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A Character Formula for the Representation of a Weyl Group in the Cohomology of the Associated Toric Variety*

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INTRODUCTION

Let R be a reduced root system in a finite-dimensional space V, M be the root lattice spanned by the roots in V, N be the dual lattice in the space V^*, and W be the Weyl group of R. Each Weyl chamber C is a polyhedral cone in the space V which is also rational with respect to the lattice N. The collection Φ of all Weyl chambers defines a complete N-fan in V^*, and we denote by $X(R)$ the corresponding complete toric variety X_Φ associated to the fan Φ (see [O]). Alternatively, $X(R)$ can be characterized as the closure of a general orbit of the maximal torus T of a semi-simple simply connected algebraic group of type R in its natural action on the homogeneous space G/B, where B is a Borel subgroup of G containing T[A]. The Weyl group W, being the symmetry group of Φ, acts naturally on $X(R)$. This defines a representation of W in the graded space of cohomology $H^*(X(R)) = H^*(X(R), \mathbb{Q})$. In this paper we give a formula for the corresponding graded character. The essential ingredient of this formula is a natural resolution of T-equivariant cohomology $H_T^*(X(R))$ which takes place for arbitrary toric varieties with quotient singularities [BL]. Another formula was proved by Procesi at the 1985 Durham Symposium on the Symmetric Group [P]. The equivalence of the two formulas was shown by Stembridge [Ste2]. In the cases when R is of

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type A_n, B_n, or C_n, by a purely geometric method, we prove an equivalent formula in terms of symmetric functions. It is based on a construction of a W-equivariant birational mapping from the toric space to the projective space $\mathbb{P}^r$ (resp. $(\mathbb{P}^1)^n$) which allows one to give a simple recurrent decomposition of the cohomology. In the case A_n, this formula was derived by Stanley (cf. [S], p. 529), who used a formula of Procesi [P], whose proof uses the same geometric method (suggested by DeConcini). Essentially the same formula was obtained by Stembridge [Ste1] for some permutation representation of S_{n+1} on the set of certain types of Young tableaux with marked entries proving the isomorphism of the two representations.

It is well-known that the even-dimensional (resp. odd-dimensional) Betti numbers of smooth projective algebraic varieties provide examples of symmetric unimodal sequences of positive integers, i.e., sequences $(a_0 \ldots, a_n)$ with the properties $a_i = a_{n-i}$ and $a_i < a_{i+1}$ for $i \leq [n/2]$. These properties of the Betti numbers are easy consequences of the Poincaré Duality and the Hard Lefschetz theorems. The most notorious example is the sequence of binomial coefficients realized as the sequence of even-dimensional Betti numbers of the product of n copies of the projective line. As was noted by Stanley, more examples of symmetric unimodal sequences can be obtained by considering isotypic Betti numbers, i.e., the multiplicities of an irreducible representation in the cohomology of a smooth projective variety on which a finite group acts biregularly. The interesting combinatorial aspect of this problem is explained by the fact that in case $W = S_{n+1}$ the kth Betti number of $X(R)$ equals the $(k + 1)$th Eulerian number, i.e., the number of permutations with k descents. Thus isotypic Betti numbers of $X(R)$ for any root system R give both a generalization and a refinement of Eulerian numbers.

A natural problem is to extend the results of this paper to the case of the torus orbit in any homogeneous space G/P, where P is a parabolic group containing T (cf. [FH]). The appearance of non-quotient singularities of the corresponding varieties makes it natural to replace the rational cohomology by the intersection cohomology. The projection maps $G/B \to G/P$ define some natural direct summands in $H^*(X(R), \mathbb{Q})$ which might lead to interesting recurrent decomposition formulas.

This work was finished almost three years ago; however, we held its publication until the appearance in print of [BL]. Since that time much progress has been done in the work of Stembridge [Ste2], making some of our results almost obsolete. Among his results is an elementary proof of our formula which can be extended to the case of more general toric varieties. Also, for the cases A_n, B_n, and C_n, he translates this formula into the language of symmetric functions without using the geometric method. Finally, for the root systems of exceptional type, he made our formula more explicit by calculating its essential ingredients. However, it seems that our approach is still the only available one which may lead to a generalization of our formula to the case of representations of groups in the intersection cohomology of toric varieties.

1. A Resolution for the Equivariant Cohomology

Recall that the rational equivariant cohomology $H_T^*(X)$ of a space X acted on by a group T is defined to be the cohomology $H^*(X \times_T \mathbf{E}T, \mathbb{Q})$, where $\mathbf{E}T \to \mathbf{B}T$ is the universal classifying principal T-bundle. In our situation when T is an r-dimensional torus, one can take for $\mathbf{E}T$ the space $(\mathbb{C}^\infty\backslash\{0\})^r$ with the quotient $\mathbf{B}T = \mathbf{E}T/T$ isomorphic to $(\mathbb{P}^\infty)^r$, where $\mathbb{P}^\infty$ is the "infinite-dimensional" projective space. The canonical homomorphism $H^*(\mathbf{B}T, \mathbb{Q}) \to H_T^*(X)$ corresponding to the fibration $X_T = X \times_T \mathbf{E}T \xrightarrow{p} \mathbf{E}T/T = \mathbf{B}T$ makes $H_T^*(X)$ a module over the ring $H^*(\mathbf{B}T, \mathbb{Q})$. Put $H = H^*(\mathbf{B}T, \mathbb{Q})$. The latter is naturally isomorphic to the polynomial ring

$$H = H^*((\mathbb{P}^\infty)^r, \mathbb{Q}) \cong H^*(\mathbb{P}^\infty, \mathbb{Q})^{\otimes r} \cong \mathbb{C}[x_1]\otimes \dots \otimes\mathbb{C}[x_r] \cong \mathbb{C}[x_1, \dots, x_r].$$

Here the polynomial ring is graded in such a way that each x_i is of degree 2.

One can compute the equivariant cohomology $H_T^*(X)$ in the following way. Let $\mathbb{C}_{X_T}$ denote the constant sheaf on X_T. Consider its direct image $p_*(\mathbb{C}_{X_T})$ (in the derived category) which is a complex of sheaves on $\mathbf{B}T$. Then $H^*(X_T)$ equals the hypercohomology $H^*(\mathbf{B}T, p_*(\mathbb{C}_{X_T}))$.

In a similar way we can also define the equivariant cohomology with compact supports $H_{T,c}^*(X)$ (see [BL]). Namely, consider the direct image with compact supports $p_!(\mathbb{C}_{X_T})$, and again take the hypercohomology $H^*(\mathbf{B}T, p_!(\mathbb{C}_{X_T}))$. Then, by definition, $H_{T,c}^*(X) = H^*(\mathbf{B}T, p_!(\mathbb{C}_{X_T}))$.

In our case X is a projective rationally smooth variety which implies that $H^*(X) \cong H_c^*(X)$. In this case $H_T^*(X)$ (resp. $H_{T,c}^*(X)$) is a free H-module (see [BL]), with a basis given by $H^*(X)$ (resp. by $H_c(X)$). Hence we have also $H_T^*(X) \cong H_{T,c}^*(X)$.

Let us recall from [BL] a canonical complex $K^\cdot$ which computes $H_{T,c}^*(X)$.

The cohomology ring H can be canonically identified with the graded ring A of polynomial functions on V^*, where linear functions have degree 2. We describe the complex $K^\cdot$ using the fan Φ. Namely for each cone $\sigma \in \Phi$, let A_σ denote the ring of polynomial functions on σ. Let $P_\sigma = A_\sigma \cdot e_\sigma$ be the free A_σ-module with formal generator e_σ in degree 0. Then P_σ is naturally a graded A-module. Note that the Weyl group W acts naturally on the ring A.

THEOREM 1.1.

(1) *There exists a natural complex of graded A-modules*

$$K^{\bullet}: 0 \to K^0 \xrightarrow{d^0} K^1 \xrightarrow{d^1} K^2 \to \cdots \to K^r \to 0$$

with the following properties:

(i) $K^i = \bigoplus_{\operatorname{codim}(\sigma)=i} P_\sigma$;

(ii) $K^{\bullet}$ *is acyclic except in degree* 0;

(iii) $\operatorname{Ker}(d^0) \cong H^*_{T,c}(X))$ *as graded A-modules.*

(2) *The complex $K^{\bullet}$ is W-equivariant in the following sense:*

(i) *W acts naturally on each A-module K^i and differentials d^i commute with this action;*

(ii) *the W-action is compatible with the A-module structure, that is,*

$$g(a \cdot k) = g(a) \cdot g(k), \qquad g \in W, a \in A, k \in K^i.$$

Part 1 of the theorem is a result of [BL]. So it remains to prove part (2). This is done in the Appendix. The action of W on each module K^i is described in Lemma 3.1 below, which is also proved in the Appendix.

2. THE FORMULA

Since the complex $K^{\bullet}$ is a W-resolution of the equivariant cohomology module $H^*_{T,c}(X)$ we can express the character of W on $H^*_{T,c}(X)$ as the alternating sum of the characters on each K^i. Finally, since $H^*_{T,c}(X)$ is a free A-module, we will be able to compute the character of W on the non-equivariant cohomology $H^*_c(X) = H^*(X)$ (see Theorem 2.1 below).

Let us introduce some notation.

