Lecture 14. YANG-MILLS EQUATIONS

These equations arise as the Euler-Lagrange equations for gauge fields with a special Lagrangian. A particular case of these equations is the Maxwell equations for electromagnetic fields.

14.1 Let G be a Lie group and P be a principal G-bundle over a d-dimensional manifold M. Recall that a *gauge field* is a connection on P. It is given by a 1-form A on M with values in the adjoint affine bundle $Ad(P) \subset End(\mathfrak{g})$. Here affine means that the transition functions belong to the affine group of the Lie algebra $\mathfrak{g}$ of G. More precisely, the transition functions are given by

$$A_e = g \cdot A_{e'} \cdot g^{-1} + g^{-1}dg,$$

where A_e and $A_{e'}$ are the $\mathfrak{g}$-valued 1-forms on M corresponding to A via two different trivializations of P (*gauges*).

The curvature of a gauge field A is a section F_A of the vector bundle $\Lambda^2(ad(E)) = \bigwedge^2 T^*(M) \otimes Ad(E)$. It can be defined by the formula

$$F_A = dA + \frac{1}{2}[A, A].$$

If we fix a gauge, then $F = F_A$ is given by an $n \times n$ matrix whose entries F_i^j are smooth 2-forms on U. If $(x^1, \ldots, x^d)$ is a system of local parameters in U, then we can write

$$F_i^j = \sum_{\mu,\nu=1}^{d} F_{i,\mu\nu}^j(x) dx^\mu \wedge dx^\nu$$

Equivalently we can write

$$F = (F_{\mu\nu})_{1 \le \mu,\nu \le d},$$

where $F_{\mu\nu}$ is the matrix with (ij)-entry equal to $F_{i,\mu\nu}^j(x)$. We can start with any $Ad(P)$-valued 2-form F as above, and say that a gauge field A is a *potential* gauge field for F if $F = F_A$.

Let $E = E(\rho) \to M$ be an associated vector bundle on M with respect to some linear representation $\rho : G \to GL(V)$. The corresponding representation of the Lie algebra $d\rho : \mathfrak{g} \to End(V)$ defines a homomorphism of vector bundles $Ad(P) \to End(E)$. Applying this homomorphism to the values of A and F_A, we obtain the notion of a connection and its curvature in E.

For example, let us take $G = U(1)$ and let P be the trivial principal G-bundle. In this case $Ad(P)$ is the trivial bundle $M \times \mathbb{R}$. So a gauge field is a 1 form $\sum_{\mu=0}^{3} A_\mu dx^\mu$ which can be identified with a vector function

$$A = (A_0, A_1, A_2, A_3).$$

For any smooth function $\phi : M \to U(1)$, the 1-form

$$A' = A + d \log \phi$$

defines the same connection. Its curvature is a smooth 2-form

$$F = \sum_{\lambda,\nu=0}^{3} F_{\mu\nu} dx^\mu \wedge dx^\nu.$$

We shall identify it with the skew symmetric matrix $(F_{\mu\nu})$ whose entries are smooth functions in (t, x):

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & H_3 & -H_2 \\ E_2 & -H_3 & 0 & H_1 \\ E_3 & H_2 & -H_1 & 0 \end{pmatrix}. \tag{14.1}$$

For a reason which will be clear later, this is called the *electromagnetic tensor*. Since $[A, A] = 0$, we have

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{14.2}$$

Let us write

$$A = (\phi, \mathbf{A}) \in C^\infty(\mathbb{R}^4) \times (C^\infty(\mathbb{R}^4))^3.$$

This gives

$$F_{0\nu} = \partial_0 A_\nu - \partial_\nu \phi, \nu = 0, 1, 2, 3,$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \mu, \nu = 1, 2, 3.$$

If we set

$$\mathbf{E} = (E_1, E_2, E_3), \quad \mathbf{H} = (H_1, H_2, H_3),$$

we can rewrite the previous equalities in the form

$$\mathbf{E} = \text{grad}_x \phi - \partial_0 \mathbf{A} = (\partial_1 \phi, \partial_2 \phi, \partial_3 \phi) - (\partial_0 A_1, \partial_0 A_2, \partial_0 A_3), \tag{14.3}$$

$$\mathbf{H} = \text{curl } \mathbf{A} = \nabla \times (A_1, A_2, A_3), \tag{14.4}$$

where

$$\nabla = (\partial_1, \partial_2, \partial_3).$$

Of course, equation (14.2) means that the differential form F_A satisfies

$$F_A = d(A_0 dx^0 + A_1 dx^1 + A_2 dx^2 + A_3 dx^3).$$

Since $\mathbb{R}^4$ is simply-connected, this occurs if and only if

$$dF = d(-E_1 dx^0 \wedge dx^1 - E_2 dx^0 \wedge dx^2 - E_3 dx^0 \wedge dx^3 + H_3 dx^1 \wedge dx^2 - H_2 dx^1 \wedge dx^3 + H_1 dx^2 \wedge dx^3)$$

$$= (-\partial_2 E_1 + \partial_1 E_2 + \partial_0 H_3) dx^0 \wedge dx^1 \wedge dx^2 + (-\partial_3 E_1 + \partial_1 E_3 - \partial_0 H_2) dx^0 \wedge dx^1 \wedge dx^3 +$$
$$+ (-\partial_3 E_2 + \partial_1 E_3 + \partial_0 H_3) dx^0 \wedge dx^2 \wedge dx^3 + \mathrm{div} H dx^1 \wedge dx^2 \wedge dx^3 = 0.$$

This is equivalent to

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{H}}{\partial t} = 0, \qquad (M1)$$

$$\mathrm{div}\mathbf{H} = \nabla \cdot \mathbf{H} = 0. \qquad (M2)$$

This is the first pair of Maxwell's equations. The second pair will follow from the Euler-Lagrange equation for gauge fields.

14.2 Let us now introduce the Yang-Mills Lagrangian defined on the set of gauge fields. Its definition depends on the choice of a pseudo-Riemannian metric g on M which is locally given by

$$g = \sum_{\nu,\mu=1}^{n} g_{\mu\nu} dx^\nu \otimes dx^\mu.$$

Its value at a point $x \in M$ is a non-degenerate quadratic form on $T(M)_x$. The dual quadratic form on $T^*(M)_x$ is given by the inverse matrix $(g^{\mu\nu}(x))$. It defines a symmetric tensor

$$g^{-1} = \sum_{\nu,\mu=1}^{n} g^{\mu\nu} \frac{\partial}{\partial x^\nu} \otimes \frac{\partial}{\partial x^\mu}.$$

We can use g to transform the $\mathfrak{g}$-valued 2-form $F = (F_{\mu\nu})$ to the $\mathfrak{g}$-valued vector field

$$\hat{F} = (F^{\mu\nu}) = (g^{\mu\alpha} g^{\nu\beta} F_{\beta\alpha}) = \sum_{\mu,\nu=1}^{n} F^{\mu\nu} \frac{\partial}{\partial x^\nu} \otimes \frac{\partial}{\partial x^\mu}.$$

Let $\mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ be the Killing form on $\mathfrak{g}$. It is defined by

$$\langle A, B \rangle = -Tr(ad(A) \circ ad(B)),$$

where $ad : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is the adjoint representation, $ad(A) : X \to [A, X]$. This is a bilinear form which is invariant with respect to the adjoint representation $Ad : G \to GL(\mathfrak{g})$. In the case when G is semisimple, the Killing form is non-degenerate (in fact, this is one of many equivalent definitions of the semi-simplicity of a Lie group). If furthermore, G is compact (e.g. $G = SU(n)$), it is also positive definite. The latter explains the minus sign in the formula.

Now we can form the scalar function

$$\langle F, \hat{F}\rangle := \sum_{\mu,\nu=1}^{n} \langle F_{\mu\nu}, F^{\mu\nu}\rangle. \tag{14.5}$$

It is easy to see that this expression does not depend on the choice of coordinate functions. Neither does it depend on the choice of the gauge. Now if we choose the volume form $\mathrm{vol}(g)$ on M associated to the metric g (we recall its definition in the next section), we can integrate (14.5) to get a functional on the set of gauge fields

$$S_{YM}(A) = \int_M \langle F, \hat{F}\rangle \mathrm{vol}(g). \tag{14.6}$$

This is called the *Yang-Mills action* functional.

14.3 One can express (14.5) differently by using the star operator $*$ defined on the space of differential forms. Recall its definition. Let V be a real vector space of dimension n, and $g : V \times V \to \mathbb{R}$ (equivalently, $g : V \to V^*$) be a metric on V, i.e. a symmetric non-degenerate form on V. Let $e_1, \ldots, e_n$ be an orthonormal basis in V with respect to $\mathbb{R}$. This means that $g(e_i, e_j) = \pm\delta_{ij}$. Let $e^1, \ldots, e^n$ be the dual basis in V^*. The n-form $\mu = e^1 \wedge \ldots \wedge e^n$ is called the *volume form* associated to g. It is defined uniquely up to ± 1. A choice of the two possible volume elements is called an orientation of V. We shall choose a basis such that $\mu(e_1, \ldots, e_n) > 0$. Such a basis is called positively oriented.

Let W be a vector space with a bilinear form $B : W \times W \to \mathbb{R}$. We extend the operation of exterior product $\bigwedge^k V \times \bigwedge^s V \to \bigwedge^{k+s} V$ by defining

$$(\bigwedge^k V \otimes W) \times (\bigwedge^s V \otimes W) \to \bigwedge^{k+s} V$$

using the formula

$$(\alpha \otimes w) \wedge (\alpha' \otimes w') = B(w, w')\alpha \wedge \alpha'.$$

Also, if $g : V \to V$ is a bilinear form on V, we extend it to a bilinear form on $\bigwedge^k V \otimes W$ by setting

$$g(\alpha \otimes w, \alpha' \otimes w') = \wedge^k(g)(\alpha, \alpha')B(w, w'),$$

where we consider g as a linear map $V \to V^*$ so that $\wedge^k(g)$ is the linear map $\bigwedge^k V \to \bigwedge^k V^*$ which is considered as a bilinear form on $\bigwedge^k V$.

Lemma 1. *Let g be a metric on V and $\mu \in \bigwedge^n(V^*)$ be a volume form associated to g. Let W be a finite-dimensional vector space with a non-degenerate bilinear form $B : W \times W \to \mathbb{R}$. Then there exists a unique linear isomorphism $* : \bigwedge^k V^* \otimes W \to \bigwedge^{n-k} V^* \otimes W$, such that, for all $\alpha, \beta \in Hom(\bigwedge^k V, W)$,*

$$\alpha \wedge *\beta = g^{-1}(\alpha, \beta)\mu. \tag{14.7}$$

Here $g^{-1} : V^* \times V^* \to \mathbb{R}$ is the inverse metric.

Proof. Consider $\gamma \in \bigwedge^{n-k} V^* \otimes W$ as a linear function $\phi_\gamma : \bigwedge^k V^* \otimes W \to \mathbb{R}$ which sends $\omega \in \bigwedge^k V^* \otimes W$ to the number $\phi_\gamma(\omega)$ such that $\gamma \wedge \omega = \phi_\gamma(\omega)\mu$. It is easy to see that $\beta \to \phi_\beta$ establishes a linear isomorphism from $\bigwedge^{n-k} V^* \otimes W$ to $(\bigwedge^k V^* \otimes W)^* \cong \bigwedge^k V \otimes W$. On the other hand, we have an isomorphism $\wedge^k(g^{-1}) : \bigwedge^k V^* \otimes W \to \bigwedge^k V \otimes W$. Thus if, for a fixed $\beta \in \bigwedge^k V^* \otimes W$ we define a linear function $\alpha \to g^{-1}(\alpha, \beta)$, there exists a unique $*\beta \in \bigwedge^{n-k} V^* \otimes W$ such that (14.7) holds.

The assertion of the previous lemma can be "globalized" by considering metrics $g : T(M) \to T^*(M)$ and vector bundles over oriented manifolds. We obtain the definition of the associated volume form $\text{vol}(g)$, and the star operator

$$* : \Lambda^k(E) \to \Lambda^{n-k}(E)$$

satisfies (14.7) at each point $x \in M$. Here we assume that the vector bundle E is equipped with a bilinear map $B : E \times E \to 1_M$ such that the corresponding morphism $E \to E^*$ is bijective. We also can define the star operator on the sheaves of sections of $\Lambda^k(E)$ to get a morphism of sheaves:

$$* : \mathcal{A}^k(E) \to \mathcal{A}^{n-k}(E).$$

We shall apply this to the special case when E is the adjoint bundle. Its typical fibre is a Lie algebra $\mathfrak{g}$ and the bilinear form is the Killing form. The non-degeneracy assumption requires us to assume further that $\mathfrak{g}$ is a semi-simple Lie algebra.

In particular, taking $\mathfrak{g} = \mathbb{R}$, we obtain that $\mathcal{A}^k(E)(M) = \mathcal{A}^k(M)$ is the space of k-differential forms and $*$ is the usual star operator introduced in analysis on manifolds. Note that in this case

$$*1 = \text{vol}(g). \tag{14.8}$$

In our situation, $k = 2$ and we have $F_A \in \mathcal{A}^2(Ad(P)(M), *F_A \in \mathcal{A}^{n-2}(Ad(P)(M)$ and

$$F_A \wedge *F_A = g^{-1}(F_A, F_A)\text{vol}(g) = \langle F, \hat{F} \rangle \text{vol}(g).$$

Here we use the following explicit expression for $g^{-1}(\alpha, \beta), \alpha, \beta \in \bigwedge^k(V^*) \otimes W$:

$$g^{-1}(\alpha, \beta) = \sum g^{i_1 j_1} \dots g^{i_k j_k} B(\alpha_{i_1 \dots i_k}, \beta_{j_1 \dots j_k}),$$

where $\alpha = \sum e^{i_1} \wedge \dots e^{i_k} \otimes \alpha_{i_1 \dots i_k} \in \bigwedge^k(V^*) \otimes W$, and there is a similar expression for β.

Assume that we have a coordinate system where $g_{ij} = \pm\delta_{ij}$ (a *flat* coordinate system). Then

$$g^{-1}(dx^{i_1} \wedge \dots \wedge dx^{i_k}, dx^{j_1} \wedge \dots \wedge dx^{j_k}) = g^{i_1 j_1} \dots g^{i_k j_k}.$$

This implies that

$$*(F_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}) = g^{i_1 i_1} \dots g^{i_k i_k} F_{i_1 \dots i_k} dx^{j_1} \wedge \dots \wedge dx^{j_{n-k}}, \tag{14.9}$$

where

$$dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_{n-k}} = dx^1 \wedge \ldots \wedge dx^n.$$

If $g^{ij} = \delta_{ij}$ then this formula can be written in the form

$$(*F)_{j_1 \ldots j_{n-k}} = \epsilon(i_1, \ldots, i_k, j_1, \ldots, j_{n-k}) F_{i_1 \ldots i_k} dx^{j_1} \wedge \ldots \wedge dx^{j_{n-k}},$$

where $\epsilon(i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$ is the sign of the permutation $(i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$.

If G is a compact semi-simple group we may consider the unitary inner product in the space $\mathcal{A}^k(Ad(P))(M)$

$$\langle F_1, F_2 \rangle = \int_M F_1 \wedge *F_2. \tag{14.10}$$

Lemma 2. *Assume M is compact, or F, G vanish on the boundary of M. Let A be a connection on P. Then, for any $F_1 \in \mathcal{A}^k(Ad(P))(M), F_2 \in \mathcal{A}^{k+1}(Ad(P))(M)$,*

$$\langle d^A(F_1), F_2 \rangle = (-1)^{k+1} \langle F_1, d^A * F_2 \rangle.$$

Proof. By Theorem 1 from Lecture 12, we have $d^A(F_1) = dF_1 + [A, F_1]$. Since $d(F_1 \wedge *F_2)$ is a closed n-form, by Stokes' theorem, its integral over M is equal to the integral of $F_1 \wedge *F_2$ over the boundary of M. By our assumption, it must be equal to zero. Also, since the Killing form $\langle\ ,\ \rangle$ is invariant with respect to the adjoint representation, we have $\langle [a, b], c \rangle = \langle [c, a], b \rangle$ (check it in the case $\mathfrak{g} = \mathfrak{gl}_n$ where we have $Tr((ab - ba)c) = Tr(abc - bac) = Tr(abc) - Tr(bac) = Tr(bca) - Tr(bac) = Tr(b(ca - ac)) = \langle b, [c, a] \rangle$). Using this, we obtain

$$[A, F_1] \wedge *F_2 = (-1)^{k+1} F_1 \wedge [A, *F_2].$$

Collecting this information, we get

$$\langle d^A(F_1), F_2 \rangle = \langle dF_1 + [A, F_1], F_2 \rangle = \int_M dF_1 \wedge *F_2 + [A, F_1] \wedge *F_2 =$$

$$\int_M d(F_1 \wedge *F_2) - (-1)^k (F_1 \wedge d * F_2 + F_1 \wedge [A, *F_2])$$

$$= \int_M dF_1 \wedge *F_2 - (-1)^k \int_M F_1 \wedge d * F_2 + F_1 \wedge [A, *F_2] =$$

$$= (-1)^{k+1} \int_M F_1 \wedge d^A * F_2 = (-1)^{k+1} \langle F_1, d^A * F_2 \rangle.$$

14.4 Let us find the Euler-Lagrange equation for the Yang-Mills action functional

$$S_{YM}(A) = \int_M \langle F_A, \hat{F}_A \rangle \mathrm{vol}(g) = \int_M F_A \wedge *F_A = ||F_A||^2,$$

where the norm is taken in the sense of (14.10). We know that the difference of two connections is a 1-form with values in $Ad(P)$. Applying Theorem 1 (iii) from lecture 12, we have, for any $h \in \mathcal{A}^1(Ad(P))$ and any $s \in \Gamma(E)$,

$$F_{A+h} = d(A+h) + \frac{1}{2}[A+h, A+h] = F_A + dh + [A,h] + o(||h||) = F_A + d^A(h).$$

Now, ignoring terms of order $o(||h||)$, we get

$$S_{YM}(A+h) - S_{YM}(A) = ||F_{A+h}||^2 - ||F_A||^2 = ||F_{A+h} - F_A||^2 + 2\langle F_A, F_{A+h} - F_A\rangle =$$

$$= 2\int_M d^A h \wedge *F_A = -2\int_M h \wedge d^A(*F_A).$$

Here, at the last step, we have used Lemma 2. This implies the equation for a critical connection A:

$$d^A(*F_A) = 0 \tag{14.11}$$

It is called the *Yang-Mills* equation. Note, by Bianchi's identity, we always have $d^A(F_A) = 0$.

In flat coordinates we can use (14.9) to find an explicit expression for the coordinate functions $F_{\mu\nu}$ of $*F$. Then explicitly equation (14.11) reads

$$\sum_\mu \frac{\partial F_{\mu\nu}}{\partial x^\nu} + [A_\nu, F_{\mu\nu}] = 0. \tag{14.12}$$

This is a second-order differential equation in the coordinate functions A_ν of the connection A.

Definition. A connection satisfying the Yang-Mills equation (14.11) is called a *Yang-Mills connection.*

14.5 Note the following property of the star operator $* : \bigwedge^k V^* \to \bigwedge^{n-k} V^*$:

$$* \circ * = \epsilon(g)(-1)^{k(n-k)}\mathbf{id}, \tag{14.13}$$

where $\epsilon(g)$ is equal to the determinant of the Gram-Schmidt matrix of the metric g with respect to some orthonormal basis.

In particular, if $n = 4$ and $k = 2$, and g is a Riemannian metric, we have

$$*^2 = \mathbf{id}.$$

This allows us to decompose the space $\mathcal{A}^2(Ad(P))(M)$ into two eigensubspaces

$$\mathcal{A}^2(Ad(P))(M) = \mathcal{A}^2(Ad(P))(M)_+ \oplus \mathcal{A}^2(Ad(P))(M)_-.$$

A connection A with $F_A \in \mathcal{A}^2(Ad(P))(M)_+$ (resp. $F_A \in \mathcal{A}^2(Ad(P))(M)_-$) is called *self-dual* (resp. *anti-self-dual*) connection. Let us write $F_A = F_A^+ + F_A^-$, where

$$*(F_A^{\pm}) = \pm F_A.$$

Thus we have

$$S_{YM}(A) = ||F_A^+ + F_A^-||^2 = ||F_A^+||^2 + ||F_A^-||^2 + 2\langle F_A^+, F_A^-\rangle = ||F_A^+||^2 + ||F_A^-||^2, \quad (14.14)$$

where we use that $\langle F_A^+, F_A^-\rangle = -\langle F_A^+, *F_A^-\rangle = -\langle *F_A^+, F_A^-\rangle = -\langle F_A^+, F_A^-\rangle$.

Recall that by the De Rham theorem

$$H^k(M,\mathbb{R}) = \mathrm{Ker}(d : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M))/\mathrm{Im}(d : \mathcal{A}^{k-1}(M) \to \mathcal{A}^k(M)).$$

Let E be a complex rank r vector bundle associated to a principal G-bundle P. Pick a connection form A on P and let Let F_A be the curvature form of the associated connection. When we trivialize E over an open subset U, we can view F_A as a $r \times r$ matrix X with coefficients in $\mathcal{A}^2(U)$. Let $T = (T_{ij}), i, j = 1, \ldots, r$ be a matrix whose entries are variables T_{ij}. Consider the characteristic polynomial

$$\det(T - \lambda I_r) = (-\lambda)^r + a_1(T)(-\lambda)^{r-1} + \ldots + a_r(T).$$

Its coefficients are homogeneous polynomials in the variables T_{ij} which are invariant with respect to the conjugation transformation $T \to C \cdot T \cdot C^{-1}, C \in GL(n, K)$, where K is any field. If we plug in the entries of the matrix F_A^{ρ} in $a_k(T)$ we get a differential form $a_k(F_A)$ of degree $2k$. By the invariance of the polynomials a_k, the form $a_k(F_A)$ does not depend on the choice of trivialization of E. Moreover, it can be proved that the form $a_k(F_A)$ is closed. After we rescale F_A by multiplying it by $\frac{i}{2\pi}$ we obtain the cohomology class

$$c_k(E) = [a_k(\frac{i}{2\pi}F_A)] \in H^{2k}(M,\mathbb{R}).$$

It is called the kth Chern class of E and is denoted by $c_k(E)$. One can prove that it is independent of the choice of the connection A, so it can be denoted by $c_k(E)$. Also one proves that

$$c_k(E) \in H^{2k}(M,\mathbb{Z}) \subset H^{2k}(M,\mathbb{R}).$$

Assume $G = SU(2)$ and take $E = E(\rho)$, where ρ is the standard representation of G in $\mathbb{C}^2$. Then $\mathrm{Lie}(G)$ is isomorphic to the algebra of matrices $X \in M_2(\mathbb{C})$ with $X^t + \bar{X} = 0$. The Killing form is equal to $-4Tr(AB)$. Then

$$F_A^{\rho} = \begin{pmatrix} F_1^1 & F_1^2 \\ F_2^1 & F_2^2 \end{pmatrix},$$

where $F_1^1 + F_2^2 = 0, F_1^2 = \bar{F}_2^1, F_1^1 \in i\mathbb{R}$. This shows that

$$c_1(E) = 0,$$

$$c_2(E) = -\frac{1}{4\pi^2}[\det F_A] = -\frac{1}{32\pi^2}[F_A \wedge F_A],$$

Here we use that $Tr(X^2) = -2\det(X)$ for any 2×2 matrix X with zero trace. Now, omitting the subscript A,

$$F \wedge F = (F^+ + F^-) \wedge (F^+ + F^-) = F^+ \wedge F^+ + F^- \wedge F^- + F^+ \wedge F^- + F^- \wedge F^+ =$$

$$= F^+ \wedge F^+ + F^- \wedge F^- = F^+ \wedge *F^+ - F^- \wedge *F^-.$$

Here we use that

$$F^+ = F_{01}(dx^0 \wedge dx^1 + dx^2 \wedge dx^3) + F_{02}(dx^0 \wedge dx^2 + dx^3 \wedge dx^1) + F_{03}(dx^0 \wedge dx^3 + dx^1 \wedge dx^2),$$

$$F^- = F_{01}(dx^0 \wedge dx^1 - dx^2 \wedge dx^3) + F_{02}(dx^0 \wedge dx^2 - dx^3 \wedge dx^1) - F_{03}(dx^0 \wedge dx^3 + dx^1 \wedge dx^2),$$

which easily implies that $F^+ \wedge F^- = 0$. From this we deduce

$$-32\pi^2 k(A) = \int_M (F_A^+ \wedge *F_A^+ - F_A^- \wedge *F_A^-) = ||F_A^+||^2 - ||F_A^-||^2,$$

where

$$k(E) = \int_M c_2(E). \tag{14.15}$$

Hence, using (14.14), we obtain

$$S_{YM}(A) = 2||F_A^+||^2 + (||F_A^-||^2 - ||F_A^+||^2) \geq 32\pi^2 k(A) \quad \text{if } k(E) > 0$$

with equality iff $F^+ = 0$. Similarly,

$$S_{YM}(A) = 2||F_A^-||^2 + (||F_A^+||^2 - ||F_A^-||^2) \geq -32\pi^2 \quad \text{if } k(E) < 0$$

with equality iff $F^- = 0$. Also, we see that A is a Yang-Mills connection if it is self-dual or anti-self-dual. The condition for this is the first order *self-dual (anti-self-dual) Yang-Mills equation*

$$F = \pm * F. \tag{14.16}$$

Definition. An *instanton* is a principal $SU(2)$-bundle P over a four-dimensional Riemannian manifold with $k(Ad(P)) > 0$ together with an anti-self-dual connection. The number $k(Ad(P)))$ is called the *instanton number.*

Note that all complex vector G-bundles of the same rank and the same Chern classes are isomorphic. So, let us fix one principal $SU(2)$-bundle P with given instanton number k and consider the set of all anti-self-dual connections on it. The group of gauge transformations acts naturally on this set, we can consider the moduli space

$$\mathcal{M}_k(M) = \{\text{anti-self-dual connections on } P\}/\mathcal{G}.$$

The main result of Donaldson's theory is that this moduli space is independent of the choice of a Riemannian metric and hence is an invariant of the smooth structure on M. Also, when M is a smooth structure on a nonsingular algebraic surface, the moduli space $\mathcal{M}_k(M)$ can be identified with the moduli space of stable holomorphic rank 2 bundles with first Chern class zero and second Chern class equal to k.

