Lecture 18. PATH INTEGRALS

The path integral formalism was introduced by R. Feynman. It yields a simple method for quantization of non-free fields, such as gauge fields. It is also used in conformal field theory and string theory.

18.1 The method is based on the following postulate:

The probability $P(a, b)$ of a particle moving from point a to point b is equal to the square of the absolute value of a complex valued function $K(a, b)$:

$$P(a, b) = |K(a, b)|^2. \tag{18.1}$$

The complex valued function $K(a, b)$ is called the *probability amplitude* . Let γ be a path leading from the point a to the point b. Following a suggestion from P. Dirac, Feynman proposed that the amplitude of a particle to be moved along the path γ from the point a to b should be given by the expression $e^{iS(\gamma)/\hbar}$, where

$$S(\gamma) = \int_{t_a}^{t_b} L(\mathbf{x}, \dot{\mathbf{x}}, t)dt$$

is the action functional from classical mechanics. The total amplitude is given by the integral

$$K(a, b) = \int_{\mathcal{P}(a,b)} e^{iS(\gamma)/\hbar} D[\gamma], \tag{18.2}$$

where $\mathcal{P}(a, b)$ denotes the set of all paths from a to b and $D[\gamma]$ is a certain measure on this set. The fact that the absolute value of the amplitude $e^{iS(\gamma)/\hbar}$ is equal to 1 reflects Feynman's principle of the democratic equality of all histories of the particle.

The expression (18.2) for the probability amplitude is called the *Feynman propagator*. It is an analog of the integral

$$F(\lambda) = \int_\Omega f(x)e^{i\lambda S(x)}dx,$$

where $x \in \Omega \subset \mathbb{R}^n, \lambda >> 0$, and $f(x), S(x)$ are smooth real-valued functions. Such integrals are called the integrals of rapidly oscillating functions. If f has compact support and $S(x)$ has no critical points on the support of $f(x)$, then $F(\lambda) = o(\lambda^{-n})$, for any $n > 0$, when $\lambda \to \infty$. So, the main contribution to the integral is given by critical (or stationary) points, i.e. points $x \in \Omega$ where $S'(x) = 0$. This idea is called the method of stationary phase. In our situation, x is replaced with a path γ. When $\hbar$ is sufficiently small, the analog of the method of stationary phase tells us that the main contribution to the probablity $P(a, b)$ is given by the paths with $\frac{\delta S}{\delta \gamma} = 0$. But these are exactly the paths favored by classical mechanics!

Let $\psi(t, x)$ be the wave function in quantum mechanics. Recall that its absolute value is interpreted as the probability for a particle to be found at the point $x \in \mathbb{R}^3$ at time t. Thus, forgetting about the absolute value, we can think that $K((t_b, x_b), (t, x))\psi(t, x)$ is equal to the probability that the particle will be found at the point x_b at time t_b if it was at the point x at time t. This leads to the equation

$$\psi(t_b, x_b) = \int_{\mathbb{R}^3} K((t_b, x_b), (t, x))\psi(t, x)d^3x.$$

From this we easily deduce the following important property of the amplitude function

$$K(a, c) = \int_{t_a}^{t_b} \int_{\mathbb{R}^3} K((t_c, x_c), (t, x))K((t, x), (t_a, x_a))d^3x dt. \tag{18.3}$$

18.2 Before we start computing the path integral (18.2) let us try to explain its meaning. The space $\mathcal{P}(a, b)$ is of course infinite-dimensional and the integration over such a space has to be defined. Let us first restrict ourselves to some special finite-dimensional subspaces of $\mathcal{P}(a, b)$. Fix a positive integer N and subdivide the time interval $[t_a, t_b]$ into N equal parts by inserting intermediate points $t_1 = t_a, t_2, \ldots, t_N, t_{N+1} = t_b$. Choose some points $x_1 = x_a, x_2, \ldots, x_N, x_{N+1} = x_b$ in $\mathbb{R}^n$ and consider the path $\gamma : [t_a, t_b] \to \mathbb{R}^n$ such that its restriction to each interval $[t_i, t_{i+1}]$ is the linear function

$$\gamma_i(t) = \gamma_i(t) = x_i + \frac{x_{i+1} - x_i}{t_{i+1} - t_i}(t - t_i).$$

It is clear that the set of such paths is bijective with $(\mathbb{R}^n)^{N-1}$ and so can be integrated over. Now if we have a function $F : \mathcal{P}(a, b) \to \mathbb{R}$ we can integrate it over this space to get the number J_N. Now we can define (18.2) as the limit of integrals J_N when N goes to infinity. However, this limit may not exist. One of the reasons could be that J_N contains a factor C^N for some constant C with $|C| > 1$. Then we can get the limit by redefining J_N, replacing it with $C^{-N}J_N$. This really means that we redefine the standard measure on $(\mathbb{R}^n)^{N-1}$ replacing the measure d^nx on $\mathbb{R}^n$ by $C^{-1}d^nx$. This is exactly what we are going to do. Also, when we restrict the functional to the finite-dimensional space $(\mathbb{R}^n)^{N-1}$ of piecewise linear paths, we shall allow ourselves to replace the integral $\int_{t_a}^{t_b} S(\gamma)dt$ by its

Riemann sum. The result of this approximation is by definition the right-hand side in (18.2). We should immediately warn that the described method of evaluating of the path integrals is not the only possible. It is only meanigful when we specify the approximation method for its computation.