The Weyl group W acts on the real vector space V^* as well as on the space $A^1 = V$ of complex linear functions on V^*. Let $s(w)$ be the linear operator on A^1 corresponding to an element w of W. Consider the subspace $V^{*w} \subset V^*$ of vectors fixed by w, and let $V_w = (V^{*w})^{\perp} \subset A^1$ be its annihilator. Denote by $\bar{s}(w)$ the restriction of $s(w)$ to V_w. Let $d(w) = \operatorname{codim} V_w = \dim V^{*w}$. For each $w \in W$ we introduce the characteristic polynomial $\det(1 - \bar{s}(w)t)$ in the variable t. Note that

$$\det(1 - s(w)t) = (1 - t)^{d(w)} \det(1 - \bar{s}(w)t). \tag{1}$$

The intersection of V^{*w} with the fan Φ defines a fan Φ^w in V^{*w} which corresponds to some toric variety $X(R; w)$.

Denote by

$$P_w(t) = \sum_{i=0}^{d(w)} b_{2i}(X(R; w))t^i$$

the usual Poincaré polynominal of $X(R; w)$, and let

$$\chi(w) = \sum_{i=0}^{r} \mathrm{Tr}^i(w)t^i$$

be the graded character of $w \in W$ on the cohomology $H^*(X(R))$ of $X(R)$. Here $\mathrm{Tr}^i(w)$ is the trace of w of its action on $H^{2i}(X(R))$.

Our main result is the following:

THEOREM 2.1. *With the above notation, we have*

$$\chi(w) = P_w(t) \cdot \det(1 - \bar{s}(w)t).$$

3. Proof of Theorem 2.1

Let $s^{(n)}(w)$ be the operator on the nth symmetric power $A^n = \mathrm{Sym}^n(A^1)$ of the space A^1 corresponding to an element $w \in W$. Recall the following formula for the graded character of w on A_n ([Bo], Chap. V, Section 5, no. 3, Lemma 4.3):

$$\chi_s(w) := \sum_{n=0}^{\infty} \mathrm{Tr}(s^{(n)}(w))t^n = \frac{1}{\det(1 - s(w)t)}. \tag{2}$$

Set

$$\chi_{\mathrm{eq}}(w) = \sum_{j=0}^{\infty} \mathrm{Tr}^j_{\mathrm{eq}}(w)t^j$$

to be the graded character of w in the equivariant cohomology $H^*_{T_c}(X(R))$. Since the latter is a free A-module with a basis given by $H^*_c(X(R))$, we obtain:

$$\chi(w) \cdot \chi_s(w) = \chi_{\mathrm{eq}}(w). \tag{3}$$

Therefore, using formulas (1)–(3), our theorem is equivalent to the following formula

$$\chi_{\mathrm{eq}}(w) = \frac{P_w(t)}{(1-t)^{d(w)}}, \tag{4}$$

which we shall now prove.

Let

$$\chi^i(w) = \sum_{j=0}^{\infty} \mathrm{Tr}^j_{K^i}(w)t^j$$

be the graded character of w in its action on the ith component K^i of the complex $K^\cdot$. In view of the discussion in the beginning of Section 2, we have

$$\chi_{\mathrm{eq}}(w) = \sum_{i=0}^{r} (-1)^i \chi^i(w).$$

So we have to prove the following identity

$$\sum_{i=0}^{r} (-1)^i \chi^i(w) = \frac{P_w(t)}{(1-t)^{d(w)}}. \tag{*}$$

Let us analyze the representation of W on a fixed module K^i. Again this is a graded representation which is a sum of induced representations in the following way.

Fix a Weyl chamber C_0 in Φ. The Weyl group acts on Φ by permuting cones σ, and each W-orbit contains a unique cone which lies in the closure of C_0. Choose a cone $\sigma \subset C_0$, and let W_σ be its stabilizer subgroup. We have the following formula for the W-module K^i:

$$K^i = \bigoplus_{\substack{\sigma \subset C_0 \\ \mathrm{codim}(\sigma) = i}} \mathrm{ind}^W_{W_\sigma} P_\sigma. \tag{5}$$

Recall that $P_\sigma = A_\sigma \cdot e_\sigma$. The action of W_σ on A_σ is trivial (since W_σ acts trivially on the cone σ). It remains to describe the one-dimensional representation of W_σ on $\mathbb{C} \cdot e_\sigma$.

LEMMA 3.1. *The representation of W_σ on $\mathbb{C} \cdot e_\sigma$ is the sign representation:*

$$w \cdot e_\sigma = \det(w) \cdot e_\sigma, \qquad w \in W_\sigma.$$

Proof. See Appendix.

We claim that for each $w \in W$ there is the following equality of the power series in t:

$$\chi^i(w) = \sum_{\substack{\sigma \subset V^{*w} \\ \operatorname{codim}(\sigma) = i}} \frac{1}{(1-t)^{r-i}} \det(w).$$

This formula follows immediately from the formula for the character of an induced representation. Indeed each cone $\sigma \subset V^{*w}$ of codimension i contributes to the graded character the term $\det(w)/(1-t)^{r-i}$, where $1/(1-t)^{r-i}$ is the Hilbert series of A_σ (cf. [St], Section 9). In particular, $\chi^i(w) = 0$ if $\dim V^{*W} < r - i$. Taking the alternating sum of terms as in (*), we find

$$\sum_{i=0}^{r} (-1)^i \chi^i(w) = \det(w) \left(\sum_{i=r-d(w)}^{r} (-1)^i \sum_{\substack{\sigma \subset V^{*w} \\ \operatorname{codim}(\sigma) = i}} \frac{1}{(1-t)^{r-i}} \right). \tag{6}$$

Note that $\det(w) = (-1)^{r-d(w)}$, hence the last sum is equal to

$$\sum_{i=r-d(w)}^{r} (-1)^{i+r-d(w)} \sum_{\substack{\sigma \subset V^{*w} \\ \operatorname{codim}(\sigma) = i}} \frac{1}{(1-t)^{r-i}} = \sum_{i=0}^{d(w)} (-1)^i \left[\sum_{\substack{\sigma \subset V^{*w} \\ \operatorname{codim}(\sigma) = i + d(w)}} \frac{1}{(1-t)^{d(w)-i}} \right]. \tag{7}$$

We claim that (7) is the Poincaré series of the T^w-equivariant cohomology $H^*_{T^w,c}(X(R; w))$. Indeed, this last cohomology may again be computed using the corresponding complex $K(w)^\cdot$ constructed for the toric variety $X(R; w)$, and (7) is the alternating sum of the terms of this complex. But again $H^*_{T^w,c}(X(R; w))$ is a free module over the ring of functions on V^{*w} with the basis $H^*_c(X(R; w)) = H^*(X(R; w))$. Hence from (6) we infer

$$\sum_{i=0}^{r} (-1)^i \chi^i(w) = \frac{P_w(t)}{(1-t)^{d(w)}}. \tag{8}$$

This proves (*) and Theorem 2.1.

Remark 3.2. Our formula is quite explicit. For each conjugacy class of w in W the polynomial $\det(1 - s(w)t)$ is computed in [Ca]. The Poincaré polynomial $P_w(t)$ of $X(R;\, w)$ can be computed by applying the following formula of Ehlers,

$$b_{2j}(X(R;\, w)) = \sum_{i=0}^{d(w)} (-1)^{i-j}\binom{i}{j} d_i$$

for the Betti numbers of any non-singular toric variety [E], where d_i is the number of cones of codimension i in V^{*w}. In the next section we will carry out the explicit computations for the root systems of types A, B, and C. The computations for other series are more involved.

EXAMPLE 3.3. Take $R = A_1$, then $V^* = \mathbb{R}$. The fan Φ has 3 cones: the positive ray σ_+, the negative ray σ_-, and the origin σ_0. The associated toric variety $X(R)$ is isomorphic to the projective line $\mathbb{P}^1$. The Poincaré polynomial of $X(R)$ is $1 + t$ ($\deg(t) = 2$). The Weyl group W has 2 elements $W = \{1, s\}$, where s interchanges σ_+ and σ_-.

Let us compute the T-equivariant cohomology $H^*_{T,c}(X(R))$. Let $A = \mathbb{C}[x]$ be the ring of polynomial functions on V^* where $\deg(x) = 2$ (so that $\mathbb{C}[x] \cong H^*(\mathbf{B}T, \mathbb{C})$). Then the complex $K^\cdot$ is

$$K^\cdot\colon 0 \to \mathbb{C}[x]\cdot e_+ \oplus \mathbb{C}[x]\cdot e_- \xrightarrow{d^0} \mathbb{C}\cdot e_0 \to 0,$$

where e_+, e_-, e_0 are formal generators in degree 0, and d_0 is a map of degree 0. The kernel of d^0 is a free $C[x]$-module with generators in degrees 0 and 2. Denote by e its generator in degree 0. This module is canonically isomorphic to the equivariant cohomology $H^*_{T,c}(X(R))$ which, by the general result, is a free $C[x]$-module with a basis given by the usual cohomology $H^*_c(X(R)) = H^*(X(R))$. The reflection $s \in W$ induces an algebraic automorphism of $X = \mathbb{P}^1$, hence its graded character on the cohomology of $X(R)$ is

$$\chi(s) = \mathrm{Tr}^0(s) + \mathrm{Tr}^1(s)t = 1 + t.$$

We wish to verify this using the complex $K^\cdot$ as in the theorem.

By Lemma 3.1 we have

$$s\cdot e_0 = (-1)^{\det(s)}\cdot e_0 = -e_0.$$

The trace of s on $K^0 = \mathbb{C}[x]\cdot e_+ \oplus \mathbb{C}[x]\cdot e_-$ is zero since s interchanges the two summands. Since d^0 is W-equivariant

$$s\cdot e = (0 - (-1))e = e$$

so $\mathrm{Tr}^0(s) = 1$. The trace $\mathrm{Tr}^1(s)$ is equal to the trace of s on the quotient space

$$\mathbb{C}\cdot xe_+ \oplus \mathbb{C}\cdot xe_- / \mathbb{C}\cdot xe. \qquad (*)$$

However, $s(x) = -x$, hence $s(xe) = (-1)x\cdot 1\cdot e = -x\cdot e$, so the trace of s on (*) is $\mathrm{Tr}^1(s) = 0 - (-1) = 1$.

