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On Elements of Order p^s in the Plane Cremona Group over a Field of Characteristic p

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To the memory of Vasily Iskovskikh

Abstract—We show that the plane Cremona group over a field of characteristic $p > 0$ does not contain elements of order of power of p larger than 2. We also describe conjugacy classes of elements of order p^2.

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1. INTRODUCTION

The classification of conjugacy classes of elements of finite order ℓ in the plane Cremona group $\mathrm{Cr}_2(k)$ over an algebraically closed field k of characteristic 0 has been known for more than a century. The possible orders of elements not conjugate to a projective transformation are $2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 15, 18, 20, 24$, and 30, and any even order is realized by a de Jonquières transformation (see [3] and historic references there). Much less is known in the case when k is of positive characteristic p and the order is divisible by p.

In this note we prove the following main theorem.

Theorem 1. *Let k be a field of characteristic $p > 0$. Then the group $\mathrm{Cr}_2(k)$ does not contain elements of order p^s with $s > 2$.*

We will also describe conjugacy classes of elements of order p^2 over an algebraically closed field of characteristic $p > 0$.

I thank J.-P. Serre for asking about the existence of elements of order 8 in $\mathrm{Cr}_2(k)$ over a field of characteristic 2 and his numerous comments on the previous versions of the paper. The question has initiated the present paper.

For more than 45 years, Vasya Iskovskikh had been a friend, a collaborator on several papers and an inspiring guide in the area of birational geometry. He will be greatly missed.

2. CONIC BUNDLES

It is clear that in the proof of the main theorem, we may assume that k is an algebraically closed field of characteristic $p > 0$. On several occasions I refer to [3], where the ground field was assumed to be the field of complex numbers. The proofs of the facts that I will use extend to our case.

Let $\sigma \in \mathrm{Cr}_2(k)$ be of order p^s. A standard argument (see [3]) shows that σ acts biregularly on one of the following rational surfaces X:

(i) X has a structure of a conic bundle $f\colon X \to \mathbb{P}^1_k$ with $m \geq 0$ singular fibres,

(ii) X is a Del Pezzo surface of degree d.

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Moreover, we may assume that X is σ-minimal, i.e., $\mathrm{Pic}(X)^\sigma$ is of rank 2 in the first case and of rank 1 in the second case. This is equivalent to the fact that any σ-equivariant birational morphism $X \to X'$ must be an isomorphism. When X is σ-minimal, we say that σ acts minimally on X.

We start with the first case. Recall the following well-known fact.

Lemma 2. *Let σ be an element of order p^s in $\mathrm{Aut}(\mathbb{P}^r_k)$. Then $s < 1 + \log_p(r+1)$.*

Proof. Let $A \in \mathrm{GL}_{r+1}(k)$ represent σ and $A^{p^s} = cI_{r+1}$ for some constant c. Multiplying A by c^{1/p^s}, we may assume that $A^{p^s} = I_{r+1}$ but $A^{p^{s-1}} \neq I_{r+1}$. Since k^* does not contain nontrivial pth roots of unity, we can reduce A to the Jordan form with 1 at the diagonal. Obviously $A^{p^{s-1}} = I_{r+1} + (A - I_{r+1})^{p^{s-1}}$. Since for any Jordan block-matrix J with zeros at the diagonal we have $J^{r+1} = 0$, we get $p^{s-1} < r+1$. The assertion follows. □

Corollary 3. *Let $f\colon X \to \mathbb{P}^1_k$ be a conic bundle and σ be an automorphism of X of order p^s preserving the conic bundle. Then $s \le 2$.*

Proof. Let $\bar{g}$ be the image of σ in the automorphism group of the base of the fibration. By the previous lemma $\bar{\sigma}^p = 1$. Thus σ^p acts identically on the base and hence acts on the general fibre of f. By Tsen's theorem, the latter is isomorphic to the projective line over the function field of the base. Applying the lemma again, we obtain $\sigma^{p^2} = 1$. □

This checks the theorem in the case of a conic bundle. Let us give a closer look at elements of order p^2.

