ON THE ARITHMETIC-GEOMETRIC MEAN FOR CURVES OF GENUS 2

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Abstract. We study the relationship between two genus 2 curves whose jacobians are isogenous with kernel equal to a maximal isotropic subspace of p-torsion points with respect to the Weil pairing. When $p = 2$ this relationship is a generalization of Gauss's arithmetic-geometric mean for elliptic curves studied by Richelot (1837) and Humbert (1901), and in modern terms by Bost-Mestre (1988) and Donagi-Livné (1999).

1. Introduction

As is well-known the arithmetic-geometric mean (agM) $(a, b) \mapsto (\frac{a+b}{2}, \sqrt{ab})$ was used by Gauss to calculate numerically elliptic integrals. By doubling the period τ of an elliptic curve one obtains another elliptic curve whose Weierstrass equation can be explicitly expressed via the Weierstrass equation of the original curve via the arithmetic-geometric mean. Iterating this process, Gauss reduced the computation of elliptic integrals to integrals of rational functions. A modern interpretation of this construction consists of replacing an elliptic curve E with the isogenous elliptic curve E' such that the kernel of the isogeny is equal to the subgroup of order 2 (see a modern exposition of Gauss's agM in [?]). An immediate generalization of this construction is to consider the quotient of the Jacobian variety of a curve C of genus g by a maximal isotropic subspace of p-torsion points with respect to the Weil pairing defined by the principal polarization. The quotient variety is a principally polarized abelian variety, and, in lower genus, one hopes to realize it as the Jacobian variety of another curve C' of genus g. This turns out to be the case when $p = 2$ and $g \leq 3$, and the explicit geometric moduli relationship between the two curves was found by Richelot [?] and Humbert [?] for $g = 2$, and extended to the case $g = 3$ by Donagi-Livné [?] and Lehavi-Ritzenthaler [?]. The case $g = 1$ and $p > 2$ was considered by Jacobi (p = 3,5), Cayley (p = 7,11) and by many others (see a survey of these results in [?], Chapter 4). This has found application in computational number theory for finding counting algorithms for rational points of elliptic curves over finite fields (see [?]).

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In the present paper we study the case $g = 2$ and $p > 2$ and assume that the ground field $\mathbb{K}$ is an algebraically closed field of characteristic $\neq 2$. Our main result is the following

Theorem 1.1. *Let C be a smooth genus 2 curve. Assume that $\mathrm{Jac}(C)$ is an ordinary abelian variety. Let F be a maximal isotropic étale group subscheme of the group scheme $\mathrm{Jac}(C)[p]$ of p-torsion points. For any $e, -e \in F \setminus \{0\}$ let $\{x_e, y_e\}$ be a unique, up to the hyperelliptic involution ι, pair of points on C such that $x_e - y_e = \pm e$. Let $\phi : C \to R_{2p} \subset \mathbb{P}^{2p}$ be the degree two map onto a rational norm curve given by the linear system $|2pK_C|^{\iota}$. Let (c_e, d_e) be the images of the pairs (x_e, y_e) in R_{2p} and $\ell_e = \overline{c_e, d_e}$ be the corresponding secant line of R_{2p}. There exists a unique hyperplane $\mathcal{H}$ in $\mathbb{P}^{2p}$ containing the images $w_1, \ldots, w_6$ of the six Weierstrass points such that the intersection points of $\mathcal{H}$ with the secants ℓ_e are contained in a subspace L of $\mathcal{H}$ of codimension 3. The images of the points w_i under a projection from L to $\mathbb{P}^3$ are contained on a conic (maybe reducible), and the double cover of the conic ramified at these points is a stable curve C' of arithmetic genus 2 such that $J(C') \cong \mathrm{Jac}(C)/F$.*

In the case $\mathbb{K} = \mathbb{C}$ and $p = 3$ we give an effective algorithm for determining the curve C' in terms of C.

2. Preliminaries

2.1. **Polarized abelian varieties.** Let A be a g-dimensional abelian variety over an algebraically closed field $\mathbb{K}$. Let $\mathcal{L}$ be an invertible sheaf on A and $\pi : \mathbb{V}(\mathcal{L}) \to A$ be the corresponding line bundle, the total space of $\mathcal{L}$. One defines the *theta group scheme* $G(\mathcal{L})$ whose S-points are lifts of translation automorphisms $t_a, a \in A(S)$, of $A_S = A \times_{\mathbb{K}} S$ to automorphisms of $\mathbb{V}(\mathcal{L})_S$. It fits in the canonical central extension of group schemes

$$1 \to \mathbb{G}_m \to G(\mathcal{L}) \to K(\mathcal{L}) \to 1, \tag{1}$$

where $K(\mathcal{L})(S)$ is the subgroup of $A(S)$ of translations which send $\mathcal{L}_S$ to isomorphic invertible sheaf on A_S. The extension is determined by the *Weil pairing*

$$e^{\mathcal{L}} : K(\mathcal{L}) \times K(\mathcal{L}) \to \mathbb{G}_m$$

defined by the commutator in $G(\mathcal{L})$. A subgroup K of $K(L)$ is isotropic with respect to the Weil pairing if and only if the extension splits over K.

From now we assume that $\mathcal{L}$ is ample. In this case $K(\mathcal{L})$ is a finite group scheme and the Weil pairing is non-degenerate. Recall that the algebraic equivalence class of an ample invertible sheaf on A is called a *polarization* on A. An abelian variety equipped with a polarization is called a *polarized abelian variety*.

Let $A^{\vee}$ be the dual abelian variety representing the connected component of the Picard scheme of A. Any invertible sheaf $\mathcal{L}$ defines a homomorphism of abelian varieties

$$\phi_{\mathcal{L}} : A \to A^{\vee}, a \mapsto t_a^*(\mathcal{L}) \otimes \mathcal{L}^{-1}.$$

It depends only on the algebraic equivalence class of $\mathcal{L}$ and its kernel is isomorphic to the group $K(\mathcal{L})$. In particular, λ is an isogeny if and only $\mathcal{L}$ is ample. We say that $\mathcal{L}$ defines a *principal polarization* if $\phi_{\mathcal{L}}$ is an isomorphism. This is also equivalent to that $\mathcal{L}$ is ample and $h^0(\mathcal{L}) = 1$.

The proof of the following proposition can be found in [**?**], §23.

Proposition 2.1. *Let $\lambda : A \to B$ be a separable isogeny of abelian varieties. There is a natural bijective correspondence between the following sets*

- *the set of isomorphism classes of invertible ample sheaves $\mathcal{M}$ such that $\lambda^*\mathcal{M} \cong \mathcal{L}$;*
- *the set of homomorphisms $\ker(\lambda) \to G(\mathcal{L})$ lifting the inclusion $\ker(\lambda) \hookrightarrow A$.*

Under this correspondence $K(\mathcal{M}) = \ker(\lambda)^{\perp}/\ker(\lambda)$. In particular, $\mathcal{M}$ defines a principal polarization on B if and only if $\ker(\lambda)$ is a maximal isotropic subgroup.

Assume that $\mathcal{L}$ defines a principal polarization on A. Then $K(\mathcal{L}^n) = A[n] = \ker([n]_A)$, where $[n]_A$ is the multiplication map $x \mapsto nx$ in A. Applying the previous proposition to $\mathcal{L}^n$, we obtain

Corollary 2.2. *Assume $(n, \operatorname{char} \mathbb{K}) = 1$. Let K be a maximal isotropic subspace of $A[n]$ and $\lambda : A \to B = A/K$ be the quotient map. Then B admits a principal polarization $\mathcal{M}$ such that $\lambda^*\mathcal{M} \cong \mathcal{L}^n$.*

2.2. **Kummer varieties.** Let A be a principally polarized abelian variety (i.e. A is equipped with $\mathcal{L}$ defining a principal polarization). Since $h^0(\mathcal{L}) = 1$, there exists a unique effective divisor Θ such that $\mathcal{L} \cong \mathcal{O}_A(\Theta)$. The divisor Θ is called a *theta divisor* associated to the polarization. It is defined only up to a translation. One can always choose a theta divisor satisfying $[-1]_A^*\Theta = \Theta$, a *symmetric theta divisor.* Two symmetric theta divisors differ by a translation $t_a, a \in A[2](\mathbb{K})$.

The proof of the following result over $\mathbb{K} = \mathbb{C}$ can be found in [**?**], Chapter IV, §8 and in [**?**] in the general case.

Proposition 2.3. *Let A be a principally polarized abelian variety and Θ be a symmetric theta divisor. Then the map $\phi_{2\Theta} : A \to \mid 2\Theta \mid^*$ factors through the projection $\phi : A \to A/\langle[-1]_A\rangle$ and a morphism $j : A/\langle[-1]_A\rangle \hookrightarrow \mid 2\Theta \mid \cong \mathbb{P}^{2^g-1}$. If A is not the product of principally polarized varieties of smaller dimension and $\operatorname{char} \mathbb{K} \neq 2$, then j is a closed embedding.*

We assume that $\operatorname{char} \mathbb{K} \neq 2$. The quotient variety $A/\langle[-1]_A\rangle$ is denoted by $\operatorname{Km}(A)$ and is called the *Kummer variety* of A. In the projective embedding $\operatorname{Km}(A) \hookrightarrow \mathbb{P}^{2^g-1}$ its degree is equal to $2^{g-1}g!$. The image of any $e \in A[2](\mathbb{K})$ in $\operatorname{Km}(A)$ is a singular point P_e, locally (formally) isomorphic to the affine cone over the second Veronese variety of $\mathbb{P}^{g-1}$. For any $e \in A[2](\mathbb{K})$, the image of $\Theta_a := t_e^*(\Theta)$ in $\operatorname{Km}(A) \subset \mathbb{P}^{2^g-1}$ is a subvariety T_e cut out by a hyperplane $2\Theta_e$ with multiplicity 2. It is called a *trope.*

Since each Θ_e is symmetric, the corresponding trope T_e is isomorphic to the quotient $\Theta_e/\langle[-1]_A\rangle$.

The configuration of the singular points P_e and the tropes T_e form an abstract symmetric configuration $(2^{2g}, 2^{g-1}(2^g-1))$. This means that each trope contains $2^{g-1}(2^g-1)$ singular points and each singular point is contained in the same number of tropes.

The Kummer variety $\mathrm{Km}(A)$ admits a resolution of singularities

$$\pi : \mathcal{K}(A) \to \mathrm{Km}(A)$$

with the exceptional locus equal to the union of $E_e = \pi^{-1}(P_e), e \in A[2](\mathbb{K})$. Each E_e is isomorphic to $\mathbb{P}^{g-1}$ and the self-intersection E_e^g is equal to the degree of the Veronese variety $\nu_2(\mathbb{P}^{g-1})$ taken with the sign $(-1)^{g-1}$, that is, the number $(-2)^{g-1}$.