4. Explicit Computations

To compute explicitly the values of the character $\chi(w)$ we need to know the Poincaré polynomials of the varieties $X(R;\, w)$. We start with the following general result about the Betti numbers of the toric varieties $X(R)$.

Theorem 4.1 ([K], [*Bj*]). *Let $X = X(R)$ be the toric variety associated with a root system R. Let $B = \{\alpha_1, \ldots, \alpha_n\}$ be a root basis of R. Then the odd Betti numbers of X are zero, and its even Betti numbers are given by the formula:*

$$b_{2i}(X) = \#\{w \in W\colon \#\{j\colon w(\alpha_j) < 0\} = i\}.$$

Remark 4.2. By the Ehlers formula the sequence $(b_0(X), \ldots, b_{2n}(X))$ is the h-vector of the simplicial Coxeter polytope $P(R)$ associated with Weyl chambers. By definition,

$$\sum_{i=0}^{n} b_{2i}(X)t^{n-i} = \sum_{i=0}^{n} f_{i-1}(t-1)^{n-i},$$

where $(f_0, \ldots, f_{n-1})$ is the f-vector of $P(R)$ defined by setting f_i equal to the number of i-dimensional faces of $P(R)$, $f_{-1} = 1$. The interpretation of the h-vector of the Coxeter polytope as in Theorem 4.1 is well-known for combinatorialists (cf. [Bj, Thm. 2.1]).

Case A_n. The set R of roots of type A_n can be identified with the set of vectors $e_i - e_j$, $1 \le i \ne j \le n+1$ in

$$V = \{(x_1, \ldots, x_{n+1} \in \mathbb{R}^{n+1}\colon x_1 + \ldots + x_{n+1} = 0\}.$$

The Weyl group is the symmetric group S_{n+1} that acts on $\mathbb{R}^{n+1}$ by permuting the coordinates. The fan $\Phi(A_n)$ in the dual space V^* consists of the Weyl chambers and its faces. Each Weyl chamber is equal to $w(C)$, $w \in W$, where C is given by the inequalities

$$\alpha_1 \ge 0, \ldots, \alpha_n \ge 0,$$

where $\{\alpha_1, \ldots, \alpha_n\} = \{e_1 - e_2, \ldots, e_n - e_{n+1}\}$ is the root basis.

LEMMA 4.3.

$$X(A_n; w) \cong X(A_{d(w)}).$$

Proof. Fix a fundamental chamber C in V^*, and let σ be the face of C^* of maximal dimension which is contained in V^{*w}. Let $I \subset \{1, \dots, n\}$ be such that σ is given by the equalities $\alpha_1 = 0$, $i \in I$. It is clear that $\#I = n - d(w)$. Then V^{*w} is given by the equalities $\alpha_i = 0$, $i \in I$. We may identify it with the dual space of the factor space

$$V/\Sigma_{i\in i}\mathbb{R}\alpha_i \cong \Sigma_{i\notin I}\mathbb{R}\alpha_i \cong \{(x_1, \dots x_{d(w)+1}) \in \mathbb{R}^{d(w)+1}: x_1 + \dots + x_{d(w)+1} = 0\},$$

and the images of the roots $\alpha \in \Phi$ in this subspace form a root system of type $A_{d(w)}$. The maximal cones of $\Phi(w)$ are connected components of $V^{*w} \backslash \cup_{\alpha\in\Phi}(H(\alpha) \cap V^{*w})$, where $H(\alpha)$ denotes the hyperplane $\alpha = 0$. We have

$$H(\alpha) \cap V^{*w} = H(\alpha') \cap V^{*w} \Leftrightarrow \alpha - \alpha' \in \Sigma_{i\in I}\mathbb{R}\alpha_i.$$

This easily implies that the fan $\Phi(w)$ is combinatorially isomorphic to the fan Φ corresponding to the root system $A_{d(w)}$. This proves the assertion.

For every $w \in S_{n+1}$ let des(w) denote the number of descents in w, i.e.,

$$\text{des}(w) = \#\{i: w(i) > w(i+1)\}.$$

Since a root $e_i - e_j$ is positive if and only if $i > j$, we have

$$\text{des}(w) = \#\{i: w(\alpha_i) = e_{w(i)} - e_{w(i+1)} < 0\},$$

and, by Theorem 4.1,

$$b_{2k}(X(A_n)) = \#\{w \in W: \text{des}(w) = k\}.$$

This number is known to combinatorialists as the *Eulerian number* $A(n+1, k+1)$.

Applying Theorem 2.1, we immediately get

COROLLARY 4.4.

$$P_w(t) = \sum_{i=0}^{d(w)} A(n+1, i+1)t^i.$$

Instead of using a somewhat inexplicit formula from Theorem 9.1 one may compute the Betti numbers of varieties X by using directly the Ehlers formula

$$b_{2k}(X(A_n)) = \sum_{i=0}^{n} (-1)^{i-k} \binom{i}{k} d_i(n),$$

where $d(n)$ is the number of cones of codimension i. This allows one to deduce an explicit formula for the Eulerian numbers certainly known to combinatorialists. It is given in Proposition 4.7 below.

Let $S(N, m)$ denote the Stirling numbers of the second kind defined as the number of equivalence relations with m equivalence classes defined on a set of cardinality N. The next lemma collects some of the well-known properties of Stirling numbers of the second kind:

LEMMA 4.5.

(i) $S(N, m) = S(N-1, m-1) + mS(N-1, m)$;

(ii) $S(N, m) = \sum_{i=m-1}^{N-1} \binom{N-1}{i} S(i, m-1)$;

(iii) $x^N = \sum_{i=0}^{n} S(N, i)(x)_i$, *where* $(x)_i = x(x-1)\ldots(x-i+1)$, $(x)_0 = 1$.

Proof. See [Co, pp. 208, 209].

LEMMA 4.6. *Let $d_k(n)$ denote the number of cones of codimension k in the fan $\Phi(A_n)$. Then*

$$d_k(n) = (n-k+1)!S(n+1, n+1-k).$$

Proof. We shall give two proofs. One is an inductive proof that may be applied to any root system. The other due to Stembridge, is a direct one that uses the specific combinatorics of A_n.

1st proof. Each cone of codimension k is W-equivalent to a unique face of the fundamental chamber given by a subset I of the root basis S. The isotropy subgroup of this face is the Weyl group W_I of the root system defined by the Dynkin subdiagram associated to the subset I. Therefore

$$d_k(n) = \sum_{\#I=k} [W: W_I].$$

Let $T(k)$ be the set of subsets I of S of cardinality k. Then

$$T(k) = \{I \in T(k): \alpha_1 \notin I\} \amalg \{I \in T(k): \alpha_1 \in I, \alpha_2 \notin I\} \amalg \ldots \amalg \{I \in T(k): \alpha_1, \ldots, \alpha_k \in I\}.$$

For every I from the first set, W_I is a subgroup of the Weyl group of type A_{n-1} associated with the root basis $S - \{\alpha_1\}$. So

$$\sum_{\#I=k,\alpha_1\notin I} [W\colon W_I = [W(A_n)\colon W(A_{n-1})]\, d_k(n-1) = \binom{n+1}{n} d_k(n-1).$$

For every I from the second set, W_I is a subgroup of the Weyl group of type $A_1 \times A_{n-2}$ associated with the root basis S-$\{\alpha_2\}$ that is the product of the Weyl group generated by the simple reflection associated with α_1 and the Weyl subgroup generated by $k-1$ simple reflections associated to α_i, $i > 2$. So

$$\sum_{\#I=k,\alpha_1\in I,\alpha_2\notin I} [W\colon W_I] = [W(A_n)\colon W(A_1)\times W(A_{n-2})]\, d_{k-1}(n-2)$$
$$= \binom{n+1}{2} d_{k-1}(n-1).$$

Continuing in this way, we find

$$d_k(n) = \sum_{i=1}^{k+1} \binom{n+1}{i} d_{k+1-i}(n-i).$$

By induction on n, using Lemma 4.5, we can write this as

$$\sum_{i=1}^{k+1} \binom{n+1}{i} (n-k)!S(n-i+1, n-k) = (n-k)! \sum_{i=n-k}^{n} \binom{n+1}{i} S(i, n-k)$$
$$= (n-k)!\left(\sum_{i=n-k}^{n+1} \binom{n+1}{i} S(i, n-k) - S(n+1, n-k)\right)$$
$$= (n-k)!(S(n+2, n-k) - S(n+1, n-k))$$
$$= (n-k)!(n-k+1)S(n+1, n-k+1)$$
$$= (n-k+1)!S(n+1, n-k+1).$$

This proves the claimed formula.