14.6 It is time to consider an example. Let

$$M = S^4 := \{(x_1, \ldots, x_5) \in \mathbb{R}^5 : \sum_{i=1}^{5} x_i^2 = 1\}$$

be the four-dimensional unit sphere with its natural metric induced by the standard Euclidean metric of $\mathbb{R}^5$. Let $N_+ = (0,0,0,0,1)$ be its "north pole" and $N_- = (0,0,0,0,-1)$ be its "south pole". By projecting from the poles to the hyperplane $x_5 = 0$, we obtain a bijective map $S^4 \setminus \{N_\pm\} \to \mathbb{R}^4$. The precise formulae are

$$(u_1^\pm, u_2^\pm, u_3^\pm, u_4^\pm) = \frac{1}{1 \pm x_5}(x_1, x_2, x_3, x_4).$$

This immediately implies that

$$(\sum_{i=1}^{4}(u_i^+)^2)(\sum_{i=1}^{4}(u_i^-)^2 = \frac{\sum_{i=1}^4 x_i^2}{(1-x_5^2)} = \frac{1-x_5^2}{1-x_5^2} = 1.$$

After simple transformations, this leads to the formula

$$u_i^- = \frac{u_i^+}{\sum_{i=1}^4 (u_i^+)^2.}, \quad i = 1,2,3,4. \tag{14.17}$$

Thus we can cover S^4 with two open sets $U_+ = S^4 \setminus \{N_+\}$ and $U_- = S^4 \setminus \{N_-\}$ diffeomeorphic to $\mathbb{R}^4$. The transition functions are given by (14.17). We can think of S^4 as a compactification of $\mathbb{R}^4$. If we identify the latter with U_+, then the point N_+ corresponds to the point at infinity ($u_i^+ \to \infty$). This point is the origin in the chart U_-.

Let P be a principal $SU(2)$-bundle over S^4. It trivializes over U_+ and U_-. It is determined by a transition function $g : U_+ \cap U_- \to SU(2)$. Restricting this function to the sphere $S^3 = S^4 \cap \{x_5 = 0\}$, we obtain a smooth map

$$g : S^3 \to SU(2).$$

Both S^3 and $SU(2)$ are compact diffeomorphic three-dimensional manifolds. The homotopy classes of such maps are classified by the degree $k \in \mathbb{Z}$ of the map. Now, $H^2(S^4, \mathbb{Z}) = 0$, so the first Chern class of P must be zero. One can verify that the number k can be identified with the second Chern number defined by (14.15).

Let us view $\mathbb{R}^4$ as the algebra $\mathbb{H}$ of quaternions $x = x_1 + x_2\mathbf{i} + x_2\mathbf{j} + x_4\mathbf{k}$. The Lie algebra of $SU(2)$ is equal to the space of complex 2×2-matrices X satisfying $A^* = -A$. We can write such a matrix in the form

$$\begin{pmatrix} ix_2 & x_3 + ix_4 \\ -(x_3 - ix_4) & -ix_2 \end{pmatrix}$$

and identify it with the pure quaternion $x_2\mathbf{i} + x_2\mathbf{j} + x_4\mathbf{k}$. Then we can view the expression

$$dx = dx_1 + dx_2\mathbf{i} + df\mathbf{j} + dx_4\mathbf{k}$$

as quaternion valued differential 1-form. Then

$$d\bar{x} = dx_1 - dx_2\mathbf{i} - df\mathbf{j} - dx_4\mathbf{k}$$

has clear meaning. Now we define the gauge potential (=connection) A as the 1-form on U_+ given by the formula

$$A(x) = \mathrm{Im}\Big(\frac{xd\bar{x}}{1+|x|^2}\Big) = \frac{1}{2}\frac{xd\bar{x} - dx\bar{x}}{1+|x|^2}. \tag{14.18}$$

Here we use the coordinates x_i instead of u_i^+. The coordinates A_μ of A are given by

$$A_1 = \frac{x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}}{1+|x|^2}, \quad A_2 = \frac{x_1\mathbf{i} + x_4\mathbf{j} - x_3\mathbf{k}}{1+|x|^2},$$

$$A_3 = \frac{-x_4\mathbf{i} - x_1\mathbf{j} + x_2\mathbf{k}}{1+|x|^2}, \quad A_4 = \frac{x_3\mathbf{i} + x_2\mathbf{j} - x_1\mathbf{k}}{1+|x|^2}.$$

Computing the curvature F_A of this potential, we get

$$F = \frac{d\bar{x} \wedge dx}{(1+|x|^2)^2},$$

where

$$d\bar{x} \wedge dx = -2[(dx_1 \wedge dx_2 + dx_3 \wedge dx_4)\mathbf{i} + (dx_1 \wedge dx_3 + dx_4 \wedge dx_2)\mathbf{j} + (dx_1 \wedge dx_4 + dx_2 \wedge dx_3)\mathbf{k}].$$

Now formula (14.9) shows that F is anti-self-dual.

On the open set U_- the gauge potential is given by

$$A'(y) = \mathrm{Im}\Big(\frac{yd\bar{y}}{1+|y|^2}\Big).$$

In quaternion notation, the coordinate change (14.7) is

$$x = \bar{y}^{-1}.$$

It is easy to verify the following identity

$$\bar{x}\Big(\frac{xd\bar{x}}{1+|x|^2}\bar{x}\Big)+\bar{x}d\bar{x}^{-1}=\frac{yd\bar{y}}{1+|y|^2}. \tag{14.19}$$

Let us define the map $U_+\cap U_-\to SU(2)$ by the formula

$$\phi(x)=\frac{\bar{x}}{|x|}$$

Then

$$\mathrm{Im}(\bar{x}d\bar{x}^{-1})=\phi(x)^{-1}d\phi(x)$$

and taking the imaginary part in (14.19), we obtain

$$A'(x)=\phi(x)\circ A(x)\circ\phi^{-1}(x)+\phi(x)^{-1}d\phi(x).$$

Note that the stereographic projection $S^4\to\mathbb{R}^4$ which we have used maps the subset $x_5=0$ bijectively onto the unit 3-sphere $S^3=\{x:|x|=1\}$. The restriction of the gauge map ϕ to S^3 is the identity map. Thus, its degree is equal to 1, and hence we have constructed an instanton over S^4 with instanton number $k=1$.

There is interesting relation between instantons over S^4 and algebraic vector bundles over $\mathbf{P}^3(\mathbb{C})$. To see it, we have to view S^4 as the quaternionic projective line $\mathbb{P}^1(\mathbb{H})$. For this we identify $\mathbb{C}^2$ with $\mathbb{H}$ by the map $(a+bi,c+di)\to(a+bi)+(c+di)j=a+bi+cj+dk$, then

$$S^4=\mathbb{H}^2\setminus\{0\}/\mathbb{H}^*=\mathbb{H}\cup\{\infty\}=\mathbb{R}^4\cup\{\infty\},$$

using this, we construct the map

$$\pi:\mathbb{P}^3(\mathbb{C})=\mathbb{C}^4\setminus\{0\}/\mathbb{C}^*=\mathbb{H}^2\setminus\{0\}/\mathbb{C}^*\to S^4=\mathbb{H}^2\setminus\{0\}/\mathbb{H}^*$$

with the fibre

$$\mathbb{H}^*/\mathbb{C}^*=\mathbb{C}^2\setminus\{0\}/\mathbb{C}^*=\mathbb{P}^1(\mathbb{C})=S^2.$$

Now the pre-image of the adjoint vector bundle $Ad(P)$ of an instanton $SU(2)$-bundle over S^4 is a rank 3 complex vector bundle over $\mathbb{P}^3(\mathbb{C})$. One shows that this bundle admits a structure of an algebraic vector bundle. We refer to [**Atiyah**] for details and further references.

14.7 Let us take $G=U(1),M=\mathbb{R}^4$. Take the Lorentzian metric g on M. Let

$$F=dA=\sum(\partial_\mu A_\nu-\partial_\nu A_\mu)dx^\mu\wedge dx^\nu,$$

and, by (14.7)

$$*F=F_{23}dx^0\wedge dx^1-F_{01}dx^2\wedge dx^3+F_{31}dx^0\wedge dx^2-F_{02}dx^3\wedge dx^1+F_{12}dx^0\wedge dx^3-F_{03}dx^1\wedge dx^2=$$

$$= H_1 dx^0 \wedge dx^1 + E_1 dx^2 \wedge dx^3 + H_2 dx^0 \wedge dx^2 + E_2 dx^3 \wedge dx^1 + H_3 dx^0 \wedge dx^3 + E_3 dx^1 \wedge dx^2.$$

It corresponds to the matrix

$$*F = \begin{pmatrix} 0 & H_1 & H_2 & H_3 \\ -H_1 & 0 & E_3 & -E_2 \\ -H_2 & -E_3 & 0 & E_1 \\ -H_3 & E_2 & -E_1 & 0 \end{pmatrix}.$$

It is obtained from matrix (14.1) of F by replacing $(\mathbf{E}, \mathbf{H})$ with $(\mathbf{H}, -\mathbf{E})$. Equivalently, if we form the complex electromagnetic tensor $\mathbf{H} + i\mathbf{E}$, then $*F$ corresponds to $i(\mathbf{H} + i\mathbf{E})$. The Yang-Mills equation

$$0 = d^A(*F) = d(*F)$$

gives the second pair of Maxwell equations (in vacuum):

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{E}}{\partial t} = 0, \tag{M3}$$

$$\text{div}\mathbf{E} = 0, \tag{M4}$$

Remark 1. The two equations $dF = 0$ and $d(*F) = 0$ can be stated as one equation

$$(dd^* + d^*d)F = 0,$$

where d^* is the operator adjoint to the operator d with respect to the unitary metric on the space of forms defined by (14.10).

Let us try to solve the Maxwell equations in vacuum. Since we can always add the form $d\psi$ to the potential A, we may assume that the scalar potential $\phi = A_0 = 0$. Thus equation (14.3) gives us $\mathbf{E} = -\frac{\partial A}{\partial t}$. Substituting this and (14.4) into (M3), we get

$$\text{curl curl } \mathbf{A} = -\Delta \mathbf{A} + \text{grad div } \mathbf{A} = -\frac{\partial^2 \mathbf{A}}{\partial t^2},$$

where Δ is the Laplacian operator applied to each coordinate of $\mathbf{A}$. We still have some freedom in the choice of $\mathbf{A}$. By adding to $\mathbf{A}$ the gradient of a suitable time-independent function, we may assume that $\text{div}\mathbf{A} = 0$. Thus we arrive at the equation

$$-\Box \mathbf{A} = \Delta \mathbf{A} - \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0. \tag{14.20}$$

Thus in the gauge where $\text{div}\mathbf{A} = 0, A_0 = 0$, the Maxwell equations imply that each coordinate function of the gauge potential $\mathbf{A}$ satisfies the d'Alembert equation from the previous lecture. Conversely, it is easy to see that (14.20) implies the Maxwell equations in vacuum (in the chosen gauge). Let us use this opportunity to explain how the d'Alembert equation can be solved if $\mathbf{A}$ depends only on one coordinate (plane waves). We rewrite (14.20) in the form

$$(\frac{\partial}{\partial x} - \frac{\partial}{\partial x})(\frac{\partial}{\partial x} + \frac{\partial}{\partial x})f = 0.$$

After introducing new variables $\xi = x - t, \eta = x + t$, it transforms to the equation

$$\frac{\partial^2 f}{\partial \xi \partial \eta} = 0.$$

Integrating this equation with respect to ξ, and then with respect to η, we get

$$f = f_1(\xi) + f_2(\eta) = f_1(x - t) + f_2(x + t).$$

This represents two plane waves moving in opposite directions.

To get the general Maxwell equations, we change the Yang-Mills action. The new action is

$$S(A) = S_{YM}(A) - j_e \cdot A,$$

where

$$j_e = (\rho_e, \mathbf{j}_e)$$

is the electric 4-vector current. The Euler-Lagrange equation is

$$d(*F_A) = \mathbf{j}_e.$$

This gives

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}_e, \qquad (M3')$$

$$\mathrm{div}\mathbf{E} = \rho_e, \qquad (M4')$$

Remark 2. If we change the metric $g = dt^2 - \sum_\mu (dx^\mu)^2$ to $c^2 dt^2 - \sum_\mu (dx^\mu)^2$, where c is the light speed, the Maxwell equations will change to

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = 0, \qquad (M1)$$

$$\mathrm{div}\mathbf{H} = \nabla \cdot \mathbf{H} = 0, \qquad (M2)$$

$$\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c}\mathbf{j}_e, \qquad (M3)$$

$$\mathrm{div}\mathbf{E} = \rho_e. \qquad (M4)$$

14.8 Let $\mathbf{q}(\tau) = x^\mu(t), \mu = 0, 1, 2, 3$ be a path in the space $\mathbb{R}^4$. Consider the natural (classical) Lagrangian on $T(M)$ defined by

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{m}{2}\sum_{i,j=1}^{4} g_{ij}\dot{q}_i\dot{q}_j = \frac{m}{2} g_{\mu\nu} dx^\mu dx^\nu.$$

Consider the action defined on the set of particles with *charge* e in the electromagnetic field by

$$S(\mathbf{q}) = \frac{m}{2}\int_{\mathbb{R}}(g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau})dt - e\int_{\mathbb{R}}\frac{dx^{\mu}}{d\tau}A_{\mu}(x^{\mu}(\tau))dt + \frac{1}{2}||F_A||^2.$$

Here we think of j_e as the derivative of the delta-function $\delta(x - x(\tau))$. The Euler Lagrange equation gives the equation of motion

$$m\frac{d^2x^{\mu}}{d\tau^2} = eg_{\nu\sigma}\frac{dx^{\nu}}{d\tau}F^{\sigma\mu}. \tag{14.21}$$

Set

$$\mathbf{v}(\tau) = \beta(\tau)^{-1}(\frac{dx^1}{d\tau}, \frac{dx^2}{d\tau}, \frac{dx^3}{d\tau}),$$

where

$$\beta(\tau) = \frac{dx^0}{d\tau}.$$

If $g(\mathbf{x}', \mathbf{x}') = 1(\tau$ is proper time), then

$$\beta(\tau) = (1 - ||v||^2)^{-1/2}.$$

The we can write

$$\mathbf{x}(\tau) = \beta(1, \mathbf{v})$$

and rewrite (14.21) in the form

$$\frac{dm\beta}{d\tau} = e\mathbf{E}\cdot\mathbf{v}, \quad \frac{d(m\beta\mathbf{v})}{d\tau} = e(\mathbf{E} + \mathbf{v}\times\mathbf{H}). \tag{14.22}$$

the first equation asserts that the rate of change of energy of the particle is the rate at which work is done on it by the electric field. The second equation is the relativistic analogue of Newton's equation, where the force is the so-called *Lorentz force*.

14.9 Let E be a vector bundle with a connection A. In Lecture 13 we have defined the general Lagrangian $\mathcal{L} : E \oplus (E \otimes T^*(M)) \to \mathbb{R}$ by the formula

$$\mathcal{L} = -u \circ q + \lambda g^{-1} \otimes q.$$

Here $g : T(M) \to \mathbb{R}$ is the non-degenerate quadratic form defined by a metric on M, and q is a positive definite quadratic form on E, u and λ are some scalar non-negative valued functions on $\mathbb{R}$ and M, respectively. The Lagrangian $\mathcal{L}$ defines an action on $\Gamma(E)$ by the formula

$$S(\phi) = \int_M (\lambda(x)g^{-1} \otimes q(\nabla^A(\phi)) - u(q(\phi))d^nx.$$

Now we can extend this action by throwing in the Yang-Mills functional. The new action is

$$S(\phi) = \int_M (\lambda(x)g^{-1} \otimes q(\nabla^A(\phi)) - u(q(\phi)))\mathrm{vol}(g) + c\int_M F_A \wedge *F_A \tag{14.23}$$

for some positive constant c. For example, if we choose a gauge such that $g^{-1}(dx_\mu, dx_\nu) = g^{\mu\nu}$ and $||\phi|| = \sum_{i,j=1}^{r} a^{ij}\phi_i\phi_j$, then

$$\nabla(\phi)_i = \partial_\mu\phi_i dx^\mu + A_i^j{}_\mu \phi_j dx^\mu$$

and

$$\mathcal{L}(\phi)(x) = \lambda(x)a^{ij}(g^{\mu\nu}(\partial_\mu\phi_i + A_i^k{}_\mu\phi_k)(\partial_\nu\phi_j + A_i^l{}_\nu\phi_l) - u(a^{ij}\phi_i\phi_j) - cTr(F_{\mu\nu}F^{\mu\nu}). \quad (14.24)$$

Exercises.

1. Show that the introduction of the complex electromagnetic tensor defines a homomorphism of Lie groups $O(1,3) \to GL(3,\mathbb{C})$. Show that the pre-image of $O(3,\mathbb{C})$ is a subgroup of index 4 in $O(1,3)$.

2. Show that the star operator is a unitary operator with respect to the inner product $\langle F, G\rangle$ given by (14.10).

3. Show that the operator $\delta^A = (-1)^k * d^A * : \mathcal{A}^k(Ad(P)) \to \mathcal{A}^{k-1}(Ad(P))$ is the adjoint to the operator $d^A : \mathcal{A}^k(Ad(P)) \to \mathcal{A}^{k+1}(Ad(P))$.

4. The operator $\Delta^A = d^A \circ \delta^A + \delta^A \circ d^A$ is called the *covariant Laplacian*. Show that A is a Yang-Mills connection if and only if $\Delta^A(F_A) = 0$.

5. Prove that the quantities $||\mathbf{E}||^2 - ||\mathbf{H}||^2$ and $(\mathbf{H}\cdot\mathbf{E})^2$ are invariant with respect to the choice of coordinates.

6. Show that applying a Lorentzian transformation one can reduce the electromagnetic tensor $F \neq 0$ to the form where
 (i) $\mathbf{E}\times\mathbf{H} = 0$ if $\mathbf{H}\cdot\mathbf{E} \neq 0$;
 (ii) $\mathbf{H} = 0$ if $\mathbf{H}\cdot\mathbf{H} = \mathbf{0}, ||\mathbf{E}||^{\mathbf{2}} > ||\mathbf{H}||^{\mathbf{2}}$;
 (iii) $\mathbf{E} = 0$ if $\mathbf{H}\cdot\mathbf{H} = \mathbf{0}, ||\mathbf{E}||^{\mathbf{2}} < ||\mathbf{H}||^{\mathbf{2}}$.

7. Show that the Lagrangian (20) is invariant with respect to gauge transformations. Find the corresponding conservation laws.

8. Replacing the quaternions with complex numbers follow section 14.6 to construct a non-trivial principal $SU(1)$-bundle over two-dimensional sphere S^2. Show that its total space is diffeomorphic to S^3.

9. Show that the total space of the $SU(2)$-instanton over S^4 constructed in section 14.6 is diffeomorphic to S^7.

Lecture 15. SPINORS

15.1 Before we embark on the general theory, let me give you the idea of a spinor. Suppose we are in three-dimensional complex space $\mathbb{C}^3$ with the standard quadratic form $Q(x) = x_1^2 + x_2^2 + x_3^2$. The set of zeroes (=isotropic vectors) of Q considered up to proportionality is a conic in $\mathbb{P}^2(\mathbb{C})$. We know that it must be isomorphic to the projective line $\mathbb{P}^1(\mathbb{C})$. The isomorphism is given, for example, by the formulas

$$x_1 = t_0^2 - t_1^2, \quad x_2 = i(t_0^2 + t_1^2), \quad x_3 = -2t_0t_1.$$

The inverse map is given by the formulas

$$t_0 = \pm\sqrt{\frac{x_1 - ix_2}{2}}, \quad t_1 = \pm\sqrt{\frac{-x_1 - ix_2}{2}}.$$

It is not possible to give a consistent choice of sign so that these formulas define an isomorphism from $\mathbb{C}^2$ to the set I of isotropic vectors in $\mathbb{C}^3$. This is too bad because if it were possible, we would be able to find a linear representation of $O(3,\mathbb{C})$ in $\mathbb{C}^2$, using the fact that the group acts naturally on the set I. However, in the other direction, if we start with the linear group $GL(2,\mathbb{C})$ which acts on (t_0, t_1) by linear transformations

$$(t_0, t_1) \to (at_0 + bt_1, ct_0 + dt_1),$$

then we get a representation of $GL(2,\mathbb{C})$ in $\mathbb{C}^3$ which will preserve I. This implies that, for any $g \in GL(2,\mathbb{C})$, we must have $g^*(Q) = \xi(g)Q$ for some homomorphism $\xi : GL(2,\mathbb{C}) \to \mathbb{C}^*$. Since the restriction of ξ to $SL(2,\mathbb{C})$ is trivial, we get a homomorphism

$$s : SL(2,\mathbb{C}) \to SO(3,\mathbb{C}).$$

It is easy to see that the homomorphism s is surjective and its kernel consists of two matrices equal to $\pm$ the identity matrix. The vectors (t_0, t_1) are called *spinors* representing isotropic vectors in $\mathbb{C}^3$. Although the group $SO(3,\mathbb{C})$ does not act on spinors, its double cover $SL(2,\mathbb{C})$ does. The action is called the *spinor representation* of $SO(3,\mathbb{C})$.

Now let us look at the Lorentz group $G = O(1,3)$. Via its action on $\mathbb{R}^4$, it acts naturally on the space $\bigwedge^2(\mathbb{R}^4)$ of skew-symmetric 4×4-matrices $A = (a_{ij})$. If we set

$$\mathbf{v} = (ia_{12} + a_{34}, ia_{13} + a_{24}, ia_{14} + a_{23}) = (a_1, a_2, a_3) + i(b_1, b_2, b_3) \in \mathbb{C}^3,$$

we obtain an isomorphism of real vector spaces $\bigwedge^2(\mathbb{R}^4) \to \mathbb{C}^3$. Since

$$\mathbf{v} \cdot \mathbf{v} = \mathbf{a} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b} + i\mathbf{a} \cdot \mathbf{b},$$

the real part of $\mathbf{v} \cdot \mathbf{v}$ coincides with $\langle A, A \rangle$, where $\langle\ ,\ \rangle$ is the scalar product on $\bigwedge^2(\mathbb{R}^4)$ induced by the Lorentz metric in $\mathbb{R}^4$. Now if we switch from $O(1,3)$ to $SO(1,3)$, we obtain that the determinant of A is preserved. But the determinant of A is equal to $(\mathbf{a} \cdot \mathbf{b})^2$ (the number $\mathbf{a} \cdot \mathbf{b}$ is the Pfaffian of A). To preserve $\mathbf{a} \cdot \mathbf{b}$ we have to replace $O(1,3)$ with $SO(1,3)$ which guarantees the preservation of the square of the Pfaffian. To preserve the Pfaffian itself we have to go further and choose a subgroup of index 2 of $SO(1,3)$. This is called the *proper Lorentz group*. It is denoted by $SO(1,3)_0$. Thus under the isomorphism $\bigwedge^2(\mathbb{R}^4) \cong \mathbb{C}^3$, the group $SO(1,3)_0$ is mapped to $SO(3,\mathbb{C})$. One can show that this is an isomorphism. Combining with the above, we obtain a homomorphism

$$s' : SL(2,\mathbb{C}) \to SO(1,3)_0$$

which is the spinor representation of the proper Lorentz group.