Theorem 4. *Let σ be a minimal automorphism of order p^2 of a conic bundle $X \to \mathbb{P}^1_k$. Then $p = 2$.*

Proof. Let $m = K_X^2 - 8$ be the number of singular fibres of the conic bundle. Assume first that $m = 0$, i.e., $\pi\colon X \to \mathbb{P}^1_k$ is a minimal ruled surface $\mathbf{F}_n$. If $n = 1$, the surface is not σ-minimal. If $n = 0$, the automorphism group of $\mathbf{F}_0 \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$ preserving one of the rulings is isomorphic to $\mathrm{Aut}(\mathbb{P}^1_k) \times \mathrm{Aut}(\mathbb{P}^1_k)$. It does not contain elements of order p^2.

So we may assume that $n \ge 2$. The automorphism group $\mathrm{Aut}(X)$ of the surface $\mathbf{F}_n$ is well-known (see [3, § 4.4]). By blowing down the exceptional section, we find that $\mathrm{Aut}(X)$ is isomorphic to the group of automorphisms of the weighted projective plane $\mathbb{P}(1,1,n)$ with coordinates t_0, t_1 of degree 1 and coordinate t_2 of degree n. Any automorphism g of $\mathbb{P}(1,1,n)$ can be given by the formula

$$\sigma\colon (t_0, t_1, t_2) \mapsto \big(at_0 + bt_1, ct_0 + dt_1, et_2 + f_n(t_0, t_1)\big),$$

where f_n is a binary form of degree n. In our case we can change the coordinates to assume that $a = b = d = 1$ and $c = 0$. By iterating, we get $e^{p^s} = 1$; hence $e = 1$. Also

$$\sigma^p\colon (t_0, t_1, t_2) = \left(t_0, t_1, t_2 + \sum_{j=0}^{p-1} f_n(t_0 + jt_1, t_1)\right).$$

Let $\bar{\sigma}$ be the transformation $(t_0, t_1) \mapsto (t_0 + t_1, t_1)$. Since $\sum_{i=0}^{p-1} \bar{\sigma}^i = 0$, it follows that the sum in the above expression is equal to zero; hence $\sigma^p = 1$. Thus there are no automorphisms of order p^2.

Assume now that $m > 0$, i.e., X is obtained from a minimal ruled surface $\mathbf{F}_n$ by blowing up m points. If $n > 0$, the proper transform of the exceptional section of $\mathbf{F}_n$ is a section of the conic bundle with negative self-intersection. If $n = 0$, the proper transform of a section of $\mathbf{F}_0$ passing through a point we blow up is a section with negative self-intersection. So, in any case we have a section of the conic bundle with negative self-intersection. It intersects a component of a singular fibre at its nonsingular point. Since X is σ-minimal, σ cannot fix this component, so $\sigma(E) \neq E$. By Lemma 2, σ^p acts identically on the base of the conic bundle. Since $p > 2$, σ^p cannot switch components of singular fibres; hence it must act identically on $\mathrm{Pic}(X)$. Since an irreducible curve with negative self-intersection does not move in a linear system, σ^p fixes E and $\sigma(E)$. But in characteristic $p > 0$ an automorphism of order p of a general fibre has only one fixed point. This shows that $\sigma^p = 1$ if $p > 2$. □

Example 1. Recall that $\mathrm{Cr}_2(k)$ contains a subgroup of de Jonquières transformations of the form $(x, y) \mapsto \left(\frac{\alpha x+\beta}{\gamma x+\delta}, \frac{a(x)y+b(x)}{c(x)y+d(x)}\right)$. Each element of finite order in this subgroup is realized as an automorphism of a conic bundle. Assume $p = 2$. Without loss of generality we may assume that $x \mapsto x + 1$.