Let $p > 2$ be a prime number and K be a maximal isotropic subgroup in $A[p](\mathbb{K})$. If $p \neq \mathrm{char}\ \mathbb{K}$, then $A[p](\mathbb{K}) \cong \mathbb{F}_p^{2g}$ and the number of such K's is equal to $\prod_{i=1}^{g}(p^i+1)$. If $p = \mathrm{char}\ \mathbb{K}$, we assume that A is an ordinary abelian variety, i.e. $A[p](\mathbb{K}) \cong \mathbb{F}_p^g$. In this case $K = A[p](\mathbb{K})$ is unique.

Proposition 2.4. *Let $\lambda : A \to B = A/K$ be the quotient isogeny defined by $\mathcal{L}^p$. There exists a symmetric theta divisor Θ on A and a symmetric theta divisor Θ' on B such that $\lambda^*\Theta' \in \mid p\Theta \mid$ and $\lambda(\Theta) \in \mid p^{g-1}\Theta' \mid$. Let D be the proper transform of $\lambda(\Theta)$ in $\mathcal{K}(B)$. Let m_e be the multiplicity of Θ at 2-torsion point e. Then*

$$2D \in \left| p^{g-1}H - \sum_{e \in A[2](\mathbb{K})} m_e E_e \right|,$$

where H is the divisor class of the pre-image of a hyperplane in $\mathbb{P}^{2^g-1}$ under the composition map $\sigma : \mathcal{K}(B) \to \mathrm{Km}(B)$ and the map $j : \mathrm{Km}(B) \to \mathbb{P}^{2^g-1}$ induced by the map $\phi_{2\Theta}$.

Proof. As we observed earlier there exists an ample invertible sheaf $\mathcal{M}$ on B defining a principal polarization such that $\lambda^*\mathcal{M} \cong \mathcal{L}^p$. Let Θ' be a theta divisor on B defined by $\mathcal{M}$. We have $\lambda^*\Theta' \in \mid p\Theta \mid$ and

$$\lambda^*(\lambda(\Theta)) = \sum_{e \in K} t_e^*(\Theta) \equiv p^g\Theta.$$

Since the canonical map of the Neron-Severi groups $\lambda^* : \mathrm{NS}(B) \to \mathrm{NS}(A)$ is injective, we obtain that $p^{g-1}\Theta'$ and $\lambda(\Theta)$ are algebraically equivalent divisors on B. Since they are both symmetric divisors, they differ by a translation with respect to a 2-torsion point e. Replacing Θ' by $t_a^*(\Theta)$ we obtain the linear equivalence of the divisors.

It remains to prove the second assertion. Let $\sigma : B' \to B$ be the blow-ups of 2-torsion points on B. We have a commutative diagram

$$\begin{array}{ccc} B' & \xrightarrow{\phi'} & \mathcal{K}(B) \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\pi} \\ B & \xrightarrow{\phi} & \mathrm{Km}(B). \end{array}$$

It is clear that λ defines a bijection $A[2](\mathbb{K}) \to B[2](\mathbb{K})$. Since λ is a local isomorphism, for any $e \in A[2](\mathbb{K})$, the multiplicity m_e of Θ at e is equal to the multiplicity of $\lambda(\Theta)$ at $\lambda(e)$. Thus $2\lambda(\Theta)$ belongs to the linear system $\left|p^{g-1}(2\Theta') - 2\sum_{e\in B[2](\mathbb{K})} m_e e\right|$ of divisors in $|\ 2p^{g-1}\Theta'\ |$ passing through the 2-torsion points e with multiplicities $2m_e$. Let D' be the proper transform of $\lambda(\Theta)$ in B'. Then $D' \sim \sigma^*(p^{g-1}\Theta' - \sum m_e\sigma^{-1}(e))$. On the other hand, since ϕ' ramifies over each E_e with multiplicity 2, we have

$$2\sigma^*(p^{g-1}\Theta' - 2\sum m_e\sigma^{-1}(e)) \sim \phi'^*(p^{g-1}\pi^*(H) - \sum_e m_e E_e).$$

This shows that the proper transform of the image of $2\lambda(\Theta)$ in $\mathcal{K}(B)$ is linearly equivalent to $p^{g-1}\pi^*(H) - \sum_e m_e E_e$. □

Remark 2.5. It is known that a theta divisor on general principally polarized abelian variety has no singular points at 2-torsion points. Thus $m_e = 1$ for $2^{g-1}(2^g - 1)$ points and $m_e = 0$ at the remaining 2-torsion points. Also note that $\mathrm{Pic}(\mathcal{K}(B))$ has no 2-torsion, so there is only one half of the divisor class $p^{g-1}\pi^*(H) - \sum_e m_e E_e$.

2.3. **Theta level structure.** The main reference here is [?] (see also [?], [?]). Let A be an ordinary abelian variety of dimension g and $\mathcal{L} \cong \mathcal{O}_A(\Theta)$ be an ample invertible sheaf defining a symmetric principal polarization. The theta divisor Θ defines a function

$$q_\Theta : A[2](\mathbb{K}) \to \mu_2, \qquad x \mapsto (-1)^{\mathrm{mult}_x(\Theta)+\mathrm{mult}_0(\Theta)}.$$

This function is a quadratic form whose associated bilinear form is the Weil pairing. We call Θ *even* (resp. *odd*) if the quadratic form is even (resp. odd). Recall that the latter means that $\#q^{-1}(0) = 2^{g-1}(2^g+1)$ (resp. $\#q^{-1}(1) = 2^{g-1}(2^g-1)$). One can show that Θ is even if and only $\mathrm{mult}_0(\Theta)$ is even. Also, if we normalize the isomorphism $\mathcal{L} \to [-1]_A^*\mathcal{L}$ to assume that is equal to the identity on the fibres over the zero point, then Θ is even if and only if $[-1]_A^*$ acts as the identity on $\Gamma(\mathcal{L})$.

Let $G(\mathcal{L}^n)$ be the theta group of $\mathcal{L}^n$. A *level n theta structure* on A is a choice of an isomorphism of group schemes over $\mathbb{K}$ that is the identity on the centers

$$\theta : G(\mathcal{L}^n)(\mathbb{K}) \to \mathcal{H}_g(n),$$

where $\mathcal{H}_d(n)$ is the *Heisenberg group scheme* defined by the exact sequence

$$1 \to \mathbb{K}^* \to \mathcal{H}_g(n) \to (\mathbb{Z}/n\mathbb{Z})^g \oplus \mu_n^g \to 1.$$

If $(\lambda, a, b) \in \mathbb{K}^* \times (\mathbb{Z}/n\mathbb{Z})^g \times \mu_n(\mathbb{K})^g$ represents a point of $\mathcal{H}_g(n)(\mathbb{K})$, then the law of composition is

$$(\lambda, a, b) \cdot (\lambda', a', b') = (\lambda\lambda' b'(a), a + a', bb'),$$

where we identify $\mu_n(\mathbb{K})^g(\mathbb{K})$ with $\mathrm{Hom}((\mathbb{Z}/n\mathbb{Z})^g, \mathbb{K}^*)$.

A theta level n structure defines an n-level on A, i.e. an isomorphism of symplectic group schemes

$$\bar{\theta} : (A[n], e^{\mathcal{L}^n}) \to \big((\mathbb{Z}/n\mathbb{Z})^g \times \mu_n^g, E\big),$$

where

$$E : (\mathbb{Z}/n\mathbb{Z})^{2g} \times (\mathbb{Z}/n\mathbb{Z})^{2g} \to \mathbb{K}^*$$

is the standard symplectic form $((a, b), (a', b')) \mapsto b'(a)/a'(b)$. In particular, $\bar{\alpha}^{-1}((\mathbb{Z}/n\mathbb{Z})^g)$ is a maximal isotropic subgroup in $A[n](\mathbb{K})$.

The choice of a theta structure of level n defines a representation of the Heisenberg group $\mathcal{H}_g(n)$ on the linear space $V_n(g) = \Gamma(A, \mathcal{L}^n)$, called the *Schrödinger representation*. In this representation the space $V_n(g)$ admits a basis $\eta_\sigma, \sigma \in (\mathbb{Z}/n\mathbb{Z})^g$, such that $(\lambda, a, b) \in \mathcal{H}_g(n)$ acts by sending η_σ to $\lambda b(\sigma + a)\eta_{\sigma+a}$. We will explain how to built such a basis from theta functions when we discuss the complex base field case.

If $n \geq 3$, the map $\phi : A \to \mathbb{P}(V_n(g)^*)$ given by the complete linear system $|\mathcal{L}^n|$ is a closed embedding and the Schrödinger representation defines a projective linear representation of the abelian group scheme $(\mathbb{Z}/n\mathbb{Z})^g \oplus \mu_n(\mathbb{K})^g$ in $\mathbb{P}(V_n(g)^*) \cong \mathbb{P}^{g^n-1}$ such that the image of A is invariant, and the action on the image is the translation by n-torsion points.

Let $\mathcal{L}$ be a symmetric principal polarization. The automorphism $[-1]_A$ of $A[n]$ lifts to a normalized automorphism $[-1]_A^*$ of $\mathcal{L}$ and hence defines an automorphism of $G(\mathcal{L})$ defined by $\phi \mapsto [-1]_A^* \circ \phi [-1]_A^*$ inducing the negation on $A[n](\mathbb{K})$. A theta structure is called *symmetric* if, under the isomorphism $G(\mathcal{L}) \to \mathcal{H}_g(n)$, the automorphism δ_{-1} corresponds to the automorphism D_{-1} of $\mathcal{H}_g(n)$ defined by $(t, a, b) \mapsto (t, -a, b^{-1})$. This defines the action of D_{-1} in $V_n(g)$.

From now on we assume that $n > 1$ is odd.

Since D_{-1} is of order 2, the vector space $V_n(g)$ decomposes into the direct sum of two eigensubspaces $V_n(g)^+$ and $V_n(g)^-$ with eigenvalue 1 and -1, respectively. If $\mathcal{L}$ is defined by an even theta divisor Θ, then $D_{-1}(\eta_\sigma) = \eta_{-\sigma}$ and we can choose a basis $y_\sigma = \eta_\sigma + \eta_{-\sigma}, \sigma \in (\mathbb{Z}/n\mathbb{Z})^g$ in V_n^+ and the basis $z_\sigma = \eta_\sigma - \eta_{-\sigma}, \sigma \in (\mathbb{Z}/n\mathbb{Z})^g$ in V_n^-. In particular,

$$\dim V_n(g)^\pm = (n^g \pm 1)/2.$$

If Θ is odd, then $D_{-1}(\eta_\sigma) = -\eta_{-\sigma}$ and we have

$$\dim V_n(g)^\pm = (n^g \mp 1)/2.$$

The two projectivized subspaces form the fixed loci of the projective involution D_{-1}. We will call the subspace of dimension $(n^g - 1)/2$ the *Burhardt* *space* and denote it by $\mathbb{P}_{Bu}$. The other subspace of dimension $(n^g-3)/2$ we call the *Masche subspace* and denote it by $\mathbb{P}_{Ma}$.