15.2 The general theory of spinors is based on the theory of Clifford algebras. These were introduced by the English mathematician William Clifford in 1876. The notion of a spinor was introduced by the German mathematician R. Lipschitz in 1886. In the twenties, they were rediscovered by B. L. van der Waerden to give a mathematical explanation of Dirac's equation in quantum mechanics. Let us first explain Dirac's idea. He wanted to solve the D'Alembert equation

$$(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2})\phi = 0.$$

This would be easy if we knew that the left-hand side were the square of an operator of first order. This is equivalent to writing the quadratic form $t^2 - x_1^2 - x_2^2 - x_3^2$ as the square of a linear form. Of course this is impossible as the latter is an irreducible polynomial over $\mathbb{C}$. But it is possible if we leave the realm of commutative algebra. Let us look for a solution

$$t^2 - x_1^2 - x_2^2 - x_3^2 = (tA_1 + x_2A_2 + x_3A_3 + x_4A_4)^2.$$

where A_i are square complex matrices, say of size two-by-two. By expanding the right-hand side, we find

$$A_1^2 = I_2, \quad A_2^2 = A_3^2 = A_4^2 = -I_2, \quad A_iA_j + A_jA_i = 0, \quad i \neq j. \tag{15.1}$$

One introduces a non-commutative unitary algebra C given by generators A_1, A_2, A_3, A_4 and relations (15.1). This is the Clifford algebra of the quadratic form $t^2 - x_1^2 - x_2^2 - x_3^2$ on

$\mathbb{R}^4$. Now the map $X_i \to A_i$ defines a linear (complex) two-dimensional representation of the Clifford algebra (the spinor representation). The vectors in $\mathbb{C}^2$ on which the Clifford algebra acts are spinors.

Let us first recall the definition of a Clifford algebra. It generalizes the concept of the exterior algebra of a vector space.

Definition. Let E be a vector space over a field K and $Q : E \to K$ be a quadratic form. The *Clifford algebra* of (E, Q) is the quotient algebra

$$C(Q) = T(E)/I(Q)$$

of the tensor algebra $T(E) = \oplus_n T^n E$ by the two-sided ideal $I(Q)$ generated by the elements $v \otimes v - Q(v), v \in T^1 = E$.

It is clear that the restrictions of the factor map $T(E) \to C(Q)$ to the subspaces $T^0(E) = K$ and $T^1(E) = E$ are bijective. This allows us to identify K and E with subspaces of $C(Q)$. By definition, for any $v \in E$,

$$v \cdot v = Q(v). \tag{15.2}$$

Replacing $v \in V$ with $v + w$ in this identity, we get

$$v \cdot w + w \cdot v = B(v, w), \tag{15.3}$$

where $B : E \times E \to K, (x, y) \to Q(x + y) - Q(x) - Q(y)$, is the symmetric bilinear form associated to Q. Using the *anticommutator* notation

$$\{a, b\} = ab + ba$$

one can rewrite (15.3) as

$$\{v, w\} = B(v, w). \tag{15.3$'$}$$

Example 1. Take Q equal to the zero function. By definition $v \cdot v = 0$ for any $v \in E$. Let us show that this implies that $C(Q)$ is isomorphic to the exterior (or Grassmann) algebra of the vector space E

$$\bigwedge(E) = \bigoplus_{n \geq 0} \bigwedge^n(E).$$

In fact, the latter is defined as the quotient of $T(E)$ by the two-sided ideal J generated by elements of the form $v \otimes x \otimes v$ where $v \in E, x \in T(E)$. It is clear that $I(Q) \subset J$. It is enough to show that $J \subset I(Q)$. For this it suffices to show that $v \otimes x \otimes v \in I(Q)$ for any $v \in E, x \in T^n(E)$, where n is arbitrary. Let us prove it by induction on n. The assertion is obvious for $n = 0$. Write $x \in T^n(E)$ in the form $x = w \otimes y$ for some $y \in T^{n-1}(E), w \in E$. Then

$$v \otimes w \otimes y \otimes v = (v + w) \otimes (v + w) \otimes y \otimes v - v \otimes v \otimes y \otimes v - w \otimes v \otimes y \otimes v - w \otimes w \otimes y \otimes v.$$

By induction, each of the four terms in the right-hand side belongs to the ideal $I(Q)$.

Example 2. Let $\dim\ E = 1$. Then $T(E) \cong K[t]$ and $C(Q) \cong K[t]/(t^2 - Q(e))$, where E is spanned by e. In particular, any quadratic extension of K is a Clifford algebra.

Theorem 1. *Let $(e_i)_{i\in I}$ be a basis of E. For any finite ordered subset $S = (i_1, \ldots, i_k)$ of I, set*

$$e_S = e_{i_1} \ldots e_{i_k} \in C(Q).$$

Then there exists a basis of $C(Q)$ formed by the elements e_S. In particular, if $n = \dim\ E$,

$$\dim_K C(Q) = 2^n.$$

Proof. Using (15.3), we immediately see that the elements e_S span $C(Q)$. We have to show that they are linearly independent. This is true if $Q = 0$, since they correspond to the elements $e_{i_1} \wedge \ldots \wedge e_{i_k}$ of the exterior algebra. Let us show that there is an isomorphism of vector spaces $C(Q) \cong C(0)$. Take a linear function $f \in E^*$ on E. Define a linear map

$$\phi_f : T(E) \to T(E) \tag{15.4}$$

as follows. By linearity it is enough to define $\phi_f(x)$ for decomposable tensors $x \in T^n$. For $n = 0$, set $\phi_f = 0$. For $n = 1$, set $\phi_f(v) = f(v) \in T^0(E)$. Now for any $x = v \otimes y \in T^n(E)$, set

$$\phi_f(v \otimes y) = f(v)y - v \otimes \phi_f(y). \tag{15.5}$$

We have $\phi_f(v \otimes v - Q(v)) = f(v)v - v \otimes f(v) = 0$. We leave to the reader to check that moreover $\phi_f(x \otimes (v \otimes v - Q(x)) \otimes y) = 0$ for any $x, y \in T(E)$. This implies that the map ϕ_f factors to a linear map $C(Q) \to C(Q)$ which we also denote by ϕ_f. Notice that the map $E^* \to \mathrm{End}(T(E)), f \to \phi_f$ is linear.

Now let $F : E \times E \to K$ be a bilinear form, and let $i_F : E \to E^*$ be the corresponding linear map. Define a linear map

$$\lambda_F : T(E) \to T(E) \tag{15.6}$$

by the formula

$$\lambda_F(v \otimes y) = v \otimes \lambda_F(y) + \phi_{i_F(v)}(\lambda_F(y)), \tag{15.7}$$

where $v \in E, y \in T^{n-1}(E)$ and $\lambda_F(1) = 1$.

Obviously the restriction of this map to $E = T^0(E) \oplus T^1(E)$ is equal to the identity. Also the map λ_F is the identity when $F = 0$. We need the following

Lemma 1.

(i) for any $f \in E^*$,

$$\phi_f^2 = 0;$$

(ii) for any $f, g \in E^*$,

$$\{\phi_f, \phi_g\} = 0;$$

(iii) for any $f \in E^*$,

$$[\lambda_F, \phi_f] = 0.$$

Proof. (i) Use induction on the degree of the decomposable tensor. We have

$$\phi_f^2(v \otimes y) = \phi_f(\phi_f(v)y - v \otimes \phi_f(y)) = \phi_f(v)\phi_f(y) - \phi_f(v)\phi_f(y) - v \otimes \phi_f^2(y) = 0.$$

(ii) We have

$$\phi_f \circ \phi_g(v \otimes y) = \phi_f(g(v)y - v \otimes \phi_g(y)) = \phi_f(g(v))y) - g(v)\phi_f(y) + v \otimes \phi_f(\phi_g(y)) =$$

$$= v \otimes \phi_f(\phi_g(y)).$$

(iii) Using (ii) and induction, we get

$$\lambda_F(\phi_f(v \otimes y)) = \lambda_F(f(v)y - v \otimes \phi_f(y)) = f(v)\lambda_F(y) - v \otimes \lambda_F(\phi_f(y)) - \phi_{i_F(v)}(\lambda_F(\phi_f(y)) =$$

$$= f(v)\lambda_F(y) - v \otimes \phi_f(\lambda_F(y)) - \phi_{i_F(v)}\phi_f(\lambda_F(y)) =$$

$$= \phi_f(v \otimes \lambda_F(y)) + \phi_f(\phi_{i_F(v)}(\lambda_F(y)) = \phi_f(\lambda_F(v \otimes y)).$$

Corollary. *For any other bilinear form G on E, we have*

$$\lambda_{F+G} = \lambda_F \circ \lambda_G,$$

Proof. Using (iii) and induction, we have

$$\lambda_G \circ \lambda_F(v \otimes y) = \lambda_G(v \otimes \lambda_F(y) + \phi_{i_F(v)}(\lambda_F(y)) = v \otimes \lambda_G(\lambda_F(y)) + \phi_{i_G}(v)(\lambda_F(y)))+$$

$$+\lambda_G(\phi_{i_F(v)}(\lambda_F(y)) = v \otimes \lambda_G(\lambda_F(y)) + (\phi_{i_G(v)} + \phi_{i_G(v)})(\lambda_G(\lambda_F(y))) =$$

$$= v \otimes \lambda_{G+F}(y) + \phi_{i_{F+G}(v)}(\lambda_{G+F}(y)) = \lambda_{F+G}(v \otimes y).$$

It is clear that

$$\lambda_{-F} = (\lambda_F)^{-1}. \tag{15.8}$$

Let Q' be another quadratic form on E defined by $Q'(v) = Q(v) + F(v,v)$. Let us show that λ_F defines a linear isomorphism $C(Q) \cong C(Q')$. By (15.8), it suffices to verify that $\lambda_F(I(Q')) \subset I(Q)$. Since $\phi_{i_F(v)}(I(Q)) \subset I(Q)$, formula (15.5) shows that the set of $x \in T(E)$ such that $\lambda_F(x) \in I(Q)$ is a left ideal. So, it is enough to verify that $\lambda_F(v \otimes v \otimes x - Q'(v)x) \in I(Q)$ for any $v \in E, x \in T(E)$. We have

$$\lambda_F(v \otimes v \otimes x - Q'(v)x) = v \otimes \lambda_F(v \otimes x) + \phi_{i_F(v)}(\lambda_F(v \otimes x)) - Q'(v)\lambda_F(x) =$$

$$= v \otimes v \otimes \lambda_F(x) + v \otimes \phi_{i_F(v)}(\lambda_F(x)) + \lambda_F(\phi_{i_F(v)}(v \otimes x)) - Q'(v)\lambda_F(x) =$$

$$= v \otimes v \otimes \lambda_F(x) + v \otimes \phi_{i_F(v)}(\lambda_F(x)) + \lambda_F(F(v,v)x - v \otimes \phi_{i_F(v)}(x)) - Q'(v)\lambda_F(x) =$$

$$= (v \otimes v - F(v,v) - Q'(v)) \otimes \lambda_F(x) = v \otimes v - Q(v).$$

To finish the proof of the theorem, it suffices to find a bilinear form F such that $Q(x) = -F(x,x)$. If $\mathrm{char}(K) \neq 2$, we may take $F = -\frac{1}{2}B$ where B is the associated symmetric bilinear form of Q. In the general case, we may define F by setting

$$F(e_i, e_j) = \begin{cases} -B(e_i, e_j) & \text{if } i > j, \\ Q(e_i) & \text{if } i = j, \\ 0 & \text{if } i < j. \end{cases}$$

15.3 The structure of the Clifford algebra $C(Q)$ depends very much on the property of the quadratic form. We have seen already that $C(Q)$ is isomorphic to the Grassmann algebra when Q is trivial. If we know the classification of quadratic forms over K we shall find out the classification of Clifford algebras over K.

Let $C^+(Q), C^-(Q)$ be the subspaces of $C(Q)$ equal to the images of the subspaces $T^+(E) = \oplus_k T^{2k}(E)$ and $T^-(E) = \oplus_k T^{2k+1}(E)$ of $T(E)$, respectively. Since $T^+(E)$ is a subalgebra of $T(E)$, the subspace $C^+(Q)$ is a subalgebra of $C(Q)$. Since $I(Q)$ is spanned by elements of $T^+(E)$, $I(Q) = I(Q) \cap T^+(E) \oplus I(Q) \cap T^-(E)$. This implies that

$$C(Q) = C^+(Q) \oplus C^-(Q). \tag{15.9}$$

We call elements of $C^+(Q)$ (resp. $C^-(Q)$) *even* (resp. *odd*). Let $C(Q_1)$ and $C(Q_2)$ be two Clifford algebras. We define their tensor product using the formula

$$(a \otimes b) \cdot (a' \otimes b') = \epsilon aa' \otimes bb', \tag{15.10}$$

where $\epsilon = 1$ unless a, a' are not both even or odd, and b, b' are not both even or odd. In the latter case $\epsilon = -1$. We assume here that each a, a', b, b' is either even or odd.

Theorem 2. *Let $Q_i : E_i \to K, i = 1, 2$ be two quadratic forms, and $Q = Q_1 \oplus Q_2 : E = E_1 \oplus E_2 \to K$. Then there exists an algebra isomorphism*

$$C(Q_1) \otimes C(Q_2) \cong C(Q).$$

Proof. We have a canonical map $p : C(Q_1) \otimes C(Q_2) \to C(Q)$ induced by the bijective canonical map $T(E_1) \otimes T(E_2) \to T(E)$. Since the basis of E is equal to the union of bases in E_1 and E_2, we can apply Theorem 1 to obtain that p is bijective. We have $vw + wv = B(v, w) = 0$ if $v \in E_1, w \in E_2$. From this we easily find that

$$(v_1 \ldots v_k)(w_1 \ldots w_s) = (-1)^{ks}(w_1 \ldots w_s)(v_1 \ldots v_k),$$

where $v_i \in E_1, w_j \in E_2$. This implies that the map $C(Q_1) \otimes C(Q_2) \to C(Q)$ defined by $x \otimes y \to xy$ is an isomorphism. Here is where we use the definition of the tensor product given in (15.10).

Corollary. *Assume $n =$ dim E. Let $(e_1, \ldots, e_n)$ be a basis in E such that the matrix of Q is equal to a diagonal matrix diag$(d_1, \ldots, d_n)$ (it always exists). Then $C(Q)$ is an associative algebra with a unit, generated by $1, e_1, \ldots, e_n$, with relations*

$$e_i^2 = d_i, \quad e_i e_j = -e_j e_i, \quad i, j = 1, \ldots, n, i \neq j.$$

Example 3. Let $E = \mathbb{R}^2$ and Q have signature $(0, 2)$. Then there exists a basis such that $Q(\sum x_1 e_1 + x_2 e_2) = -x_1^2 - x_2^2$. Then, setting $e_1 = \mathbf{i}, e_2 = \mathbf{j}$, we obtain that $C(Q)$ is generated by $1, \mathbf{i}, \mathbf{j}$ with relations

$$\mathbf{i}^2 = \mathbf{j}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji}.$$

Thus $Q(E)$ is isomorphic to the algebra of quaternions $\mathbb{H}$.

Recall from linear algebra that a non-degenerate quadratic form $Q : V \to K$ on a vector space V of dimension $2n$ is called *neutral* (or of *Witt index* n) if $V = E \oplus F$ with $Q|E \equiv 0, Q|F \equiv 0$. It follows that dim $E =$ dim $F = n$. For example, every non-degenerate quadratic form on a complex vector space of dimension $2n$ is neutral.

Theorem 3. *Let* dim $E = 2r, Q : E \to K$ *be a neutral quadratic form on* V. *There exists an isomorphism of algebras*

$$s : C(Q) \cong \mathrm{Mat}_{2^r}(K).$$

The image of $C^+(Q)$ *under* s *is isomorphic to the sum of two left ideals in* $\mathrm{Mat}_{2^r}(K)$, *each isomorphic to* $\mathrm{Mat}_{2^{r-1}}(K)$.

Proof. Let $E = F \oplus F'$, where $Q|F \equiv 0, Q|F' \equiv 0$. Let S be the subalgebra of $C(Q)$ generated by F. Then $S \cong \bigwedge(F)$. We can choose a basis $(f_1, \ldots, f_r)$ of F and a basis $f_1^*, \ldots, f_r^*$ of F' such that $B(f_i, f_j^*) = \delta_{ij}$. Thus we can consider F' as the dual space F^* and $(f_1^*, \ldots, f_r^*)$ as the dual basis to $(f_1, \ldots, f_r)$. For any $f \in F$ let $l_f : x \to fx$ be the left multiplication endomorphism of S (*creation operator*). For any $f' \in F'$ let $\phi_{f'} : S \to S$ be the endomorphism defined in the proof of Theorem 1 (*annihilation operator*). Using formula (15.5), we have

$$l_f \circ \phi_{f'} + \phi_{f'} \circ l_f = B(f, f')\mathbf{id}. \tag{15.11}$$

For $v = f + f' \in E$ set

$$s_v = l_f + \phi_{f'} \in \mathrm{End}(F).$$

Then $v \to s_v$ is a linear homomorphism from V to $\mathrm{End}(S) \cong \mathrm{Mat}_{2^r}(K)$. From the equality

$$s_v^2(x) = (l_f + \phi_{f'})^2(x) = e \otimes e \otimes x + B(f, f')x = Q(f)x + B(f, f')x = Q(v)x,$$

we get that $v \to s_v$ can be extended to a homomorphism $s : C(Q) \to \mathrm{End}(S)$. Since both algebras have the same dimension ($= 2^n$), it is enough to show that the obtained homomorphism is surjective. We use a matrix representation of $\mathrm{End}(S)$ corresponding to the natural basis of S formed by the products $e_I = e_{i_1} \wedge \ldots \wedge e_{i_k}, i_1 < \ldots < i_k$. We need to construct an element $x \in C(Q)$ such that $s(x)$ is the unit matrix $E_{KH} = (\delta_{KH})$, where $K = (1 \le k_1 < \ldots < k_s \le r), H = (1 \le h_1 < \ldots < h_t \le r)$ are ordered subsets of $[r] = \{1, \ldots, r\}$. One takes $x = e_H f_{[r]} e_K$, where $e_H = e_{k_1} \ldots e_{k_s}, e_K = e_{h_1} \ldots e_{h_s}, f_{[r]} = f_1 \ldots f_r \in C(Q)$. We skip the computation.

For the proof of the second assertion, we set $S^+ = S \cap C^+(Q)$ and $S^- = S \cap C^-(Q)$. Each of them is a subspace of S of dimension 2^{r-1}. Then it is easy to see that S^+ and S^- are left invariant under $s(C^+(Q))$. Since s is injective, s maps $C^+(Q)$ isomorphically onto the subalgebra of $\mathrm{End}(S)$ isomorphic to $\mathrm{End}(S^+) \times \mathrm{End}(S^-)$.

Example 4. Let $E = \mathbb{R}^2, Q(x_1, x_2) = x_1 x_2$. Then we can take $F = Ke_1 = \{x_2 = 0\}$, $F' = Ke_2 = \{x_1 = 0\}$. The algebra S is two-dimensional with basis consisting of $1 \in \bigwedge^0(F)$ and $e_1 \in \bigwedge^1(F) = F$. The Clifford algebra is spanned by $1, e_1, e_2, e_1e_2$ with $e_i^2 = 0, e_1e_2 = -e_2e_1$. The map $C(Q) \to \mathrm{End}(S) \cong \mathrm{Mat}_2(K)$ is defined by

$$s(e_1) = l_{e_1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad s(e_2) = \phi_{e_2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$s(e_1e_2) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad s(1 - e_1e_2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

This checks the theorem.

Corollary 1. *Assume $n = 2r$, Q is non-degenerate and K is algebraically closed, or $K = \mathbb{R}$ and Q is of signature (r, r). Then*

$$C(Q) \cong \mathrm{Mat}_{2^r}(K).$$

Corollary 2. *Let $E = \mathbb{R}^{2r}$ and Q be of signature $(r, r+2)$. Then*

$$C(Q) \cong \mathrm{Mat}_r(\mathbb{H}),$$

where H is the algebra of quaternions.

Proof. We can write $E = E_1 \oplus E_2$, where E_1 is orthogonal to E_2 with respect to B, and $Q|E_1$ is of signature $(1, 1)$ and $Q|E_2$ is of signature $(0, 2)$. By Theorem 2 $C(Q) \cong C(Q_1) \otimes C(Q_2)$. By Theorem 3, $C(Q_1) \cong \mathrm{Mat}_2(\mathbb{R})$. By Example 3, $C(Q_2) \cong \mathbb{H}$. It is easy to check that $\mathrm{Mat}_2(\mathbb{R}) \otimes \mathbb{H} \cong \mathrm{Mat}_2(\mathbb{H})$.

Corollary 3. *Assume Q is non-degenerate and $n = 2r$. Then there exists a finite extension K'/K of the field K such that*

$$C(Q) \otimes_K K' \cong \mathrm{Mat}_{2^r}(K').$$

In particular, $C(Q)$ is a central simple algebra (i.e., has no non-trivial two-sided ideals and its center is equal to K). The center Z of $C^+(Q)$ is a quadratic extension of K. If Z is a field $C^+(Q)$ is simple. Otherwise, $C^+(Q)$ is the direct sum of two simple algebras.

Proof. Let $e_1, \ldots, e_n$ be a basis in E such that the matrix of B is equal to $d_i \delta_{ij}$ for some $d_i \in K$. Let K' be obtained from K by adjoining square roots of d_i's and $\sqrt{-1}$. Then we can find a basis $f_1, \ldots, f_n$ of $E \otimes_K K'$ such that $B(f_i, f_j) = \delta_{ii+r}$. Then we can apply Theorem 3 to obtain the assertion.

The algebra $\mathrm{Mat}_{2^r}(K')$ is central simple. This implies that $C(Q)$ is also central simple.

In the case when dim E is odd, the structure of $C(Q)$ is a little different.

Theorem 4. *Assume* dim $E = 2r + 1$ *and there exists a hyperplane H in E such that $(H, Q|H)$ is neutral. Then*

$$C^+(Q) \cong \mathrm{Mat}_{2^r}(K).$$

The center Z of $C(Q)$ is a quadratic extension of K and

$$C(Q) \cong \mathrm{Mat}_{2^r}(Z).$$

Proof. Let $v_0 \in E$ be a non-zero vector orthogonal to H. It is non-isotropic since otherwise Q is degenerate. Define the quadratic form Q' on H by $Q'(h) = -Q(v_0)Q(h)$. Since $v_0 h = -h v_0$ for all $h \in H$, we have $(v_0 h)^2 = -v_0^2 h^2 = -Q(v_0)Q(h) = Q'(h)$. This easily implies that the homomorphism $T(H) \to T(E), h_1 \otimes \ldots \otimes h_n \to (v_0 \otimes h_1) \otimes \ldots \otimes (v_0 \otimes h_n)$ factors to define a homomorphism $f : C(Q') \to C^+(Q)$. Since dim H is even, by Theorem 3, the algebra $C(Q')$ is a matrix algebra, hence does not have non-trivial two-sided ideals. This implies that the homomorphism f is injective. Since the dimension of both algebras is the same, it is also bijective. Applying again Theorem 3, we obtain the first assertion.

Let $v_1, \ldots, v_{2r}$ be an orthogonal basis of $Q|H$. Then, it is immediately verified that $z = v_0 v_1 \ldots v_{2r}$ commutes with any $v_i, i = 0, \ldots, 2r$, hence belongs to the center of $C(Q)$. We have $z^2 = (-1)^r Q(v_0) \ldots Q(v_{2r}) \in K^*$. So $Z' = K + Kz$ is a quadratic extension of K contained in the center. Since $z \in C^-(Q)$ and is invertible, $C(Q) = zC^+(Q) + C^+(Q)$. This implies that the image of the homomorphism $\alpha : Z' \otimes C^+(Q) \to C(Q), x \otimes y \to xy$, contains $C^+(Q)$ and $C^-(Q)$, hence is surjective. Thus, $Z' \otimes C^+(Q) \cong \mathrm{Mat}_{2^r}(Z')$. Since the center of $\mathrm{Mat}_{2^r}(Z')$ can be identified with Z', we obtain that $Z' = Z$. and the center hence α is injective. Thus $C(Q) \cong \mathrm{Mat}_{2^r}(Z)$.

