AUTOMORPHIC FORMS AND QUASIHOMOGENEOUS SINGULARITIES

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In this paper we announce several results related to a variant and a generalization of a construction for normal singularities with a $\mathbb{C}^*$-action (see [3]).

1. Definition. Let $U$ be a homogeneous complex manifold, $\Gamma$ a group of analytic automorphisms of the manifold $U$, and $L$ a one-dimensional vector (or line) $\Gamma$-bundle on $U$ (see [8]). An automorphic form with weight $k$ with respect to $L$ is a cross section $\varphi \in H^0(U, L^\otimes k)$ of $k$-th order tensors for the $\Gamma$-bundle $L$ which is invariant with respect to the natural action of $\Gamma$. The graded $\mathbb{C}$-algebra $A(L) = \bigoplus_{k \in \mathbb{Z}} H^0(U, L^\otimes k)$ will be called the algebra of automorphic forms with respect to $L$.

THEOREM 1. We shall assume that the triple $(U, \Gamma, L)$ is admissible, i.e., that the following assumptions hold:

A1. There is a normal subgroup of finite index $\Gamma' \subset \Gamma$ which acts freely and discretely on $U$.

A2. The factor space $U/\Gamma$ is a compact analytic space.

A3. For some subgroup $\Gamma' \subset \Gamma$ satisfying A1, the factor $L/\Gamma'$ determines a positive (in the sense of Kodaira) line bundle over the manifold $U/\Gamma'$.

Under these assumptions, the algebra of automorphic forms $A(L)$ is a normal $\mathbb{C}$-algebra of finite type and dimension $\dim U + 1$, with nonnegative grading.

2. Definition. The affine algebraic manifold $X$ over the algebraically closed field $k$ is said to be a quasi-cone if the one-dimensional algebraic torus $\mathbb{G}_m$ acts effectively on $X$ and there is a unique point $x_0 \in X$ which belongs to the closure of every orbit. The point $x_0$ is called the vertex of the quasi-cone $X$.

PROPOSITION. Let $X$ be an affine algebraic manifold over $k$. The following are equivalent:

1) $X$ is a quasi-cone;

2) the coordinate ring $k[X]$ has nonnegative grading and $k[X]_{\geq 0}$;

3) there is a closed inclusion $j: X \rightarrow \mathbb{A}^n$ such that $j(X)$ is invariant with respect to the action of $\mathbb{G}_m$ on $\mathbb{A}^n$ where the action is defined by the formula $(x_1, \ldots, x_n) \mapsto (x_1^{q_1 t}, \ldots, x_{n}^{q_n t})$, with $t \in \mathbb{G}_m(k)$ and $q_1, \ldots, q_n$ being positive integers;

4) there is an inclusion $j: X \rightarrow \mathbb{A}^n$ such that the ideal giving $j(X)$ is generated by weighted-homogeneous polynomials with positive rational weights [4].

The proof of this proposition is based on standard arguments about the actions of algebraic tori on affine manifolds (cf. [9]).

THEOREM 2. Let $(U, \Gamma, L)$ be an admissible triple. Then the affine algebraic manifold $\text{Spec } A(L)$ is a normal quasi-cone with vertex $x_0$ defined by the maximal ideal $A(L)_{x_0} = \bigoplus \mathbb{C} (L)_{x_0}$. Conversely, each normal two-dimensional quasi-cone is isomorphic to the manifold $\text{Spec } A(L)$ for some admissible triple $(U, \Gamma, L)$.
While the first part of this theorem follows immediately from the preceding results, the proof of the second part is very specialized and uses the idea of a "singular Seifert bundle" from [9].

**Definition.** A singularity in this article is the jet \((Y, y)\) of the analytic space \(Y\) at the point \(y\). A singularity will be called a normal singularity if \(Y\) is normal at the point \(y\). Isomorphism of singularities means an analytic isomorphism between the corresponding jets. A singularity is called quasi-homogeneous if it is isomorphic to the jet of some quasi-cone at its vertex.

**COROLLARY.** Each admissible triple \((U, \Gamma, L)\) determines a normal quasi-homogeneous singularity \(S(L)\). Each normal two-dimensional quasi-homogeneous singularity is isomorphic to a singularity of the form \(S(L)\).

3. **Examples.** 1) Let \(G\) be a finite subgroup of the group \(LG(n + 1, C)\), \(\Gamma\) its image under the canonical homomorphism \(\varphi: GL(n + 1, C) \to PL(n, C)\), \(m\) the order of the subgroup \(G \cap Ker \varphi\). The bundle \(L = H \otimes m\), where \(H\) corresponds to the hyperplane cross section of \(P^n(C)\), is a \(\Gamma\)-bundle with respect to the natural action of \(\Gamma\) on \(P^n(C)\). The triple \((P^n(C), \Gamma, L)\) is admissible, and the corresponding singularity \(S(L)\) is isomorphic to the factor-singularity \((C^n/G, 0)\), where 0 is the image of the coordinate origin.

When \(n = 1\) and \(G \subset SL(2, C)\) the singularity obtained in this way is a Klein singularity (in other terms it is a double rational singularity, a platonic singularity, a singularity of type \(A, D, E\); see [4], §9).

2) Let \(U\) be a bounded homogeneous region in \(C^n\), \(\Gamma\) a discrete group of analytic automorphisms of \(U\) with compact factor \(U/\Gamma\). Each \(\Gamma\)-bundle over \(U\) is given by the trivial bundle \(U \times C\) with \(\Gamma\)-action \((z, \alpha) \to (g(z), h(g) \cdot z, \alpha)\), specified by the automorphicity factor \(h \in \mathfrak{Z}(\Gamma, \Theta(U)^*)\). In particular, the automorphicity factor is defined as \(h = J^{-1}\), where \(J(g; z)\) is the Jacobian of \(g \in \Gamma\) at the point \(z\). The well-known results of Borel [5] and Kodaira [7] show that the triple \((U, \Gamma, J)\) is admissible. The quasi-homogeneous singularity associated with it is called canonical and is denoted by \(S(\Gamma)\).

In particular, let \(U = \{z \in C | |z| < 1\}, \Gamma\) the Fuchsian group of the first kind with signature \((0, m; n_0, \ldots, n_m)\). When \(m = 3\) the singularities \(S(\Gamma)\) were called canonical triangular singularities in [3], and in that article there were listed those singularities which occurred in \(C^3\) (the 14 unimodular singularities of Arnol’d). If \(r\) is a positive integer relatively prime to each of the \(n_i\), then there is not more than one automorphicity factor \(h\) with \(h^r = J^{-1}\). When such a factor exists (the appropriate conditions can be obtained through explicit calculation of the group of the cohomologies \(H^2(\Gamma, Z)\); see [6]), we denote the singularity corresponding to the triple \((U, \Gamma, h)\) by \(S(\Gamma, r)\). The sets \((n_0, \ldots, n_m)\) and factors \(r\) for the level surfaces of the bimodal critical points of Arnol’d are given in Table 1 (notation from [2]).

3) Arnol’d’s [1] parabolic two-dimensional singularities can be obtained from the appropriate automorphicity factor for the lattice \(\Gamma\) in \(C\).

**LITERATURE CITED**