Two different theta structures of level n differ by an automorphism of $\mathcal{H}_g(n)$ which is the identity on $\mathbb{K}^*$. Let $A(\mathcal{H}_g(n))$ be the group of such automorphisms. Let $(\mathbb{Z}/n\mathbb{Z})^{2g} \rtimes \mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z})$ be the semi-direct product defined by the natural action of $\mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z}$ on $(\mathbb{Z}/n\mathbb{Z})^{2g}$. There is a natural isomorphism

$$(\mathbb{Z}/n\mathbb{Z})^{2g} \rtimes \mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z}) \to A(\mathcal{H}_g(n))$$

defined by sending (e,σ) to $(t,u)\mapsto(t[e,u],\sigma(u))$. The group $A(\mathcal{H}_g(n))$ acts simply transitively on the set of theta structures of level n with fixed even symmetric theta divisor. However, if n is odd, the subgroup of $A(\mathcal{H}_g(n))$ preserving the set of symmetric structures consists of elements $(0,\sigma)$, hence isomorphic to $\mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z})$. This shows that a symmetric theta structure (with fixed Θ) is determined uniquely by the level n structure. If n is even there are 2^{2g} symmetric theta structures over a fixed level structure.

Since the Schrödinger representation of $\mathcal{H}_g(n)$ is known to be irreducible, by Schur's Lemma, the group $A(\mathcal{H}_g(n))$ has a projective representation in $V_n(g)$. Under this representation, the normal subgroup $(\mathbb{Z}/n\mathbb{Z})^{2g} \cong \mathcal{H}_g(n)/\mathbb{K}^*$ acts via the projectivized Schrödinger representation. We will identify $\mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z})$ with the subgroup of $A(\mathcal{H}_g(n))$ equal to the centralizer of D_{-1}. The Burhardt and the Maschke subspaces are invariant with respect to the action of the group $\mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z})$ in $\mathbb{P}(V_n(g))$ with the kernel equal to $\langle D_{-1}\rangle$ and hence define two projective representations of $\mathrm{PSp}(2g,\mathbb{Z}/n\mathbb{Z})$ of dimensions $(n^g-1)/2$ and $(n^g-3)/2$.

2.4. **The theta map.** We assume that char $\mathbb{K}\neq 2$ and n is invertible in $\mathbb{K}$. Let $\mathcal{A}_g(n)^{\pm}$ be the moduli space of principally polarized abelian varieties of dimension g with a symmetric even (resp. odd) theta structure of odd level $n\geq 2$. It is known to exist and it is a fine moduli space, so it admits a universal family $\mathcal{X}_g(n)^+$. There is a canonical forgetful morphism

$$f_{\pm}:\mathcal{A}_g(n)^{\pm}\to\mathcal{A}_g(n) \tag{2}$$

to the moduli space of principally polarized abelian varieties of dimension g with level n structure. The fibres are bijective to the set of even (resp. odd) theta divisors, hence the degree of the forgetful map is equal to $2^{g-1}(2^g\pm 1)$.

A theta structure defines a basis in $V_n(g)=\Gamma(A,\mathcal{L}^n)$ which is independent of A. This defines a $(\mathbb{Z}/n\mathbb{Z}\times\mu_n)^g\rtimes\mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z})$-equivariant morphism

$$\widetilde{\mathrm{Th}}^{\pm}:\mathcal{X}_g(n)^{\pm}\to\mathbb{P}^{n^g-1}=\mathbb{P}(V_n(g)^*), \tag{3}$$

where the group $(\mathbb{Z}/n\mathbb{Z}\times\mu_n)^g$ acts by translations on the image of each A. These maps are called the even and the odd *extended theta maps*. By composing these map with the zero section $\mathcal{A}_g(n)^{\pm}\to\mathcal{X}_g(n)^{\pm}$ we get a $\mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z})$-equivariant morphisms, the *even theta map* and the *odd theta map*

$$\mathrm{Th}^+:\mathcal{A}_g(n)^+\to\mathbb{P}_{Bu},\quad \mathrm{Th}^-:\mathcal{A}_g(n)^-\to\mathbb{P}_{Ma}. \tag{4}$$

Here we use that the value at the origin of any section from the subspace $V_n(g)^-$ is equal to zero.

Recall that over $\mathbb{C}$ the coarse moduli space $\mathcal{A}_g$ of principally polarized abelian varieties is isomorphic to the orbit space $\mathcal{Z}_g/\Gamma_g$, where $\mathcal{Z}_g$ is the Siegel moduli space of complex symmetric $g \times g$-matrices $\tau = X + iY$ such that $Y > 0$, and $\Gamma_g = \mathrm{Sp}(2g,\mathbb{Z})$ acting on $\mathcal{Z}_g$ in a well-known manner. The moduli space $\mathcal{A}_g(n)$ is isomorphic to $\mathcal{Z}_g/\Gamma_g(n)$, where $\Gamma_g(n) = \{M \in \Gamma_g : M - I_{2g} \equiv 0 \mod n\}$. The quotient group is $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$. The moduli space $\mathcal{A}_g(n,2n)$ is isomorphic to $\mathcal{Z}_g/\Gamma_g(n,2n)$, where $\Gamma_g(n,2n)$ is the subgroup of $\Gamma_g(n)$ of matrices $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_g(n)$ (where A,B,C,D are $g \times g$-submatrices) such that the vectors of diagonal elements in the matrices $A \cdot {}^tB$ and $C \cdot {}^tD$ are divisible by $2n$. When n is even, $\Gamma_g(n,2n)$ is a normal subgroup of $\Gamma_g(n)$ with quotient isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{2g}$. If n is odd, it is a non-normal subgroup.

Assume n is odd. It is known that the index of $\Gamma_g(n)$ in Γ_g is equal to $2^{n^2}\prod_{i=1}^{g}(2^{2i}-1)$, the order of the finite symplectic group $\mathrm{Sp}(2g,\mathbb{Z}/n\mathbb{Z})$. We have a canonical exact sequence

$$1 \to \Gamma_g(2n) \to \Gamma_g(n) \to \mathrm{Sp}(2g,\mathbb{F}_2) \to 1$$

defined by the natural inclusion of the groups $\Gamma_g(2n) \subset \Gamma_g(n)$. Comparing the indices with the order of $\mathrm{Sp}(2g,\mathbb{F}_2)$, we see that the last map is surjective. It is well-known that the group $\mathrm{Sp}(2g,\mathbb{F}_2)$ contains the even and the odd orthogonal subgroups $\mathrm{O}(2g,\mathbb{F}_2)^\pm$ of indices $2^{g-1}(2^g \pm 1)$. Let $\Gamma_g(n)^\pm$ be the pre-image in $\Gamma_g(n)$ of the subgroup $\mathrm{O}(2g,\mathbb{F}_2)^\pm$. Then

$$\mathcal{Z}_g/\Gamma_g(n)^\pm \cong \mathcal{A}_g(n)^\pm.$$

A choice of a symmetric theta structure is defined by a line bundle $\mathcal{L}$ whose space of sections is generated by a Riemann theta function $\vartheta\left[\begin{smallmatrix} m \\ m' \end{smallmatrix}\right](z;\tau)$ with theta characteristic $(m,m') \in (\mathbb{Z}/n\mathbb{Z})^g \times (\mathbb{Z}/n\mathbb{Z})^g$. The even (resp. odd) structure corresponds to the case when $m \cdot m' \equiv 0 \mod 2$ (resp. $\equiv 1 \mod 2$). A basis of the space $\Gamma(\mathcal{L}^n)$ is given by the functions $\vartheta\left[\begin{smallmatrix} m+\sigma \\ m' \end{smallmatrix}\right](nz,n\tau)$, where $\sigma \in (\mathbb{Z}/n\mathbb{Z})^g$. It follows from the standard properties the Riemann theta function that

$$\vartheta\left[\begin{smallmatrix} m+\sigma \\ m' \end{smallmatrix}\right](-z;\tau) = (-1)^{m\cdot m'}\vartheta\left[\begin{smallmatrix} m-\sigma \\ m' \end{smallmatrix}\right](z;\tau). \tag{5}$$

The theta map (**??**) is defined by the theta constants $x_\sigma = \vartheta\left[\begin{smallmatrix} m+\sigma \\ m' \end{smallmatrix}\right](0,n\tau)$. They span the space of modular forms of weight 1/2 with respect to the group $\Gamma(n)^\pm$ and some character $\chi : \Gamma(n,2n) \to \mathbb{C}^*$. It follows from (**??**) that the functions $y_\sigma = x_\sigma + x_{-\sigma}$ (resp.$z_\sigma = x_\sigma - x_{-\sigma}$) are identical zero if (m,m') is odd (resp. even). This shows that the theta maps have the same target spaces as in (**??**).

Proposition 2.6. *Assume n is odd. The theta map*

$$\mathrm{Th} : \mathcal{A}_g(n)^+ \to \mathbb{P}_{Bu}$$

is an embedding for $n \equiv 0 \mod 3$.

The proof can be found in [?], p. 240 for $n \geq 4$ and in [?], p. 235 for $n = 3$.

3. Abelian surfaces

3.1. **Kummer surfaces.** Now we specialize to the case when A is a principally polarized abelian surface. It is known that A is not the product of two elliptic curves if and only if Θ is an irreducible divisor. In this case Θ is a smooth curve of genus 2 and A is isomorphic to its Jacobian variety $\mathrm{Jac}(\Theta)$. By adjunction formula $K_\Theta \cong \mathcal{O}_\Theta(\Theta)$ and the map $\phi_{2\Theta}$ restricts to the bicanonical map of Θ onto the corresponding trope of $\mathrm{Km}(A)$. Let C be a genus 2 curve and $\mathrm{Jac}^1(C)$ be its Picard scheme of degree 1. Fix a Weierstrass point w_0 to identify $\mathrm{Jac}^1(C)$ with $\mathrm{Jac}(C)$. Then one can take for Θ the translate of the divisor W of effective divisors of degree 1, naturally identified with C. Under this identification Θ contains the six 2-torsion points $w_i - w_0$, where $w_0 = w_1, w_2, \ldots, w_6$ are the six Weierstrass points on C. None of them is a singular point of Θ.

Assume $A = \mathrm{Jac}(C)$. In this case $\mathrm{Km}(A)$ is isomorphic to a quartic surface in $\mathbb{P}^3$. It has 16 nodes as singularities and its tropes are conics passing through 6 nodes. The surface $\mathcal{K}(A)$ is a K3 surface with 16 disjoint smooth rational curves $E_e, e \in A[2](\mathbb{K})$. The proper transform of a trope T is a smooth rational curve $\overline{T}$ in the divisor class $\frac{1}{2}(H - \sum_{e\in T} E_e)$.