Let $a(x) = d(x) = xP(x)$, where $P(x)$ is a polynomial of degree n without multiple roots. Let $b(x) = P(x)P(x + 1)$ and $c(x) = x(x + 1)$. We have

$$a(x)a(x + 1) + b(x)c(x + 1) = a(x)a(x + 1) + b(x + 1)c(x) = 0.$$

With this choice, we have $\sigma^2\colon (x, y) \mapsto (x, R(x)/y)$, where

$$R(x) = \frac{a(x + 1)b(x) + a(x)b(x + 1)}{a(x)c(x + 1) + a(x + 1)c(x)} = \frac{P(x)P(x + 1)}{x(x + 1)}.$$

Replacing y with $x(x + 1)y$, we obtain the de Jonquières involution $(x, y) \mapsto (x, P(x)P(x + 1)/y)$. It is known that it is realized as a minimal automorphism of a conic bundle with the number m of singular fibres equal to the degree of $P(x)P(x + 1)$. On the other hand, it is known that for $m \geq 8$ a minimal automorphism of such a conic bundle is conjugate to neither a projective automorphism, nor a minimal automorphism of a Del Pezzo surface, nor a minimal automorphism of a conic bundle with number of singular fibres different from m (see Corollary 7.11 in [3]). Thus we have constructed a countable set of conjugacy classes of elements of order 4 in $\mathrm{Cr}_2(k)$.

3. DEL PEZZO SURFACES OF DEGREE ≥3

Now we consider the case when σ is an automorphism of order p^s of a Del Pezzo surface X of degree $d := K_X^2 \geq 4$.

If $d = 9$, then $X = \mathbb{P}_k^2$ and by Lemma 2 we get $s \leq 2$. All elements of order p^2 are conjugate in $\mathrm{Aut}(\mathbb{P}_k^2)$.

If $d = 8$, then $X \cong \mathbb{P}_k^1 \times \mathbb{P}_k^1$ because the ruled surface $\mathbf{F}_1$ is not σ-minimal. We know that $\mathrm{Aut}(\mathbf{F}_0)$ contains a subgroup of index 2 isomorphic to $\mathrm{Aut}(\mathbb{P}_k^1) \times \mathrm{Aut}(\mathbb{P}_k^1)$. Applying Lemma 2, we obtain $s = 1$ if $p \neq 2$ and $s \leq 2$ otherwise. The automorphism of X given in affine coordinates by $(x, y) \mapsto (y + 1, x)$ is of order 4.

If $d = 7$, the surface is not σ-minimal since it is obtained by blowing up two points in $\mathbb{P}_k^2$; the proper transform of the line joining the points is a σ-invariant (-1)-curve.

Assume $d = 6$. Then $\mathrm{Aut}(X)$ is isomorphic to the semidirect product $T \rtimes G$, where $T \cong k^{*2}$ is a 2-dimensional torus and G is a dihedral group $D_{12} \cong (\mathbb{Z}/2\mathbb{Z}) \times S_3$. Since T does not contain elements of order p and D_{12} does not contain elements of order $p^s, s > 1$, we find that the only possibility is $s = 1$ and $p = 2, 3$.

Assume $d = 5$. It is known that $\mathrm{Aut}(X)$ acts faithfully on the Picard group of X of a Del Pezzo surface of degree ≤ 5. Via this action it becomes isomorphic to a subgroup of the Weyl group $W(A_4) \cong S_5$. Thus $s = 1$ unless $p = 2$ and $s = 2$. The group $W(A_4)$ acts on $K_X^\perp \cong \mathbb{Z}^4$ via its standard irreducible representation on $\{(a_1, \ldots, a_5) \in \mathbb{Z}^5\colon a_1 + \ldots + a_5 = 0\}$. A cyclic permutation of order 4 has a fixed vector. This shows that X is not σ-minimal.