Corollary. *Assume n is odd and Q is non-degenerate. Then $C(Q)$ is either simple (if Z is a field), or the product of two simple algebras (if Z is not a field). The algebra $C^+(Q)$ is a central simple algebra.*

If n is even, $C(Q)$ is simple and hence has a unique (up to an isomorphism) irreducible linear representation. It is called *spinor representation*. The elements of the corresponding space are called *spinors*. In the case when Q satisfies the assumptions of Theorem 3, we may realize this representation in the maximal isotropic subspace S of E. We can do the same if we allow ourselves to extend the field of scalars K. The restriction of the spinor representation to $C^+(Q)$ is either irreducible or isomorphic to the direct sum of two non-isomorphic irreducible representations. The latter happens when Q is neutral. In this case the two representations are called the *half-spinor representations* of $C^+(Q)$. The elements of the corresponding spaces are called *half-spinors*.

If n is odd, $C^+(Q)$ has a unique irreducible representation which is called a spinor representation. The elements of the corresponding space are spinors.

15.4 Now we can define the spinor representations of orthogonal groups. First we introduce the *Clifford group* of the quadratic form Q. By definition, it is the group $G(Q)$ of invertible elements x of the Clifford algebra of $C(Q)$ such that $xvx^{-1} \in E$ for any $v \in E$. The subgroup $G^+(Q) = G(Q) \cap C^+(Q)$ of $G(Q)$ is called the *special Clifford group*.

Let $\phi : G(Q) \to GL(E)$ be the homomorphism defined by

$$\phi(x)(v) = xvx^{-1}. \tag{15.12}$$

Since

$$Q(xvx^{-1}) = xvx^{-1}xvx^{-1} = xvvx^{-1} = xQ(v)x^{-1} = Q(v),$$

we see that the image of ϕ is contained in the orthogonal group $O(Q)$.

From now on, we shall assume that $\mathrm{char}(K) \neq 2$ and Q is a nondegenerate quadratic form on E. We continue to denote the associated symmetric bilinear form by B.

Let $v \in E$ be a non-isotropic vector (i.e., $Q(v) \neq 0$). The linear transformation of E defined by the formula

$$r_v(w) = w - \frac{B(v,w)}{Q(v)} v$$

is called the *reflection* in the vector v. Since

$$B(r_v(w), r_v(w)) = B(w,w) - 2\frac{B(v,w)}{Q(v)}B(w,v) + 2(\frac{B(v,w)}{Q(v)})^2 Q(v) = B(w,w),$$

the transformation r_v is orthogonal. It is immediately seen that the restriction of r_v to the hyperplane H_v orthogonal to v is the identity, and $r_v(v) = -v$. Conversely, any orthogonal transformation of (E, Q) with this property is equal to r_v.

We shall use the following result from linear algebra:

Theorem 5. *$O(Q)$ is generated by reflections.*

Proof. Induction on dim E. Let $T : E \to E$ be an element of $O(Q)$. Let v be a non-isotropic vector. Assume $T(v) = v$. Then T leaves invariant the orthogonal complement H of v and induces an orthogonal transformation of H. It is clear that the restriction of Q to H is a non-degenerate quadratic form. By induction, $T|H = r_1 \dots r_k$ is the product of reflections in some vectors $v_1, \dots, v_k$ in H. Let $L_i = (Kv_i)^{\perp}_{H}$. If we extend r_i to an orthogonal transformation $\tilde{r}_i$ of E which fixes v, then $\tilde{r}_i$ fixes pointwisely the hyperplane $Kv_i + L_i = (Kv_i)^{\perp}_{E}$ and coincides with the reflection in E in the vector v_i. It is clear that $T = \tilde{r}_1 \dots \tilde{r}_k$.

Now assume that $T(v) = -v$. Then $r_v \circ T(v) = v$. By the previous case, $r_v \circ T$ is the product of reflections, so T is also.

Finally, let us consider the general case. Take $w = T(v)$, then $Q(w) = Q(v)$ and $Q(v+w) + Q(v-w) = 2Q(v) + 2Q(w) = 4Q(v) \neq 0$. So, one of the two vectors $v + w$ and $v - w$ is non-isotropic. Assume $h = v - w$ is non-isotropic. Then $B(w,h) = Q(w+h) - Q(h) - Q(w) = Q(v) - Q(v) - Q(h) = -Q(h)$. Thus $r_h(w) = w - \frac{B(w,x)}{Q(h)}h = w + h = v$. This implies that $T \circ r_h(w) = T(v) = w$. By the first case, $T \circ r_h$ is the product of reflections, so is T. Similarly we consider the case when $v + w$ is non-isotropic. We leave this to the reader.

Proposition 1. *The set $G(Q) \cap E$ is equal to the set of vectors $v \in E$ such that $Q(v) \neq 0$. For any $v \in G(Q) \cap E$, the map $-\phi(v) : E \to E$ is equal to the reflection r_v.*

Proof. Since $v^2 = Q(v)$, the vector v is invertible in $C(Q)$ if and only if $Q(v) \neq 0$. So, if $v \in G(Q)$, we must have $v^{-1} = Q(v)^{-1}v$, hence, for any $w \in E$,

$$vwv^{-1} = vwQ(v)^{-1}v = Q(v)^{-1}vwv = Q(v)^{-1}v(B(v,w) - vw) =$$

$$= Q(v)^{-1}vB(v,w) - Q(v)^{-1}v^2w = Q(v)^{-1}vB(v,w) - w.$$

Corollary. *Any element from $G(Q)$ can be written in the form*

$$g = zv_1 \dots v_k,$$

where $v_1, \dots, v_k \in E \setminus Q^{-1}(0)$ and z is an invertible element from the center Z of $C(Q)$.

Proof. Let $\phi : G(Q) \to O(Q)$ be the map defined in (15.12). Its kernel consists of elements x which commute with all elements from E. Since $C(Q)$ is generated by elements from E, we obtain that $\mathrm{Ker}(\phi)$ is a subset of the group Z^* of invertible elements of Z. Obviously, the converse is true. So

$$\mathrm{Ker}(\phi) = Z^*.$$

By Theorem 5, for any $x \in G(Q)$, the image $\phi(x)$ is equal to the product of reflections r_i. By Proposition 1, each $r_i = -\phi(v_i)$ for some non-isotropic vector $v_i \in E$. Thus x differs from the product of v_i's by an element from $\mathrm{Ker}(\phi) = Z^*$. This proves the assertion.

Theorem 6. *Assume* $\mathrm{char}(K) \neq 2$. *If* $n = \dim\, E$ *is even, the homomorphism* $\phi : G(Q) \to O(Q)$ *is surjective. In this case the image of the subgroup* $G^+(Q)$ *is equal to* $SO(Q)$. *If* n *is odd, the image of* $G(Q)$ *and* $G^+(Q)$ *is* $SO(Q)$. *The kernel of* ϕ *is equal to the group* Z^* *of invertible elements of the center of* $C(Q)$.

Proof. The assertion about the kernel has been proven already. Also we know that the image of ϕ consists of transformations $(-r_{v_1}) \circ \ldots \circ (-r_{v_k}) = (-1)^k r_{v_1} \circ \ldots \circ r_{v_k}$, where $v_1, \ldots, v_k$ are non-isotropic vectors from E.

Assume n is even. By Theorem 5, any element T from $O(Q)$ is a product of N reflections. If N is even, we take $k = N$ and find that T is in the image. If N is odd, we write $-\mathbf{id}_E$ as the product of even number n of reflections. Then we take $k = N + n$ to obtain that T is in the image.

Assume n is odd. As above we show that each product of even number of reflections in $O(Q)$ belongs to the image of ϕ. Now the product of odd number of reflections cannot be in the image because its determinant is -1 (the determinant of any reflection equals -1.) But the determinant of $(-r_{v_1}) \circ \ldots \circ (-r_{v_k})$ is always one. However, $-\mathbf{id}_E$ is equal to the product of n reflections. So, multiplying a product of odd number of reflections by $-\mathbf{id}_E$ we get a product of even reflections which, by the above, belongs to the image. This proves the assertion about the image of ϕ.

If n is even, the restriction of the spinor representation of $C(Q)$ to $G(Q)$ is a linear representation of the Clifford group $G(Q)$. It is called the *spinor representation*. One can prove that it is irreducible. Abusing the terminology this is also called the spinor representation of $O(Q)$. The restriction of the spinor representation to $G^+(Q)$ is isomorphic to the direct sum of two non-isomorphic irreducible representations. They are called the *half-spinor representations* of $G^+(Q)$ (or of $SO(Q)$).

If n is odd, the restriction of the spinor representation of $C^+(Q)$ to $G^+(Q)$ is an irreducible representation. It is called the spinor representation of $G^+(Q)$ (or of $SO(Q)$).

Example 5. Take $E = \mathbb{C}^3$ and $Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$. This is the odd case. Changing the basis, we can transform Q to the form $-x_1x_2 + x_3^2$. Obviously, Q satisfies the assumption of Theorem 4 (take $H = \mathbb{C}e_1 + \mathbb{C}e_2$.) Thus $C^+(Q) \cong M_2(\mathbb{C})$. Hence $G^+(Q) \subset M_2(\mathbb{C})^* = GL(2, \mathbb{C})$. By the proof of Theorem 4, $C^+(Q) \cong C(Q')$ where $Q' : \mathbb{C}^2 \to \mathbb{C}, (z_1, z_2) \to z_1 z_2$. The maximal isotropic subspace of Q' is $F = \mathbb{C}e_1$. The Grassmann algebra $S = \bigwedge^0(F) \oplus \bigwedge^1(F) = \mathbb{C} + \mathbb{C}e_1 \cong \mathbb{C}^2$. Since $e_1e_2 + e_2e_1 = B(e_1, e_2) = 1, e_i^2 = Q'(e_i) = 0$, we can write any element $x \in C(Q')$ in the form $x = a + be_1 + ce_2 + de_1e_2$.

Let us find the corresponding matrix. It is enough to find it for $x = e_1, e_2$, and e_1e_2. By the proof of Theorem 3,

$$s_{e_1}(1) = e_1, \quad s_{e_1}(e_1) = e_1e_1 = 0;$$

$$s_{e_2}(1) = \phi_{e_2}(1) = 0, \quad s_{e_2}(e_1) = \phi_{e_2}(e_1) = B'(e_1, e_2) = 1;$$

$$s_{e_1e_2}(1) = s_{e_1}(s_{e_2}(1)) = 0, \quad s_{e_1e_2}(e_1) = s_{e_1}(s_{e_2}(e_1)) = s_{e_1}(1) = e_1.$$

So, in the basis $(1, e_1)$ of S, we get the matrices

$$s_{e_1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad s_{e_2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad s_{e_1e_2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad s_x = \begin{pmatrix} a+d & c \\ b & a \end{pmatrix}.$$

At this point it is better to write each x in the form $x = ae_1e_2 + be_2 + ce_1 + de_2e_1$ to be able to identify x with the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Now $C^+(Q)$ consists of elements

$$y = a(e_3e_1)(e_3e_2) + be_3e_2 + ce_3e_1 + d(e_3e_2)(e_3e_1) = -ae_1e_2 + be_3e_2 + ce_3e_1 - de_2e_1.$$

and $y \in G^+(Q)$ iff $y \in C(Q)^*$ and $ye_iy^{-1} \in E, i = 1, 2, 3$. Before we check these conditions, notice that

$$e_3^2 = 1, (e_1e_2)^2 = e_1e_2(-1 - e_2e_1) = -e_1e_2, \quad e_2e_1e_2 = e_2(-1 - e_2e_1)$$

$$= -e_2, \quad e_1e_2e_1 = e_1(-1 - e_1e_2) = -e_1.$$

Notice that here we use the form B whose restriction to H equals $-B'$. Clearly,

$$y^{-1} = (ad - bc)^{-1}(-de_1e_2 - be_3e_2 - ce_3e_1 - ae_2e_1),$$

where, of course, we have to assume that $ad - bc \neq 0$. We have

$$(-ae_1e_2 + be_3e_2 + ce_3e_1 - de_2e_1)e_3(-de_1e_2 - be_3e_2 - ce_3e_1 - ae_2e_1) =$$

$$= (-ae_1e_2e_3 - be_2 - ce_1 - de_2e_1e_3)(-de_1e_2 - be_3e_2 - ce_3e_1 - ae_2e_1) =$$

$$= -(ad + bc)e_1e_2e_3 - (ad + bc)e_2e_1e_3 - 2bde_2 - 2ace_1 = -(ad + bc)e_3 - 2bde_2 - 2ace_1.$$

Just to confirm that it is correct, notice that

$$Q(\phi(y)(e_3)) = \frac{1}{(ad - bc)^2}((ad + bc)^2 - 2bdac) = \frac{1}{(ad - bc)^2}(ad - bc)^2 = 1 = Q(e_3),$$

as it should be, because $\phi(y)$ is an orthogonal transformation. Similarly, we check that $ye_iy^{-1} \in E, i = 1, 2$. Thus

$$G^+(Q) \cong GL(2, \mathbb{C}).$$

The kernel of $\phi : G^+(Q) \to SO(Q)$ consists of scalar matrices. Restricting this map to the subgroup $SL(2,\mathbb{C})$ has the kernel ± 1. This agrees with our discussion in 15.1.

By Theorem 4, $C(Q) \cong C^+(Q) \otimes_{\mathbb{C}} Z$, where $Z \cong \mathbb{C} \oplus \mathbb{C}$. Hence $C(Q) \cong C^+(Q) \oplus C^+(Q) \cong \mathrm{Mat}_2(\mathbb{C})^2$. This shows that $C(Q)$ admits two isomorphic 2-dimensional representations. To find one of them explicitly, we need to assign to each $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{C}^3$ a matrix $A(\mathbf{x})$ such that

$$A(\mathbf{x})^2 = ||\mathbf{x}||^2 I_2 = (x_1^2 + x_2^2 + x_3^2) I_2, \quad A(\mathbf{x})A(\mathbf{y}) + A(\mathbf{y})A(\mathbf{x}) = 2\mathbf{x}\cdot\mathbf{y} I_2.$$

Here it is:

$$A(\mathbf{x}) = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}.$$

Let

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = A(e_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_2 = A(e_2) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = A(e_3) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{15.13}$$

In the physics literature these matrices are called the *Pauli matrices*. Then

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I_2, \quad \sigma_i\sigma_j = -\sigma_j\sigma_i, \quad i = 1,2,3, \quad i \neq j.$$

Also, if we write

$$a_0\sigma_0 + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 = \begin{pmatrix} a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_0 - a_3 \end{pmatrix} \tag{15.14}$$

with real a_0, a_1, a_2, a_3, we obtain an isomorphism from $\mathbb{R}^4$ to $\mathrm{Mat}_2(\mathbb{C})$ such that

$$a_0^2 - a_1^2 - a_2^2 - a_3^2 = \det(a_0\sigma_0 + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3). \tag{15.15}$$

Notice that $SL(2,\mathbb{C})$ acts $\mathbb{R}^4$ by the formula

$$X \cdot A(\mathbf{x}) = X \cdot A(\mathbf{x}) \cdot X^*. \tag{15.16}$$

Notice that the set of matrices of the form $A(\mathbf{x})$ is the subset of Hermitean matrices in $\mathrm{Mat}_2(\mathbb{C})$. So the action is well defined. We will explain this homomorphism in the next lecture.

15.5 Finally let us define the *spinor group* $Spin(Q)$. This is a subgroup of the Clifford group $G^+(Q)$ such the kernel of the restriction of the canonical homomorphism $G^+(Q) \to SO(Q)$ to $Spin(Q)$ is equal to ± 1.

Consider the natural anti-automorphism of the tensor algebra $\rho' : T(E) \to T(E)$ defined on decomposable tensors by the formula

$$v_1 \otimes \ldots \otimes v_k \to v_k \otimes \ldots \otimes v_1.$$

Recall that an anti-homomorphism of rings $R \to R'$ is a homomorphism from R to the opposite ring R'^o (same ring but with the redefined multiplication law $x * y = y \cdot x$). It is clear that $\rho'(I(Q)) = I(Q)$ so ρ' induces an anti-automorphism $\rho : C(Q) \to C(Q)$. For any $x \in G(Q)$, we set

$$N(x) = x \cdot \rho(x). \tag{15.17}$$

By Corollary to Proposition 1, each $x \in G(Q)$ can be written in the form $x = zv_1 \dots v_k$ for some $z \in Z(C(Q))^*$ and non-isotropic vectors $v_1, \dots, v_k$ from E. We have

$$N(x) = zv_1 \dots v_k \rho(zv_1 \dots v_k) = z^2 v_1 \dots v_k v_k \dots v_1 = z^2 Q(v_1) \dots Q(v_k) \in K^*. \tag{15.18}$$

Also

$$N(x \cdot y) = xy\rho(xy) = xy\rho(y)\rho(x) = xN(y)\rho(x) = N(y)x\rho(x) = N(y)N(x).$$

This shows that the map $x \to N(x)$ defines a homomorphism of groups

$$N : G(Q) \to K^*. \tag{15.19}$$

It is called the *spinor norm* homomorphism. We define the *spinor group* (or *reduced Clifford group*) $Spin(Q)$ by setting

$$Spin(Q) = Ker(N) \cap G^+(Q). \tag{15.20}$$

Let

$$SO(Q)_0 = \mathrm{Im}(G^+(Q) \xrightarrow{\phi} SO(Q)). \tag{15.21}$$

This group is called the *reduced orthogonal group.*

From now on we assume that $\mathrm{char}(K) \neq 2$.

Obviously, $\mathrm{Ker}(\phi) \cap Spin(Q)$ consists of central elements z with $z^2 = 1$. This gives us the exact sequence

$$1 \to \{\pm 1\} \to Spin(Q) \to SO(Q)_0 \to 1. \tag{15.22}$$

It is clear that $N(K^*) \subset (K^*)^2$ so $\mathrm{Im}(N)$ contains the subgroup $(K^*)^2$ of K^*. If $K = \mathbb{R}$ and Q is definite, then (15.18) shows that $N(G^+) \subset (\mathbb{R}^*)^2 = \mathbb{R}_{>0}$. Since $N(\rho(x)) = N(x)$, we have $N(x/\sqrt{N(x)}) = N(x)/N(x) = 1$. Thus we can always represent an element of $SO(Q)$ by an element $x \in G^+(Q)$ with $N(x) = 1$. This shows that the canonical homomorphism $Spin(Q) \to SO(Q)$ is surjective, i.e., $SO(Q)_0 = SO(Q)$. In this case, if $n = \dim\ E$, the group $Spin(Q)$ is denoted by $Spin(n)$, and we obtain an exact sequence

$$1 \to \{\pm 1\} \to Spin(n) \to SO(n) \to 1. \tag{15.23}$$

One can show that $Spin(n), n \geq 3$ is the universal cover of the Lie group $SO(n)$.

If K is arbitrary, but E contains an isotropic vector $v \neq 0$, we know that $Q(w) = a$ has a solution for any $a \in K^*$ (because E contains a hyperbolic plane $Kv + Kv'$, with $Q(v) = Q(v') = 0, B(v, v') = b \neq 0$, and we can take $w = av + b^{-1}v'$). Thus if we choose $w, w' \in E$ with $Q(w) = a, Q(w') = 1$, we get $N(ww') = Q(w)Q(w') = a$. This shows that

$\text{Im}(N(G^+(Q))) = K^*$. Under the homomorphism N the factor group $G^+(Q)/Spin(Q)$ is mapped isomorphically onto K^*. On the other hand, under ϕ this group is mapped surjectively to $SO(Q)/SO(Q)_0$ with kernel $(K^*)^2$. This shows that

$$SO(Q)/SO(Q)_0 \cong K^*/(K^*)^2. \tag{15.24}$$

In particular, when $K = \mathbb{R}$, and Q is indefinite, we obtain that $SO(Q)_0$ is of index 2 in $SO(Q)$. For example, when $E = \mathbb{R}^4$ with Lorentzian quadratic form, we see that $SO(1,3)_0$ is the proper Lorentz group, so even our notation agrees. As we shall see in the next lecture, $Spin(1,3) \cong SL(2,\mathbb{C})$. This agrees with section 15.1 of this lecture.

Exercises.

1. Show that the Lorentz group $O(1,3)$ has 4 connected components. Identify $SO(1,3)_0$ with the connected component containing the identity.

2. Assume $Q = 0$ so that $C(Q) \cong \bigwedge(E)$. Let $f \in E^*$ and $i_f : \bigwedge(E) \to \bigwedge(E)$ be the map defined in (15.3). Show that

$$i_f(x_1 \wedge \ldots \wedge x_k) = \sum_{i=1}^{k} (-1)^{i-1} f(x_i)(x_1 \wedge \ldots \wedge x_{i-1} \wedge x_{i+1} \wedge \ldots \wedge x_k).$$

3. Using the classification of quadratic forms over a finite field, classify Clifford algebras over a finite field.

4. Show that $C(Q)$ can be defined as follows. For any linear map $f : E \to D$ to some unitary algebra $C(Q)$ satisfying $f(v) = Q(v) \cdot 1$, there exists a unique homomorphism f' of algebras $C(Q) \to D$ such that its restriction to $E \subset C(Q)$ is equal to f.

5. Let Q be a non-degenerate quadratic form on a vector space of dimension 2 over a field K of characteristic different from 2. Prove that $C(Q) \cong M_2(K)$ if and only if there exists a vector $x \neq 0$ and two vectors y, z such that $Q(x) + Q(y)Q(z) = 0$.

6. Using the theory of Clifford algebras find the known relationship between the orthogonal group $O(3)$ and the group of pure quaternions $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}, a^2 + b^2 + c^2 \neq 0$.

7. Show that $Spin(2) \cong SU(1), \quad Spin(3) \cong SU(2)$.

8. Prove that $Spin(1,2) \cong SL(2,\mathbb{R})$.

Lecture 16. THE DIRAC EQUATION

As we have seen in Lecture 13, the Klein-Gordon equation does not give the right relativistic picture of the 1-particle theory. The right approach is via the Dirac equation. Instead of a scalar field we shall consider a section of complex rank 4 bundle over $\mathbb{R}^4$ whose structure group is the spinor group $Spin(1,3)$ of the Lorentzian orthogonal group.

16.1 Let $O(1,3)$ be the Lorentz group. The corresponding Clifford algebra C is isomorphic to $\mathrm{Mat}_2(\mathbb{H})$ (Corollary 2 to Theorem 3 from Lecture 15). If we allow ourselves to extend $\mathbb{R}$ to $\mathbb{C}$, we obtain a 4-dimensional complex spinor representation of C. It is given by the *Dirac matrices*:

$$\gamma^0 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad j = 1,2,3, \tag{16.1}$$

where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

are the Pauli matrices (15.13). It is easy to see that

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} I_4,$$

where $g^{\mu\nu}$ is the inverse of the standard Lorentzian metric. Using (15.3), this implies that $e_i \to \gamma^i$ is a 4-dimensional complex representation $S = \mathbb{C}^4$ of C. Let us restrict it to the space $V = \mathbb{R}^4$. Then the image of V consists of matrices of the form

$$\begin{pmatrix} 0 & X \\ \mathrm{adj}(X) & 0 \end{pmatrix}, \tag{16.2}$$

where

$$X = \begin{pmatrix} x_0 - x_1 & ix_3 - x_2 \\ -x_2 - ix_3 & x_0 + x_1 \end{pmatrix} \tag{16.3}$$

is a Hermitian 2×2-complex matrix (i.e., $X^* = X$) and $\mathrm{adj}(X)$ is its adjugate matrix (i.e., $\mathrm{adj}(X) \cdot X = \det(X) I_2$). The Clifford group $G(Q) \subset C(Q)^*$ acts on the set of such matrices by conjugation (see (15.12)). Its subgroup $G^+(Q)$ must preserve the subspaces $S_L = \mathbb{R}e_1 + \mathbb{R}e_2$ and $S_R = \mathbb{R}e_3 + \mathbb{R}e_4$. Hence any $g \in G^+(Q)$ can be given by a block-diagonal matrix of the form

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}. \tag{16.4}$$

It must satisfy

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} 0 & X \\ \mathrm{adj}(X) & 0 \end{pmatrix} \cdot \begin{pmatrix} A_1^{-1} & 0 \\ 0 & A_2^{-1} \end{pmatrix} = \begin{pmatrix} 0 & Y \\ \mathrm{adj}(Y) & 0 \end{pmatrix}$$

for some Hermitian matrix Y. This immediately implies that

$$A = \begin{pmatrix} B & 0 \\ 0 & (B^*)^{-1} \end{pmatrix}, \quad Y = BXB^*, \tag{16.5}$$

where $B \in GL(2,\mathbb{C})$ satisfies $|\det B| = 1$. Now it is easy to see that any $A \in G^+$ must satisfy the additional property that $\det B = 1$. This follows from the fact that G^+ is mapped surjectively onto the group $SO(1,3)$ and the kernel consists only of the matrices ± 1. Thus

$$G^+ = Spin(1,3) \cong SL(2,\mathbb{C}). \tag{16.6}$$

Also, we see that, if we identify vectors from V with Hermitian matrices X, the group $G^+ = SL(2,\mathbb{C})$ acts on V via transformations $X \to B \cdot X \cdot B^*$, which is in accord with equality (15.15) from Lecture 15.