Assume A is the product of two elliptic curves $F \times F'$. In this case $\mathrm{Km}(A)$ is the double cover of a nonsingular quadric $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ branched over the union B of eight lines, four in each family. Their preimages on $\mathrm{Km}(A)$ is the union of two sets of four disjoint smooth rational curves. The tropes T on $\mathrm{Km}(A)$ are the unions of a curve T_1 from one set and a curve T_2 from another set. Each component of a trope has four 2-torsion points, one point is common to both components. The proper transform of a trope on $\mathcal{K}(A)$ is the disjoint union of two smooth rational curves from the divisor class $\frac{1}{2}(H - \sum_{e\in T_1+T_2} E_e - 2E_{T_1\cap T_2})$.

3.2. **Main result.** We employ the notations of Proposition **??**.

Proposition 3.1. *Assume $A = \mathrm{Jac}(C)$. Then $D = \lambda(\Theta)$ is an irreducible curve of arithmetic genus p^2+1 with p^2-1 ordinary double points. Its image in $\mathrm{Km}(B)$ is a rational curve of arithmetic genus $\frac{1}{2}(p^2-1)$ with $\frac{1}{2}(p^2-1)$ ordinary double points.*

Proof. We know that $D \in \mid p\Theta' \mid$. Thus $D^2 = 2p^2$ and the first assertion follows from the adjunction formula. Since the isogeny $A \to B$ is a local isomorphism in étale topology, the curve D has only ordinary multiple points corresponding to the intersection of an orbit of K with Θ. Let $\Gamma_a \subset A \times A$ be the graph of the translation map $t_a, a \in K$. It is algebraically equivalent to the diagonal Δ_A of $A \times A$. Let $C \times C \subset A \times A$ embedded via the Cartesian product of the Abel-Jacobi map. A point in the intersection $(C \times C) \cap \Gamma_a$

is a pair of points (x,y) on C such that $[x-y]=a$. By the intersection theory,

$$(C\times C)\cdot\Gamma_a=(C\times C)\cdot\Delta_A=\Delta_C^2=2.$$

Thus, for any nonzero $a\in K$, there exists two ordered pairs of points on C such that the difference is linearly equivalent to a. Since $x-y\sim x'-y'$ implies that $x+y'\sim x'+y\sim K_C$, the two pairs differ by the hyperelliptic involution. If we forget about the order we get p^2 unordered pairs of points in a coset of K. This shows that D has p^2-1 ordinary double points.

The last assertion follows from the Hurwitz formula since the map $D\to D/\langle[-1]_B\rangle$ ramifies at six 2-torsion points contained in D. □

3.2. Let us restrict the isogeny $\lambda:\mathrm{Jac}(C)\to B$ to Θ and compose it with the map $B\to\mathrm{Km}(B)\subset\mathbb{P}^3$ given by $\mid 2\Theta'\mid$ to obtain a map $f:\Theta\to\mathbb{P}^3$. Since $\lambda^*(\Theta')\in|p\Theta|$, the map f is given by a linear system contained in $\mid 2p\Theta\mid$ restricted to Θ. This is the linear system $|2pK_\Theta|$. Since Θ is invariant with respect to the involution $[-1]_A$, the image of f is equal to the projection of a rational norm curve R_{2p} of degree $2p$ in $\mathbb{P}^{2p}=\mathbb{P}(H^0(2pK_\Theta)^{\iota_A})$ from a subspace L of dimension $2p-4$. Let $v_1,\ldots,v_6$ be the images of the six Weierstrass point of Θ in R_{2p}. The divisor $2\lambda^*(\Theta')$ belongs to $\mid 2p\Theta\mid$ and defines a hyperplane $\mathcal{H}$ in $\mathbb{P}^{2p}$ which cuts out R_{2p} at $2p$ points containing the points $v_1,\ldots,v_6$. This is because $\lambda^*(\Theta')$ contains $\Theta\cap A[2](\mathbb{K})$ which we identified with the Weierstrass points. Our main observation is the following.

Theorem 3.3. *Let $(z_i,z_i'),i=1,\ldots,\frac{1}{2}(p^2-1)$, be the images on R_{2p} of the pairs of points on Θ belonging to the same coset of K and $\ell_i=\overline{z_i,z_i'}$ be corresponding secant lines of R_{2p}. Then the hyperplane $\mathcal{H}$ intersect the secants at $(p^2-1)/2$ points which span a linear subspace contained in $L\cong\mathbb{P}^{2p-4}$. The projection of R_{2p} from L maps the points $v_1,\ldots,v_6$ to a reduced conic Q in $\mathbb{P}^3$. If Q is irreducible, the double cover of Q branched along the points $v_1,\ldots,v_6$ is a nonsingular curve C' of genus 2 such that* $\mathrm{Jac}(C')\cong B=A/K$. *If Q is the union of lines then each component has three of the points v_i's and the double covers of each line branched along the three points and the intersection point of the line components define two elliptic curves F and F' such that $B=F\times F'$.*

Proof. Assume first that $B\cong\mathrm{Jac}(C')$ for some nonsingular curve C'. By Proposition **??**, the image of the Veronese curve R_{2p} in $\mathbb{P}^3$ is a rational curve with $(p^2-1)/2$ ordinary nodes, the images of the points z_i,z_i'. This means that each secant ℓ_i intersects the center of the projection $L\cong\mathbb{P}^{2p-4}$. Since the divisor $\lambda^*(\Theta')$ is the pre-image of a trope in $\mathbb{P}^3$, the hyperplane $\mathcal{H}$ must contain the center of the projection L. This implies that L intersects the secants ℓ_i's at the points, where $\mathcal{H}$ intersects them. The image of R_{2p} in $\mathbb{P}^3$ lies on the Kummer quartic surface $\mathrm{Km}(B)$ and intersects the trope $T=\Theta'/\langle[-1]_B\rangle$ at six nodes. The nodes are the images of the Weierstrass points $w_1,\ldots,w_6$. The conic T and the six nodes determine the isomorphism class of the curve C' such that $\mathrm{Jac}(C')\cong B$.

Next assume that B is the product of elliptic curves $F \times F'$. The argument is the same, only this time the image of R_{2p} lies on the quadric Q, the image of $\mathrm{Km}(B)$ in $\mathbb{P}^3$. The trope $T = \Theta'/\langle[-1]_B\rangle$ is mapped to the union of two lines $l_1 \cup l_2$ intersecting at a point. Each line contains the images of three nodes of $\mathrm{Km}(B)$. The image of R_{2p} intersects each line at these three points. Again this reconstructs the isomorphism classes of the elliptic curves F and F'. □

3.3. **The four secants.** The four secants $\ell_1, \ldots, \ell_4$ of the Veronese curve R_6 in $\mathbb{P}^6$ has a remarkable property: there exists a hyperplane $\mathcal{H}$ such that the points $q_i = \mathcal{H} \cap \ell_i$ are coplanar. Recall that the secants are parametrized by $R_6^{(2)} \cong \mathbb{P}^2$. This raises an interesting question. Fixing a hyperplane $\mathcal{H}$ in $\mathbb{P}^6$ intersecting R_6 transversally, what is the subvariety X of $(\mathbb{P}^2)^{(4)}$ parametrizing the quadruples of secants with the property from above. Note that the set of hyperplanes in $\mathbb{P}^6$ intersecting R_6 transversally, modulo the action of $\mathrm{SL}(2)$ by means of the projective representation in the space of binary forms of degree 6, is the moduli space of genus 2 curves. So the orbits of X modulo $\mathrm{SL}(2)$ define a special structure on a genus 2 curve. What is the dimension of X?

Recall that the secant variety $\mathrm{Sec}_1(R_n)$ is a 3-fold of degree $(n-1)(n-2)/2$ in $\mathbb{P}^n$. It contains the curve R_n embedded naturally, as a zero section of the corresponding tangent scroll. The multiplicity of the curve R_n in $\mathrm{Sec}_1(R_n)$ is equal to $n-2$. A hyperplane H intersecting R_n transversally intersects $\mathrm{Sec}_1(R_n)$ along a surface S_H isomorphic to the blow-up of the plane at the $n(n-1)/2$ intersection points p_{ij} of n tangent lines l_i of a conic. The map $S_H \to H \cong \mathbb{P}^{2p-1}$ is defined by the proper transform of the linear system $|\mathcal{O}_{\mathbb{P}^2}(n-1) - \sum p_{ij}|$ of curves of degree $n-1$ passing through the points p_{ij}. The proper transforms of the lines l_i are mapped to singular points of S_H of multiplicity $n-2$. The proper transforms of the exceptional curves of the blow-up map $S_H \to \mathbb{P}^2$ are mapped to the $n(n-1)/2$ pairs of intersection points of H with R_n. A set of N of secants of R_n not contained in H intersect H at N points $P_1, \ldots, P_N$ on S_H corresponding to a set of N points $p_1, \ldots, p_N$ on the plane different from the points p_{ij}. The codimension of the linear span of the points P_i in H is equal to the dimension of the linear system $V = |\mathcal{O}_{\mathbb{P}^2}(n-1) - \sum p_{ij} - \sum p_i|$. Its expected codimension in H is equal to $\frac{1}{2}(n+1)n - \frac{1}{2}n(n-1) - N - 1 = n - N - 1$.

Let us specialize to our case when $n = 6$ and we are dealing with a set of 4 secants such that the corresponding points in S_H span a plane. A condition that a curve of degree 5 passes through 19 points is a system of 19 linear equation with 21 variables. When we fix the 15 points p_{ij}, the condition on the remaining four points $p_1, \ldots, p_4$ in order the set of solutions is of projective dimension 2 is expressed by vanishing of maximal minors of the matrix of the system of the linear equations. They are multi-homogeneous polynomials of multi-degree $(5, 5, 5, 5)$ in the coordinates of the four points.

It follows from the theory of determinantal varieties that the variety of $a\times b$-matrices of rank $\le k$ is of codimension $(a-k)(b-k)$ in the affine space of all matrices. In our situation $a=21, b=19$ and $k=18$. Thus the expected dimension of the subvariety of $(\mathbb{P}^2)^{(4)}$ of point sets $\{p_1,\dots,p_4\}$ as above is equal to 3.

Our final remark is that the Kummer surface $\mathrm{Km}(B)$ is birationally isomorphic to the double cover of $S_{\mathcal{H}}$ unramified outside of singular points.