Let us compute something simple. Assume that the action is given by

$$S = \int_{t_a}^{t_b} \frac{1}{2} m\dot{x}^2 dt.$$

As we have explained before we subdivide $[t_a, t_b]$ into N parts with $\epsilon = (t_b - t_a)/N$ and put

$$K(a,b) = \lim_{N\to\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \exp[\frac{im}{2\epsilon}\sum_{i=1}^{N}(x_i - x_{i+1})^2]C^{-N}dx_2\ldots dx_N. \tag{18.4}$$

Here the number C should be chosen to guarantee convergence in (18.4). To compute it we use the formula

$$\int_{-\infty}^{\infty} x^{2s-1}e^{-zx^2}dx = \frac{\Gamma(s)}{z^s}. \tag{18.5}$$

This integral is convergent when $\mathrm{Re}(z) > 0, \mathrm{Re}(s) > 0$, and is an analytic function of z when s is fixed (see, for example, [**Whittaker**], 12.2, Example 2). Taking $s = \frac{1}{2}$, we get, if $\mathrm{Re}(z) > 0$,

$$\int_{-\infty}^{\infty} e^{-zx^2}dx = \frac{\Gamma(1/2)}{z^{1/2}} = \sqrt{\pi/z}. \tag{18.6}$$

Using this formula we can define the integral for any $z \neq 0$. We have

$$\int_{-\infty}^{\infty} \exp[-a(x_{i-1} - x_i)^2 - a(x_i - x_{i+1})^2]dx_i =$$

$$= \int_{-\infty}^{\infty} \exp[-2a(x_i + \frac{x_{i-1}+x_{i+1}}{2})^2 - \frac{a}{2}(x_{i-1} - x_{i+1})^2]dx_i =$$

$$= \exp[-\frac{a}{2}(x_{i-1} - x_{i+1})^2] \int_{-\infty}^{\infty} \exp(-2ax^2)dx = \sqrt{\frac{\pi}{2a}}\exp[-\frac{a}{2}(x_{i-1} - x_{i+1})^2].$$

After repeated integrations, we find

$$\int_{-\infty}^{\infty} \exp[-a\sum_{i=1}^{N}(x_i - x_{i+1})^2]dx_2\ldots dx_N = \sqrt{\frac{\pi^{N-1}}{Na^{N-1}}}\exp[-\frac{a}{N}(x_1 - x_{N+1})^2],$$

where $a = m/2i\epsilon$. If we choose the constant C equal to

$$C = \left(\frac{m}{2\pi i\epsilon}\right)^{\frac{1}{2}},$$

then we can rewrite (18.4) in the form

$$K(a,b) = \left(\frac{m}{2\pi iN\epsilon}\right)^{\frac{1}{2}} e^{\frac{mi(x_b-x_a)^2}{2N\epsilon}} = \left(\frac{m}{2\pi i(t_b-t_a)}\right)^{\frac{1}{2}} e^{\frac{mi(x_b-x_a)^2}{2(t_b-t_a)}}, \tag{18.7}$$

where $a = (t_a, x_a), b = (t_b, x_b)$.

The function $b = (t, x) \to K(a,b) = K(a;t,x)$ satisfies the *heat equation*

$$\frac{1}{2m}\frac{\partial^2}{\partial x^2}K(b,a) = i\frac{\partial}{\partial t}K(a,b), \quad b \neq a.$$

This is the Schrödinger equation (5.6) from Lecture 5 corresponding to the harmonic oscillator with zero potential energy.

18.3 The propagator function $K(a,b)$ which we found in (18.7) is the Green function for the Schrödinger equation. The *Green function* of this equation is the (generalized) function $G(x,y)$ which solves the equation

$$(i\frac{\partial}{\partial t} - \frac{1}{2m}\sum_\mu \frac{\partial^2}{\partial x_\mu^2})G(x,y) = \delta(x-y). \tag{18.8}$$

Here we view x as an argument and y as a parameter. Notice that the homogeneous equation

$$(i\frac{\partial}{\partial t} - \frac{1}{2m}\sum_\mu \frac{\partial^2}{\partial x_\mu^2})\phi = 0$$

has no non-trivial solutions in the space of test functions $C_0^\infty(\mathbb{R}^3)$ so it will not have solutions in the space of distributions. This implies that a solution of (18.8) will be unique. Let us consider a more general operator

$$D = \frac{\partial}{\partial t} - P(i\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}), \tag{18.9}$$

where P is a polynomial in n variables. Assume we find a function $F(x)$ defined for $t \geq 0$ such that

$$DF(t,\mathbf{x}) = 0, \quad F(0,\mathbf{x}) = \delta(\mathbf{x}).$$

Then the function

$$F'(x) = \theta(t)F(x)$$

(where $\theta(t)$ is the Heaviside function) satisfies

$$DF'(x) = \delta(x). \tag{18.10}$$

Taking the Fourier transform of both sides in (18.10), it suffices to show that

$$(-ik_0 - P(\mathbf{k}))\hat{F}'(k_0, \mathbf{k}) = (1/2\pi)^{n+1}. \tag{18.11}$$