It is clear that the restriction of the spinor representation S to G^+ splits into the direct sum $S_L \oplus S_R$ of two complex representations of dimension 2. One is given by $g \to A$, another one by $g \to (A^*)^{-1}$. The elements of S_L (resp. of S_R) are called *left-handed half-spinors* (resp. *right-handed half-spinors*). They are dual to each other and not isomorphic (as real 4-dimensional representations of G^+). We denote the vectors of the 4-dimensional space of spinors $S = \mathbb{C}^4$ by

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix},$$

where $\phi_L \in S_L, \phi_R \in S_R$. Note that the spinor representation S is self-dual, since

$$(g^*)^{-1} = \gamma^0 g (\gamma^0)^{-1}.$$

We can express the projector operators $p_R : S \to S_R$, $p_L : S \to S_L$ by the matrices

$$p_R = \frac{1+\gamma^5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad p_L = \frac{1-\gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

where

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{16.7}$$

(the reason for choosing superscript 5 but not 4 is to be able to use the same notation for this element even if we start indexing the Dirac matrices by numbers $1, 2, 3, 4$). The spinor group $Spin(1,3)$ acts in S leaving S_R and S_L invariant. These are the half-spinor representations of G^+.

16.2 Now we can introduce the *Dirac equation*

$$(\gamma^\mu \partial_\mu + imI_4)\psi = 0. \tag{16.8}$$

The left-hand side, is a 4×4-matrix differential operator, called the *Dirac operator*. The constant m is the mass (of the electron). The function ψ is a vector function on the space-time with values in the spinor representation S. If we multiply both sides of (16.8) by $(\gamma^\mu \partial_\mu - imI_4)$, we get

$$(\gamma^\mu \partial_\mu)^2 \psi = (\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2) I_4 \psi + m^2 \psi = 0.$$

This means that each component of ψ satisfies the Klein-Gordon equation.

In terms of the right and left-handed half-spinors the Dirac equation reads

$$\begin{pmatrix} (\sigma_0 \partial_0 - \sigma_j \partial_j)\psi_R \\ (\sigma_0 \partial_0 + \sigma_j \partial_j)\psi_L \end{pmatrix} = -im \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}. \tag{16.9}$$

Notice that the Dirac operator maps S_R to S_L.

Now let us see the behavior of the Dirac equation (16.8) under the proper Lorentz group $SO(1,3)_0$. We let it act on $V = \mathbb{R}^4$ via its spinor cover $Spin(1,3)$ by identifying $\mathbb{R}^4$ with the space of matrices (16.2). Now we let $g \in Spin(1,3)$ act on the solutions $\psi : \mathbb{R}^4 \to S = \mathbb{C}^4$ by acting on the source by means of $\phi(g) \in SO(1,3)_0$ and on the target by means of the spinor representation. We identify the arguments $x = (t = x_0, x_1, x_2, x_3)$ with matrices $M(x) = x_i \gamma^i$. An element $g \in Spin(1,3)$ is identified with a matrix $A(g)$ from (16.4). If $\Lambda(g)$ is the corresponding to $SO(1,3)_0$, then we have

$$A(g) \cdot M(x) \cdot A(g)^{-1} = M(\Lambda(g) \cdot x).$$

Thus the function $\psi(x)$ is transformed to the function ψ' such that

$$\psi'(x') = A(g) \cdot \psi(x), \quad x' = \Lambda(g) \cdot x \tag{16.10}$$

We have

$$\partial'_\mu \phi'(x') = \frac{\partial \phi'(x')}{\partial x'_\mu} = A(g) \Lambda(g)_{\mu\nu} \partial_\nu \phi(x),$$

$$A(g)^{-1}(\gamma^\mu \partial'_\mu + imI_4)\phi'(x') = A(g)^{-1} M(e_\mu) A(g) (\Lambda(g)^{-1} \partial_\mu)\phi + mA(g)^{-1}\phi'(x') =$$

$$= M(\Lambda^{-1}(g) \cdot e_\mu)(\Lambda(g)^{-1}\partial_\mu)\partial_\mu \phi + m\phi(x) = \gamma^\mu \partial_\mu \phi + m\phi(x) = 0.$$

This shows that the Dirac equation is invariant under the Lorentz group when we transform the function $\psi(x)$ according to (16.10). There are other nice properties of invariance. For example, let us set

$$\psi^* = (\psi_L, \psi_R)^* = (\bar{\psi}_R, \bar{\psi}_L) = \gamma^0 \bar{\psi} \tag{16.11}$$

Then

$$\psi^* \cdot \psi = \bar{\psi}_R \cdot \psi_L + \bar{\psi}_L \cdot \psi_R$$

is invariant. Indeed

$$\psi'(x')^* \cdot \psi'(x') = \bar{\psi}'(x')\gamma^0\psi'(x') = \bar{\psi}(x)A(g)^*\gamma^0 A(g) \cdot \psi =$$

$$= \bar{\psi}(x)\gamma^0 A(g)^*\gamma^0 A(g) \cdot \psi = \psi(x)^*\psi(x).$$

Here we used that

$$\gamma^0 A(g)^*\gamma^0 A(g) = \begin{pmatrix} 0 & I \\ 1 & 0 \end{pmatrix} \begin{pmatrix} B^* & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} 0 & I \\ 1 & 0 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{*-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

16.3 Now we shall forget about physics for a while and consider the analog of the Dirac operator when we replace the Lorentizian metric with the usual Euclidean metric. That is, we change the group $O(1,3)$ to $O(4)$, i.e. we consider $V = \mathbb{R}^4$ with the standard quadratic form $Q = x_1^2 + x_2^2 + x_3^2 + x_4^2$. The spinor complex representation $s : C(Q) \to M_4(\mathbb{C})$ is given by the matrices

$$\gamma'^0 = \begin{pmatrix} 0 & \sigma_0' \\ \sigma_0' & 0 \end{pmatrix}, \quad \gamma'^j = \begin{pmatrix} 0 & -\sigma_j' \\ \sigma_j' & 0 \end{pmatrix}, \quad j = 1, 2, 3, \tag{16.12}$$

obtained by replacing the Pauli matrices $\sigma_i, i = 1, 2, 3$ with

$$\sigma_0' = \sigma_0, \quad \sigma_1' = i\sigma_1, \quad \sigma_2' = i\sigma_2, \quad \sigma_3' = i\sigma_3.$$

The image $s(V)$ of V consists of matrices of the form

$$A = \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix},$$

where

$$X = \begin{pmatrix} a - bi & -c - di \\ c - di & a + bi \end{pmatrix} = a\sigma_0' + b\sigma_1' + c\sigma_2' + d\sigma_3', \quad a, b, c, d \in \mathbb{R}. \tag{16.13}$$

These are characterized by the condition $X^* = \mathrm{adj}(X)$. Note that

$$a^2 + b^2 + c^2 + d^2 = \det(s(a, b, c, d)).$$

If we identify (a, b, c, d) with the quaternion $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, this corresponds to the well-known identification of quaternions with complex matrices (16.13).

Similar to (16.5) in the previous section we see that elements of $s(G^+(Q))$ are matrices of the form

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where $A_1, A_2 \in U(2), \det(A_1) = \det(A_2)$. In particular, the two half-spinor representations in this case are isomorphic. This is explained by the general theory. The center of the algebra $C^+(Q)$ is equal to $Z = \mathbb{R} + \gamma'^5\mathbb{R}$, where

$$\gamma'^5 = \gamma'_0\gamma'_1\gamma'_2\gamma'_3.$$

Since $(\gamma'^5)^2 = -I_4$, the center Z is a field. This implies that $C^+(Q)$ is a simple algebra, hence the restriction of the spinor representation to $C^+(Q)$ is isomorphic to the direct sum of two isomorphic 2-dimensional representations. Notice that in the Lorentzian case, the center is spanned by 1 and $\gamma^5 = i\gamma'_0\gamma'_1\gamma'_2\gamma'_3$ so that $(\gamma^5)^2 = I_4$. This implies that C^+ is not simple.

Now, arguing as in the Lorentzian case, we easily get that

$$Spin(4) \cong G^+(Q) \cong SU(2) \times SU(2) \cong Sp(1) \times Sp(1), \tag{16.14}$$

where $Sp(1)$ is the group of quaternions of norm 1. The isomorphism $Sp(1) \cong SU(2)$ is induced by the map $\mathbb{R}^4 \to GL_2(\mathbb{C})$ given by formula (16.13).

Let

$$S^+ = \mathbb{C}e_1 + \mathbb{C}e_2, \quad S^- = \mathbb{C}e_3 + \mathbb{C}e_4$$

They are our half-spinor representations of $G^+(Q)$. The Dirac matrices γ'^μ define a linear map

$$s : V \to \mathrm{Hom}_{\mathbb{C}}(S^+, S^-).$$

In the standard bases of $S^\pm$ this is given by the Pauli matrices

$$e_i \to \sigma'_i, \quad i = 0, \ldots, 3.$$

Let us view $S^\pm$ as the space of quaternions by writing

$$(a + bi, c + di) \to a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = (a + bi) + j(c - di).$$

Then the image $s(V)$ is the set of linear maps over $\mathbb{H}$. It is a line over the quaternion skew field. Equivalently, if we define the (anti-linear) operator $J : S^\pm \to S^\pm$ by

$$J(z_1, z_2) = (-\bar{z}_2, \bar{z}_1),$$

then $s(V)$ consists of linear maps satisfying $f(Jx) = Jf(x)$.

Let us equip S^+ and S^- with the standard Hermitian structure

$$\langle (z_1, z_2), (w_1, w_2) \rangle = z_1\bar{w}_1 + z_2\bar{w}_2.$$

Then for any $\psi : S^+ \to S^-$ we can define the adjoint map $\psi^* : S^- \to S^+$. It is equal to the composition $S^- \to S^+ \to S^- \to S^+$, where the first and the third arrow are defined by using the Hermitian pairing. The middle arrow is the map ψ. It is easy to see that

$$s(e_i) + s(e_i)^* : S^+ \oplus S^- \to S^- \oplus S^+$$

is given by the Dirac matrix γ'^i. This implies that, for any $v \in V$, the image of v in the spinor representation $\tilde{s} : C(Q) \to \mathrm{Mat}_4(\mathbb{C})$ is given by $s(v) + s(v)^*$. Since s is a homomorphism, we get, for any $v, v' \in C(Q)$,

$$\tilde{s}(vv') = \tilde{s}(v)\tilde{s}(v') = (s(v) + s(v)^*) \circ (s(v') + s(v')^*) = s(v)^*s(v') + s(v')^*s(v) = s(vv').$$

In particular, if v is orthogonal to v', we have $vv' = -v'v$, hence

$$s(v)^*s(v') + s(v')^*s(v) = 0 \quad \text{if } v \cdot v' = 0. \tag{16.15}$$

Also, if $||v|| = Q(v) = 1$, we must have $v^2 = 1$, hence

$$s(v)^*s(v) = 1 \quad \text{if } v \cdot v = 1. \tag{16.16}$$

Let $\phi : \bigwedge^2(V) \to \mathrm{End}(S^+)$ be defined by sending $v \wedge v'$ to $s(v)^*s(v') - s(v')^*s(v)$. If v, v' are orthogonal, we have by (8),

$$(s(v)^*s(v') - s(v')^*s(v))^* = -(s(v)^*s(v') - s(v')^*s(v)).$$

Since any $v \wedge v'$ can be written in the form $w \wedge w'$, where w, w' are orthogonal, we see that the image of ϕ is contained in the subspace of $\mathrm{End}(S^+)$ which consists of operators A such that $A^* + A = 0$. From Lecture 12, we know that this subspace is the Lie algebra of the Lie group of unitary transformations of S^+. So, we denote it by $\mathfrak{su}(S^+)$. Thus we have defined a linear map

$$\phi^+ : \bigwedge^2 V \to \mathfrak{su}(S^+). \tag{16.17}$$

Similarly we define the linear map

$$\phi^- : \bigwedge^2 V \to \mathfrak{su}(S^-). \tag{16.18}$$

To summarize, we make the following definition

Definition. Let V be a four-dimensional real vector space with the Euclidean inner product. A *spinor structure* on V is a homomorphism

$$s : V \to \mathrm{Hom}_{\mathbb{C}}(S^+, S^-)$$

where $S^\pm$ are two-dimensional complex vector spaces equipped with a Hermitian inner product. Additionally, it is required that the homomorphism preserves the inner products.

Here the inner product in $\mathrm{Hom}(S^+, S^-)$ is defined naturally by

$$||\phi||^2 = \left| \frac{\phi(e_1) \wedge \phi(e_2)}{e_1' \wedge e_2'} \right|,$$

where e_1, e_2 (resp. (e_1', e_2')) is an orthonormal basis in S^+ (resp. S^-).

Given a spin-structure on V, we can consider the action of $Spin(4) \cong SU(2) \times SU(2)$ on $\mathrm{Hom}_{\mathbb{C}}(S^+, S^-)$ via the formula

$$(A, B) \cdot f(\sigma) = A \cdot f(B^{-1} \cdot \sigma).$$

This action leaves $s(V)$ invariant and induces the action of $SO(4)$ on V.

16.4 Now let us globalize. Let M be an oriented smooth Riemannian 4-manifold. The tangent space $T(M)_x$ of any point $x \in M$ is equipped with a quadratic form $g_x : T(M)_x \to \mathbb{R}$ which is given by the metric on M. Thus we are in the situation of the previous lecture. First of all we define a *Hermitian vector bundle* over a manifold M. This is a complex vector bundle E together with a Hermitian form $Q : E \to 1_M$. The structure group of a Hermitian bundle can be reduced to $U(n)$, where n is the rank of E.

Definition. A *spin structure* on M is the data which consists of two complex Hermitian rank 2 vector bundles S^+ and S^- over M together with a morphism of vector bundles

$$s : T(M) \to \mathrm{Hom}(S^+, S^-)$$

such that for any $x \in M$, the induced map of fibres $s_x : T(M)_x \to Hom(S^+_x, S^-_x)$ preserves the inner products.

The Riemannian metric on M plus its orientation allows us to reduce the structure group of $T(M)_{\mathbb{C}}$ to the group $SO(4)$. A choice of a spin-structure allows us to lift it to the group $Spin(4) \cong SU(2) \times SU(2)$. Conversely, given such a lift, the two obvious homomorphisms $Spin(4) \to SU(2)$ define the associated vector bundles $S^{\pm}$ and the morphism $s : T(M) \to \mathrm{Hom}(S^+, S^-)$.

Theorem 1. *A spin-structure exists on M if and only if for any $\gamma \in H^2(M, \mathbb{Z}/2\mathbb{Z})$*

$$\gamma \cap \gamma = 0.$$

Moreover, it is unique if $H^1(M, \mathbb{Z}/2\mathbb{Z}) = 0$.

Proof. We give only a sketch. Recall that there is a natural bijective correspondence between the isomorphism classes of principal G-bundles and the cohomology set $H^1(M, \mathcal{O}_M(G))$. Let

$$1 \to (\mathbb{Z}/2\mathbb{Z})_M \to \mathcal{O}_M(Spin(4)) \to \mathcal{O}_M(SO(4)) \to 1$$

be the exact sequence of sheaves corresponding to the exact sequence of groups

$$1 \to \{\pm 1\} \to Spin(4) \to SO(4) \to 1.$$

Applying cohomology, we get an exact sequence

$$H^1(M, \mathbb{Z}/2\mathbb{Z}) \to H^1(M, \mathcal{O}_M(Spin(4)) \to H^1(M, \mathcal{O}_M(SO(4))) \to H^2(M, \mathbb{Z}/2\mathbb{Z}). \quad (16.18)$$

If $T(M)$ corresponds to an element $t \in H^1(M, \mathcal{O}_M(SO(4)))$, then its image $\delta(t)$ in $H^2(M, \mathbb{Z}/2\mathbb{Z})$ is equal to zero if and only if t is the image of an element t' from the cohomology group $\in H^1(M, \mathcal{O}_M(Spin(4))$ representing the isomorphism class of a principal $Spin(4)$-bundle. Clearly, this means that the structure group of $T(M)$ can be lifted to $Spin(4)$. One shows that $\delta(t)$ is the Stieffel-Whitney characteristic class $w_2(M)$ of $T(M)$. Now, by Wu's formula

$$\gamma \cap \gamma = \gamma \cap w_2(M).$$

This shows that the condition $w_2(M) = 0$ is equivalent to the condition from the assertion of Theorem 1. The uniqueness assertion follows immediately from the exact sequence (16.18).

Example. A compact complex 2-manifold M is called a K3-surface if $b_1(M) = 0, c_1(M) = 0$, where $c_1(M)$ is the Chern class of the complex tangent bundle $T(M)$. An example of a K3 surface is a smooth projective surface of degree 4 in $\mathbb{P}^3(\mathbb{C})$ of degree 4. One shows that M is simply-connected, so that $H^1(M, \mathbb{Z}) = 0$. It is also known that for any complex surface M, $w_2(M) \equiv c_1(M)$ mod 2. This implies that $w_2(M) = 0$, hence every K3 surface together with a choice of a Riemannian metric admits a unique spin structure.

16.5 To define the global Dirac operator, we have to introduce the Levi-Civita connection on the tangent bundle $T(M)$. Let A be any connection on the tangent bundle $T(M)$. It defines a covariant derivative

$$\nabla_A : \Gamma(T(M)) \to \Gamma(T(M)) \otimes T^*(M)).$$

Thus for any vector field τ, we have an endomorphism

$$\nabla_A^\tau : \Gamma(T(M)) \to \Gamma(T(M)), \quad \eta \to \nabla_\tau^A(\eta),$$

defined by

$$\nabla_A^\tau(\eta) = \langle \nabla_A(\eta), \tau \rangle,$$

where $\langle\ ,\ \rangle$ denotes the contraction $\Gamma(T(M) \otimes T^*(M)) \times \Gamma(T(M)) \to \Gamma(T(M)$. Recall now that we have an additional structure on $\Gamma(T(M))$ defined by the Lie bracket of vector fields. For any $\tau, \eta \in \Gamma(T(M))$ define the *torsion operator* by

$$T_A(\tau, \eta) = \nabla_A^\tau(\eta) - \nabla_A^\eta(\tau) - [\tau, \eta] \in \Gamma(T(M)). \tag{16.20}$$

Obviously, T_A is a skew-symmetric bilinear map on $\Gamma(T(M)) \times \Gamma(T(M))$ with values in $\Gamma(T(M))$. In other words,

$$T_A \in \Gamma(\bigwedge^2(T^*(M) \otimes T(M)) = \mathcal{A}^2(T(M))(M).$$

One should not confuse it with the curvature

$$F_A \in \mathcal{A}^2(\mathrm{End}(T(M)))(M)$$

defined by (see (12.9))

$$F_A(\tau,\eta) = [\nabla_A^\tau, \nabla_\eta] - \nabla_A^{[\tau,\eta]}.$$

The connection ∇_A on $T(M)$ naturally yields a connection on $T^*(M)$ and on $T^*(M)^* \otimes T^*(M)$ (use the formula $\nabla(s\otimes s') = \nabla(s)\otimes s' + s\otimes\nabla(s')$). Locally, if $(\partial_1,\ldots,\partial_n)$ is a basis of vector fields on M, and $dx^1,\ldots,dx^n$ is the dual basis of 1-forms, we can write

$$T_A = T^i_{jk}\partial_i \otimes dx^j \otimes dx^k, \quad F_A = R^i_{j\ kl}\partial_i \otimes dx^j \otimes dx^k \otimes dx^l. \tag{16.21}$$

We shall keep the same notation ∇_A for it. Now, we can view a Riemannian metric g on M as a section of the vector bundle $T^*(M)\otimes T^*(M)$.

Lemma-Definition. Let g be a Riemannian metric on M. There exists a unique connection A on $T(M)$ satisfying the properties

(i) $T_A = 0$;

(ii) $\nabla_A(g) = 0$.

This connection is called the *Levi-Civita* (or *Riemannian*) connection.

The Levi-Civita connection ∇_{LC} is defined by the formula

$$g(\eta,\nabla^\tau_{LC}(\xi)) = -\eta g(\tau,\xi)+\xi g(\eta,\tau)+\tau g(\xi,\eta)+g(\xi,[\eta,\tau])+g(\eta,[\xi,\tau])-g(\tau,[\xi,\eta]), \tag{16.22}$$

where η,τ,ξ are arbitrary vector fields on M. Once checks that $\xi\to\nabla^\tau_{LC}(\xi)$ defined by this formula satisfies the Leibnitz rule, and hence is a covariant derivative. Using (16.20) we easily obtain

$$g(\xi,\nabla^\tau_{LC}(\eta)) + g(\eta,\nabla^\tau_{LC}(\xi)) = \tau g(\eta,\xi). \tag{16.23}$$

If locally

$$\nabla^{\partial_i}_{LC}(\partial_j) = \sum_k \Gamma^k_{ij}\partial_k, \quad g_{ij} = g(\partial_i,\partial_j),$$

then we get (*Christoffel's identity*)

$$\Gamma^l_{ij} = \frac{1}{2}\sum_k(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})g^{kl}. \tag{16.24}$$

The vanishing of the torsion tensor is equivalent to

$$\Gamma^k_{ij} = \Gamma^k_{ji}.$$

The meaning of (16.23) is the following. Let $\gamma:(a,b)\to M$ be an integral curve of a vector field τ. We say that a vector field ξ is *parallel* over γ if $\nabla^\tau_{LC}(\xi)_x = 0$ for any $x\in\gamma((0,1))$. Let $(x_1,\ldots,x_n)$ be a system of local parameters in an open subset $U\subset M$, and let $\gamma(t) = (x_1(t),\ldots,x_n(t)), \xi = \sum_j a^j\partial_j, \tau = \sum_i b^i\partial_i$. Since $\gamma(T)$ is an integral curve of τ, we have $b^i = \frac{dx^i}{dt}$. Then ξ is parallel over $\gamma(t)$ if and only if

$$\nabla^\tau_{LC}(\xi)_{\gamma(t)} = \nabla^\tau_{LC}(a^j\partial_j)_{\gamma(t)} = \frac{da^k(\gamma(t))}{dt}\partial_k + a^j\frac{dx^i(t)}{dt}\Gamma^k_{ij}\partial_k = 0.$$

This is equivalent to the following system of differential equations:

$$\frac{da^k(\gamma(t))}{dt} + a^j \frac{dx^i(t)}{dt}\Gamma^k_{ij} = 0, \quad k = 1, \dots, n$$

In particular, τ is parallel over its integral curve γ if and only if

$$\frac{d^2x^k(t)}{dt^2} + \frac{dx^j(t)}{dt}\frac{dx^i(t)}{dt}\Gamma^k_{ij} = 0 \quad k = 1, \dots, n.$$

Comparing this and (16.24) with Exercise 1 in Lecture 1, we find that this happens if and only if γ is a geodesic, or a critical path for the natural Lagrangian on the Riemannian manifold M.

Now formula (16.23) tells us that for any two vector fields η and ξ parallel over $\gamma(t)$, we have

$$\frac{d(g(\eta_{\gamma(t)}, \xi_{\gamma(t)}))}{dt} = 0.$$

In other words, the scalar product $g(\eta_x, \xi_x)$ is constant along $\gamma(t)$.

Remark. Let

$$F \in \Gamma(\mathrm{End}(T(M)) \otimes \bigwedge^2(T^*(M)) \subset \Gamma(T^*(M) \otimes T(M) \otimes T^*(M) \otimes T^*(M))$$

be the curvature tensor of the Levi-Civita connection. Let $Tr : T(M) \otimes T^*(M) \to 1_M$ be the natural contraction map. If we apply it to the product of the second and the third tensor factor in above, we obtain the linear map $\Gamma(\mathrm{End}(T(M) \otimes \bigwedge^2(T^*(M)) \to \Gamma(\bigwedge^2(T^*(M))$. The image of F is a 2-form R on M. It is called the *Ricci tensor* of the Riemannian manifold M. One proves that R is a symmetric tensor. In coordinate expression (16.21), we have

$$R = R_{kl}dx^k \otimes dx^l, \quad R_{kl} = \sum_i R^i_{k\ il}.$$

We can do other things with the curvature tensor. For example, we can use the inverse metric $g^{-1} \in \gamma(T(M) \otimes T(M))$ to contract R to get the scalar function $K : M \to \mathbb{R}$. It is called the *scalar curvature* of the Riemannian manifold M. We have

$$K = \sum_{k,l=1}^{n} g^{kl} R_{kl}. \tag{16.25}$$

When

$$R = \lambda g \tag{16.26}$$

for some scalar function $\lambda : M \to \mathbb{R}$, the manifold M is called an *Einstein space*. By contracting both sides, we get

$$K = g^{ij}R_{ij} = \lambda g^{ij}g_{ij} = n\lambda$$

One can show that K is constant for an Einstein space of dimension $n \geq 3$. The equation

$$R - \frac{K}{n} g = \frac{8\pi G}{c^4} T \qquad (16.27)$$

is the *Einstein equation* of general relativity. Here c is the speed of light, G is the gravitational constant, and T is the energy-momentum tensor of the matter.