4. The case $p=3$ and $\mathbb{K}=\mathbb{C}$

4.1. **The Burkhardt quartic and the Coble cubic.** We specialize the discussion from subsection **??** to the case $g=2$ and $n=3$. In this case we have the theta maps

$$\begin{aligned}\mathrm{Th}^+ :&\mathcal{A}(3)^+ \to \mathbb{P}_{Bu} \cong \mathbb{P}^4\\ \mathrm{Th}^- :&\mathcal{A}(3)^- \to \mathbb{P}_{Ma} \cong \mathbb{P}^3.\end{aligned}$$

According to Proposition **??**, the first map is an embedding. The second map is an embedding of the open subset of jacobians [**?**].

It is also known that the restriction of the extended theta maps

$$\widetilde{\mathrm{Th}}^{\pm} : \mathcal{X}(3)^{\pm} \to \mathbb{P}(V_3(2)) \cong \mathbb{P}^8$$

restricted to any fibre (A,Θ,θ) defines a closed embedding

$$\phi_{\pm} : A \hookrightarrow \mathbb{P}^8 = \mid 3\Theta \mid^*$$

where $\mathcal{L}=\mathcal{O}_A(\Theta)$. This embedding is $\mathcal{H}_2(3)$-equivariant, where $\mathcal{H}_2(3)$ acts on A via an isomorphism $\bar{\alpha} : A[3](\mathbb{K}) \to \mathbb{F}_3^4$ compatible with the symplectic structures and acts in $\mathbb{P}^8$ by means of the projectivized Schrödinger representation.

We have the following theorem due to A. Coble (for a modern exposition see, for example, [**?**], 5.3.1).

Theorem 4.1. *Assume* $\mathrm{char}\mathbb{K} \ne 2,3$. *Choose the new coordinates in* $\mathbb{P}^8$ *as follows.*

$$y_0=\eta_{00},\ 2y_1=\eta_{01}+\eta_{02},\ 2y_2=\eta_{10}+\eta_{20},\ 2y_3=\eta_{11}+\eta_{22},\ 2y_4=\eta_{12}+\eta_{21}.$$
$$2z_1=\eta_{01}-\eta_{02},\ 2z_2=\eta_{10}-\eta_{20},\ 2z_3=\eta_{11}-\eta_{22},\ 2z_4=\eta_{12}-\eta_{21}.$$

Then the image $\phi(A)$ *is defined by the equations*

$$\begin{pmatrix} y_0^2 & 2(y_1^2-z_1^2) & 2(y_2^2-z_2^2) & 2(y_3^2-z_3^2) & 2(y_4^2-z_4^2)\\ y_1^2+z_1^2 & 2y_0y_1 & 2(y_3y_4-z_3z_4) & 2(y_2y_4-z_2z_4) & 2(y_2y_3-z_2z_4)\\ y_2^2+z_2^2 & 2(y_3y_4-z_3z_4) & 2y_0y_2 & 2(y_1y_4+z_1z_4) & 2(y_1y_3-z_1z_3)\\ y_3^2+z_3^2 & 2(y_2y_4+z_2z_4) & 2(y_1y_4-z_1z_4) & 2y_0y_1 & 2(y_1y_2+z_1z_2)\\ y_4^2+z_4^2 & 2(y_2y_3+z_2z_3) & 2(y_1y_3+z_1z_3) & 2(y_1y_2-z_1z_2) & 2y_0y_4 \end{pmatrix}\cdot\begin{pmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{pmatrix}=0, \tag{6}$$

$$\begin{aligned} z_1\pi_{01}+z_2\pi_{43}+z_3\pi_{24}+z_4\pi_{32} &= 0,\\ z_1\pi_{43}+z_2\pi_{02}+z_3\pi_{14}+z_4\pi_{13} &= 0,\\ z_1\pi_{24}+z_2\pi_{14}+z_3\pi_{03}+z_4\pi_{12} &= 0,\\ z_1\pi_{32}+z_2\pi_{13}+z_3\pi_{12}+z_4\pi_{04} &= 0, \end{aligned} \tag{7}$$

where $\pi_{ij} = \alpha_i y_j - \alpha_j y_i$ and the the vector of the parameters $(\alpha_0, \ldots, \alpha_4)$ is a point α on the Burkhardt quartic

$$\mathcal{B}_4 : T_0^4 + 8T_0(T_1^3 + T_2^3 + T_3^3 + T_4^3) + 48T_1T_2T_3T_4 = 0. \tag{8}$$

The vector α depends only the choice of a 3-level structure on A and its coordinates can be identified with explicit modular forms of weight 2 with respect to $\Gamma_2(3)$ [**?**], p. 253. As we will review bellow, the coordinates T_i may be naturally identified with the coordinates y_i in $\mathbb{P}_{Bu}$. One easily notice that the 9 quadratic forms are the partials of a unique cubic form (surprisingly it was missed by Coble). It defines a cubic hypersurface $\mathcal{C}_3$ in $\mathbb{P}^8$, the *Coble cubic*. Thus, the previous theorem expresses the fact that $\phi(A)$ is the singular locus of the Coble hypersurface $\mathcal{C}_3$.

The negation involution $[-1]_A$ acts on $\phi_+(A)$ via the projective transformation $\eta_\sigma \mapsto \eta_{-\sigma}$ which gives in the new coordinates $y_i \mapsto y_i,\ z_j \mapsto -z_j$. Its fixed locus in $\mathbb{P}^8$ is the union of two subspaces

$$\mathbb{P}_{Ma} = \{y_0 = \ldots = y_4 = 0\},\ \mathbb{P}_{Bu} = \{z_1 = \ldots = z_4 = 0\}.$$

Intersecting $\phi_-(A)$ with $\mathbb{P}_{Ma}$ we find 6 points in $A[2]$ lying on Θ. One of them is the origin of A. The remaining 10 points in $A[2]$ is the intersection of $\phi_+(A)$ with $\mathbb{P}_{Bu}$. Let us compute this intersection: Plugging $y_i = 0$ in the equations in Theorem **??**, we obtain that the parameters $(\alpha_0, \ldots, \alpha_4)$ satisfy the equations

$$\begin{pmatrix} 0 & -2z_1^2 & -2z_2^2 & -2z_3^2 & -2z_4^2 \\ z_1^2 & 0 & -2z_3z_4 & -2z_2z_4 & -2z_2z_4 \\ z_2^2 & -2z_3z_4 & 0 & 2z_1z_4 & -2z_1z_3 \\ z_3^2 & 2z_2z_4 & -2z_1z_4 & 0 & 2z_1z_2 \\ z_4^2 & 2z_2z_3 & 2z_1z_3 & -2z_1z_2 & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = 0, \tag{9}$$

As is well-known the coordinates of a non-trivial solution of a skew-symmetric matrix of corank 1 can be taken to be the pffafians of the principal matrices. This gives a map

$$c_- : \mathbb{P}_{Ma} \to \mathcal{B}_4, \tag{10}$$

$$\begin{aligned} \alpha_0 &= 6z_1z_2z_3z_4 \\ \alpha_1 &= z_1(z_2^3 + z_3^3 - z_4^3) \\ \alpha_2 &= -z_2(z_1^3 + z_3^3 + z_4^3) \\ \alpha_3 &= z_3(-z_1^3 - z_2^3 + z_4^3) \\ \alpha_4 &= z_4(z_1^3 + z_2^3 - z_3^3) \end{aligned} \tag{11}$$

We now go back to compute the intersection $\phi_-(A)\cap\mathbb{P}_{Bu}$: Plugging $z_i=0$ in (**??**) we obtain that α satisfies the equations

$$\begin{pmatrix} y_0^2 & 2y_1^2 & 2y_2^2 & 2y_3^2 & 2y_4^2\\ y_1^2 & 2y_0y_1 & 2y_3y_4 & 2y_2y_4 & 2y_2y_3\\ y_2^2 & 2y_3y_4 & 2y_0y_2 & 2y_1y_4 & 2y_1y_3\\ y_3^2 & 2y_2y_4 & 2y_1y_4 & 2y_0y_1 & 2y_1y_2\\ y_4^2 & 2y_2y_3 & 2y_1y_3 & 2y_1y_2 & 2y_0y_4 \end{pmatrix}\cdot\begin{pmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{pmatrix}=0, \tag{12}$$

Recall that the Hessian hypersurface $\mathrm{Hess}(V(F))\subset\mathbb{K}^m$ of a hypersurface $V(F)$ is defined by the determinant of the matrix of the second partials of F. It parametrizes the locus of points x such that the polar quadric $P_{x^{m-2}}(V(F))$ of $V(F)$ is singular. The locus of singular points of such quadrics is parametrized by the Steinerian hypersurface $\mathrm{St}(V(F))$. It coincides with the locus of points x such that the first polar $P_x(V(F))$ is singular. One immediately recognize that if we make the identification $y_i=T_i$, then the matrix of the coefficients in (**??**), after multiplying the last four rows by 2, coincides with the matrix of the second partials of a polynomial defining the Burkhardt quartic (**??**). Thus α is a point on the Steinerian hypersurface of the Burkhardt quartic. On the other hand we know that it lies on the Burkhardt quartic. This makes $\mathcal{B}_4$ a very exceptional hypersurface: it coincides with its own Steineirian. This fact was first discovered by A. Coble [**?**].

The first polar of $\mathcal{B}_4$ at a nonsingular point is a cubic hypersurface with 10 nodes at nonsingular points of the Hessian surface. Any such cubic hypersurface is projectively isomorphic to the *Segre cubic primal* $\mathcal{S}_3$ given by the equations in $\mathbb{P}^5$ exhibiting the S_6-symmetry:

$$T_0^3+\cdots+T_5=T_0+\cdots+T_5=0. \tag{13}$$

The map from the nonsingular locus of $\mathrm{Hess}(\mathcal{B}_4)$ to $\mathcal{B}_4=\mathrm{St}(\mathcal{B}_4)$ which assigns to a point x the singular point α of the polar quadric $P_{x^2}(\mathcal{B}_4)$ is of degree 10. Its fibres are the sets of singular points of the first polars. We will give its moduli-theoretical interpretation in the next section.

Let

$$c_+:\mathrm{Hess}(\mathcal{B}_4)^{nsg}\to\mathcal{B}_4,\ (y_0,\dots,y_4)\mapsto\alpha \tag{14}$$

be the map given by the cofactors of any column of the matrix of coefficients in (**??**).

Theorem 4.2. *The image of the map* Th^+ *is equal to* $\mathrm{Hess}(\mathcal{B}_4)^{nsg}$ *and the composition of this map with the map* c_+ *is the forgetful map* $\mathcal{A}_2(3)^+\overset{10:1}{\to}\mathcal{A}_2(3)$. *The composition of the map* Th^- *with* c_- *is the forgetful map* $\mathcal{A}_2(3)^-\overset{6:1}{\to}\mathcal{A}_2(3)$.

The first assertion is proved in [**?**] (see also [**?**]). The second assertion is proved in [**?**] (see also [**?**]).