Assume $d = 4$. In this case $\mathrm{Aut}(X)$ is isomorphic to a subgroup of the Weyl group $W(D_5) \cong (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$. Thus an automorphism of order p^s with $s > 1$ may exist only if $p = 2$.

It is known that X is isomorphic to the blow-up of five points $p_1, \ldots, p_5$ in the plane, no three among them being collinear. The surface admits five pairs $(|C_i|, |C_i'|)$ of pencils of conics in the anticanonical embedding $X \hookrightarrow \mathbb{P}_k^4$. The pencil $|C_i|$ is the proper transform of the pencil of lines through the point p_i, and the pencil $|C_i'|$ is the proper transform of the pencil of conics through the points p_j, $j \neq i$. Since $C_i + C_i' \sim -K_X$, the Weyl group permutes the five pairs of the divisor classes $[C_i], [C_i]'$ and switches $[C_i]$ with $[C_i']$ in the even pairs of them (see [3, Proposition 6.6]). It is known that the anticanonical linear system $|-K_X|$ maps X isomorphically onto the intersection of two quadrics in $\mathbb{P}_k^4$. Under the multiplication map $|C_i| \times |C_i'| \to |-K_X|$, the two pencils generate a hyperplane H_i in $|-K_X|$ and the map $f_i \times f_i' \colon X \to \mathbb{P}_k^1 \times \mathbb{P}_k^1$ defined by the two pencils is equal to the composition of the anticanonical map and the projection from the point $h_i \in |-K_X|^*$ corresponding to the hyperplane H_i. Since the image of X under this projection is a nonsingular quadric, we see that the center of the projection lies on a singular quadric Q_i of corank 1 in the pencil $\mathcal{Q}$ of quadrics containing X. Conversely, every such quadric defines a degree 2 map $f \colon X \to \mathbb{P}_k^1 \times \mathbb{P}_k^1$, and the preimages of the ruling define a pair of pencils of conics on X. Thus we see that the pencil of quadrics $\mathcal{Q}$ contains exactly five singular quadrics. Any automorphism σ of X acts on the pencil $\mathcal{Q}$ leaving the set of five quadrics invariant. Its square σ^2 acts identically on the pencil and hence leaves invariant all pairs of conic pencils. Since the divisor classes $[C_i]$ together with K_X generate $\mathrm{Pic}(X)$, it follows that σ^4 acts identically on $\mathrm{Pic}(X)$; hence it is the identity.

Remark 1. Another proof of the nonexistence of an automorphism of order 8 on a Del Pezzo surface of degree 4 was suggested by J.-P. Serre. It is known that an element of order 8 in $W(D_5)$ has trace equal to -1 in the root lattice. Since the latter is isomorphic to $K_X^\perp$, the automorphism of order 8 has trace 0 in $\mathrm{Pic}(X)$ and hence in the second cohomology group with ℓ-adic coefficients. Thus the Lefschetz number of σ is equal to 2, and hence, by the Lefschetz fixed-point formula, σ has a fixed point. Blowing it up, we get an automorphism of order 8 of a cubic surface. Since any automorphism of a cubic surface is the restriction of an automorphism of $\mathbb{P}_k^3$, applying Lemma 2 we find a contradiction.

Let us summarize what we have learnt.

Theorem 5. *A Del Pezzo surface of degree ≥ 4 does not contain elements of order p^3. An automorphism of order p^2 not conjugate to a projective automorphism in $\mathrm{Cr}_2(k)$ exists only if $p = 2$. It is minimally realized on $X = \mathbb{P}_k^1 \times \mathbb{P}_k^1$ or on a Del Pezzo surface of degree 4.*

Note that any automorphism of order 4 of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ has a fixed point, and the projection from this fixed point makes it conjugate to a projective transformation.