16.6 The notion of the Levi-Civita connection extends to any Hermitian bundle E. We require that $\nabla_A(h) = 0$, where $h : E \to 1_M$ is the positive definite Hermitian form on E. Such a connection is called a *unitary connection.*

Let us use the linear map (16.18) to define a unitary connection on the spinor bundles $S^{\pm}$ such that the induced connection on $\mathrm{Hom}(S^+, S^-)$ coincides with the Levi-Civita connection on $T(M)$ (after we identify $T(M)$ with its image $s(T(M))$. Let $\mathfrak{so}(n)$ be the Lie algebra of $SO(n)$. From Lecture 12 we know that $\mathfrak{so}(n)$ is isomorphic to the space of skew-symmetric real $n \times n$-matrices. This allows us to define an isomorphism of Lie algebras

$$\mathfrak{so}(n) \cong \bigwedge^2 (V).$$

More explicitly, if we take the standard basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$, then $e_i \wedge e_j, i < j$, is identified with the skew-symmetric matrix $E_{ij} - E_{ji}$, so that

$$[e_i \wedge e_j, e_k \wedge e_l] = [E_{ij} - E_{ji}, E_{kl} - E_{lk}] = [E_{ij}, E_{kl}] - [E_{ji}, E_{kl}] - [E_{ij}, E_{lk}] + [E_{ji}, E_{lk}]$$

$$= \begin{cases} 0 & \text{if } \#\{i,j\} \cap \{k,l\} \text{ is even,} \\ -E_{jl} & \text{if } i = k, j \neq l, \\ -E_{ik} & \text{if } j = l, i \neq k, \\ E_{jk} & \text{if } i = l, \\ E_{il} & \text{if } j = k. \end{cases}$$

We use this to verify that the linear maps (16.17) and (16.18) are homomorphism of Lie algebras

$$\psi^{\pm} : \mathfrak{so}(4) \to \mathfrak{su}(S^{\pm}).$$

Now the half-spinor representations of $Spin(4)$ define two associated bundles over M which we denote by $S^{\pm}$. If we identify the Lie algebra of $Spin(4)$ with the Lie algebra of $SO(4)$, then $\psi^{\pm}$ becomes the corresponding representation of the Lie algebras. Let A be the connection on the principal bundle of orthonormal frames which defines the Levi-Civita connection on the associated tangent bundle $T(M)$. Then A defines a connection on the associated vector bundles $S^{\pm}$.

Given a spin structure on M we can define the *Dirac operator.*

$$D : \Gamma(S^+) \to \Gamma(S^-). \qquad (16.28)$$

Locally, we choose an orthonormal frame $e_1, \ldots, e_4$ in $T(M)$, and define, for any $\sigma \in \Gamma(S^+)$,

$$D\sigma = \sum_i s(e_i) \nabla_i(\sigma). \qquad (16.28)$$

where $\nabla_i = \partial_i + A_i$ is the local expression of the Levi-Civita connection. We leave to the reader to verify that this definition is independent of the choice of a trivialization. We can also define the *adjoint Dirac operator*

$$D^* : \Gamma(S^-) \to \Gamma(S^+),$$

by setting

$$D^*\sigma = -\sum_i s(e_i)^* \nabla_i(s).$$

When M is equal to $\mathbb{R}^4$ with the standard Euclidean metric, the Christoffel identity (16.24) tells us that $\Gamma^k_{ij} = 0$, hence

$$\nabla_i(a^\mu \partial_\mu) = \partial_i(a^\mu)\partial_\mu.$$

We can identify $S^\pm$ with trivial Hermitian bundles $M \times \mathbb{C}^2$. A section of $S^\pm$ is a function $\psi^\pm : M \to \mathbb{C}^2$. We have

$$D\phi^+ = \sum_{i=1}^4 s(e_i)\partial_i(\phi^+) = (\sigma'_0\partial_0 + \sigma'_1\partial_1 + \sigma'_2\partial_2 + \sigma'_3\partial_3)\phi^+$$

Similarly,

$$D^*\phi^- = -\sum_{i=1}^4 s(e_i)^*\partial_i(\phi^-) = (-\sigma'_0\partial_0 + \sigma'_1\partial_1 + \sigma'_2\partial_2 + \sigma'_3\partial_3)\phi^-.$$

We verify immediately that

$$D^* \circ D = D \circ D^* = -\sum_{i=1}^4 (\frac{\partial}{\partial x^i})^2. \tag{16.30}$$

A solution of the *Dirac equation*

$$D\psi = 0 \tag{16.31}$$

is called a *harmonic spinor*. It follows from (16.30) that each coordinate of a harmonic spinor satisfies the Laplace equation.

An equivalent way to see the Dirac operator is to consider the composition

$$D : \Gamma(S^+) \xrightarrow{\nabla} \Gamma(S^+) \otimes T^*(M) \xrightarrow{c} \Gamma(S^-)$$

Here c is the Clifford multiplication arising from the spin-structure $T(M) \to Hom(S^+, S^-)$.

Finally, we can generalize the Dirac operator by throwing in another Hermitian vector bundle E with a unitary connection A and defining the Dirac operators

$$D_A : \Gamma(E \otimes S^+) \to \Gamma(E \otimes S^-), \quad D^*_A : \Gamma(E \otimes S^-) \to \Gamma(E \otimes S^+).$$

We shall state the next theorem without proof (see [**Lawson**]).

Theorem (Weitzenböck's Formula). *Let A be a unitary connection on a bundle over a 4-manifold M with a fixed spinor structure. For any section σ of $E \otimes S^+$ we have*

$$D_A^* D_A \sigma = (\nabla_A)^* \nabla_A \sigma - F_A^+ \sigma + \frac{1}{4} K \sigma,$$

where K is the scalar curvature of M.

16.7 As we saw not every Riemannian manifold admits a spin-structure. However, there is no obstruction to defining a *complex spin-structure* on any Riemannian manifold of even dimension. Let $Spin^c(4)$ denote the group of complex 4×4 matrices of the form

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where $A_1, A_2 \in U(2), \det(A_1) = \det(A_2)$. The map $A \to \det(A_1)$ defines an exact sequence of groups

$$1 \to Spin(4) \to Spin^c(4) \to U(1) \to 1. \qquad (16.32)$$

As we saw in section 15.3, the group $Spin^c(4)$ acts on the space $V = \mathbb{R}^4$ identified with the subspace $s(V)$, where $s : V \to \mathrm{Mat}_4(\mathbb{C})$ is the complex spinor representation. This defines a homomorphism of groups $Spin^c(4) \to SO(4)$. Since $Spin^c(4)$ contains $Spin(4)$, it is surjective. Its kernel is isomorphic to the subgroup of scalar matrices isomorphic to $U(1)$. The exact sequence

$$1 \to U(1) \to Spin^c(4) \to SO(4) \to 1 \qquad (16.33)$$

gives an exact cohomology sequence

$$H^0(M, \underline{Spin}^c(4)) \to H^0(M, \underline{SO}(4)) \to H^1(M, \underline{U}(1)) \to$$

$$\to H^1(M, \underline{Spin}^c(4)) \to H^1(M, \underline{SO}(4)) \to H^2(M, \underline{U}(1)).$$

Here the underline means that we are considering the sheaf of smooth functions with values in the corresponding Lie group.

One can show that the map

$$H^0(M, \underline{Spin}^c(4)) \to H^0(M, \underline{SO}(4))$$

is surjective. This gives us the exact sequence

$$1 \to H^1(M, \underline{U}(1)) \to H^1(M, \underline{Spin}^c(4)) \to H^1(M, \underline{SO}(4)) \to H^2(M, \underline{U}(1)). \qquad (16.34)$$

Consider the homomorphism $U(1) \to U(1)$ which sends a complex number $z \in U(1)$ to its square. It is surjective, and its kernel is isomorphic to the group $\mathbb{Z}/2$. The exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to U(1) \to U(1) \to 1$$

defines an exact sequence of cohomology groups

$$H^1(M, \underline{U}(1)) \to H^2(M, \mathbb{Z}/2\mathbb{Z}) \to H^2(M, \underline{U}(1)).$$

Consider the inclusion $Spin(4) \subset Spin^c(4)$. We have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{Z}/2 & \to & Spin(4) & \to & SO(4) & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & U(1) & \to & Spin^c(4) & \to & SO(4) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & U(1) & = & U(1) & \to & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Here the middle vertical exact sequence is the sequence (16.32) and the middle horizontal sequence is the exact sequence (16.33). Applying the cohomology functor we get the following commutative diagram

$$\begin{array}{ccccc}
 & & & & H^1(M, \underline{U}(1)) \\
 & & & & \downarrow \\
 & & H^1(M, \underline{SO}(4)) & \to & H^2(M, \mathbb{Z}/2) \\
 & & \| & & \downarrow \\
H^1(M, \underline{Spin^c}(4))) & \to & H^1(M, \underline{SO}(4)) & \to & H^2(M, \underline{U}(1)).
\end{array}$$

From this we infer that the cohomology class $c \in H^1(M, \underline{SO}(4))$ of the principal $SO(4)$-bundle defining $T(M)$ lifts to a cohomology class $\tilde{c} \in H^1(M, \underline{Spin^c}(4))$ if and only if its image in $H^2(M, \underline{U}(1))$ is equal to zero. On the other hand, if we look at the previous horizontal exact sequence, we find that c is mapped to the second Stiefel-Whitney class $w_2(M) \in H^2(M, \mathbb{Z}/2\mathbb{Z})$. Thus, $\tilde{c}$ exists if and only if the image of $w_2(M)$ in $H^2(M, \underline{U}(1))$ in the right vertical exact sequence is equal to zero.

Now let us use the isomorphism $U(1) \cong \mathbb{R}/\mathbb{Z}$ induced by the map $\phi \to e^{2\pi i \phi}$. It defines the exact sequence of abelian groups

$$1 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 1, \tag{16.35}$$

and the corresponding exact sequence of sheaves

$$1 \to \mathbb{Z} \to \mathcal{O}_M \to \underline{U}(1)) \to 1, \tag{16.36}$$

It is known that $H^i(M, \mathcal{O}_M) = 0, i > 0$. This implies that

$$H^i(M, \underline{U}(1)) \cong H^{i+1}(M, \mathbb{Z}), \quad i > 0. \tag{16.37}$$

The right vertical exact sequence in the previous commutative diagram is equivalent to the exact sequence

$$H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{Z}/2) \to H^3(M, \mathbb{Z})$$

defined by the exact sequence of abelian groups

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0.$$

It is clear that the image of $w_2(M)$ in $H^3(M, \mathbb{Z})$ is equal to zero if and only if there exists a class $w \in H^2(M, \mathbb{Z})$ such that its image in $H^2(M, \mathbb{Z}/2)$ is equal to $w_2(M)$. In particular, this is always true if $H^3(M, \mathbb{Z}) = 0$. By Poincaré duality, $H^3(M, \mathbb{Z}) \cong H^1(M, \mathbb{Z})$. So, any compact oriented 4-manifold with $H^1(M, \mathbb{Z}) = 0$ admits a spin^c-structure.

The exact sequence (16.34) and isomorphism $H^2(M, \mathbb{Z}) \cong H^1(M, \underline{U}(1))$ from (16.38) tells us that two lifts $\tilde{c}$ of c differ by an element from $H^2(M, \mathbb{Z})$. Thus the set of spin^c-structures is a principal homogeneous space with respect to the group $H^2(M, \mathbb{Z})$.

This can be interpreted as follows. The two homomorphisms $Spin^c(4) \to U(2)$ define two associated rank 2 Hermitian bundles $W^\pm$ over M and an isomorphism

$$c : T(M) \otimes \mathbb{C} \to Hom(W^+, W^-).$$

We have

$$\bigwedge^2(W^+) \cong \bigwedge^2(W^-) \cong L,$$

where L is a complex line bundle over M. The Chern class $c_1(L) \in H^2(M, \mathbb{Z})$ is the element $\tilde{c}$ which lifts $w_2(M)$. If one can find a line bundle M such that $M^{\otimes 2} \cong L$, then

$$W^\pm \cong S^\pm \otimes M.$$

Otherwise, this is true only locally. A unitary connection A on L allows one to define the Dirac operator

$$D_A : \Gamma(W^+) \to \Gamma(W^-). \tag{16.38}$$

Locally it coincides with the Dirac operator $D : \Gamma(S^+) \to \Gamma(S^-)$.

The *Seiberg-Witten equations* for a 4-manifold with a spin^c-structure $W^\pm$ are equations for a pair (A, ψ), where

(i) A is a unitary connection on $L = \bigwedge^2(W^+)$;
(ii) ψ is a section of W^+.

They are

$$D_A\psi = 0, \quad F_A^+ = -\tau(\psi), \tag{16.39}$$

where $\tau(\psi)$ is the self-dual 2-form corresponding to the trace-free part of the endomorphism $\psi^* \otimes \psi \in W^+ \otimes W^+ \cong \text{End}(W^+)$ under the natural isomorphism $\Lambda^+ \to \text{End}(W^+)$ which is analogous to the isomorphism from Exercise 4.

Exercises.

1. Show that the reversing the orientation of M interchanges D and D^*.

2. Show that the complex plane $\mathbb{P}^2(\mathbb{C})$ does not admit a spinor structure.

3. Prove the Christoffel identity.

4. Let $\Lambda^\pm$ be the subspace of $\bigwedge^2 V$ which consists of symmetric (resp. anti-symmetric) forms with respect to the star operator $*$. Show that the map ϕ^+ defined in (16.17) induces an isomorphism $\Lambda^+ \to \mathfrak{su}(S^+)$.

5. Show that the contraction $\sum_i R^i_{i\ kl}$ for the curvature tensor of the Levi-Civita connection must be equal to zero.

6. Prove that $Spin^c(4) \cong (U(1) \times Spin(4)/\{\pm 1\}$, where $\{\pm\}$ acts on both factors in the obvious way.

7. Show that the group $SO(4)$ admits two homomorphisms into $SO(3)$, and the associated rank 3 vector bundles are isomorphic to the bundles $\Lambda^\pm$ of self-dual and anti-self-dual 2-forms on M.

8. Show that the Dirac equation corresponds to the Lagrangian $\mathcal{L}(\psi) = \psi^*(i\gamma^\mu\partial_\mu - m)\psi$.

9. Prove the formula $Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$.

10. Let $\sigma_{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. Show that the matrices of the spinor representation of $Spin(3,1)$ can be written in the form $A = e^{-i\sigma_{\mu\nu}a_{\mu\nu}/4}$, where $(a_{\mu\nu}) \in \mathfrak{so}(1,3)$.

11. Show that the current $J = (j^0, j^1, j^2, j^3)$, where $j^\mu = \bar{\psi}\gamma^\mu\psi$ is conserved with respect to Lorentz transformations.

Lecture 17. QUANTIZATION OF FREE FIELDS

In the previous lectures we discussed classical fields; it is time to make the transition to the quantum theory of fields. It is a generalization of quantum mechanics where we considered states corresponding to a single particle. In quantum field theory (QFT) we will be considering multiparticle states. There are different approaches to QFT. We shall start with the canonical quantization method. It is very similar to the canonical quantization in quantum mechanics. Later on we shall discuss the path integral method. It is more appropriate for treating such fields as the Yang-Mills fields. Other methods are the Gupta-Bleuler covariant quantization or the Becchi-Rouet-Stora-Tyutin (BRST) method, or the Batalin-Vilkovsky (BV) method. We shall not discuss these methods. We shall start with quantization of free fields defined by Lagrangians without interaction terms.

17.1 We shall start for simplicity with real scalar fields $\psi(t,\mathbf{x})$ described by the Klein-Gordon Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\psi)^2 - \frac{1}{2}m^2\psi^2.$$

By analogy with classical mechanics we introduce the conjugate *momentum field*

$$\pi(t,\mathbf{x}) = \frac{\delta\mathcal{L}}{\delta\partial_0\psi(t,\mathbf{x})} = \partial_0\psi(t,\mathbf{x}) := \dot{\psi}.$$

We can also introduce the Hamiltonian functional in two function variables ψ and π:

$$H = \frac{1}{2}\int(\pi\dot{\psi} - \mathcal{L})d^3x. \tag{17.1}$$

Then the Euler-Lagrange equation is equivalent to the *Hamilton equations* for fields

$$\dot{\psi}(t,\mathbf{x}) = \frac{\delta H}{\delta\pi(t,\mathbf{x})}, \quad \dot{\pi}(t,\mathbf{x}) = -\frac{\delta H}{\delta\psi(t,\mathbf{x})}. \tag{17.2}$$

Here we use the partial derivative of a functional $F(\psi,\pi)$. For each fixed ψ_0,π_0, it is a linear functional F'_ψ defined as

$$F(\psi_0+h,\pi_0)-F(\pi_0)=F'_\psi(\psi_0,\pi_0)(h)+o(||h||),$$

where h belongs to some normed space of functions on $\mathbb{R}^3$, for example $L_2(\mathbb{R}^3)$. We can identify it with the kernel $\frac{\delta F}{\delta\phi}$ in its integral representation

$$F'_\psi(\psi_0,\pi_0)(h)=\int\frac{\delta F}{\delta\phi}(\psi_0,\pi_0)h(\mathbf{x})d^3x.$$

Note that the kernel function has to be taken in the sense of distributions. We can also define the Poisson bracket of two functionals $A(\psi,\pi),B(\psi,\pi)$ of two function variables ψ and π (the analogs of $\mathbf{q},\mathbf{p}$ in classical mechanics):

$$\{A,B\}=\int\limits_{\mathbb{R}^3}\Big(\frac{\delta A}{\delta\psi(\mathbf{z})}\frac{\delta B}{\delta\pi(\mathbf{z})}-\frac{\delta A}{\delta\pi(\mathbf{z})}\frac{\delta B}{\delta\psi(\mathbf{z})}\Big)d^3z. \tag{17.3}$$

To give it a meaning we have to understand $A(\psi,\pi),B(\psi,\pi)$ as bilinear distributions on some space of test functions K defined by

$$(f,g)\to\int\limits_{\mathbb{R}^3}A(\psi,\pi)(\mathbf{z})f(z)g(\mathbf{z})d^3z,$$

and similar for $B(\psi,\pi)$. The product of two generalized functionals is understood in a generalized sense, i.e., as a bilinear distribution.

In particular, if we take B equal to the Hamiltonian functional H, and use (17.2), we obtain

$$\{A,H\}=\int\limits_{\mathbb{R}^3}\Big(\frac{\delta A}{\delta\psi(\mathbf{z})}\dot\psi+\frac{\delta A}{\delta\pi(\mathbf{z})}\dot\pi(\mathbf{z})\Big)d^3z=\dot A. \tag{17.4}$$

Here

$$\dot A(\psi,\pi):=\frac{dA(\psi(t,\mathbf{x}),\pi(t,\mathbf{x}))}{dt}.$$

This is the Poisson form of the Hamilton equation (17.2).

Let us justify the following identity:

$$\{\psi(t,\mathbf{x}),\pi(t,\mathbf{y})\}=\delta(\mathbf{x}-\mathbf{y}). \tag{17.5}$$

Here the right-hand-side is the Dirac delta function considered as a bilinear functional on the space of test functions K

$$(f(\mathbf{x}),g(\mathbf{y}))\to\int\limits_{\mathbb{R}^3}f(\mathbf{z})g(\mathbf{z})d^3z$$

(see Example 10 from Lecture 6). The functional $\psi(\mathbf{x})$ is defined by $F(\psi,\pi) = \psi(t,\mathbf{x})$ for a fixed $(t,\mathbf{x}) \in \mathbb{R}^4$. Since it is linear, its partial derivative is equal to itself. Similar interpretation holds for $\pi(\mathbf{y})$. If we denote the functional by $\psi(\mathbf{a})$, then we get

$$\frac{\delta\psi(\mathbf{x})}{\delta\psi(\mathbf{z})}(\psi,\pi) = \psi(t,\mathbf{x}) = \int_{\mathbb{R}^3} \psi(t,\mathbf{z})\delta(\mathbf{z}-\mathbf{x})d^3z.$$

Thus we obtain

$$\frac{\delta\psi(\mathbf{x})}{\delta\psi(\mathbf{z})} = \delta(\mathbf{z}-\mathbf{x}), \quad \frac{\delta\pi(\mathbf{y})}{\delta\pi(\mathbf{z})} = \delta(\mathbf{z}-\mathbf{y}), \quad \frac{\delta\psi(\mathbf{x})}{\delta\pi(\mathbf{z})} = \frac{\delta\pi(\mathbf{y})}{\delta\psi(\mathbf{z})} = 0.$$

Following the definition (17.3), we get

$$\{\psi(\mathbf{x}),\pi(\mathbf{y})\} = \int \delta(\mathbf{z}-\mathbf{x})\delta(\mathbf{z}-\mathbf{y})d^3z = \delta(\mathbf{x}-\mathbf{y}).$$

The last equality is the definition of the product of two distributions $\delta(\mathbf{z}-\mathbf{x})$ and $\delta(\mathbf{z}-\mathbf{y})$. Similarly, we get

$$\{\psi(\mathbf{x}),\psi(\mathbf{y})\} = \{\pi(\mathbf{x}),\pi(\mathbf{y})\} = 0.$$

Since it is linear, its partial derivative is equal to itself. Similar interpretation holds for $\pi(\mathbf{y})$. If we denote the functional by $\Psi(\mathbf{a})$, then we get

$$\frac{\delta\psi(\mathbf{x})}{\delta\psi(\mathbf{z})} \equiv \delta(\mathbf{z}-\mathbf{x}), \quad \frac{\delta\pi(\mathbf{y})}{\delta\Pi(\mathbf{z})} \equiv \delta(\mathbf{z}-\mathbf{y}), \quad \frac{\delta\Psi(\mathbf{x})}{\delta\Pi(\mathbf{z})} = \frac{\delta\Pi(\mathbf{y})}{\delta\Psi(\mathbf{z})} \equiv 0.$$

where we understand the delta function $\delta(\mathbf{z}-\mathbf{x})$ as the linear functional $h(\mathbf{z}) \to h(\mathbf{x})$. Thus we have

$$\{\psi(\mathbf{x}),\pi(\mathbf{y})\} = \int \delta(\mathbf{z}-\mathbf{x})\delta(\mathbf{z}-\mathbf{y})d^3z = \delta(\mathbf{x}-\mathbf{y})$$

Similarly, we get

$$\{\psi(\mathbf{x}),\psi(\mathbf{y})\} = \{\pi(\mathbf{x}),\pi(\mathbf{y})\} = 0.$$

17.2 To quantize the fields ψ and π we have to reinterpret them as Hermitian operators in some Hilbert space $\mathcal{H}$ which satisfy the commutation relations

$$[\Psi(t,\mathbf{x}),\Pi(t,\mathbf{y})] = \frac{i}{\hbar}\delta(\mathbf{x}-\mathbf{y}),$$

$$[\Psi(t,\mathbf{x}),\Psi(t,\mathbf{y})] = [\Pi(t,\mathbf{x}),\Pi(t,\mathbf{y})] = 0. \tag{17.6}$$

This is the quantum analog of (17.5). Here we have to consider Ψ and Π as operator valued distributions on some space of test functions $K \subset C^\infty(\mathbb{R}^3)$. If they are regular distributions, i.e., operators dependent on t and $\mathbf{x}$, they are defined by the formula

$$\Psi(t,f) = \int f(\mathbf{x})\Psi(t,\mathbf{x})d^3x.$$

Otherwise we use this formula as the formal expression of the linear continuous map $K \to \mathrm{End}(\mathcal{H})$. The operator distribution $\delta(\mathbf{x}-\mathbf{y})$ is the bilinear map $K \times K \to \mathrm{End}(\mathcal{H})$ defined by the formula

$$(f,g) \to (\int_{\mathbb{R}^3} f(\mathbf{x})g(\mathbf{x})d^3x)\mathbf{id}_{\mathcal{H}}.$$

Thus the first commutation relation from (17.6) reads as

$$[\Psi(t,f),\Pi(t,g)] = \Big(i\int_{\mathbb{R}^3} f(\mathbf{x})g(\mathbf{x})d^3x\Big)\mathbf{id}_{\mathcal{H}},$$

The quantum analog of the Hamilton equations is straightforward

$$\dot{\Pi} = \frac{i}{\hbar}[\Pi,H], \quad \dot{\Psi} = \frac{i}{\hbar}[\Psi,H], \tag{17.7}$$

where H is an appropriate Hamiltonian operator distribution.