4.2. **The 3-canonical map of a genus 2 curve.** Let (A,Θ,θ) be a member of the universal family $\mathcal{X}_2(3)^-$. We assume that the divisor Θ is irreducible, i.e. $A\cong\mathrm{Jac}(C)$ for some smooth genus 2 curve $C\cong\Theta$. By the adjunction formula, the restriction of the map $\phi_-:A\to\mathbb{P}^8$ to Θ is the 3-canonical map

$$\phi_{3K_C}:C\to|3K_C|^*\subset\mathbb{P}^8.$$

Here the identification of $|3K_C|^*$ with the subspace of $\mathbb{P}^8 = |3\Theta|^*$ is by means of the canonical exact sequence

$$0 \to \mathcal{O}_A(2\Theta) \to \mathcal{O}_A(3\Theta) \to \mathcal{O}_\Theta(3\Theta) \to 0. \tag{15}$$

Denote the subspace $|3K_C|^* \cong \mathbb{P}^4$ by $\mathbb{P}^4_\Theta$. The hyperelliptic involution ι_C acts naturally on $\mathbb{P}^4_\Theta$ and its fixed locus set consists of the union of a hyperplane H_0 and an isolated point x_0. The dual of H_0 is the divisor $W = w_1 + \cdots + w_6$, where w_i are the Weierstrass points. It coincides with $\phi_-(A) \cap \mathbb{P}_{Ma}$ and hence

$$H_0 = \mathbb{P}_{Ma}.$$

The dual of x_0 is the hyperplane spanned by the image of the Veronese map $|K_C| \to |3K_C|$. The projection map $C \to H_0$ from the point x_0 is the degree 2 map onto a rational normal curve R_3 of degree 3 in H_0. It is ramified at the Weierstrass points.

Since Θ is an odd divisor, the image of $\Gamma(\mathcal{O}_A(2\Theta)) \cong \mathbb{C}^4$ in $\Gamma(\mathcal{O}_A(3\Theta))$ is contained in $V_3(2)^- \cong \mathbb{C}^5$. Thus the image of $V_3(2)^-$ in $\Gamma(\mathcal{O}_\Theta(3\Theta))$ is the one-dimensional subspace corresponding to the point x_0. The projectivization of the image of $V_3(2)^+$ in $\Gamma(\mathcal{O}_\Theta(3\Theta))$ the subspace H_0.

Observe that

$$\{x_0\} = \mathbb{P}^4_\Theta \cap \mathbb{P}_{Bu}.$$

It is known that the subspace $\mathbb{P}^4_\Theta$ is contained in the Coble cubic $\mathcal{C}_3$ and $\mathbb{P}_{Bu} \cap \mathcal{C}_3$ is equal to the polar cubic $P_\alpha(\mathcal{B}_4)$ [**?**], Proposition 4.3 and section 5.3. A natural guess is that $x_0 = \alpha$. This turns out to be true.

Lemma 4.3. *Let $\alpha = c_-(W) \in \mathcal{B}_4$. Then considering $\mathcal{B}_4$ as a subspace of $\mathbb{P}_{Bu}$ we have*

$$x_0 = \alpha.$$

Proof. For simplicity of the notation let us denote $\phi_-(A)$ by A. Let $I_A(2)$ be the space of $Q \in S^2V_3(2)$ such that $V(Q)$ contains A. As we know it is spanned by the partial derivatives of the Coble cubic $V(F_3)$. Let $I_\Theta(2)$ be the space of quadrics in $\mathbb{P}^4_\Theta$ vanishing on Θ. The polar map $v \mapsto P_v(F_3)$ defines a $\mathcal{H}_3(2) \rtimes \langle D_{-1}\rangle$-equivariant isomorphism $V_3(2) \to I_A(2)$.

Consider the restriction map

$$r : I_A(2) \to I_\Theta(2).$$

By [**?**], Proposition 4.7, the map is surjective. By Riemann-Roch, its kernel L is of dimension 5. We know that $I_A(2) = I_A(2)^+ \oplus I_A(2)^- = \mathbb{C}^5 \oplus \mathbb{C}^4$ with the obvious notation. The subspace $I_A(2)^+$ is spanned by the four quadrics from (**??**). Obviously they vanish on $\mathbb{P}_{Ma} \subset \mathbb{P}^4_\Theta$. Since they also contain a non-degenerate curve Θ they vanish on the whole space $\mathbb{P}^4_\Theta$. Thus $L = I_A(2)^+ \oplus L^-$, where $L^- = L \cap I_A(2)^-$ is of dimension 1. In other words, there exists a unique point $x \in \mathbb{P}_{Bu}$ such that the polar quadric $P_x(\mathcal{C}_3)$ vanishes on $\mathbb{P}^4_\Theta$. It remains to prove that x_0 and α both play the role of the x.

Recall the important property of the polar

$$P_x(\mathcal{C}_3) \cap \mathcal{C}_3 = \{c \in \mathcal{C}_3 : x \in \mathbb{T}_c(\mathcal{C}_3)\},$$

where $\mathbb{T}_c(\mathcal{C}_3)$ denotes the embedded Zariski tangent space. Since $\mathbb{P}^4_\Theta$ is contained in $\mathcal{C}_3$, for any $c \in \mathbb{P}^4_\Theta$ we have $\mathbb{P}^4_\Theta \subset \mathbb{T}_c(\mathcal{C}_3)$. But x_0 belongs to $\mathbb{P}^4_\Theta$, therefore $c \in P_{x_0}(\mathcal{C}_3)$. This proves that $\mathbb{P}^4_\Theta \subset P_{x_0}(\mathcal{C}_3)$.

Now consider the polar quadric $P_\alpha(\mathcal{C}_3)$. It is defined by the quadratic form

$$(\alpha_0,\ldots,\alpha_4)\cdot M(y,z)\cdot \begin{pmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{pmatrix}=0,$$

where $M(y,z)$ is the matrix from (**??**). Restricting the quadric to the subspace $\mathbb{P}_{Ma}$ we see that it is equal to ${}^t\alpha\cdot M(0,z)\cdot\alpha$, where $M(0,z)$ is the skew-symmetric matrix from (**??**). Therefore, it is identically zero on the Maschke subspace, and, and as above, since it also contains Θ, it must contain the whole $\mathbb{P}^4_\Theta$. □

4.3. **Moduli space of genus 2 curves.** It is well-known that

$$\mathcal{M}_2\cong\mathbb{C}^3/\mu_5, \tag{16}$$

where a generator ϵ of μ_5 acts via the diagonal matrix $(\epsilon,\epsilon^2,\epsilon^3)$. Let us show how to arrive to this construction in four different ways.

Clebsch-Gordan:

Let $V(6)$ be the vector space of binary sextics

$$f_6=\sum_{i=0}^{6}a_i\tbinom{6}{i}t_1^{6-i}t_0^i$$

and $\Delta_{10}(a_0,\ldots,a_6)$ be the discriminant polynomial of degree 10. We use the fact

$$\mathcal{M}_2\cong(\mathbb{P}(V(6))\setminus\Delta=0)//\mathrm{SL}(2).$$

According to A. Clebsch and P. Gordan, the algebra of invariant polynomials is generated by polynomials $I_2,I_4,I_6,I_{10},I_{15}$ in the coefficients a_i's. The square of I_{15} is a polynomial in the remaining invariants. The discriminant Δ is a homogeneous polynomial in the first 4 invariants, of degree 1 in I_{10}. This implies that the GIT-quotient, isomorphic to the projective spectrum of the subalgebra of invariant polynomials of even degree, is isomorphic to $\mathbb{P}(2,4,6,10)$ and the open subset $\Delta\neq 0$ is isomorphic to the quotient of $\mathbb{C}^3$ as in (**??**).

Coble:

Let $r_1,\ldots,r_6\in\mathbb{P}^1$ be an ordered set of distinct roots of f_6. Consider the Veronese map $\nu_3:\mathbb{P}^1\to\mathbb{P}^3$ such that the images of the first 5 roots are the reference points in $\mathbb{P}^3$. The image of the sixth root lies on a unique rational norm cubic curve through the reference points. Consider a rational map $\mathbb{P}^3-\to\mathbb{P}^4$ given by the linear system of quadrics through the reference points. Its image is a cubic hypersurface isomorphic to the Segre cubic $\mathcal{S}_3$. Its ten nodes are the images of the lines joining a pair of the reference points. Coble shows (see later) that one can choose 6 quadrics $A,B,\ldots,F$ such that $A+\cdots+F=0$ and $A^3+\cdots+F^3=0$ so that the image is given by (**??**). He also shows that permuting the roots is equivalent to the natural action of Σ_6 on $\mathcal{S}_3$. The open subset of the images of the sixth root r_6 in $\mathbb{P}^3$ is mapped isomorphically onto the open subset of the Segre cubic defined by $\sigma_5\neq 0$, where σ_i are elementary symmetric polynomials in $Z_1,\ldots,Z_6$. Applying the Fundamental Theorem on symmetric polynomials we obtain that

$$\mathcal{M}_2\cong(\mathcal{S}_3\setminus V(\sigma_5))/\Sigma_6\cong\mathbb{P}(2,4,5,6)\setminus V(t_2)\cong\mathbb{C}^3/\mu_5$$

as in (**??**), where t_2 is the second symmetric polynomial.

Burkhardt:

Let G be a simple group of order 25,920 isomorphic to $\mathrm{PSp}(4,\mathbb{F}_3)$. The group $(\mathbb{Z}/2\mathbb{Z})\times G \ncong \mathrm{Sp}(4,\mathbb{F}_3)$ is isomorphic to a complex reflection group in $\mathbb{C}^5$ (Number 33 in Shepherd-Todd's list). It acts projectively in $\mathbb{P}^4$, the action is defined by one of its two 5-dimensional irreducible representation. The algebra of invariants of the complex reflection group $(\mathbb{Z}/2\mathbb{Z})\times G$ was computed by Burkhardt [?]. It is freely generated by 5 polynomials of degrees $J_4, J_6, J_{10}, J_{12}, J_{18}$. The invariant J_4 of degree 4 in $\mathbb{P}^4$ defines the equation of the Burkhardt quartic (??) $\mathcal{B}_4$. This implies that

$$\mathcal{B}_4/G \cong \mathbb{P}(6,10,12,18) \cong \mathbb{P}(3,5,4,6)$$

and the quotient of the open subset $J_{10}\neq 0$ is isomorphic to $\mathbb{C}^3/\mu_5$ as in (??). The moduli interpretation is as follows.