2nd *proof.* There is a bijection between the set of cones in $\Phi(A_n)$ of codimension 0 and the Weyl group S_{n+1}. Each $w \in S_{n+1}$ defines the corresponding translate $w(C)$, of the fundamental chamber C. It is given by the inequalities:

$$w(C)\colon e_{w(1)} > e_{w(2)} > \cdots > e_{w(n+1)}.$$

So the formula is obviously true for $k = 0$. A face of $w(C)$ of codimension k is a w-translate of some face of C. It is given by a choice of a subset $I = \{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$ of the set $\{\alpha_1, \ldots, \alpha_n\}$. The corresponding face C_I is determined by the following system of inequalities and equalities:

$$e_1 > \cdots > e_{i_1} = e_{i_1+1} > \cdots > e_{i_k} = e_{i_k+1} > \cdots > e_{n+1}.$$

So, each cone of codimension k is given by a system of inequalities between e_i, where exactly $n + 1 - k$ are strict inequalities. It defines an equivalence relation in the set $\{1, \ldots, n + 1\}$: $i \sim j$ *iff* $e_i = e_j$ in this set of inequalities. There are exactly $n + 1 - k$ equivalence classes and they come with a natural order. So the total number of such faces is $(n + 1 - k)!S(n + 1, n + 1 - k)$.

PROPOSITION 4.7.

$$\sum_{i=0}^{n} b_{2i}(X(A_n))t^i = \sum_{i=0}^{n} (n - i + 1)!S(n + 1, n - i + 1)(t - 1)^i.$$

Proof. This is known as the Frobenius theorem for the Eulerian numbers ([Co, p. 244]). We shall deduce it from the Ehlers formula and Lemma 4.6:

$$\begin{aligned}\sum_{i=0}^{n} b_{2i}(X(A_n))t^i &= \sum_{i=0}^{n} t^i \sum_{k=0}^{n} (-1)^{k-i} \binom{k}{i} (n - k + 1)!S(n + 1, n - k + 1)\\ &= \sum_{k=0}^{n} \sum_{i=0}^{n} (-1)^{k-i} \binom{k}{i} t^i(n - k + 1)!S(n + 1, n - k + 1)\\ &= \sum_{k=0}^{n} (t - 1)^k(n - k + 1)!S(n + 1, n - k + 1).\end{aligned}$$

It remains to change the summation index k to i.

Now everything is ready to give an explicit formula for $\chi(w)$. It is well known that there is a bijective correspondence between the conjugacy classes in S_{n+1} and partitions λ of number $n + 1$. If $\lambda = (\lambda_1, \ldots, \lambda_k)$ is such a partition, i.e., $n + 1 = \lambda_l + \cdots + \lambda_k$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$, then the conjugacy class $C(\lambda)$ corresponding to this partition consists of all permutations which can be written as the product of cycles of length λ_i. Computing the characteristic polynomial of a cyclic permutation, we easily obtain for any $w \in C(\lambda)$

$$d(w) = k - 1$$

and

$$\det(1 - \bar{s}(w)t) = \prod_{i=1}^{k} \frac{1 - t^{\lambda_i}}{1 - t}.$$

Collecting everything together we get:

THEOREM 4.8. *Let R be of type A_n, and let w correspond to a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of $n + 1$. Then*

$$\chi(w) = \left(\sum_{i=0}^{k-1} A(k, i+1)t^i\right) \prod_{i=1}^{k} \frac{1 - t^{\lambda_i}}{1 - t} = \left(\sum_{i=0}^{k-1} (i+1)!S(k, i+1)\right.$$

$$\left.(t-1)^i\right) \prod_{i=1}^{k} \frac{1 - t^{\lambda_i}}{1 - t}.$$

This is the formula from Proposition 3.3 of [Ste 1].

Cases B_n, C_n. In these cases the root system consists of vectors $\pm e_i$, $i = 1, \ldots, n$, $\pm e_i \pm e_j$ $1 \leq i < j \leq n$, (Case B_n) and $\pm 2e_i$, $i = 1, \ldots, n$, $\pm e_i \pm e_j$ $1 \leq i < j \leq n$, (Case C_n) in $\mathbb{R}^n$. We consider only the Case B_n; the other case is no different and leads to the same results. The root basis is the set of vectors $\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n$, $\alpha_n = e_n$. Positive roots are $e_i \pm e_j$, $1 \leq i < j \leq n$, e_i, $i = 1, \ldots, n$. The Weyl group is the semi-direct product of S_n and the group $(\mathbb{Z}/2\mathbb{Z})^n$ that acts by permuting the coordinates and multiplying them by ± 1.

LEMMA 4.9. *Let $w \in W$, then*

$$X(B_n; w) \cong X(B_{d(w)}).$$

Proof. It is similar to the proof of Lemma 4.3 and is left to the reader.

Theorem 4.1 gives a formula for the Betti numbers of the varieties $X(B_n)$ in terms of the Eulerian B-numbers:

$$b_{2k}(X(B_n)) = B(n, k+1) = \#\{w \in W : \#\{i : w(\alpha_i) \leq 0\} = k\}.$$

If w acts as the composition of a permutation $\sigma = (\sigma_1(1), \ldots, \sigma(n))$ and the transformation $e_i \mapsto (-1)^{\varepsilon(i)} e_i$, where $\varepsilon(i) = 0$ or 1, then it is easy to compute $\#\{i : w(\alpha_i) < 0\} = k$. For $i \neq n$, we have

$$w(\alpha_i) = w(e_i - e_{i+1}) = (-1)^{\varepsilon(\sigma(i))} e_{\sigma(i)} - (-1)^{\varepsilon(\sigma(i+1))} e_{\sigma(i+1)}$$

is negative if and only if the following conditions are satisfied:

(i) if σ has descent at i, i.e., $\sigma(i) > \sigma(i+1)$, then $\varepsilon(\sigma(i+1)) = 0$;
(ii) if σ has no descent at i, then $\varepsilon(\sigma(i)) = 1$.

Since

$$w(\alpha_n) = w(e_n) = (-1)^{\varepsilon(\sigma(n))} e_{\sigma(n)} < 0 \Leftrightarrow \varepsilon(\sigma(n)) = 1,$$

this case can be included in (ii) (by definition, σ has no descent at n).

We shall give an explicit formula for the Betti numbers in terms of modified Stirling numbers $\tilde{S}(N, m)$. They will be referred to as *Stirling numbers of type B*.

Let $(S, *)$ be a set of cardinality $N + 1$ with a marked element $*$. We set $\tilde{S}(N, m) = \#\{$equivalence relations in S with $m + 1$ equivalence classes together with a choice of a subset in each equivalence class except the one containing the element $*\}$. We immediately observe that

$$\tilde{S}(N, N) = 2^N, \tilde{S}(N, 0) = 1, \tilde{S}(N, m) = 0 \qquad \text{if } N < m.$$

One computes the numbers $\tilde{S}(N, m)$ in terms of ordinary Stirling numbers by using the following inductive formula:

LEMMA 4.10.

$$\tilde{S}(N, m) = \sum_{i=0}^{N-m} 2^{N-i}\binom{N}{i} S(N-i, m)).$$

Proof. Each summand counts the part of the number $\tilde{S}(N, m)$ which is responsible for the equivalence relations that contain i elements in the equivalence class of $*$.

Other properties of $\tilde{S}(N, m)$ are similar to the properties of the ordinary Stirling numbers:

LEMMA 4.11.

(i) $\tilde{S}(N, m) = \tilde{S}(N-1, m) + \sum_{i=m-1}^{N-1} 2^{N-i}\binom{N-1}{i}\tilde{S}(i, m-1))$;

(ii) $\tilde{S}(N, m) = 2\tilde{S}(N-1, m-1) + (2m+1)\tilde{S}(N-1, m)$;

(iii) *each* $\tilde{S}(N, m)$ *is divisible by* 2^m. *If we write* $\tilde{S}(N, m) = 2^m\tilde{S}(N, m)_0$, *then*

$$x^N = \sum_{k=0}^{N} \tilde{S}(N, k)_0((x))_m,$$

where $((x))_m = (x-1)(x-3)\cdots(x-2m+1)$, $((x))_0 = 1$.

Proof. (i) Choose an arbitrary element $x_0 \neq *$. Then $\tilde{S}(N-1, m)$ counts the part of $\tilde{S}(N, m)$ responsible for the equivalence relations with $x_0 \sim *$. Each summand $2^{N-i}\binom{N-1}{i}\tilde{S}(i, m-1)$ is the number which is responsible for the equivalence relations with $x_0 \nsim *$ and $\# \{x \in S\colon x \sim x_0\} = N - i$.

(ii) We use (i) where we replace $\binom{N}{i}$ by $\binom{N-1}{i} + \binom{N-1}{i-1}$. After some obvious simplifications we obtain the needed formula.

(iii) Since $\tilde{S}(m, m) = 2^m$, by induction, the divisibility property follows immediately from (ii). The latter implies also that

$$\tilde{S}(N, m)_0 = \tilde{S}(N-1, m-1)_0 + (2m+1)\tilde{S}(N-1, m)_0.$$

The set of polynomials $((x))_m$, $m = 0, \ldots, N$, form a basis in the space of polynomials of degree $\leq N$. Write

$$x^N = xx^{N-i} = x \sum_{m=0}^{N} \alpha(N, m)((x))_m$$

for some coefficients $\alpha(N, m)$. Then use that

$$\begin{aligned}
x^N = xx^{N-1} &= x \sum_{m=0}^{N-1} \alpha(N-1, m)((x))_m \\
&= \sum_{m=0}^{N-1} \alpha(N, m)(x - 2m - 1 + 2m + 1)((x))_m \\
&= \sum_{m=0}^{N-1} \alpha(N, m)((x))_{m+1} + \sum_{m=0}^{N-1} (2m+1)\alpha(N, m)((x))_m \\
&= \sum_{m=0}^{N} \alpha(N-1, m-1) + (2m+1)\alpha(N-1, m)((x))_m .
\end{aligned}$$

This shows that $\alpha(N, m)$ satisfy the same recurrency relation as $\tilde{S}(N, m)_0$. They also satisfy the same initial conditions showing that $\alpha(N, m) = \tilde{S}(N, m)_0$ for all N and m.