Let us first consider the discrete version of quantization. Here we assume that the operator functions $\phi(t,\mathbf{x})$ belong to the space $L_2(B)$ of square integrable functions on a box $B = [-l_1,l_1]\times[-l_2,l_2]\times[-l_3,l_3]$ of volume $\Omega = 8l_1l_2l_3$. At fixed time t we expand the operator functions $\Psi(t,\mathbf{x})$ in Fourier series

$$\Psi(t,\mathbf{x}) = \frac{1}{\sqrt{\Omega}}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}q_{\mathbf{k}}(t). \tag{17.8}$$

Here

$$\mathbf{k} = (k_1,k_2,k_3) \in \frac{\pi}{l_1}\mathbb{Z}\times\frac{\pi}{l_2}\mathbb{Z}\times\frac{\pi}{l_3}\mathbb{Z}.$$

The reason for inserting the factor $\frac{1}{\sqrt{\Omega}}$ instead of the usual $\frac{1}{\Omega}$ is to accommodate with our definition of the Fourier transform: when $l_i \to \infty$ the expansion (17.8) passes into the Fourier integral

$$\Psi(t,\mathbf{x}) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} e^{i\mathbf{k}\cdot\mathbf{x}}q_{\mathbf{k}}(t)d^3k. \tag{17.9}$$

Since we want the operators Ψ and Π to be Hermitian, we have to require that

$$q_{\mathbf{k}}(t) = q^*_{-\mathbf{k}}(t).$$

Recalling that the scalar fields $\psi(t,\mathbf{x})$ satisfied the Klein-Gordon equation, we apply the operator $(\partial_\mu\partial^\mu + m^2)$ to the operator functions Ψ and Π to obtain

$$(\partial_\mu\partial^\mu + m^2)\Psi = \sum_{\mathbf{k}}(\partial_\mu\partial^\mu + m^2)e^{i\mathbf{k}\cdot\mathbf{x}}q_{\mathbf{k}}(t) =$$

$$= \sum_{\mathbf{k}}(|\mathbf{k}|^2 + m^2 - E_{\mathbf{k}}^2)e^{i\mathbf{k}\cdot\mathbf{x}}q_{\mathbf{k}}(t) = 0,$$

provided that we assume that

$$q_{\mathbf{k}}(t) = q_{\mathbf{k}} e^{-iE_{\mathbf{k}}t}, \quad q_{-\mathbf{k}}(t) = q_{-\mathbf{k}} e^{iE_{\mathbf{k}}t},$$

where

$$E_{\mathbf{k}} = \sqrt{|\mathbf{k}|^2 + m^2}. \tag{17.10}$$

Similarly we consider the Fourier series for $\Pi(t, \mathbf{x})$

$$\Pi(t, \mathbf{x}) = \frac{1}{\sqrt{\Omega}} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} p_{\mathbf{k}}(t)$$

to assume that

$$p_{\mathbf{k}}(t) = p_{\mathbf{k}} e^{-iE_{\mathbf{k}}t}, \quad p_{-\mathbf{k}}(t) = p_{-\mathbf{k}} e^{iE_{\mathbf{k}}t}.$$

Now let us make the transformation

$$q_{\mathbf{k}} = \frac{1}{\sqrt{2E_{\mathbf{k}}}} (a(\mathbf{k}) + a(-\mathbf{k})^*),$$

$$p_{-\mathbf{k}} = -i\sqrt{E_{\mathbf{k}}/2}(a(\mathbf{k}) - a(-\mathbf{k})^*).$$

Then we can rewrite the Fourier series in the form

$$\Psi(t, \mathbf{x}) = \sum_{\mathbf{k}} (2\Omega E_k)^{-1/2} (e^{i(\mathbf{k}\cdot\mathbf{x} - E_k t)} a(\mathbf{k}) + e^{-i(\mathbf{k}\cdot\mathbf{x} - E_k t)} a(\mathbf{k})^*), \tag{17.11a}$$

$$\Pi(t, \mathbf{x}) = \sum_{\mathbf{k}} -i(E_k/2\Omega)^{1/2} (e^{i(\mathbf{k}\cdot\mathbf{x} - E_k t)} a(\mathbf{k}) - e^{-i(\mathbf{k}\cdot\mathbf{x} - E_k t)} a(\mathbf{k})^*). \tag{17.11b}$$

Using the formula for the coefficients of the Fourier series we find

$$q_{\mathbf{k}}(t) = \frac{1}{\Omega^{1/2}} \int_B e^{-i\mathbf{k}\cdot\mathbf{x}} \Psi(t, \mathbf{x}) d^3x,$$

$$p_{\mathbf{k}}(t) = \frac{1}{\Omega^{1/2}} \int_B e^{i\mathbf{k}\cdot\mathbf{x}} \Pi(t, \mathbf{x}) d^3x.$$

Then

$$[q_{\mathbf{k}}, p_{\mathbf{k}'}] = \frac{1}{\Omega} \int_B \int_B e^{-i(\mathbf{k}\cdot\mathbf{x} - \mathbf{k}'\cdot\mathbf{x}')} [\Psi(t, \mathbf{x}), \Pi(t, \mathbf{x}')] d^3x d^3x' =$$

$$= \frac{i}{\Omega} \int_B e^{-i(\mathbf{k}\cdot\mathbf{x} - \mathbf{k}'\cdot\mathbf{x})} \delta(\mathbf{x} - \mathbf{x}') d^3x d^3x' = \frac{i}{\Omega} \int_B e^{-i(\mathbf{k} - \mathbf{k}')\cdot\mathbf{x}} d^3\mathbf{x} = i\delta_{\mathbf{k}\mathbf{k}'},$$

where

$$\delta_{\mathbf{k}\mathbf{k}'} = \begin{cases} 1 & \text{if } \mathbf{k} = \mathbf{k}', \\ 0 & \text{if } \mathbf{k} \neq \mathbf{k}'. \end{cases}$$

is the Kronecker symbol. Similarly, we get

$$[p_{\mathbf{k}}, p_{\mathbf{k}'}] = [q_{\mathbf{k}}, q_{\mathbf{k}'}] = 0.$$

This immediately implies that

$$[a(\mathbf{k}), a(\mathbf{k}')^*] = \delta_{\mathbf{k}\mathbf{k}'},$$

$$[a(\mathbf{k}), a(\mathbf{k}')] = [a(\mathbf{k})^*, a(\mathbf{k}')^*] = 0. \tag{17.12}$$

This is in complete analogy with the case of the harmonic oscillator, where we had only one pair of operators a, a^* satisfying $[a, a^*] = 1, [a, a] = [a^*, a^*] = 0$. We call the operators $a(\mathbf{k}), a(\mathbf{k})^*$ the *harmonic oscillator annihilation* and *creation operators*, respectively.

Conversely, if we assume that (17.12) holds, we get

$$[\Psi(t,\mathbf{x}), \Pi(t,\mathbf{y})] = \frac{i}{\Omega^{1/2}} \sum_{\mathbf{k}} e^{i(\mathbf{x}-\mathbf{y})\mathbf{k}}.$$

Of course this does not make sense. However, if we replace the Fourier series with the Fourier integral (17.9), we get (17.6)

$$[\Psi(t,\mathbf{x}), \Pi(t,\mathbf{y})] = \frac{i}{(2\Pi)^{3/2}} \int_{\mathbb{R}^3} e^{i(\mathbf{x}-\mathbf{y})\mathbf{k}} d^3k = i\delta(\mathbf{x}-\mathbf{y})$$

(see Lecture 6, Example 5).

The Hamiltonian (17.1) has the form

$$H = \int (\Pi\dot{\Psi} - \mathcal{L})d^3x = \int (\Pi^2 - \frac{1}{2}\Pi^2 + \frac{1}{2}(\partial_1^2 + \partial_2^2 + \partial_3^2)\Psi + \frac{1}{2}m^2\Psi^2)d^3x =$$

$$= \int (\Pi^2 + \frac{1}{2}(\partial_1^2 + \partial_2^2 + \partial_3^2)\Psi + \frac{1}{2}m^2\Psi^2)d^3x.$$

To quantize it, we plug in the expressions for Π and Ψ from (17.11) and use (17.10). We obtain

$$H = \sum_{\mathbf{k}} E_k(a(\mathbf{k})^*a(\mathbf{k}) + \frac{1}{2}).$$

However, it obviously diverges (because of the $\frac{1}{2}$). Notice that

$$\frac{1}{2} : a(\mathbf{k})^*a(\mathbf{k}) + a(\mathbf{k})a(\mathbf{k})^* := a(\mathbf{k})^*a(\mathbf{k}), \tag{17.13}$$

where $: P(a^*, a) :$ denotes the *normal ordering*; we put all $a(\mathbf{k})^*$'s on the left pretending that the a^* and a commute. So we can rewrite, using (17.12),

$$: a(\mathbf{k})^*a(\mathbf{k}) + \frac{1}{2} :=: a(\mathbf{k})^*a(\mathbf{k}) + \frac{1}{2}(a(\mathbf{k})^*a(\mathbf{k}) - a(\mathbf{k})a(\mathbf{k})^*) :=$$

$$\frac{1}{2}:a(\mathbf{k})^*a(\mathbf{k})+a(\mathbf{k})a(\mathbf{k})^*:=a(\mathbf{k})^*a(\mathbf{k}).$$

This gives

$$:H:=\sum_{\mathbf{k}}E_{\mathbf{k}}a(\mathbf{k})^*a(\mathbf{k}). \tag{17.14}$$

Let us take this for the definition of the Hamiltonian operator H. Notice the commutation relations

$$[H,a(\mathbf{k})^*]=E_{\mathbf{k}}a(\mathbf{k})^*, \quad [H,a(\mathbf{k})]=E_{\mathbf{k}}a(\mathbf{k}). \tag{17.15}$$

This is analogous to equation (7.3) from Lecture 7.

We leave to the reader to check the Hamilton equations

$$\dot{\Psi}(t,\mathbf{x})=i[\Psi(t,\mathbf{x}),H].$$

Similarly, we can define the *momentum vector operator*

$$\mathbf{P}=\sum_{\mathbf{k}}\mathbf{k}a(\mathbf{k})^*a(\mathbf{k})=(P_1,P_2,P_3). \tag{17.16}$$

We have

$$\partial_\mu\Psi(t,\mathbf{x})=i[P_\mu,\Psi]. \tag{17.17}$$

We can combine H and $\mathbf{P}$ into one 4-momentum operator $P=(H,\mathbf{P})$ to obtain

$$[\Psi,P^\mu]=i\partial^\mu\Psi, \tag{17.18}$$

where as always we switch to superscripts to denote the contraction with $g^{\mu\nu}$. Let us now define the *vacuum state* $|0\rangle$ as follows

$$a(\mathbf{k})|0\rangle=0, \quad \forall\mathbf{k}.$$

We define a *one-particle state* by

$$a(\mathbf{k})^*|0\rangle=|\mathbf{k}\rangle.$$

We define an *n-particle state* by

$$|\mathbf{k}_n\ldots\mathbf{k}_1\rangle:=a(\mathbf{k}_n)^*\ldots a(\mathbf{k}_1)^*|0\rangle$$

The states $|\mathbf{k}_1\ldots\mathbf{k}_n\rangle$ are eigenvectors of the Hamiltonian operator:

$$H|\mathbf{k}_n\ldots\mathbf{k}_1\rangle=(E_{\mathbf{k}_1}+\ldots+E_{\mathbf{k}_n})|\mathbf{k}_n\ldots\mathbf{k}_1\rangle. \tag{17.19}$$

This is easily checked by induction on n using the commutation relations (17.12). Similarly, we get

$$P|\mathbf{k}_n\ldots\mathbf{k}_1\rangle=(\mathbf{k}_1+\ldots+\mathbf{k}_n)|\mathbf{k}_n\ldots\mathbf{k}_1\rangle. \tag{17.20}$$

This tells us that the total energy and momentum of the state $|\mathbf{k}_n \dots \mathbf{k}_1\rangle$ are exactly those of a collection of n free particles of momenta $\mathbf{k}_i$ and energy $E_{\mathbf{k}_i}$. The operators $a(\mathbf{k})^*$ and $a(\mathbf{k})$ create and annihilate them. It is the stationary state corresponding to n particles with momentum vectors $\mathbf{k}_i$.

If we apply the *number operator*

$$N = \sum_{\mathbf{k}} a(\mathbf{k})^* a(\mathbf{k}) \tag{17.21}$$

we obtain

$$N|\mathbf{k}_n \dots \mathbf{k}_1\rangle = n|\mathbf{k}_n \dots \mathbf{k}_1\rangle.$$

Notice also the relation (17.10) which, after the physical units are restored, reads as

$$E_{\mathbf{k}} = \sqrt{m^2c^4 + |\mathbf{k}|^2c^2}.$$

The Klein-Gordon equations which we used to derive our operators describe spinless particles such as *pi mesons*.

In the continuous version, we define

$$\Psi(t, \mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \Big(\frac{1}{\sqrt{2E_{\mathbf{k}}}} a(\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{x} - E_{\mathbf{k}}t)} + a(\mathbf{k})^* e^{-i(\mathbf{k}\cdot\mathbf{x} - E_{\mathbf{k}}t)}\Big) d^3k, \tag{17.22a}$$

$$\Pi(t, \mathbf{x}) = \frac{-i}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \Big(\sqrt{E_{\mathbf{k}}/2}\big(a(\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{x} - E_{\mathbf{k}}t)} - a(\mathbf{k})^* e^{-i(\mathbf{k}\cdot\mathbf{x} - E_{\mathbf{k}}t)}\Big) d^3k. \tag{17.22b}$$

Then we repeat everything replacing $\mathbf{k}$ with a continuous parameter from $\mathbb{R}^3$. The commutation relation (17.12) must be replaced with

$$[a(\mathbf{k}), a(\mathbf{k}')^*] = i\delta(\mathbf{k} - \mathbf{k}').$$

The Hamiltonian, momentum and number operators now appear as

$$H = \int_{\mathbb{R}^3} E_{\mathbf{k}} a(\mathbf{k})^* a(\mathbf{k}) d^3k, \tag{17.23}$$

$$\mathbf{P} = \int_{\mathbb{R}^3} \mathbf{k} a(\mathbf{k})^* a(\mathbf{k}) d^3k, \tag{17.24}$$

$$N = \int a(\mathbf{k})^* a(\mathbf{k}) d^3k. \tag{17.25}$$

17.3 We have not yet explained how to construct a representation of the algebra of operators $a(\mathbf{k})^*$ and $a(\mathbf{k})$, and, in particular, how to construct the vacuum state $|0\rangle$. The Hilbert space $\mathcal{H}$ where these operators act admits different realizations. The most frequently used one is the realization of $\mathcal{H}$ as the Fock space which we encountered in Lecture 9. For simplicity we consider its discrete version. Let $V = l_2(\mathbb{Z}^3)$ be the space of square integrable complex valued functions on $\mathbb{Z}^3$. We can identify such a function with a complex formal power series

$$f = \sum_{\mathbf{k}\in\mathbb{Z}^3} c_{\mathbf{k}}\mathbf{t}^{\mathbf{k}}, \quad \sum_{\mathbf{k}} |c_{\mathbf{k}}|^2 < \infty. \tag{17.26}$$

Here $\mathbf{t}^{\mathbf{k}} = t_1^{k_1}t_2^{k_2}t_3^{k_3}$ and $c_{\mathbf{k}} \in \mathbb{C}$. The value of f at $\mathbf{k}$ is equal to $c_{\mathbf{k}}$.

Consider now the complete tensor product $T^n(V) = V\hat{\otimes}\ldots\hat{\otimes}V$ of n copies of V. Its elements are convergent series of the form

$$F_n = \sum_{i_1,\ldots,i_n=0}^{\infty} c_{i_1\ldots i_n} f_{i_1}\otimes\ldots\otimes f_{i_n}, \tag{17.27}$$

where the convergence is taken with respect to the norm defined by $\sum |c_{i_1\ldots i_n}|^2$. Let

$$T(V) = \bigoplus_{n=0}^{\infty} T^n(V)$$

be the tensor algebra built over V. Its homogeneous elements $F_n \in T^n(V)$ are functions in n non-commuting variables $\mathbf{k}_1,\ldots,\mathbf{k}_n \in \mathbb{Z}^3$. Let $\hat{T}(V)$ be the completion of $T(V)$ with respect to the norm defined by the inner product

$$\langle F, G\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_n} F_n^*(\mathbf{k}_1,\ldots,\mathbf{k}_n)G_n(\mathbf{k}_1,\ldots,\mathbf{k}_n). \tag{17.28}$$

Its elements are the Cauchy equivalence classes of convergent sequences of complex functions of the form

$$F = \{F_n(\mathbf{k}_1,\ldots,\mathbf{k}_n)\}.$$

We take

$$\mathcal{H} = \{F = \{F_n\} \in \hat{T}(V) : F_n(\mathbf{k}_{\sigma(1)},\ldots,\mathbf{k}_{\sigma(n)}) = F_n(\mathbf{k}_1,\ldots,\mathbf{k}_n), \forall\sigma\in S_n\} \tag{17.29}$$

to be the subalgebra of $\hat{T}(V)$ formed by the sequences where each F_n is a symmetric function on $(\mathbb{Z}^3)^n$. This will be the bosonic case. There is also the fermionic case where we take $\mathcal{H}$ to be the subalgebra of $\hat{T}(V)$ which consists of anti-symmetric functions. Let $\mathcal{H}_n$ denote the subspaces of constant sequences of the form $(0,\ldots,0,F_n,0,\ldots)$. If we interpret functions $f\in V$ as power series (17.26), we can write $F_n \in \mathcal{H}_n$ as the convergent series

$$F_n = \sum_{\mathbf{k}_1,\ldots,\mathbf{k}_n\in\mathbb{Z}^3} c_{\mathbf{k}_1\ldots\mathbf{k}_n}\mathbf{t}_1^{\mathbf{k}_1}\ldots\mathbf{t}_n^{\mathbf{k}_n}.$$

The operators $a(\mathbf{k})^*, a(\mathbf{k})$ are, in general, operator valued distributions on $\mathbb{Z}^3$, i.e., linear continious maps from some space $K \subset V$ of test sequences to the space of linear operators in $\mathcal{H}$. We write them formally as

$$A(\phi) = \sum_{\mathbf{k}} \phi(\mathbf{k}) A(\mathbf{k}),$$

where $A(\mathbf{k})$ is a generalized operator valued function on $\mathbb{Z}^3$. We set

$$a(\phi)^*(F)_n(\mathbf{k}_1, \ldots, \mathbf{k}_n) = \sum_{i=1}^{n} \phi(\mathbf{k}_j) F_{n-1}(\mathbf{k}_1, \ldots, \hat{\mathbf{k}}_j, \ldots, \mathbf{k}_n), \tag{17.30a}$$

$$a(\phi)(F)_n(\mathbf{k}_1, \ldots, \mathbf{k}_n) = \sum_{\mathbf{k}} \phi(\mathbf{k}) F_{n+1}(\mathbf{k}, \mathbf{k}_1, \ldots, \mathbf{k}_n), \tag{17.30b}$$

In particular, we may take ϕ equal to the characteristic function of $\{\mathbf{k}\}$ (corresponding to the monomial $\mathbf{t}^{\mathbf{k}}$). Then we denote the corresponding operators by $a(\mathbf{k}), a(\mathbf{k})^*$.

The vacuum vector is

$$|0\rangle = (1, 0, \ldots, 0, \ldots). \tag{17.31}$$

Obviously

$$a(\mathbf{k})|0\rangle = 0,$$

$$a(\mathbf{k})^*|0\rangle = (0, \mathbf{t}^{\mathbf{k}}, 0, \ldots, 0, \ldots),$$

$$a(\mathbf{k}_2)^* a(\mathbf{k}_1)^*|0\rangle(\mathbf{p}_1, \mathbf{p}_2) = (0, 0, \delta_{\mathbf{k}_2\mathbf{p}_1}\mathbf{t}_2^{k_1} + \delta_{\mathbf{k}_2\mathbf{p}_2}\mathbf{t}_1^{k_1}, 0, \ldots) = (0, 0, \mathbf{t}_1^{k_2}\mathbf{t}_2^{k_1} + \mathbf{t}_1^{k_1}\mathbf{t}_2^{k_2}, 0, \ldots).$$

Similarly, we get

$$a(\mathbf{k}_n)^* \ldots a(\mathbf{k}_1)^*|0\rangle = |\mathbf{k}_n \ldots \mathbf{k}_1\rangle = (0, \ldots, 0, \sum_{\sigma \in S_n} \mathbf{t}_1^{\mathbf{k}_{\sigma(1)}} \cdots \mathbf{t}_n^{\mathbf{k}_{\sigma(n)}}, 0 \ldots).$$

One can show that the functions of the form

$$F_{\mathbf{k}_1, \ldots, \mathbf{k}_n} = \sum_{\sigma \in S_n} \mathbf{t}_1^{\mathbf{k}_{\sigma(1)}} \cdots \mathbf{t}_n^{\mathbf{k}_{\sigma(n)}}$$

form a basis in the Hilbert space $\mathcal{H}$. We have

$$\langle F_{\mathbf{k}_1, \ldots, \mathbf{k}_n}, F_{\mathbf{q}_1, \ldots, \mathbf{q}_m} \rangle = \delta_{mn} \frac{1}{n!} \sum_{\mathbf{p}_1, \ldots, \mathbf{p}_n} (\sum_{\sigma \in S_n} \prod_{i=1}^{n} \delta_{\mathbf{k}_{\sigma(i)}\mathbf{p}_i})(\sum_{\sigma \in S_n} \prod_{i=1}^{n} \delta_{\mathbf{q}_{\sigma(i)}\mathbf{p}_i}) =$$

$$= n! \delta_{mn} \prod_{i=1}^{n} \delta_{\mathbf{k}_i\mathbf{q}_i}. \tag{17.32}$$

Thus the functions

$$\frac{1}{\sqrt{n!}} |\mathbf{k}_n \ldots \mathbf{k}_1\rangle$$

form an orthonormal basis in the space $\mathcal{H}$.

Let us check the commutation relations. We have for any homogeneous $F_n(\mathbf{k}_1,\ldots,\mathbf{k}_n)$

$$a(\phi)^*a(\phi')(F_n)(\mathbf{k}_1,\ldots,\mathbf{k}_n) = a(\phi)^*(\sum_{\mathbf{k}}\phi'(\mathbf{k})F_n(\mathbf{k},\mathbf{k}_1,\ldots,\mathbf{k}_{n-1})) =$$

$$= \sum_{i=1}^{n}\phi(\mathbf{k}_i)\sum_{\mathbf{k}}\phi'(\mathbf{k})F_n(\mathbf{k},\mathbf{k}_1,\ldots,\hat{\mathbf{k}}_i,\ldots,\mathbf{k}_n),$$

$$a(\phi')a(\phi)^*(F_n)(\mathbf{k}_1,\ldots,\mathbf{k}_n) = a(\phi')(\sum_{i=1}^{n+1}\phi(\mathbf{k}_j)F_n(\mathbf{k}_1,\ldots,\hat{\mathbf{k}}_j,\ldots,\mathbf{k}_{n+1})) =$$

$$= \sum_{\mathbf{k}}(\phi'(\mathbf{k})\phi(\mathbf{k})F_n(\mathbf{k}_1,\ldots,\mathbf{k}_n)) + \sum_{i=1}^{n}\phi'(\mathbf{k})\phi(\mathbf{k}_i)F_n(\mathbf{k},\mathbf{k}_1\ldots,\hat{\mathbf{k}}_i,\ldots,\mathbf{k}_n)).$$

This gives

$$[a(\phi)^*,a(\phi')](F_n)(\mathbf{k}_1,\ldots,\mathbf{k}_n) = \sum_{\mathbf{k}}\phi'(\mathbf{k})\phi(\mathbf{k})F_n(\mathbf{k}_1,\mathbf{k}_2,\ldots,\mathbf{k}_n)),$$

so that

$$[a(\phi)^*,a(\phi')] = \Big(\sum_{\mathbf{k}}\phi'(\mathbf{k})\phi(\mathbf{k})\Big)\mathbf{id}_{\mathcal{H}}.$$

In particular,

$$[a(\mathbf{k})^*,a(\mathbf{k}')] = \delta_{\mathbf{k}\mathbf{k}'}\mathbf{id}_{\mathcal{H}}.$$

Similarly, we get the other commutation relations.