The image of the map $\mathbb{P}_{Ma}\to\mathbb{P}_{Bu}$ is equal to the Burhardt quartic $\mathcal{B}_4 = V(J_4)$. The image of the theta map $\mathrm{Th} : \mathcal{A}_2(3,6)\to\mathbb{P}_{Bu}$ is equal to the nonsingular locus of the hypersurface $V(J_6)$ identified with the Hessian hypersurface $\mathrm{Hess}(\mathcal{B}_4)$ of the Burkhardt quartic. This follows from [?] and Proposition ??. The whole Hessian hypersurface is isomorphic to the Satake compactification of $\mathcal{A}_2(3,6)$ [?], p. 237. The degree 10 map $\mathrm{Hess}(\mathcal{B}_4)^{nsg}\to\mathcal{B}_4$ defined in the previous section can be identified with the forgetful map $\mathcal{A}_2(3,6)\to\mathcal{A}_2(3)$. This shows that Burhardt quartic is isomorphic to a compactification of the moduli space $\mathcal{A}_2(3)$ [?].

Let $\alpha\in\mathcal{B}_4$. We know that the first polar $P_\alpha(\mathcal{B}_4)$ is a cubic hypersurface S_α with 10 nodes lying on $\mathrm{Hess}(\mathcal{B}_4)$. The second polar $P^2_\alpha(\mathcal{B}_4)$ is a quadric Q_α and the third polar is the tangent hyperplane $\mathbb{T}_\alpha$ of $\mathcal{B}_4$ at the point α. The intersection of $S_\alpha\cap Q_\alpha\cap\mathbb{T}_\alpha$ consists of 6 lines passing through α. They are projected to 6 points in a $\mathbb{P}^3$ which define a binary sextic determining the isomorphism class of a genus 2 curve C (see [?], [?]).

Borchardt:

Let $G = 2^5\rtimes\Sigma_6$ act on $\mathbb{P}^5$ by permuting and changing the signs of the coordinates. Obviously it leaves the quadric $\sum_{i=1}^6 x_i^2 = 0$ invariant. As is well known the projective special orthogonal group $\mathrm{PSO}(6)$ is isomorphic to $\mathrm{PGL}(4)$ (view the quadric as the Grassmannian of lines in $\mathbb{P}^3$). Thus G acts in $\mathbb{P}^3$. The action comes via a complex reflection group of order $2^6\cdot 6!$ in $\mathbb{P}^4$ (Number 31 in Shepherd-Todd list). Following the classical terminology we call it the *Borchardt group*. Its algebra of invariants is freely generated by polynomials $\Phi_8,\Phi_{12},\Phi_{20},\Phi_{24}$ [?]. We have

$$\mathbb{P}^3/G \cong \mathbb{P}(8,12,20,24)\cong\mathbb{P}(2,3,6,5).$$

The image of the open subset $\Phi_{20}\neq 0$ is isomorphic to $\mathbb{C}^3/\mu_5$ as in (??).

The moduli interpretations is as follows. The Heisenberg subgroup $\mathcal{H}_2(2)$ of G acts on $\mathbb{P}^3$ with the quotient isomorphic to the Segre quartic primal $\mathcal{S}_4$ in $\mathbb{P}^4$, the dual hypersurface of the Segre cubic $\mathcal{S}_3$ [?], p. 505. The quartic hypersurface is isomorphic to the Bailey-Borel compactification of the moduli space $\mathcal{A}_2(2)$ [?] (see a modern account in [?], [?]). The further action by Σ_6 provides a compactification of $\mathcal{A}_2$. We are not going to use this so omit the details.

Remark 4.4. Let G be as in the "Burkhardt" case. The group G admits a projective linear representation in $\mathbb{P}^3$ arising from a 4-dimensional irreducible representation of its central extension of degree 2 isomorphic to $\mathrm{Sp}(4,\mathbb{F}_3)$. The group $\mathbb{Z}/3\mathbb{Z}\times\mathrm{Sp}(4,\mathbb{F}_3)$ is isomorphic to complex reflection group in $\mathbb{C}^4$ (Number 32 in Shepherd-Todd's list). Its algebra of invariant polynomials was computed by

Maschke [**?**]. It is freely generated by polynomials $F_{12}, F_{18}, F_{24}, F_{30}$ of degrees indicated by the subscripts. This shows that

$$\mathbb{P}^3/G \cong \mathbb{P}(12, 18, 24, 30) \cong \mathbb{P}(2, 3, 4, 5).$$

However, it looks like a coincidence and does not have a moduli interpretation. We can identify the $\mathbb{P}^3$ with the Maschke space $\mathbb{P}_{Ma}$ which is birationally $\mathcal{A}_2(3)^-$. After dividing by G, we obtain the moduli space $\mathcal{A}_2^-$ of principally polarized abelian surfaces together with an odd theta characteristic. More precisely, one can show that the complement of the 40 reflection hyperplanes in $\mathbb{P}_{Ma}$ is the moduli space of genus 2 curves together with a symmetric odd theta theta structure on its Jacobian. The product of linear equations defining these planes is a polynomial Φ_{40} of degree 40 whose cube is an invariant of G. An explicit formula for its expression in terms of the fundamental invariants F_m can be found in [**?**] or in [**?**], p. 153. This shows that the moduli space of genus 2 curves together with a choice of a Weierstrass point is isomorphic to $\mathbb{P}(2, 3, 4, 5) \setminus V(P_{20})$ for some explicit weighted homogeneous polynomial of degree 20.

A genus 2 curve together with a choice of a Weierstrass point can be represented by the equation $y^2 + f_5(x) = 0$ for some polynomial of degree 5 without multiple roots. The above discussion suggests that the quotient of the open subset of the projectivized space of binary forms of degree 5 without multiple roots by the affine group $\mathbb{C} \rtimes \mathbb{C}^*$ must be isomorphic to $\mathbb{P}(2, 3, 4, 5) \setminus V(P_{20})$. It would be interesting to find a direct proof of this fact.

4.4. **Absolute invariants.** Let $[j_1, j_2, j_3] \in \mathbb{C}^3$ be the μ_5-orbit of $(j_1, j_2, j_3) \in \mathbb{C}^3$ corresponding to the isomorphism class of a genus 2 under (**??**). We call it the Clebsch-Gordan *absolute invariants* of a genus 2 curve. Similarly, we define the Coble, the Burkhardt and the Borchardt absolute invariants. We would like to find a dictionary between these invariants representing the same projective class of a binary sextic.

From Clebsch-Gordan to Coble: We start with the Clebsch-Gordan invariants. Let $I_2, \ldots$ be the Clebsch-Gordan invariant polynomials in the coefficients a_i of a binary sextic F_6. Then

$$[j_1, j_2, j_3]^{CG} = (I_2(a)^5/\Delta(a), I_4(a)^5/\Delta(a)^2, I_6(a)^5/\Delta(a)^3).$$

Let $e_1, \ldots, e_6$ be an ordered set of roots of f_6. We use a copy of $\mathbb{P}^3$ represented as a hyperplane $y_1 + \cdots + y_4 = 0$ in $\mathbb{P}^4$ and map the roots to $\mathbb{P}^3$ via the Veronese map such that the first 5 roots are mapped to the points $p_1 = (-4, 1, 1, 1, 1), \ldots, p_5 = (1, \ldots, 1, -4)$. Recall that Σ_6 contains two conjugacy classes of elements of order 2, one is represented by a transposition (ij) and another one by the product of three commuting transpositions $s = (ij)(kl)(mn)$, where we always assume that $i < j$, and so on. For each $K = (ij)(kl)(m6)$ consider the quadratic form $Q_K = (y_i - y_j)(y_k - y_l)$. One can group together 5 involutions s as above such that no two contain the same transposition (such a group is called a *total*). There are 6 totals $A, \ldots, F$ which are transformed under Σ to define a unique exterior automorphism of Σ_6. For any total, let us sum up the quadrics Q_K where K belongs to the total. This gives a set of 6 quadrics $Q_1, \ldots, Q_6$ vanishing on the points $p_1, \ldots, p_5$. It is checked that they satisfy $\sum Q_i = \sum Q_i^3 = 0$. The rational map

$$\mathbb{P}^3 - \to BP^5,\ (y_1, \ldots, y_4) \mapsto (Q_1(y), \ldots, Q_6(y))$$

has the image equal to the Segre cubic $\mathcal{S}_3$ (**??**). Coble shows that the elementary symmetric functions of the $Q_i(y)'$s become polynomials in the differences of roots $e_i - e_j$ which can be expressed in terms of the coefficients $a = (a_0, \ldots, a_6)$ of the binary form a follows

$$
\begin{aligned}
q_2 &= \sigma_2(Q_1(y), \ldots, Q_6(y)) = \frac{1}{9} I_2(a), \\
q_4 &= \sigma_4(Q_1(y), \ldots, Q_6(y)) = \frac{1}{9} I_2(a)^2 - \frac{25}{162} I_4(a), \\
q_5 &= \sigma_4(Q_1(y), \ldots, Q_6(y)) = \frac{1}{2 \cdot 3^6} \sqrt{\Delta(a)} \\
q_6 &= \sigma_6(Q_1(y), \ldots, Q_6(y)) = \frac{-5}{3^3} I_2(a)^3 + \frac{5^3}{2 \cdot 3^3} I_2(a) I_4(a) + \frac{5^4}{3^4} I_6(a)
\end{aligned}
$$

The Coble absolute invariants $[j_1, j_2, j_3]^C = (q_2^5/q_5^2, q_4^5/q_5^4, q_6^5/q_5^6)$ can now be expressed in terms of the Clebsch-Gordan absolute invariants (see [**?**]).