PROPOSITION 4.12. *Let $d_k(n)$ denote the number of k-dimensional faces in the fan $\Phi(B_n)$. Then*

$$d_k(n) = (n-k)!\tilde{S}(n, n-k) = (n-k)!2^{n-k}\tilde{S}(n, n-k)_0 .$$

Proof. This is similar to the proof of Proposition 4.7. The fundamental chamber C is given by the inequalities:

$$e_1 > e_2 > \cdots > e_n > 0.$$

Applying $w = \varepsilon s$ we obtain that $w(C)$ is defined by the inequalities:

$$(-1)^{\varepsilon(1)} e_{s(1)} > \cdots > (-1)^{\varepsilon(n)} e_{s(n)} > 0.$$

Its codimension k faces are obtained by converting exactly k of these inequalities into the equalities. We define the corresponding equivalence relation on the marked set $(S, *) = \{1, \ldots, n\}, \{0\})$ by setting

$$\begin{aligned} i \sim j &\quad \text{if } i, j \neq 0 \text{ and } (-1)^{\varepsilon(i)} e_i = (-1)^{\varepsilon(j)} e_j, \\ i \sim 0 &\quad \text{if } (-1)^{\varepsilon(i)} e_i = 0. \end{aligned}$$

Each equivalence class different from the equivalence class of 0 contains a subset of elements with $\varepsilon(i) = 1$. The number of such equivalence classes equals the number of the strict inequalities, that is $n - k$. This explains the factor $\tilde{S}(n, n - k)$ in the formula. Also, there is a natural order on the subset of equivalence classes not containing 0 that explains the factor $(n - k)!$.

COROLLARY 4.13. *Let*

$$P_{B_n}(t) = \sum_{i=0}^{n} b_{2i}(X(B_n)) t^i.$$

Then

$$P_{B_n}(t) = \sum_{i=0}^{n} (n - i)! \tilde{S}(n, n - k)(t - 1)^i.$$

Proof. It is similar to the proof of Proposition 4.7 by using the Ehlers formula and the formula for $d_k(n)$.

COROLLARY 4.14.

$$b_{2i}(X(B_n)) = (2n - 2i + 1) b_{i-1}(X(B_{n-1})) + (2i + 1) b_i(X(B_{n-1})).$$

Proof. Applying Lemma 4.11 (ii) we obtain after simple transformations:

$$\begin{aligned} P_{B_n}(t) &= \sum_{i=0}^{n} (n-i)!\tilde{S}(n, n-k)(t-1)^i \\ &= \sum_{i=0}^{n} (n-i)!(2\tilde{S}(n-1, n-k-1) \\ &\quad + (2n-2k+1)\tilde{S}(n-1, n-k))(t-1)^i \\ &= 2nP_{B_{n-1}}(t) - 2P_{B_{n-1}}(t) + (2n-1)(t-1)P_{B_{n-1}}(t) - 2(t-1)^2P_{B_n}(t) \\ &= ((2n-1)t+1)P_{B_{n-1}}(t) - 2(t^2-t)P_{B_{n-1}}(t). \end{aligned}$$

It remains to compare the coefficients.

The conjugacy classes in the Weyl groups of type B_n or C_n are described as follows. Let w be represented by a permutation $\sigma = (i_1, \ldots, i_n)$ followed by a homothety $e_i \to \varepsilon(i)e_i$. Write σ as the product of cycles. Let $(j_1, \ldots, j_k)$ be one of these cycles. Then $\varepsilon(i)$ has the same sign for all $i = j_1, \ldots, j_k$, so that w assigns a sign for any cycle. Let $\lambda_1, \ldots, \lambda_s$ be the positive lengths of "positive" cycles and $\mu_1, \ldots, \mu_r$ be the lengths of "negative" cycles. They define a pair of partitions $\lambda = (\lambda_1, \ldots, \lambda_s)$ and $\mu = (\mu_1, \ldots, \mu_r)$ with $|\lambda| + |\mu| = n$. We have (see [Ca]):

$$d(w) = n - (\lambda_1 - 1) - \cdots - (\lambda_s - 1) - \mu_1 - \cdots - \mu_r = s.$$

$$\det(1 - \bar{s}(w)t) = \left(\prod_{i=1}^{s} \frac{1-t^{\lambda_i}}{1-t}\right) \prod_{i=1}^{r} (t^{\mu_i} + 1).$$

This gives an explicit formula for the character $\chi(w)$:

THEOREM 4.15. *Let R be a root system of type B_n or C_n. Then*

$$\chi(w) = \sum_{i=0}^{d(w)} (d(w)-i)!\tilde{S}(d(w), d(w)-i)(t-1)^i \prod_{i=1}^{r} (t^{\mu_i} + 1) \prod_{i=1}^{s} \frac{1-t^{\lambda_i}}{1-t}.$$

Remark 4.16. The fans obtained by restricting a Coxeter arrangement to the subspace invariant under some Weyl group element were first studied in [OS]. In particular, Propositions 2.1 and 2.2 of [OS] are easily seen to be equivalent to our Lemmas 4.3 and 4.9. In the case of root systems of type D_n it is not true anymore that the fan $\Phi(w)$ in V^{*w} is always combinatorially isomorphic to the fan of type $D_{d(w)}$. The correct analogue for Lemmas 4.3 and 4.9 for the D_n case is treated in Proposition 2.3 of [OS].

5. A Recurrent Formula (Case A_n)

The formula is based on the following well known geometric fact (cf. [P]):

LEMMA 5.1. *There exists a W-equivariant birational morphism of toric varieties π: $X(A_n) \to \mathbb{P}^n$, where W acts naturally on $\mathbb{P}^n$ by permuting projective coordinates. It is equal to the composition*

$$X(A_n) = X_1 \xrightarrow{\pi_1} X_2 \longrightarrow \cdots \longrightarrow X_{n-1} \xrightarrow{\pi_{n-1}} X_n = \mathbb{P}^n,$$

where each π_i is the blowing up of the proper inverse transform under $\pi_{n-1} \circ \ldots \circ \pi_{i-1} : X_{i-1} \to X_n$ of the union of the subspaces defined by vanishing of $i+1$ projective coordinates. This proper inverse transform is isomorphic to $X(A_{n-i})$.

Proof. Recall that $X(A_n)$ is given by the fan $\Phi = \Phi_1$ whose cones are S_{n+1}-translates of the faces of the fundamental chamber

$$e_1 > \cdots > e_{n+1}.$$

We consider the fans Φ_i, $i = 1, \ldots, n$, formed by the cones whose faces are S_{n+1}-translates of the faces of the cone C_i given by the following inequalities:

$$e_k > e_{i+1} > e_{i+2} > \cdots > e_{n+1}, \; k = 1, \ldots, i.$$

It is easy to see that S_i stabilizes C_i and hence its orbit under S_{n+1} consists of $[S_{n+1} : S_i]$ cones of maximal dimension n of Φ_i. One immediately recognizes the fan Φ_n as the standard fan of the toric variety $\mathbb{P}^n$. Let X_i be the toric variety associated with the fan Φ_i. Since each C_i is a convex hull of a basis of the lattice N, all X_i are non-singular varieties. Obviously each cone of Φ_i is a subset of a cone of Φ_{i+1}, and each cone of Φ_{i+1} is the union of cones from Φ_i. The standard technique of toric varieties gives us a proper birational morphism of toric varieties $\pi_i : X_i \to X_{i+1}$. Recall that for any face σ of Φ_{i+1} the closure $\bar{o}^\sigma$ of the orbit o^σ is isomorphic to the toric variety $X_{\mathrm{St}(\sigma)}$ associated with the fan $\mathrm{St}(\sigma)$ in the space $V/\mathbb{R}\sigma$ with the lattice $N/\mathbb{Z}\sigma$ whose cones are the images of the cones $\tau \in \Phi_{i+1}$ containing σ ([O], p. 11). Its inverse image of $\bar{o}^\sigma$ in X_i is equal to the union of the closures of the orbits $\bar{o}^{\sigma'}$, where σ' are minimal elements (with respect to the order defined by the closure relation) in the fan Φ_i that are contained in σ. Let us see what we have in our situation. We rewrite the chamber C_i in the form

$$C_i = \langle \alpha_1^*, -\alpha_1^* + \alpha_2^*, \ldots, -\alpha_{i-1}^* + \alpha_i^*, \alpha_{i+1}^*, \ldots, \alpha_n^* \rangle,$$

where $\{\alpha_1^*, \ldots, \alpha_n^*\}$ is the basis dual to the basis $\{\alpha_1, \ldots, \alpha_n\}$. It is immediately seen that

$$\alpha_{i+1}^* \in \langle \alpha_1^*, -\alpha_1^* + \alpha_2^*, \ldots, -\alpha_i^* + \alpha_{i+1}^* \rangle$$

so that $\sigma' = \langle \alpha_{i+1}^* \rangle \in \Phi_i$ is contained in $\sigma = \langle \alpha_1^*, -\alpha_1^* + \alpha_2^*, \ldots, -\alpha_i^* + \alpha_{i+1}^* \rangle \in \Phi_{i+1}$. We have $\bar{\sigma}^\sigma \cong X_{\mathrm{St}(\sigma)}$, and is given by the S_{n+1-i}-translates of the cone $\langle \alpha_{i+2}^*, \ldots, \alpha_n^* \rangle$ and its faces in the space $\mathbb{R}\alpha_{i+2} + \ldots + \mathbb{R}\alpha_n$. Also, $\bar{\sigma}^{\sigma'} \cong X_{\mathrm{St}(\sigma')}$ is associated with the fan in $\mathbb{R}\alpha_1 + \cdots \mathbb{R}\alpha_i + \mathbb{R}\alpha_{i+2} + \cdots + \mathbb{R}\alpha_n$ whose faces are $S_i \times S_{n+1-i}$-translates of the faces of the cone

$$\langle \alpha_1^*, -\alpha_1^* + \alpha_2^*, \ldots, -\alpha_{i-1}^* + \alpha_i^*, \alpha_{i+2}^*, \ldots, \alpha_n^* \rangle$$
$$= \langle \alpha_1^*, -\alpha_1^* + \alpha_2^*, \ldots, -\alpha_{i-1}^* + \alpha_i^* \rangle \times \langle \alpha_{i+2}^*, \ldots, \alpha_n^* \rangle.$$

It is clear that this toric variety is isomorphic to $\mathbb{P}^i \times X(A_{n-i})$, where the projection $X_{\mathrm{St}(\sigma')} \to X_{\mathrm{St}(\sigma)}$ corresponds to the second projection $\mathbb{P}^i \times X(A_{n-i}) \to X(A_{n-i})$. This verifies the assertion.