We leave to the reader to verify that the operators $a(\phi)^*$ and $a(\phi)$ are adjoint to each other, i.e.,

$$\langle a(\phi)F,G\rangle = \langle F,a(\phi)^*G\rangle$$

for any $F,G\in\mathcal{H}$.

Remark 1. One can construct the Fock space formally in the following way. We consider a Hilbert space with an orthonormal basis formed by symbols $|a(\mathbf{k}_n)\ldots a(\mathbf{k}_1)\rangle$. Any vector in this space is a convergent sequnece $\{F_n\}$, where

$$F_n = \sum_{\mathbf{k}_1,\ldots,\mathbf{k}_n} c_{\mathbf{k}_1,\ldots,\mathbf{k}_n}|\mathbf{k}_n\ldots\mathbf{k}_1\rangle.$$

We introduce the operators $a(\mathbf{k})^*$ and $a(\mathbf{k})$ via their action on the basis vectors

$$a(\mathbf{k})^*|\mathbf{k}_n\ldots\mathbf{k}_1\rangle = |\mathbf{k}\mathbf{k}_n\ldots\mathbf{k}_1\rangle,$$

$$a(\mathbf{k})|\mathbf{k}_n\ldots\mathbf{k}_1\rangle = |\mathbf{k}_n\ldots\hat{\mathbf{k}}_j\ldots\mathbf{k}_1\rangle$$

where $\mathbf{k} = \mathbf{k}_j$ for some j and zero otherwise. Then it is possible to check all the needed commutation relations.

17.4 Now we have to learn how to quantize generalized functionals $A(\Psi, \Pi)$.

Consider the space $\mathcal{H}\hat{\otimes}\mathcal{H}$. Its elements are convergent double sequences

$$F(\mathbf{k}, \mathbf{p}) = \{F_{nm}(\mathbf{k}_1, \ldots, \mathbf{k}_n; \mathbf{p}_1, \ldots, \mathbf{p}_m)\}_{m,n},$$

where

$$F_{nm}(\mathbf{k}_1, \ldots, \mathbf{k}_n; \mathbf{p}_1, \ldots, \mathbf{p}_m) = \sum_{\mathbf{k}_1, \ldots, \mathbf{k}_n, \mathbf{p}_1, \ldots, \mathbf{p}_m \in \mathbb{Z}^3} c_{\mathbf{k}_1 \ldots \mathbf{k}_n; \mathbf{p}_1 \ldots \mathbf{p}_m} \mathbf{t}_1^{\mathbf{k}_1} \ldots \mathbf{t}_n^{\mathbf{k}_n} \mathbf{s}_1^{\mathbf{p}_1} \ldots \mathbf{s}_m^{\mathbf{p}_m}.$$

This function defines the following operator in $\mathcal{H}$:

$$A(F_{nm}) = \sum_{\mathbf{k}_i, \mathbf{p}_j} c_{\mathbf{k}_1 \ldots \mathbf{k}_n; \mathbf{p}_1 \ldots \mathbf{p}_m} a(\mathbf{k}_1)^* \ldots a(\mathbf{k}_n)^* a(\mathbf{p}_1) \ldots a(\mathbf{p}_m).$$

More generally, we can define the generalized operator valued $(m+n)$-multilinear function

$$A(F_{nm})(f_1, \ldots, f_n, g_1, \ldots, g_m) = \sum_{\mathbf{k}_i, \mathbf{p}_j} c_{\mathbf{k}_1 \ldots \mathbf{k}_n; \mathbf{p}_1 \ldots \mathbf{p}_m} a(f_1)^* \ldots a(f_n)^* a(g_1) \ldots a(g_m).$$

Here we have to assume that the series converges on a dense subset of $\mathcal{H}$. For example, the Hamiltonian operator $H = A(F_{11})$, where

$$F_{11} = \sum_{\mathbf{k}} E_{\mathbf{k}} \mathbf{t}^{\mathbf{k}} \mathbf{s}^{\mathbf{k}}.$$

Now we define for any $F = \{F_{nm}\} \in \mathcal{H}\hat{\otimes}\mathcal{H}$

$$A(F) = \sum_{m,n=0}^{\infty} A(F_{nm}),$$

where the sum must be convergent in the operator topology.

Now we attempt to assign an operator to a functional $P(\phi, \Pi)$. First, let us assume that P is a polynomial functional $\sum_{ij} a_{ij} \phi^i \Pi^j$. Then we plug in the expressions (6), and transform the expression using the normal ordering of the operators a^*, a. Similarly, we can try to define the value of any analytic function in ϕ, Π, or analytic differential operator

$$P(\partial_\mu \phi, \partial_\mu \Pi) = \sum P_{\mathbf{i},\mathbf{j}}(\phi, \Pi) \partial^{\mathbf{i}}(\phi) \partial^{\mathbf{j}}(\Pi).$$

We skip the discussion of the difficulties related to the convergence of the corresponding operators.

17.5 Let us now discuss how to quantize the Dirac equation. The field in this case is a vector function $\psi = (\psi_0, \ldots, \psi_3) : \mathbb{R}^4 \to \mathbb{C}^4$. The Lagrangian is taken in such a way that the corresponding Euler-Lagrange equation coincides with the Dirac equation. It is easy to verify that

$$\mathcal{L} = \psi^*_\beta(i(\gamma^\mu)_{\beta\alpha}\partial_\mu - m\delta_{\beta\alpha})\psi_\alpha, \tag{17.33}$$

where $(\gamma^\mu)_{\beta\alpha}$ denotes the $\beta\alpha$-entry of the Dirac matrix γ^μ and ψ^* is defined in (16.11). The momentum field is $\pi = (\pi_0, \ldots, \pi_3)$, where

$$\pi_\alpha = \frac{\delta\mathcal{L}}{\delta\dot{\psi}_\alpha} = i\psi^*_\beta(\gamma^0)_{\beta\alpha} = i\psi^*_\alpha.$$

The Hamiltonian is

$$H = \int_{\mathbb{R}^3} (\pi\dot{\psi} - \mathcal{L})d^3x = \int_{\mathbb{R}^3} (\psi^*(-i\sum_{\mu=1}^{3}\gamma^\mu\partial_\mu + m\gamma^0)\psi)d^3x. \tag{17.34}$$

For any $k = (k_0, \mathbf{k}) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^3$ consider the operator

$$A(k) = \sum_{i=0}^{3}\gamma^i k_i = \begin{pmatrix} 0 & 0 & -k_0 - k_1 & -k_3 + ik_2 \\ 0 & 0 & -k_2 i - k_3 & -k_0 + k_1 \\ k_0 + k_1 & k_3 - ik_2 & 0 & 0 \\ k_2 i + k_3 & k_0 - k_1 & 0 & 0 \end{pmatrix} \tag{17.35}$$

in the spinor space $\mathbb{C}^4$. Its characteristic polynomial is equal to $\lambda^4 - 2|k|^2 + |k|^4 = (\lambda^2 - |k|^2)^2$, where

$$|k|^2 = k_0^2 - k_1^2 - k_2^2 - k_3^2.$$

We assume that

$$|k| = m. \tag{17.36}$$

Hence we have two real eigenvalues of $A(k)$ equal to $\pm m$. We denote the corresponding eigensubspaces by $V_\pm(k)$. They are of dimension 2. Let

$$(\vec{u}_\pm(\mathbf{k}), \vec{v}_\pm(\mathbf{k}))$$

denote an orthogonal basis of $V_\pm(\mathbf{k})$ normalized by

$$\vec{\bar{u}}_\pm(\mathbf{k}) \cdot \vec{u}_\mp(\mathbf{k}) = -\vec{\bar{v}}_\pm(\mathbf{k}) \cdot \vec{v}_\mp(\mathbf{k}) = 2m,$$

where the dot-product is taken in the Lorentz sense. We take indices $\mathbf{k}$ in $\mathbb{R}^3$ since the first coordinate $k_0 \geq 0$ in the vector $k = (k_0, \mathbf{k})$ is determined by $\mathbf{k}$ in view of the relation $|k| = m$. Now we can solve the Dirac equation, using the Fourier integral expansions

$$\psi(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}}\sum_\pm \int_{\mathbb{R}^3} (b_\pm(\mathbf{k})\vec{u}_\pm(\mathbf{k})e^{-k\cdot x} + d_\pm(\mathbf{k})^*\vec{v}_\pm(\mathbf{k})e^{ik\cdot x})d^3x, \tag{17.37a}$$

$$\pi(\mathbf{x}) = \frac{i}{(2\pi)^{3/2}} \sum_{\pm} \int_{\mathbb{R}^3} (b_{\pm}(\mathbf{k})^* \vec{u}_{\pm}(\mathbf{k}) e^{ik\cdot x} + d_{\pm}(\mathbf{k}) \vec{v}_{\pm}(\mathbf{k}) e^{-ik\cdot x}) d^3x. \qquad (17.37b)$$

where

$$E_{\mathbf{k}} = k_0$$

satisfies (17.10). Here $x = (t, \mathbf{x}) = (x_0, x_1, x_2, x_3)$ and the dot product is taken in the sense of the Lorentzian metric. To quantize ψ, π, we replace the coefficients in this expansion with operator-valued functions $\hat{b}_{\pm}(\mathbf{k}), \hat{b}_{\pm}(\mathbf{k})^*, \hat{d}_{\pm}(\mathbf{k}), \hat{d}_{\pm}(\mathbf{k})^*$ to obtain

$$\Psi(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \sum_{\pm} \int_{\mathbb{R}^3} (\hat{b}_{\pm}(\mathbf{k}) \vec{u}_{\pm}(\mathbf{k}) e^{-k\cdot x} + \hat{d}_{\pm}(\mathbf{k})^* \vec{v}_{\pm}(\mathbf{k}) e^{ik\cdot x}) d^3x, \qquad (17.38a)$$

$$\Pi(\mathbf{x}) = \frac{i}{(2\pi)^{3/2}} \sum_{\pm} \int_{\mathbb{R}^3} (\hat{b}_{\pm}(\mathbf{k})^* \vec{u}_{\pm}(\mathbf{k}) e^{ik\cdot x} + \hat{d}_{\pm}(\mathbf{k}) \vec{v}_{\pm}(\mathbf{k}) e^{-ik\cdot x}) d^3x. \qquad (17.38b)$$

The operators $\hat{b}_{\pm}(\mathbf{k}), \hat{b}_{\pm}(\mathbf{k})^*, \hat{d}_{\pm}(\mathbf{k}), \hat{d}_{\pm}(\mathbf{k})^*$ are the analogs of the annihilation and creation operators. They satisfy the anticommutator relations:

$$\{\hat{b}_{\pm}(\mathbf{k}), \hat{b}_{\pm}(\mathbf{k}')^*\} = \{\hat{d}_{\pm}(\mathbf{k}), \hat{d}_{\pm}(\mathbf{k}')^*\} = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\pm\pm}, \qquad (17.39)$$

while other anticommutators among them vanish. These anticommutator relations guarantee that

$$[\Psi(t, \mathbf{x})_\alpha, \Pi(t, \mathbf{x})_\beta] = i\delta(\mathbf{x} - \mathbf{y})\delta_{\alpha\beta}.$$

This is analogous to (17.6) in the case of vector field. The Hamiltonian operator is obtained from (17.34) by replacing ψ and π with Ψ, Π given by (17.38). We obtain

$$H = \sum_{\pm} \int_{\mathbb{R}^3} E_{\mathbf{k}} (\hat{b}_{\pm}(\mathbf{k})^* \hat{b}_{\pm}(\mathbf{k}) - \hat{d}_{\pm}(\mathbf{k}) \hat{d}_{\pm}(\mathbf{k})^*) d^3x. \qquad (17.40)$$

Let us formally introduce the vacuum vector $|0\rangle$ with the property

$$\hat{b}_{\pm}(\mathbf{k})|0\rangle = \hat{d}_{\pm}(\mathbf{k})|0\rangle = 0$$

and define a state of n particles and m anti-particles by setting

$$\hat{b}(\mathbf{k}_n)^* \dots \hat{b}(\mathbf{k}_1)^* \hat{d}(\mathbf{p}_m)^* \dots \hat{d}(\mathbf{p}_1)^* |0\rangle := |\mathbf{k}_n \dots \mathbf{k}_1 \mathbf{p}'_m \dots \mathbf{p}'_1\rangle. \qquad (17.41)$$

Then we see that the energy of the state $|\mathbf{p}'_m \dots \mathbf{p}'_1\rangle$ is equal to

$$H|\mathbf{p}'_m \dots \mathbf{p}'_1\rangle = -(E_{\mathbf{p}_1} + \dots + E_{\mathbf{p}_m}) < 0$$

but the energy of the state $|\mathbf{k}_n \dots \mathbf{k}_1\rangle$ is equal to

$$H|\mathbf{k}_n \dots \mathbf{k}_1\rangle = E_{\mathbf{k}_1} + \dots + E_{\mathbf{k}_m} > 0.$$

However, we have a problem. Since

$$\hat{d}_\pm(\mathbf{k})\hat{d}_\pm(\mathbf{k})^* = 1 - \hat{d}_\pm(\mathbf{k})^*\hat{d}_\pm(\mathbf{k}),$$

$$H = \sum_\pm \int_{\mathbb{R}^3} E_\mathbf{k}(\hat{b}_\pm(\mathbf{k})^*\hat{b}_\pm(\mathbf{k}) + (\hat{d}_\pm(\mathbf{k})^*\hat{d}_\pm(\mathbf{k})) - 1)d^3x.$$

This shows that $|0\rangle$ is an eigenvector of H with eigenvalue

$$-2\int_{\mathbb{R}^3} E_\mathbf{k} = -\infty.$$

To solve this contradiction we "subtract the negative infinity" from the Hamiltonian by replacing it with the normal ordered Hamiltonian

$$: H := \sum_\pm \int_{\mathbb{R}^3} E_\mathbf{k}(\hat{b}_\pm(\mathbf{k})^*\hat{b}_\pm(\mathbf{k}) + \hat{d}_\pm(\mathbf{k})^*\hat{d}_\pm(\mathbf{k}))d^3x, \tag{17.42}$$

Similarly we introduce the momentum operators and the number operators

$$: \mathbf{P} := \sum_\pm \int_{\mathbb{R}^3} \mathbf{k}(\hat{b}_\pm(\mathbf{k})^*\hat{b}_\pm(\mathbf{k}) + \hat{d}_\pm(\mathbf{k})^*\hat{d}_\pm(\mathbf{k}))d^3x. \tag{17.43}$$

$$: N := \sum_\pm \int_{\mathbb{R}^3} (\hat{b}_\pm(\mathbf{k})^*\hat{b}_\pm(\mathbf{k}) + \hat{d}_\pm(\mathbf{k})^*\hat{d}_\pm(\mathbf{k}))d^3x, \tag{17.44}$$

There is also the *charge operator*

$$: Q := e\sum_\pm \int_{\mathbb{R}^3} (\hat{b}_\pm(\mathbf{k})^*\hat{b}_\pm(\mathbf{k}) - \hat{d}_\pm(\mathbf{k})^*\hat{d}_\pm(\mathbf{k}))d^3x. \tag{17.45}$$

It is obtained from normal ordering of the operator

$$Q = \int_{\mathbb{R}^3} \Psi^*\Psi d^3x = e\sum_{\mathbf{k},\pm}(b_\pm(\mathbf{k})^*b_\pm(\mathbf{k}) + d_\pm(\mathbf{k})^*d_\pm(\mathbf{k})).$$

The operators $: N :, : \mathbf{P} :, : Q :$ commute with $: H :$, i.e., they represent the quantized conserved currents.

We conclude:

1) The operator $\hat{d}_\pm^*(\mathbf{k})$ (resp. $\hat{d}_\pm(\mathbf{k})$) creates (resp. annihilates) a positive-energy particle (electron) with *helicity* $\pm\frac{1}{2}$, momentum $(E_\mathbf{k}, \mathbf{k})$ and negative electric charge $-e$.

2) The operator $b_\pm^*(\mathbf{k})$ (resp. $b_\pm(\mathbf{k})$) creates (resp. annihilates) a positive-energy *anti-particle* (*positron*) with helicity $\pm\frac{1}{2}$, momentum $(E_\mathbf{k}, \mathbf{k})$ and charge $-e$.

Note that because of the anti-commutation relations we have

$$b_\pm(\mathbf{k})b_\pm(\mathbf{k}) = d_\pm(\mathbf{k})d_\pm(\mathbf{k}) = 0.$$

This shows that in the package $|\mathbf{k}_n \ldots \mathbf{k}_1 \mathbf{p}'_m \ldots \mathbf{p}'_1\rangle$ all the vectors $\mathbf{k}_i, \mathbf{p}_j$ must be distinct. This is explained by saying that the particles satisfy the *Pauli exclusion principle*: no two particles with the same helicity and momentum can appear in same state.

17.6 Finally let us comment about the Poincaré invariance of canonical quantization. Using (17.35), we can rewrite the expression (17.38) for $\Psi(t,\mathbf{x})$ as

$$\Psi(t,\mathbf{x}) = \int_{\mathbb{R}_{\geq 0}} \int_{\mathbb{R}^3} (2(2\pi)^3 k_0)^{-1/2} \theta(k_0)\delta(|k|^2 - k_0^2 + m^2)\times$$

$$\times (e^{-i(k_0,\mathbf{k})\cdot(t,\mathbf{x})} a(\mathbf{k}) + e^{i(k_0,\mathbf{k})\cdot(t,\mathbf{x})} a(\mathbf{k})^*) d^3k dk_0. \tag{17.46}$$

Recall that the Poincaré group is the semi-direct product $SO(3,1)\tilde{\times}\mathbb{R}^4$ of the Lorentz group $SO(3,1)$ and the group $\mathbb{R}^4$, where $SO(3,1)$ acts naturally on $\mathbb{R}^4$. We have

$$\Psi(g(t,\mathbf{x})) = \int (2(2\pi)^3 k_0)^{-1/2} \theta(k_0)\delta(|k|^2 - k_0^2 + m^2)\times$$

$$\times (e^{-i(k_0,\mathbf{k})\cdot g(t,\mathbf{x})} a(\mathbf{k}) + e^{i(k_0,\mathbf{k})\cdot(g\cdot(t,\mathbf{x}))} a(\mathbf{k})^*) d^3k dk_0.$$

Now notice that if we take g to be the translation $g : (t,\mathbf{x}) \to (t,\mathbf{x}) + (a_0,\mathbf{a})$, we obtain, with help of formula (17.18),

$$\Psi(g\dot{(}t,\mathbf{x})) = e^{ia_\mu P^\mu} \Psi(t,\mathbf{x}) e^{-ia_\mu P^\mu},$$

where $P = (P^\mu)$ is the momentum operator in the Hilbert space $\mathcal{H}$. This shows that the map $(t,\mathbf{x}) \to \Psi(t,\mathbf{x})$ is invariant with respect to the translation group acting on the arguments $(t,\mathbf{x})$ and on the values by means of the representation

$$a = (a_0,\mathbf{a}) \to (A \to e^{i(a_\mu P^\mu)} \circ A \circ e^{-i(a_\mu P^\mu)})$$

of $\mathbb{R}^4$ in the space of operators $\mathrm{End}(\mathcal{H})$.

To exhibit the Lorentzian invariance we argue as follows.

First, we view the operator functions $a(\mathbf{k}), a(\mathbf{k})^*$ as operator valued distributions on $\mathbb{R}^4$ by assigning to each test function $f \in C_0^\infty(\mathbb{R}^4)$ the value

$$a(f) = \int f(k_0,\mathbf{k})\theta(k_0)\delta(|k|^2 - k_0^2 + m^2) a(\mathbf{k}) dk_0 d^3k.$$

Now we define the representation of $SO(3,1)$ in $\mathcal{H}$ by introducing the following operators

$$M_{0j} = i\int a(\mathbf{k})^* (E_k \frac{\partial}{\partial k_j}) a(\mathbf{k}) d^3k, \quad j = 0,1,2,3,$$

$$M_{ij} = i\int a(\mathbf{k})^*(k_i\frac{\partial}{\partial k_j} - k_j\frac{\partial}{\partial k_i})a(\mathbf{k})d^3k, \quad i,j = 1,2,3.$$

Here the partial derivative of the operator $a(\mathbf{k})$ is taken in the sense of distributions

$$\frac{\partial}{\partial k_j}a(\mathbf{k})(f) = \int \frac{\partial f}{\partial k_j}a(\mathbf{k})d^3x.$$

We have

$$[\Psi, M^{\mu\nu}] = i(x_\mu\partial^\nu - x_\nu\partial^\mu)\Psi, \tag{17.47}$$

where $M^{\mu\nu} = g^{\mu\nu}M_{\mu\nu}, \quad x_0 = t.$

Now we can construct the representation of the Lie algebra $\mathfrak{so}(3,1)$ of the Lorentz group in $\mathcal{H}$ by assigning the operator $M^{\mu\nu}$ to the matrix

$$E^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu] = (g^{\mu\nu}I_4 - \gamma^\nu\gamma^\mu).$$

It is easy to see that these matrices generate the Lie algebra of $SO(3,1)$. Now we exponentiate it to define for any matrix $\Lambda = c_{\mu\nu}E^{\mu\nu} \in \mathfrak{so}(3,1)$ the one-parameter group of operators in $\mathcal{H}$

$$U(\Lambda\tau) = \exp(i(c_{\mu\nu}M^{\mu\nu})\tau).$$

This group will act by conjugation in $\mathrm{End}(\mathcal{H})$

$$A \to U(\Lambda\tau)\circ A\circ U(-\Lambda\tau).$$

Using formula (17.47) we check that for any $g = \exp(i\Lambda\tau) \in SO(3,1)$,

$$\Psi(g(t,\mathbf{x})) = U(\Lambda\tau)\Psi(t,\mathbf{x})U(-\Lambda\tau).$$

this shows that $(t,\mathbf{x}) \to \Psi(t,\mathbf{x})$ is invariant with respect to the Poincaré group.

In the case of the spinor fields the Poincaré group acts on Ψ by acting on the arguments $(t,\mathbf{x})$ via its natural representation, and also on the values of Ψ by means of the spinor representation σ of the Lorentz group. Again, one checks that the fields $\Psi(t,\mathbf{x})$ are invariant.

What we described in this lecture is a quantization of *free fields* of spin 0 (pi-mesons) and spin $\frac{1}{2}$ (electron-positron). There is a similar theory of quantization of other free fields. For example, spin 1 fields correspond to the Maxwell equations (photons). In general, spin $n/2$-fields corresponds to irreducible linear representations of the group $Spin(3,1) \cong SL(2,\mathbb{C})$. As is well-known, they are isomorphic to symmetric powers $S^n(\mathbb{C}^2)$, where $\mathbb{C}^2$ is the space of the standard representation of $SL(2,\mathbb{C})$. So, spin $n/2$-fields correspond to the representation $S^n(\mathbb{C}^2), n = 0,1,2,\ldots$.

A non-free field contains some additional contribution to the Lagrangian responsible for interaction of fields. We shall deal with non-free fields in the following lectures.

Exercises.

1. Show that the momentum operators : $\mathbf{P}$: from (17.43) is obtained from the operator $\mathbf{P}$ whose coordinates P_μ are given by

$$P_\mu = -i \int_{\mathbb{R}^3} \Psi(t,\mathbf{x})^* \partial_\mu \Psi(t,\mathbf{x}) d^3x.$$

2. Check that the Hamiltonian and momentum operators are Hermitian operators.

3. Show that the charge operator Q for quantized spinor fields corresponds to the conserved current $J^\mu = \bar{\Psi}\gamma^\mu\Psi$. Check that it is indeed conserved under Lorentizian transformations. Show that $Q = \int J^0 d^3x = \int :\Psi^*\Psi: d^3x$.

4. Verify that the quantization of the spinor fields is Poincaré invariant.

5. Prove that the vacuum vector $|0\rangle$ is invariant with respect to the representation of the Poincaré group in the Fock space given by the operators (P^μ) and $(M^{\mu\nu})$.

6. Consider $\Psi(t,\mathbf{x})$ as an operator-valued distribution on $\mathbb{R}^4$. Show that $[\Psi(f),\Psi(g)] = 0$ if the supports of f and g are spacelike separated. This means that for any $(t,\mathbf{x}) \in \mathrm{Supp}(f)$ and any $(t',\mathbf{x}') \in \mathrm{Supp}(g)$, one has $(t-t')^2 - |\mathbf{x}-\mathbf{x}'|^2 < 0$. This is called *microscopic causality.*

7. Find a discrete version of the quantized Dirac field and an explicit construction of the corresponding Fock space.

8. Explain why the process of quantizing fields is called also second quantization.