Assume now that $d = 3$, i.e., X is a cubic surface embedded in $\mathbb{P}_k^3$ by the anticanonical linear system $|-K_X|$. In this case $\mathrm{Aut}(X)$ is isomorphic to a subgroup of the Weyl group $W(E_6)$ of a simple root lattice of type E_6. By Corollary 6.11 from [3], all elements of order $p^s, s > 1$, in $W(E_6)$ have an invariant vector in the lattice $E_6 \cong K_X^\perp$ unless $p^s = 9$. Thus we have to consider the existence of an automorphism σ of order 9 of a cubic surface over a field of characteristic $p = 3$.

The following argument was suggested to me by J.-P. Serre. It follows from the classification of conjugacy classes of elements of $W(E_6)$ that the trace of σ in its action in $K_X^\perp$ is equal to 0. Thus the Lefschetz number of σ in the ℓ-adic cohomology of X is equal to 3. This implies that σ has a fixed point x_0. Since σ acts trivially on $|-K_X - x_0| \cong \mathbb{P}_k^2$, we find that it acts trivially on $|-K_X| \cong \mathbb{P}_k^3$.

We have proved the following.

Theorem 6. *A cubic surface does not admit minimal automorphisms of order p^s with $s > 1$.*

4. DEL PEZZO SURFACES OF DEGREE 2

It is known (see [2]) that the linear system $|-K_X|$ defines a degree 2 map $f \colon X \to \mathbb{P}_k^2$. The map must be finite since $-K_X$ is ample. It is also a separable map because otherwise X must be homeomorphic to $\mathbb{P}_k^2$, but comparing the l-adic Betti numbers we find this impossible. The cover f is a Galois cover with order 2 cyclic Galois group $\langle\gamma\rangle$. The automorphism γ of X is called the *Geiser* *involution*. For any divisor D we have

$$D + \gamma^*(D) \sim (D \cdot K_X)K_X.$$

This implies that γ^* acts on $K_X^\perp$ as the minus identity. The lattice $K_X^\perp$ is isomorphic to the root lattice of type E_7, and the isometry γ^* generates the center of the Weyl group $W(E_7)$.

It follows from the classification of conjugacy classes in $W(E_7)$ that for any automorphism of order p^s, $s > 1$, the rank of $\operatorname{Pic}(X)^\sigma$ is greater than 1, unless $p = s = 2$. So, it suffices to consider the latter case. All such automorphisms form one conjugacy class (of type $2A_3 + A_1$ in the notation from [3]). It follows from the description of degree 2 covers of smooth varieties (see [1, Ch. 0]) that X is isomorphic to a surface $\mathbb{P}(1, 1, 1, 2)$ given by an equation

$$u^2 + a_2(x, y, z)u + a_4(x, y, z) = 0,$$

where a_2 and a_4 are homogeneous forms of degree 2 and 4. Since the anticanonical map is separable, we have $a_2 \neq 0$. An automorphism σ of order 4 acts linearly in $\mathbb{P}^2_k = |-K_X|^*$ leaving the branch curve $V(a_2)$ invariant. If $V(a_2)$ is an irreducible conic, then σ^2 is identical on the conic, and hence it is identical on $\mathbb{P}^2_k$. This implies that σ^2 is the Geiser involution $u \mapsto u + a_2$. However, the Weyl group $W(E_7)$ does not contain square roots of the Geiser involution. Suppose now that $V(a_2)$ is reducible. If it is not a double line, we can choose projective coordinates x, y, z to assume that $a_2 = xy$. Then σ^2 must change z to $z + ax + by$ and leave x and y unchanged. This forces a_4 to be invariant with respect to this transformation. Writing