Remark 5.2. Observe that

$$C_i = \bigcup_{w \in S_i} w(C),$$

where $S_1 \subset S_2 \subset \ldots \subset S_n \subset S_{n+1}$ is the standard inclusion of the permutation groups. According to [FH] this implies that each X_i is equal to the closure of the generic torus orbit in the homogeneous space $SL(n+1)/P_i$, where P_i is the parabolic subgroup of $SL(n+1)$ corresponding to the partition $(1, \ldots, 1, i)$ of $n + 1$. The projection $SL(n+1)/P_i \to SL(n+1)/P_{i+1}$ is a $\mathbb{P}^i$-bundle, and induces, by restriction to X_i, our morphism π_i.

Let $\mathrm{Rep}(S_k)$ be the Grothendieck group of isomorphism classes of finite-dimensional complex representations of the symmetric group S_k and

$$R = \oplus_{k \geq 1} \mathrm{Rep}(S_k)$$

be the graded ring whose composition law is defined by the formula

$$f \cdot g = \mathrm{ind}_{S_k \times S_r}^{S_{k+r}}(f \otimes g),$$

where $f \in \mathrm{Rep}(S_k)$, $g \in \mathrm{Rep}(S_r)$. This is an associative commutative graded ring to which we add formally the multiplicative unity.

We denote by $\theta_{n,k}$ the element in R corresponding to the representation of S_{n+1} on $H^k(X(A_n), \mathbb{C})$. Set

$$Ch_n^A = \sum_{k=0}^{n} \theta_{n,k} q^k \in R[q]$$

$$Ch^A = 1 + \sum_{n\geq 0} Ch_n^A t^{n+1} \in R[q][[t]],$$

$$\tilde{H}(t) = \sum_{n\geq 0} \tilde{h}_{n+1} t^{n+1},$$

where $\tilde{h}_{n+1}$ is the isomorphism class of the trivial representation of S_{n+1}.

THEOREM 5.3.

$$Ch^A = \frac{(1-q)\tilde{H}(t)}{\tilde{H}(qt) - q\tilde{H}(t)}.$$

Proof. Consider the composition

$$X(A_n) = X_1 \xrightarrow{\pi_1} X_2 \longrightarrow \cdots \longrightarrow X_{n-1} \xrightarrow{\pi_{n-1}} X_n = \mathbb{P}^n,$$

from Lemma 5.1. For every subset I of $\{0, \ldots, n\}$ let L_I denote the projective subspace of $\mathbb{P}^n$ given by vanishing of the coordinates x_i with $i \in I$. Let $\bar{L}_I$ be the proper transform of L_I in X_k, $k = \#I$. Let

$$\pi_I : E_I = \pi_{k-1}^{-1}(\bar{L}_I) \to \bar{L}_I$$

be the restriction of π_{k-1} to the exceptional locus of π_{k-1} over $\bar{L}_I$. It is a $\mathbb{P}^{k-1}$-bundle (see [GH], p. 605). It follows from loc cit that the cohomology ring $H^*(E_I)$ is a free $\pi_I^* H^*(\bar{L}_I)$-module of rank k generated by powers of an element $(I) \in H^2(E_I)$. In other words, there is an isomorphism of graded modules

$$H^*(E_I) \cong \bigoplus_{i=0}^{k-1} \pi_I^* H^*(\bar{L}_I)(-i)),$$

where we use the customary notation for the shifting of the grading. The cohomology of X can be computed inductively by using

$$H^*(X_{k-1}) \cong \pi_k^*(H^*(X_k)) \oplus \bigoplus_{\#I=k} H^*(E_I)/\pi_I^*(H^*(\bar{L}_I)).$$

By Lemma 5.1, each $\overline{L}_I$ with $\#I = k$ is isomorphic to $X(A_{n-k})$. The group $W = S_{n+1}$ acts transitively on the set of subspaces L_I with the isotropy subgroup W_I isomorphic to $S_k \times S_{n+1-k}$. The group W_I acts on the cohomology of $\overline{L}_I \cong X(A_{n-k})$ via S_{n+1-k}. This shows that there is a natural isomorphism of S_{n+1}-modules:

$$\bigoplus_{\#I=k} H^*(E_I)/\pi_I^* H^*((\overline{L}_I)) \cong \sum_{i=1}^{k-1} \operatorname{ind}_{S_k \times S_{n+1-k}}^{S_{n+1}} (1 \times H^*(X(A_{n-k}))(-i)).$$

Therefore we obtain the recurrency relation in the ring R:

$$[H^r(X_{k-1})] = [H^r(X_k)] + [1_{S_k}] \cdot \sum_{i=1}^{k-1} [H^{r-i}(X(A_{n-k}))).$$

This yields

$$\theta_{n,r} = [H^r(X(A_n)] = [H^r(\mathbb{P}^n)] + \sum_{k=2}^{n} \sum_{i=1}^{k-1} \tilde{h}_k \theta_{n-k,r-i}.$$

From this we get

$$\begin{aligned} Ch_n^A &= \sum_{r=0}^{n} \theta_{n,r} q^r = \tilde{h}_{n+1}(1 + \cdots + q^n) + \tilde{h}_2 q Ch_{n-2}^A + \tilde{h}_n (q + \cdots + q^{n-1}) Ch_0^A \\ &= \frac{1 - q^{n+1}}{1 - q} \tilde{h}_{n+1} + \left(\frac{q - q^2}{1 - q}\right) Ch_{n-2}^A \tilde{h}_2 + \cdots + \left(\frac{q - q^n}{1 - q}\right) \tilde{h}_n Ch_0^A. \end{aligned}$$

Multiplying both sides by t^{n+1} we obtain:

$$\begin{aligned} Ch_n^A t^{n+1} = \tilde{h}_{n+1} t^{n+1} + \frac{qt^{n+1} - (qt)^{n+1}}{1 - q} \tilde{h}_{n+1} + \tilde{h}_2 \left(\frac{qt^2 - (qt)^2}{1 - q}\right) Ch_{n-2}^A t^{n-1} \\ + \cdots + \tilde{h}_n \left(\frac{qt^n - (qt)^n}{1 - q}\right) Ch_0^A t. \end{aligned}$$

Setting

$$\Phi(t) = \frac{q\tilde{H}(t) - \tilde{H}(qt)}{1 - q},$$

and adding up, we get

$$Ch^A = (1 + \Phi)Ch^A + \tilde{H}(t),$$

which gives us the asserted formula.

Remark 5.4. As was shown in [Ste1] the graded representation $H^*(X(A_n), \mathbb{C})$ of S_{n+1} is isomorphic to a permutation representation. The proof of Theorem 5.3 provides a geometric explanation of this fact.

Let Λ be the ring of symmetric functions in countable number of variables t_i, $i \in \mathbb{N}$. There is a ring homomorphism

$$ch: R \to \Lambda \otimes \mathbb{Q}$$

defined by the formula

$$ch(f) = (k!)^{-1} \sum_{w \in S_k} \chi(f)(w)\psi(w),$$

where $f \in \mathrm{Rep}(S_n)$, $\chi(f)$ is its character function, and $\psi(w)$ is the symmetric function defined as follows. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be the partition of k corresponding to writing w as a product of cyclic permutations. For each i we define p_i as the sum of ith powers of the variables. Then we put

$$\psi(w) = p_\lambda := p_{\lambda_1} \cdots p_{\lambda_r}.$$

It follows from this definition that the character polynomial of the trivial one-dimensional representation 1_k of S_k is equal to

$$ch(1_k) = (k!)^{-1} \sum_{w \in S_k} \chi(1_k)(w)\psi(w) = \sum_{|\lambda| = k} z_\lambda^{-1} p_\lambda,$$

where for each partition λ of k we set

$$z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!,$$

where m_i is the number of parts of λ equal to i. Equivalently, z_λ is the order of the S_k-centralizer of any $w \in S_k$ of cycle type λ. By formula (2.14) of [M] we have

$$ch(1_k) = h_k := \text{sum of monomials of total degree } k.$$

We set

$$H(t) = \sum_{k=0}^{\infty} h_k t^k \in \Lambda[[t]].$$

COROLLARY 5.5.

$$\sum_{i,n\geq 0} ch(H^i(X(A_n))q^i t^{n+1} = \frac{(1-q)H(t)}{H(qt)-qH(t)}.$$

For every irreducible representation θ of the Weyl group W(R) let

$$P_{R,\theta}(q) = \sum_{i\geq 0} b_{2i}(X(R))_\theta q^i,$$

where $b_{2i}(X(R))_\theta$ denotes the multiplicity of θ in $\mathrm{H}^i(X(R), \mathbb{C})$. Since every representation of S_n is isomorphic to its dual, the Poincaré Duality yields the symmetry property of the numbers $b_{2i}(X(R))_\theta$;

$$b_{2i}(X(R))_\theta = b_{2n-2i}(X(R)_\theta.$$

By averaging we can find a S_{n+1}-invariant ample divisor class on $X(R)$. Applying the Hard Lefschetz Theorem, we obtain the unimodality property of the numbers $b_{2i}(X(R))_\theta$:

$$b_{2i+2}(X(R))_\theta \geq b_{2i}(X(R))_\theta, \qquad i < n.$$

An explicit computation of these numbers using Theorem 2.1 and the character tables of W is a hard computational problem. In the Case A_n we can employ the theory of symmetric functions.