From Coble to Burkhardt: This is the most difficult task which was accomplished by Coble [**?**]. He gives the expression of Burkhardt invariants in terms of Coble invariants q_2, q_4, q_5, q_6 from above. We have (see [**?**], p. 368)

$$
[j_1, j_2, j_3]^C = [q_2^5/q_5^2, q_4^5/q_5^4, q_6^5/q_5^6] = [-\frac{1}{8} j^{Bu}, \frac{1}{2^{12}} j_2^{Bu}, 2^{32} j_3^{Bu}]. \tag{17}
$$

From Borchardt to Coble:

We will not need it, but give it nevertheless for completeness sake. It is known that the Segre-Igusa quartic $\mathcal{S}_4$ is the dual hypersurface of of the Segre cubic $\mathcal{S}_3$. Let $\mathbb{P}^3$ be the representation space for the Borchardt group $\mathcal{H}_2(2) \rtimes \Sigma_6$. Restrict the action to $\mathcal{H}_2(2)$ and consider the standard basis from section **??**. It is known that the quotient map $\mathbb{P}^3 \to \mathbb{P}^3/\mathcal{H}_2(2) \cong \mathcal{S}_4$ can be given by the quartic polynomials

$$
\begin{aligned}
y_0 &= x_{00}^4 + x_{01}^4 + x_{10}^4 + x_{11}^4 \\
y_1 &= x_{00}^2 x_{10}^2 + x_{11}^2 x_{01}^2 \\
y_2 &= x_{00}^2 x_{01}^2 + x_{11}^2 x_{10}^2 \\
y_3 &= x_{00}^2 x_{11}^2 + x_{10}^2 x_{01}^2 \\
y_4 &= x_{00} x_{10} x_{11} x_{01}
\end{aligned}
$$

The image satisfies the equation (see [**?**], p. 505, [**?**]).

$$
\Phi = \sigma_2(y_0, y_1, y_2, y_3, y_4, y_5)^2 - 4\sigma_4(y_0, y_1, y_2, y_3, y_4, y_5) = 0, \tag{18}
$$

where $y_5 = -\sum_{i=0}^4$ and σ_i are elementary symmetric polynomials The quartic threefold in $\mathbb{P}^5$ defined by these equations is isomorphic to the Segre-Igusa quartic primal $\mathcal{S}_4$. It is known that the dual of $\mathcal{S}_4$ is the Segre cubic primal $\mathcal{S}_3$. Thus the map

$$
(y_0, \ldots, y_5) \to (\frac{\partial \Phi}{\partial y_0}(y), \ldots, \frac{\partial \Phi}{\partial y_5}(y))
$$

has the image equal to the Segre cubic (**??**). This gives us explicit formulae expressing the variables Z_i as homogeneous polynomials of degree 12 in the variables $z_1, \ldots, z_4$. The elementary symmetric polynomials q_2, q_4, q_5, q_6 of the Z_i's become invariant polynomials with respect to the Borchardt group. This allows us to express $[j_1, j_2, j_3]^C$ in terms of $[j_1, j_2, j_3]^{Bo}$.

4.5. **The four secants again.** Let C be a genus 2 curve. We identify C with the subvariety W of $\mathrm{Pic}^1(C)$. A choice of a Weierstrass point identifies C with the translate of W in $A = \mathrm{Jac}(C)$ equal to an odd theta divisor Θ. Now choose a symmetric odd theta structure $(\mathrm{Jac}(C), \Theta, \theta)$. This defines a splitting of $A[3]$ into the direct sum of two isotropic subspaces $A[3] = (\mathbb{Z}/3\mathbb{Z})^2 \oplus \mu_3^2$ and, in particular it chooses a maximal isotropic subspace $K = (\mathbb{Z}/3\mathbb{Z})^2$ in $A[3]$. The space $\mathbb{P}^8$ is identified with $\mathrm{Hom}(K, \mathbb{K})$ on which the group $(\mathbb{Z}/3\mathbb{Z})^2$ acts by translation of the arguments of the functions. The 3-canonical image of C spans the subspace curve $\mathbb{P}^4_\Theta$ from above. Let e be a non-trivial element of K. We know from the proof of the main theorem that there are two pairs of points $(x, y), (x', y')$ on C such that $x - y = x' - y' = e$. Let $\mathbb{P}^4_\Theta + e$ denote the image of $\mathbb{P}^4_\Theta$ under the action of e via the Schrödinger representation. It is clear that $\mathbb{P}^4_\Theta$ intersects $\mathbb{P}^4_\theta + e$ along the plane spanned by the secant lines $\overline{x, y}$ and $\overline{x', y'}$. The hyperelliptic involution acting in $\mathbb{P}^4_\theta$ switches the two lines. In particular, they intersect at a point $p_e \in H_0 = \mathbb{P}_{Ma}$ (cf. [**?**], Lemma 5.1.4). Note that replacing e with $-e$ we get the same pair of secants. In this way we obtain 8 secant lines $\overline{x_e, y_e}, \overline{x'_e, y'_e}$, each pair corresponds to the pair $(e, -e)$ of 3-torsion points from K. To reconstruct these secants we intersect $\mathbb{P}^4_\Theta$ with its "translate" $\mathbb{P}^4_\Theta + e$ to get a plane Π_e. It intersects C at 4 points x_e, y_e, x'_e, y'_e. They define two pairs of concurrent secants.

Now we need the rational norm cubic curve R_3 in $\mathbb{P}_{Ma}$ containing the set $S = c_-^{-1}(\alpha)$ of six Weierstrass points of C. It is determined uniquely by S. Note that the set S can also be found by taking the pre-image of the tangent hyperplane of the Burhardt quartic at α under the map c_-; it is a quartic Weddle surface $\mathcal{W}_4(\alpha)$ with 6 nodes at S birationally isomorphic to the Kummer surface of $\mathrm{Jac}(C)$. It can be also found by intersecting $\mathbb{P}_{Ma}$ with the secant variety of $C \subset \mathbb{P}^4_\Theta$, the hyperplane intersects it doubly along the Weddle surface [**?**].

Now we use that the image of C under the ι_C-fixed part of $\mid 6K_C \mid$ is the rational norm curve equal to the image of $\mid K_C \mid^* = \mathbb{P}^1$ under the Veronese map. Projecting to $\mathbb{P}_{Ma}$ the four points x_e, y_e, x'_e, y'_e we get 2 points spanning a secant of a rational norm cubic R_3 in $\mathbb{P}^3_\omega$. The images under the Veronese map will give us a pair of points on R_6 spanning one of the four secants we used in the proof of the main theorem.

4.6. **An algorithm in the case $\mathbb{K} = \mathbb{C}$.** It consists of the following steps.

Input: A hyperelliptic curve C of genus 2 with absolute invariants $[j_1, j_2, j_3]^{CG} = (\lambda, \mu, \nu)$.

Step 1: Find the absolute invariants $[j_1, j_2, j_3]^{Bu} = (\lambda', \nu', \gamma')$.

Step 2: Find $\alpha = (y_0, \ldots, y_4) \in \mathbb{P}_{Bu}$ such that $[j_1, j_2, j_3]^{Bu}(\alpha) = (\lambda', \nu', \gamma')$. This chooses a maximal isotropic subspace in $\mathrm{Jac}(C)[3]$. We can always replace α with α' in the same orbit of the maximal subgroup of index 40 of the Burhardt group $\mathrm{PSp}(4, \mathbb{F}_3)$ that stabilizes a maximal isotropic subgroup. So, there are 40 different choices of α.

Step 3: Find the linear equations of the $\mathbb{P}^4_\Theta$ spanned by $\mathbb{P}_{Ma}$ and α.

Step 4: Find the intersection of $\mathbb{P}^4_\Theta$ with its translate by $e \in (\mathbb{Z}/3\mathbb{Z})^2$. This is a plane Π_e.

Step 5: Find the equations of the Coble cubic (you need only α to do this) and intersect it with $\mathbb{P}_{Ma}$ to obtain the Weddle quartic surface $\mathcal{W}_4$.

Step 6: Find the 6 double points of $\mathcal{W}_4(\alpha)$.

Step 7: Pass a rational norm curve R_3 through the double points of $\mathcal{W}_4$.

Step 8: Project the plane Π_e from α to $\mathbb{P}_{Ma}$. The image is the secant line $\overline{a_e, b_e}$ of R_3, the projection of the secants $\overline{x_e, y_e}, \overline{x'_e, y'_e}$.

Step 9: Find the images of a_e, b_e under the Veronese map $R_3 \to R_6 \subset \mathbb{P}^5$. The output of the previous steps is the set of four secants ℓ_i as in the proof of the main theorem

Step 10: Find the intersection points of ℓ_i with the hyperplane H in $\mathbb{P}^5$ cutting out the images of the Weierstrass points. They must span a plane π in H.

Step 11: Project H from π to $\mathbb{P}^2$. The image of the six Weierstrass points lie on a conic, which determines a stable curve of genus 2.

5. An algorithm in characteristic 3

First let us remind an explicit algorithm for finding 3-torsion points on $\mathrm{Jac}(C)$ [?]. Let $w_1, \dots, w_6$ be the Weierstrass points of C. Fix one of them, say $w = w_1$, i.e. choose to define C by equation $y^2t - f_5(x,t) = 0$ in $\mathbb{P}^2$. The plane quintic model C_0 has a triple singular point and an infinitely near cusp at $(t,x,y) = (0,0,1)$. It is the projection of the quintic curve C in $\mathbb{P}^3$ embedded by the linear system $\mid 2K_C + w \mid$ from any point, not on C, lying on the ruling of the unique quadric containing C which cuts out the divisor $3w$ on C. The pencil of lines through the singular point of C_0 cuts out the linear system $\mid K_C \mid +3w$.

A plane cubic with equation $yt^2 - f_3(x,t) = 0$ intersects C_0 at 6 nonsingular points $p_1, \dots, p_6$ and at the point $(0,0,1)$ with multiplicity 9. This implies that $p_1 + \cdots + p_6$ is linearly equivalent to $3K_C$. Using the well-known description of $\mathrm{Jac}(C)$ in terms of the symmetric square of C, we see that $[p_1+p_2] \oplus [p_3+p_4] = -[p_5+p_6]$ in the group law on $\mathrm{Jac}(C)$, where $[p+q]$ is the divisor class of the divisor $p+q-K_C$. Here $-[p+q]$ is equal to $[p'+q']$, where $p \mapsto p'$ is the hyperelliptic involution $(t,x,y) \mapsto (t,x,-y)$.

Choosing coordinates so that the affine piece of C_0 is given by $y^2 = x^5 + \sum_{i=0}^4 b_i x^i$, we have to find coefficients $a, d_0, d_1, d_2, c_0, c_1$ which solve the equation

$$(x^3 + d_2x^2 + d_1x + d_0)^2 - a(x^5 + \sum_{i=0}^{4} b_i x^i) - (x^2 + c_1x + c_0)^3 = 0. \tag{19}$$

Let $p_1 + p_2 - H$ be a 3-torsion point corresponding to a solution, then the x coordinates of the images of p_1, p_2 on C_0 are the roots of $x^2 + c_1x + c_0$. Hence, the lines l_i corresponding to the unique reduced maximal isotropic group of the 3-torsion points on the Jacobian of C, are the lines connecting the images of the two roots of the quadratic $x^2 + c_1x + c_0$ under the map $\mathbb{P}^1 \to R_3 \subset \mathbb{P}H^0(2K_C)$, where c_1, c_0 give a solution to equation (**??**).

Expanding the x^5, x^4, x^2, x coefficients in equation (**??**) and eliminating d_0, d_1, d_2, we see that the solutions occur when a is a root of the quartic equation:

$$b_4b_2 - b_1 - a(b_4^3 + b_2) + a^4 = 0.$$

It is easy to find the roots of this equation since the splitting field of the resolvent polynomial is an Artin-Schrier extension of $\mathbb{K}$. Finally, let a be a root of this quartic, then expending the x^3 and x-free coefficient of equation (**??**), we see that the corresponding c_1, c_2 are given by:

$$c_1^3 = a^3 - a^2 b_4 - ab_3 + b_1/(a-b_4), \quad c_0^3 = b_1^2/(a-b_4)^2 - ab_0.$$

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