$$a_4 = l_0z^4 + z^3l_1 + z^2l_2 + zl_3 + l_4,$$

where l_i are binary forms in x, y, we find that $l_1 = 0$. This implies that the point $(x, y, z, u) = (0, 0, 1, 0)$ is a singular point on the surface. Thus σ^2 must be the Geiser involution and we finish as in the previous case. Finally we may assume that the equation of X looks like $u^2 + x^2u + a_4 = 0$. In this case, $\sigma^*(x) = x$ and we may assume that σ acts on the variables x, y, and z by $x \mapsto x$, $y \mapsto y + x$, and $z \mapsto z + y$. The polynomial $a_4(x, y, z)$ must be invariant with respect to the coordinate change $\sigma^2\colon (x, y, z) \mapsto (x, y, z + x)$. It is easy to see that the ring of polynomials in x, z invariant with respect to $(x, z) \mapsto (x, z + x)$ is generated by x and $z(z + x)$. This implies that a_4 can be written as a polynomial in $z(z + x)$, x, and y:

$$a_4 = cz^2(z + x)^2 + z(z + x)g(x, y) + h(x, y).$$

It is immediate to check that the point $(x, y, z, u) = (0, 0, 1, \sqrt{c})$ is a singular point of the surface.

To sum up, a Del Pezzo surface of degree 2 does not contain minimal automorphisms of order p^s, $s > 1$.

Remark 2. Another argument to prove that a Del Pezzo surface X of degree 2 has no elements of order 8 was suggested by J.-P. Serre. We use the fact that $W(E_7) = W(E_7)^+ \times \langle w_0 \rangle$, where w_0 generates the center of $W(E_7)$. In the faithful representation $\rho\colon \operatorname{Aut}(X) \to W(E_7)$, the image of the Geiser involution γ is equal to w_0. This implies that a subgroup G of order 8 of $\operatorname{Aut}(X)$ is isomorphic to a subgroup of $A \times \langle \gamma \rangle$, where A is isomorphic to a subgroup of $\operatorname{Aut}(\mathbb{P}^2_k)$. Since the latter has no elements of order 8, we are done.

5. DEL PEZZO SURFACES OF DEGREE 1

This is the most difficult and interesting case. The linear system $|-2K_X|$ defines a degree 2 map $f\colon X \to Q$, where Q is a quadric cone in $\mathbb{P}^3_k$. Again, since $-K_X$ is ample, f is a finite map, and arguing as in the previous case we see that the map is separable. The Galois group of the cover is generated by an automorphism β of X known as the *Bertini involution*. For any divisor D we have

$$D + \gamma^*(D) \sim 2(D \cdot K_X)K_X. \tag{1}$$

This shows that β^* acts as the minus identity on the lattice $K_X^\perp$. The lattice $K_X^\perp$ is isomorphic to the root lattice of type E_8. The involution β^* generates the center of the Weyl group $W(E_8)$.

The automorphism group $\mathrm{Aut}(X)$ is isomorphic to a subgroup of $W(E_8)$. The possible orders p^s, $s > 1$, of minimal automorphisms are 4 and 8 (see [3]).

So we assume $p = 2$. The linear system $|-K_X|$ has one base point p_0. Blowing it up, we obtain a fibration $\pi\colon X' \to \mathbb{P}^1_k$ whose general fibre is an irreducible curve of arithmetic genus 1. Since $-K_X$ is ample, all fibres are irreducible, and this implies that a general fibre is an elliptic curve (see [1, Corollary 5.5.7]). Let S_0 be the exceptional curve of the blow-up. It is a section of the elliptic fibration. We take it as the zero in the Mordell–Weil group of sections of π. The map $f\colon X \to Q$ extends to a degree 2 separable finite map $f'\colon X' \to \mathbf{F}_2$, where $\mathbf{F}_2$ is the minimal ruled surface with the exceptional section E satisfying $E^2 = -2$. Its branch curve is equal to the union of E and a curve B from the divisor class $3f + e$, where f is the class of a fibre and $e = [E]$. We have $f'^*(E) = 2S_0$. The elliptic fibration on X' is the preimage of the ruling of $\mathbf{F}_2$. We know that $\tau = \sigma^2$ acts identically on the base of the elliptic fibration. Since it also leaves invariant the section S_0, it defines an automorphism of the generic fibre considered as an abelian curve with zero section defined by S_0. If $\tau^2 = 1$, then τ is the negation automorphism and hence defines the Bertini involution of X. The group of automorphisms of an abelian curve in characteristic 2 is of order 2 if the absolute invariant of the curve is not equal to 0 or of order 24 otherwise. In the latter case it is isomorphic to $Q_8 \rtimes \mathbb{Z}/3$, where Q_8 is the quaternion group with the center generated by the negation automorphism (see [4, Appendix A]). Thus $\tau^4 = 1$ and the Weierstrass model of the generic fibre is

