ isomorphic to $\mathbb{P}^1$.*

Proof. Let L be the unique 0-generated module of dimension $\mathbf{d}+\mathbf{e}_i$. By Ringel's formula $\dim \operatorname{Hom}(S_i, L) \geq 2$. In fact, we have the equality since the quotient would have the dimension vector $\delta - 2\mathbf{e}_i$ which is not a root.

Any module M in $E(i)$ has $\dim \operatorname{Ext}^1(M, S_i) = \dim \operatorname{Hom}(S_i, M) = 1$ (apply Lemma 8.4.5. Thus M is isomorphic to the quotient of L by a submodule isomorphic to S_i (by the uniqueness of L). This gives a map $c : \mathbb{P}^1 = \mathbb{P}(\operatorname{Hom}(S_i, L)) \to E(i)$ which is onto, and one-to-one since L has trivial endomorphism ring (because $\dim e_0 L_0 = 1$ and L is 0-generated). We skip the proof that it is a morphism of algebraic varieties.

□

8.5 Exercises

8.1 Let $\mathbb{C}^n$ be a finite-dimensional vector space equipped with a structure of a $\mathbb{C}[x]$-module by means of a matrix $A \in Mat_n(\mathbb{C})$. Show that $\mathbb{C}^n$ is semi-simple if and only if A is diagonalizable. Show that $\mathbb{C}^n$ is indecomposable if and only if A is similar to a Jordan matrix with one block. Describe its composition series and its factors.

8.2 Given a left R-module M, let ${}^gM$ be the R-module obtained from M by replacing the action $R \to \operatorname{End}_{\mathbb{Z}}(M)$ of R on M with the composition $R \xrightarrow{g} R \to \operatorname{End}_{\mathbb{Z}}(M)$. Show that $R\#G \otimes_R M \cong \oplus_{g \in G} {}^gM$.

8.3 Suppose a finite group Γ acts freely on an affine algebraic variety X. Show that the rings $\mathcal{O}(X)\#G$ and $\mathcal{O}(X)^G$ are Morita equivalent.
8.4 As above, but assume that Γ acts with only isolated points with non-trivial stabilizer. Let $R = \mathcal{O}(X)\#\Gamma$. Show that eRe is a two-sided ideal and the quotient algebra R/ReR is finite-dimensional.
8.5 Let R^* be the group of invertible elements of a ring R. Consider the map $\alpha : R^* \to R\#G$ which sends $u \in R^*$ to $\frac{1}{|G|}\sum_{g\in\Gamma} u^{-1}g(u)g$. Show that $\alpha(u)^2 = \alpha(u)$ for any $u \in R^*$.
8.6 Let $G = C_3$ and $J = (x,y)^2 \in \text{G-Hilb}(\mathbb{A}^2)$. Check directly the assertion of Theorem 8.4.1 that $\mathrm{soc}(k[x,y]/J)$ contains a direct sum of two simple modules.
8.7 Let G be a finite group of automorphisms of a quasi-projective algebraic variety X and $p : X \to Y = X/G$ be the projection to the orbit space. Define $\mathcal{O}_X\#G$ as the sheaf of $\mathcal{O}_Y$-modules associated with the pre-sheaf $U \mapsto \mathcal{O}_X(p^{-1}(U))\#G$. Show that in the case when $X = \mathrm{Spec}\ A$ is affine, $(\mathcal{O}_X\#G \cong \mathrm{End}_{\mathcal{O}_Y}\mathcal{O}_X$.