Recall that there is a bijective correspondence between partitions λ of $n+1$ and irreducible representations of S_{n+1}. Let χ^λ denote the corresponding representation. The symmetric function $ch(\chi^\lambda)$ is denoted by s_λ and is called the Schur function of type λ. Applying Theorem 5.3, we get

COROLLARY 5.6 [Stanley [S]). *For every partition* λ *of some number* $n = |\lambda|$ *let* $P_\lambda(q) = P_{A_{|\lambda|},\lambda}(q)$. *Then*

$$\sum_\lambda P_\lambda(q) s_\lambda = \frac{\sum_{i\geq 0} h_i}{1 - q\sum_{i\geq 2}(1 + \dots + q^{i-2})h_i}.$$

The ring of symmetric functions Λ has an inner product such that Schur functions s_λ form an orthonormal basis. This allows one to compute $P_\lambda(q)$ for some small values of n.

6. A Recurrency Formula (Cases B_n and C_n)

The following lemma is an analogue of Lemma 5.8 in Case $B_n(C_n)$:

Lemma 6.1. *There exists a W-equivariant birational morphism of toric varieties $\pi: X(B_n) \to (\mathbb{P}^1)^n$, where $W(B_n)$ acts via its quotient S_n on $(\mathbb{P}^1)^n$ by permuting the factors. It is equal to the composition*

$$X(B_n) = X_1 \xrightarrow{\pi_1} X_2 \longrightarrow \cdots \longrightarrow X_{n-1} \xrightarrow{\pi_{n-1}} X_n = (\mathbb{P}^1)^n,$$

where each π_i is the blowing up of the proper inverse transform under $\pi_{n-1} \circ \ldots \circ \pi_{i-1} : X_{i-1} \to X_n$ of the union of the subvarities defined by vanishing of a projective coordinate in one of $i + 1$ factors. This proper inverse transform is isomorphic to $X(B_{n-i})$.

Proof. Recall that $X(B_n)$ is given by the fan $\Phi = \Phi_1$ whose cones are W-translates of the faces of the fundamental chamber

$$e_1 > \cdots > e_n > 0.$$

We consider the fans Φ_1, $i = 1, \ldots, n$, formed by the cones whose faces are W-translates of the faces of the cone C_i given by the following inequalities:

$$e_k > e_{i+1} > e_{i+2} > \cdots > e_n > 0, k = 1, \ldots, i.$$

It is easy to see that $S_i \subset W$ stabilizes C_i and hence its orbit under W consists of $[W : S_i]$ cones of maximal dimension n of Φ_i. One immediately recognizes the fan Φ_n as the standard fan of the toric variety $(\mathbb{P}^1)^n$. The rest of the assertions are verified similarily to Case A_n, and we leave it to the reader to fill the details.

Remark 6.2. Observe that

$$C_i = \bigcup_{w \in S_i} w(C),$$

where $S_1 \subset S_2 \subset \cdots \subset S_n \subset W$ is the standard inclusion of the permutation groups. According to [FH] this implies that each X_i is equal to the closure of the generic torus orbit in the homogeneous space Spin $(2n + 1)/P_i$, where P_i is the parabolic subgroup of Spin$(2n + 1)$ corresponding to the subset $\{\alpha_1, \ldots, \alpha_{i-1})$ of the root basis $\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1}, \alpha_n = e_n$. The projection Spin$(2n + 1)/P_i \to$ Spin$(2n + 1)/P_{i+1}$ is a $\mathbb{P}^i$-bundle, and induces, by restriction to X_i, our morphism π_i.

Let $\mathrm{Rep}(W(B_n))$ be the Grothendieck group of isomorphism classes of finite-dimensional complex representations of the group $W(B_n)$ and

$$R^B = \oplus_{n\geq 0}\mathrm{Rep}(W(B_n))$$

be the graded ring whose composition law is defined by the formula

$$f\cdot g = \mathrm{ind}_{W(B_k)\times W(B_r)}^{W(B_{k+r})}(f\otimes g),$$

where $f \in \mathrm{Rep}(S_k)$, $g \in \mathrm{Rep}(S_r)$. Here we fix an embedding of $W(B_k)$ (resp. $W(B_r)$) into $W(B_{k+r})$ as the subgroup of permutations and sign changes of the first k numbers $\{1, \ldots, k\}$ (resp. the last r numbers).

This is an associative commutative graded ring with the multiplicative unit defined by the trivial representation of $W(B_0) = \{1\}$.

We denote by $\theta^B_{n,k}$ the element in R corresponding to the representation of $W(B_n)$ in $H^k(X(B_n), \mathbb{C})$. Set

$$Ch^B_n = \sum_{k=0}^{n} \theta^B_{n,k} q^k \in R_B[q],$$

$$Ch^B = 1 + \sum_{i\geq 0} Ch^B_n t^n \in R_B[q][[t]],$$

$$\tilde{H}^B(t) = \sum_{n\geq 0} \tilde{h}^B_n t^n,$$

$$G(t) = \sum_{n\geq 0} g_n t^n$$

where $\tilde{h}^B_n$ is the isomorphism class of the trivial representation of $W(B_n)$, and g_n is the isomorphism class of the representation obtained by inducing the trivial representation of $S_n \subset W(B_n)$.

THEOREM 6.3.

$$Ch^B = \frac{(1-q)\tilde{H}^B(t)\,\tilde{H}^B(qt)}{G(qt) - qG(t)}.$$

Proof. Consider the composition

$$X(B_n) = X_1 \xrightarrow{\pi_1} X_2 \longrightarrow \cdots \longrightarrow X_{n-1} \xrightarrow{\pi_{n-1}} X_n = (\mathbb{P}^1)^n,$$

from Lemma 6.1. For every subset I of $\{1, \ldots, n\}$ and a binary vector $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{0, 1\}^n$ let $L_{I,\varepsilon}$ denote the subvariety of $(\mathbb{P}^1)^n$ given by vanishing of the projective coordinate x_{ε_i} in the ith factor, where i runs through the set I. Let $\overline{L}_{I,\varepsilon}$ be the proper transform of $L_{I,\varepsilon}$ in X_k, $k = \#I$. Let

$$\pi_{I,\varepsilon}: E_{I,\varepsilon} = \pi_{k-1}^{-1}(\overline{L}_{I,\varepsilon}) \to \overline{L}_{I,\varepsilon}$$

be the restriction of π_{k-1} to the exceptional locus of π_{k-1} over $\overline{L}_{I,\varepsilon}$. As in Case A_n, we obtain an isomorphism of graded modules

$$\tilde{H}^*(E_{I,\varepsilon}) \cong \bigoplus_{i=0}^{k-1} \pi_{I,\varepsilon}^*(\tilde{H}^*(\overline{L}_{I,\varepsilon})(-i)),$$

Now the cohomology of X can be computed inductively by using

$$H^*(X_{k-1}) \cong \pi_k^*(H^*(X_k)) \oplus \bigoplus_{\#I=k} H^*(E_I)/\pi_I^*(H^*(\overline{L}_I)).$$

By Lemma 6.1, each $\overline{L}_{I,\varepsilon}$ with $\#I = k$ is isomorphic to $X(B_{n-k})$. The group $W(B_n)$ acts transitively on the set of subvarieties $L_{I,\varepsilon}$ with the isotropy subgroup $W_{I,\varepsilon}$ isomorphic to $S_k \times W(B_{n-k})$. The group $W_{I,\varepsilon}$ acts on the cohomology of $\overline{L}_{I,\varepsilon} \cong X(B_{n-k})$ via $W(B_{n-k})$. This shows that there is a natural isomorphism of $W(B_n)$-modules:

$$\bigoplus_{\#I=k} H^*(E_I)/\pi_I^* H^*((\overline{L}_I)) \cong \sum_{i=1}^{k-1} \operatorname{ind}_{S_k \times W(B_{n-k})}^{W(B_n)} (1 \times H^*(X(B_{n-k}))(-i)).$$

Therefore we obtain the recurrency relation in the ring R^B:

$$[H^r(X_{k-1})] = [H^r(X_k)] + g_k \cdot \sum_{i=1}^{k-1} [H^{r-i}(X(B_{n-k})).$$

This yields

$$\theta_{n,r}^B = [H^r(X(B_n)] = [H^r((\mathbb{P}^1)^n)] + \sum_{k=2}^{n} \sum_{i=1}^{k-1} g_k \theta_{n-k,r-i}^B.$$

To compute $[H^r((\mathbb{P}^1)^n)]$ we observe that by the Kunneth formula

$$H^r((\mathbb{P}^1)^n, \mathbb{C}) \cong \wedge^r \mathbb{C}^n.$$

The group $W(B_n)$ acts on the left-hand side via the natural permutation action on $\mathbb{C}^n$ of its quotient S_n.