$$y^2 + a_3y + x^3 + a_4x + a_6 = 0.$$

In global terms, the Weierstrass model of the elliptic fibration $\pi\colon X' \to \mathbb{P}^1_k$ is a surface in $\mathbb{P}(1,1,2,3)$ given by the equation

$$y^2 + a_3(u,v)y + x^3 + a_4(u,v)x + a_6(u,v),$$

where a_i are binary forms of degree i. It is obtained by blowing down the section S_0 to the point $(u,v,x,y) = (0,0,1,1)$ and is isomorphic to our Del Pezzo surface X. The image of the branch curve B is given by the equation $a_3(u,v) = 0$; i.e., B is equal to the preimage of an effective divisor of degree 3 on the base plus the section S_0. Since a general point of B is a 2-torsion point of a general fibre, we see that all nonsingular fibres of the elliptic fibration are supersingular elliptic curves (i.e., have no nontrivial 2-torsion points). An automorphism of order 4 of X is defined by

$$(u,v,x,y) \mapsto \bigl(u,v,x + s(u,v)^2, y + s(u,v)x + t(u,v)\bigr),$$

where s is a binary form of degree 1 and t is a binary form of degree 3 satisfying

$$a_3 = s^3, \qquad t^2 + a_3t + s^6 + a_4s^2 = 0. \tag{2}$$

In particular, this shows that a_3 must be a cube, so we can change the coordinates (u,v) to assume that $s = u$ and $a_3 = u^3$. The second equality in (2) says that t is divisible by u, so we can write it as $t = uq$ for some binary form q of degree 2 satisfying $q^2 + u^2q + u^4 + a_4 = 0$. Let α be a root of the equation $x^2 + x + 1 = 0$ and $b = q + \alpha u^2$. Then b satisfies $a_4 = b^2 + u^2b$ and $t = ub + \alpha u^3$. Conversely, any surface in $\mathbb{P}(1,1,2,3)$ with the equation

$$y^2 + u^3y + x^3 + (b(u,v)^2 + u^2b(u,v))x + a_6(u,v) = 0, \tag{3}$$

where b is a quadratic form in (u,v) and the coefficient at uv^5 in a_6 is not zero (this is equivalent to the fact that the surface is nonsingular), is a Del Pezzo surface of degree 1 admitting an automorphism of order 4

$$\tau\colon (u,v,x,y)\mapsto (u,v,x+u^2,y+ux+ub+\alpha u^3).$$

Note that $\tau^2\colon (u,v,x,y)\mapsto (u,v,x,y+u^3)$ coincides with the Bertini involution.

Theorem 7. *Let X be a Del Pezzo surface* (3). *Then it does not admit an automorphism of order* 8.

Proof. Assume $\tau=\sigma^2$. Since σ leaves invariant $|-K_X|$, it fixes its unique base point and lifts to an automorphism of the elliptic surface X' preserving the zero section S_0. Since the general fibre of the elliptic fibration $f\colon X'\to\mathbb{P}^1_k$ has no automorphism of order 8, the transformation σ acts nontrivially on the base of the fibration. Note that the fibration has only one singular fibre F_0 over $(u,v)=(0,1)$. It is a cuspidal cubic. The transformation σ leaves this fibre invariant and hence acts on $\mathbb{P}^1_k$ by $(u,v)\mapsto(u,u+cv)$ for some $c\in k$. Since the restriction of σ to F_0 has at least two distinct fixed points, the cusp and the origin $F_0\cap S_0$, it acts identically on F_0 and freely on its complement $X'\setminus F_0$.