This easily gives the following equality in R^B:

$$[H^r((\mathbb{P}^1)^n)] = [\mathrm{ind}_{W(B_r)\times W(B_{n-r})}^{W(B_n)}(1 \times 1)] = \tilde{h}_r^B \tilde{h}_{n-r}^B.$$

From this we get

$$Ch_n^B = \sum_{r=0}^{n} \theta_{n,r}^B q^r = \sum_{i=0}^{n} q^r \tilde{h}_r^B \tilde{h}_{n-r}^B) + g_2 q Ch_{n-2}^B + g_n(q + \cdots + q^{n-1}) Ch_0^B$$

$$= \sum_{i=0}^{n} q^r \tilde{h}_r^B \tilde{h}_{n-r}^B + \left(\frac{q-q^2}{1-q}\right) Ch_{n-2}^B g_2 + \cdots + \left(\frac{q-q^n}{1-q}\right) g_n Ch_0^B.$$

Multiplying both sides by t^n we obtain:

$$Ch_n^B t^n = \sum_{i=0}^{n} q^r \tilde{h}_r^B \tilde{h}_{n-r}^B t^n + g_2\left(\frac{qt^2 - (qt)^2}{1-q}\right) Ch_{n-2}^B t^{n+2}$$

$$+ \cdots + g_n\left(\frac{qt^n - (qt)^n}{1-q}\right) Ch_0^B.$$

Setting

$$\Phi(t) = \frac{qG(t) - G(qt)}{1-q},$$

and adding up, we get

$$Ch^B = (1 + \Phi) Ch^B + \tilde{H}^B(t) \tilde{H}^B(qt),$$

which gives us the asserted formula.

APPENDIX

Here we prove Theorem 1.1 and Lemma 3.1.

We shall begin with the former. Let us recall the geometric description of the complex $K^{\cdot}$. For details the reader is referred to [BL].

Let

$$X = X(R) = \prod_{\sigma\in\Phi} O^\sigma$$

be the decomposition of X into the union of the torus orbits. The correspondence $\sigma \to O^\sigma$ between the cones in the fan Φ and the orbits satisfies the following properties:

(i) $\dim_{\mathbb{C}} O^\sigma = \operatorname{codim}_{\mathbb{R}} \sigma$;

(ii) $\sigma' \subset \sigma \Leftrightarrow \overline{O}^\sigma \subset \overline{O}^{\sigma'}$, where the bar denotes the closure;

(iii) the linear span of σ in V^* equals the real part of the Lie algebra of the stabilizer of O^σ in T.

Here we identify the root lattice M with the character lattice of T, the root lattice N with the lattice of one-parametric subgroups of T, and the space V^* with the real part of the Lie algebra of T. Consider the following filtration of X by open T-invariant subvarieties

$$T = U^r \subset U^{r-1} \subset \cdots \subset U^0 = X,$$

where

$$U^i = \coprod_{\operatorname{codim}\sigma \leq i} O^\sigma, \qquad i = 0, \ldots, r.$$

Let

$$Z_i = U^i \backslash U^{i+1} = \coprod_{\operatorname{codim}\sigma = i} O^\sigma$$

be the T-invariant locally closed subvarieties in X.

Let $\mathbf{E}T \to \mathbf{B}T$ be the universal principal T-bundle introduced at the beginning of the paper. For any T-space Y we denote by Y_T the quotient space $Y \times {}_T E = Y \times E/T$ (T acts diagonally). Then Y_T is a fibre bundle over $\mathbf{B}T$ with the fibre Y. We have the filtration of X_T by open subbundles:

$$U_T^r \subset U_T^{r-1} \subset \cdots \subset U_T^0 = X_T.$$

Clearly,

$$Z_T^i = U_T^i \backslash U_T^{i+1} = \coprod_{\operatorname{codim}\sigma = i} O_T^\sigma.$$

For a subset $P \subset X_T$ we denote its inclusion morphism in X_T by i_p: $P \hookrightarrow X_T$. Let $\mathbb{C}_p$ be the constant sheaf on P. Denote by $i_{p!}\mathbb{C}_p$ its extension by zero to X_T. With the above notation we have the following filtration of the constant sheaf $\mathbb{C}_{X_T}$ by subsheaves:

$$i_{U_T^r!} \cdot \mathbb{C}_{U_T^r} \subset \cdots \subset i_{U_T^0!} \cdot \mathbb{C}_{U_T^0} \subset \mathbb{C}_{X_T}. \tag{1}$$

Note that

$$i_{U_T^k!}\mathbb{C}_{U_T^k}/i_{U_T^{k+1}!}\mathbb{C}_{U_T^{k+1}} \cong i_{Z_T^k!}\mathbb{C}_{Z_T^k} \cong \bigoplus_{\dim O=k} i_{O^!}\mathbb{C}_{O_T}.$$

If we apply the cohomological functor $H^*_{T,c}$ to the filtered sheaf (1), we get a spectral sequence whose E_1-term is the complex of graded H-modules

$$0 \to H^*_{T,c}(Z_0) \to H^*_{T,c}(Z_1)[1] \to \cdots \to H^*_{T,c}(Z_r)[r] \to 0, \qquad (2)$$

where the shifted module $M[k]$ for a graded module M is defined as usually by $M[k]_i = M_{i+k}$. Note that

$$H^*_{T,c}(Z_k) = \bigoplus_{\dim O=k} H^*_{T,c}(O)$$

so (2) becomes the following complex

$$0 \to \bigoplus_{\dim O=0} H^*_{T,c}(O) \to \bigoplus_{\dim O=1} H^*_{T,c}(O) \to \cdots \to H^*_{T,c}(O)[r] \to 0, \qquad (3)$$

where the differentials correspond to attaching maps of one orbit to another. Let $\sigma \in \Phi$ be the cone corresponding to an orbit $O \subset X$. It is shown in [BL] that $H^*_{T,c}(O) \cong P_\sigma$, and that the complex (3) is the complex $K^\cdot$ of the theorem that satisfies all properties of part 1 (after we identify H and A).

Recall that T is a maximal torus in a semi-simple algebraic group G. Let $N(T)$ be the normalizer of T in G, so that $W = N(T)/T$ is the Weyl group. In order to prove part 2 it suffices to show two things:

(i) the normalizer $N(T)$ acts naturally on the fibration $X_T \to \mathbf{B}T$, preserving all the structures that led to the filtration (1) (note that the connected component T of $N(T)$ acts trivially on the cohomology);

(ii) the induced action of W on H is compatible with its natural action on A under the identification of H with A.

Let E be the universal space for G, and hence also for T. The group $N(T)$ then acts on the T-universal fibration $E \to \mathbf{B}T$. This induces the action of W on $H = H^*(BT, \mathbb{C})$. On the other hand, W acts naturally on the ring A of polynomial functions on the Lie algebra of the torus T, and hence on its real part V^*. It is shown in [B], Section 27 that the two actions of W agree under the identification $H = A$. This proves (ii).

The Weyl group W acts on the toric variety X preserving the subspaces U^k and Z^k (since it acts on the fan Φ). Hence also the normalizer $N(T)$

acts on X via the quotient $N(T)/T = W$. Therefore $N(T)$ acts diagonally on the product $X \times E$. This action preserves T-orbits. Indeed, for $n \in N(T)$, $x \in X$, $e \in E$, $t \in t$

$$n(tx, te) = (ntx, nte) = ((ntn^{-1})nx, (ntn^{-1})ne) = (ntn^{-1})(nx, ne).$$

Hence $N(T)$ also acts on the quotient space $X \times {}_T E = X_T$, preserving the subspaces U^k and Z^k. The natural map $X_T \to BT$ commutes with the action of $N(T)$, which proves (i) and completes the proof of Theorem 1.1.

Proof of Lemma 3.1. We shall use the notation introduced in the proof of Theorem 2.1. Let $\sigma \in \Phi$ be a cone and $W_\sigma \subset W$ be its stabilizer. Let $O \subset X$ be the T-orbit corresponding to the cone σ. Then the $A(=H)$-module P_σ in the complex $K^\cdot$ is the equivariant cohomology with compact supports $H^*_{T,c}(O)[r\text{-dim}O]$(see the proof of Theorem 1 above and [BL]). The stabilizer W_σ acts on the graded module and we want to determine the action on the 1-dimensional space $H^O_{T,C}(O)[r\text{-dim}O]$. Let us recall how $H^*_{T,c}(O)$ is computed.

Consider the fibration $O_T \xrightarrow{\alpha} BT$ with the fibre O, and let $\alpha_! \mathbb{C}_{O_T}$ be the direct image with compact supports of the constant sheaf on O_T. Then we have

$$H^q_c(O) = 0 \qquad \text{if } q < \dim O, \qquad \text{and } \dim H^{\dim O}_c(O) = 1,$$

and the spectral sequence

$$E_2^{p,q} = H^p(\mathbf{B}T, H^q_c(O)) \Rightarrow H^O_{T,c}(O)$$

shows that $H^O(\mathbf{B}T, H^{\dim O}_c(O))$ maps isomorphically to $H^O_{T,c}(O))$ in the limit.

The stabilizer W_σ acts on the above spectral sequence, hence its action on $H^O_{T,c}(O)$ is isomorphic to its action on $H^{\dim O}_c(O)$. By the Poincaré duality

$$H^{\dim O}_c(O)) \cong H_{\dim O}(O)^*.$$

Hence it suffices to prove that W_σ acts on the top homology $H_{\dim O}(O)$ by the sign representation, which is obvious. This proves Lemma 3.1.

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