Recall that X' is obtained by blowing up nine points $p_1,\dots,p_9$ in $\mathbb{P}^2_k$, the base points of a pencil of cubic curves. We may assume that X is the blow-up of the first eight points and the exceptional curve over p_9 is the zero section S_0. Let S be the exceptional curve over any other point. We know that $\beta=\sigma^4$ is the Bertini involution of X. Applying formula (1), we find that $S\cdot\beta(S)=3$. Identifying $\beta(S)$ and S with their preimages in X', we see that $\beta(S)+S=S_0$ in the Mordell–Weil group of sections of $\pi\colon X'\to\mathbb{P}^1_k$. Thus S and $\beta(S)$ meet at 2-torsion points of fibres. However, all nonsingular fibres of our fibration are supersingular elliptic curves; hence S and $\beta(S)$ can meet only at the singular fibre F_0. Let $Q\in F_0$ be the intersection point. The sections S and $\beta(S)$ are tangent to each other at Q with multiplicity 3. Now consider the orbit of the pair $(S,\beta(S))$ under the cyclic group $\langle\sigma\rangle$. It consists of four pairs

$$(S,\sigma^4(S)),\qquad(\sigma(S),\sigma^5(S)),\qquad(\sigma^2(S),\sigma^6(S)),\qquad(\sigma^3(S),\sigma^7(S)).$$

Let $D_i=\sigma^i(S)+\sigma^{i+4}(S)$, $i=1,2,3,4$. We have $D_1+\ldots+D_4\sim-8K_X$; hence for $i\neq j$ we have $D_i\cdot D_j=(64-16)/12=4$. Let $Y\to X$ be the blow-up of Q. Since Q is a double point of each D_i, the proper transform $\overline{D}_i$ of each D_i in Y has self-intersection 0 and consists of two smooth rational curves intersecting at one point with multiplicity 2. Moreover, we have $\overline{D}_i\cdot\overline{D}_j=0$. Applying (1), we get $D_i\in|-2K_X|$. Since Q is a double point of D_i, we obtain $\overline{D}_i\in|-2K_Y|$. The linear system $|-2K_Y|$ defines a fibration $Y\to\mathbb{P}^1_k$ with a curve of arithmetic genus 1 as a general fibre (an elliptic or a quasielliptic fibration) and four singular fibres $\overline{D}_i$ of Kodaira's type III. The automorphism σ acts on the base of the fibration, and the four special fibres form one orbit. But the action of σ on $\mathbb{P}^1_k$ is of order 2 and this gives us a contradiction. □

Remark 3. A computational proof of Theorem 7 was given by J.-P. Serre.

6. CONJUGACY CLASSES OF ELEMENTS OF ORDER p^2

Assume that k is algebraically closed. As we have seen in the previous sections, an element of order p^2 not conjugate to a projective transformation exists only for $p=2$. It can be realized as a minimal automorphism of a conic bundle or a Del Pezzo surface of degree 1 or 4. Del Pezzo surfaces of degree 1 are super-rigid; i.e., a minimal automorphism of such a surface could be conjugate only to a minimal automorphism of the same surface. A minimal automorphism of a Del Pezzo surface of degree 4 is conjugate to a minimal automorphism of a conic bundle with five singular fibres (see [3, § 8]).

Thus we have proved the following.

Theorem 8. *An element of order p^2 not conjugate to a projective transformation exists only if $p = 2$. Assume that k is algebraically closed. An element of order 4 is conjugate to either a projective transformation or a transformation realized by a minimal automorphism of a conic bundle or of a Del Pezzo surface of degree 1.*

For the sake of completeness let us add that elements of order p not conjugate to projective transformations occur for any p. They can be realized as automorphisms of conic bundles, and if $p = 2, 3, 5$, as automorphisms of Del Pezzo surfaces.

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