Let us first take the Fourier transform of F' in coordinates x to get the function $V(t,k)$ satisfying

$$V(t, \mathbf{k}) = 0 \quad \text{if} \quad t < 0,$$

$$V(t, \mathbf{k}) = (1/\sqrt{2\pi})^n \quad \text{if} \quad t = 0,$$

$$\frac{dV(t, \mathbf{k})}{dt} - P(\mathbf{k})V(t, \mathbf{k}) = 0 \quad \text{if} \quad t > 0.$$

Then, the last equality can be extended to the following equality of distributions true for all t:

$$\frac{dV(t, \mathbf{k})}{dt} - P(\mathbf{k})V(t, \mathbf{k}) - (1/\sqrt{2\pi})^n \delta(t) = 0.$$

(see Exercise 5 in Lecture 6). Applying the Fourier transform in t, we get (18.11).

We will be looking for $G(x, y)$ of the form $G(x - y)$, where $G(z)$ is a distribution on $\mathbb{R}^4$. According to the above, first we want to find the solution of the equation

$$(i\frac{\partial}{\partial t} - \frac{1}{2m}\sum_\mu \frac{\partial^2}{\partial x_\mu^2})G'(x - y) = 0, \quad t \geq t', \quad x_i \geq y_i \tag{18.12}$$

satisfying the boundary condition

$$G(0, \mathbf{x} - \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}). \tag{18.13}$$

Consider the function

$$G'(x - y) = i\Big(\frac{m}{2\pi i(t - t')}\Big)^{3/2} e^{im||\mathbf{x}-\mathbf{y}||^2/2(t-t')}.$$

Differentiating, we find that for $x > y$ this function satisfies (18.11). Since we are considering distributions we can ignore subsets of measure zero. So, as a distribution, this function is a solution for all $x \geq y$. Now we use that

$$\lim_{t \to 0} \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t} = \delta(x)$$

(see **[Gelfand]**, Chapter 1, §2, n°2, Example 2). Its generalization to functions in 3 variables is

$$\lim_{t \to 0} \Big(\frac{1}{2\sqrt{\pi t}}\Big)^3 e^{-||\mathbf{x}||^2/4t} = \delta(\mathbf{x}).$$

After the change of variables, we see that $G'(x - y)$ satisfies the boundary condition (18.13). Thus, we obtain that the function

$$G(x - y) = i\theta(t - t')e^{\frac{im|\mathbf{x}-\mathbf{y}|^2}{2(t-t')}} \Big(\frac{m}{2\pi i(t - t')}\Big)^{3/2} \tag{18.14}$$

is the Green function of the equation (18.8). We see that it is the three-dimensional analog of $K(a,b)$ which we computed earlier.

The advantage of the Green function is that it allows one to find a solution of the inhomogeneous Schrödinger equation

$$S\phi(x) = (i\frac{\partial}{\partial t} - \frac{1}{2m}\sum_\mu \frac{\partial}{\partial x_\mu}^2)\phi(x) = J(x).$$

If we consider $G(x-y)$ as a distribution, then its value on the left-hand side is equal to the value of $\delta(x-y)$ on the right-hand side. Thus we get

$$\int G(x-y)J(y)d^4y = \int G(x-y)S(\phi(y))d^4y =$$

$$= \int S(G(x-y))\phi(y)d^4y = \int \delta(x-y)\phi(y)d^4y = \phi(x).$$

Using the formula (18.6) for the Gaussian integral we can rewrite $G(x-y)$ in the form

$$G(x-y) = i\theta(t-t')\int_{\mathbb{R}^3} \phi_p(x)\phi_p^*(y)d^3p,$$

where

$$\phi_p(x) = \frac{1}{(2\pi)^{3/2}}e^{-ik\cdot x}, \tag{18.15}$$

where $x = (t, \mathbf{x}), k = (|\mathbf{p}|^2/2m, \mathbf{p})$. We leave it to the reader.

Similarly, we can treat the Klein-Gordon inhomogeneous equation

$$(\partial_\mu\partial^\mu + m^2)\phi(x) = J(x).$$

We find that the Green function $\Delta(x-y)$ is equal to

$$\Delta(x-y) = i\theta(t-t')\int \frac{\phi_k(x)\phi_k^*(y)}{2E_k}d^3k + i\theta(t'-t)\int \frac{\phi_k^*(x)\phi_k(y)}{2E_k}d^3k, \tag{18.16}$$

where $k = (E_k, \mathbf{k}), |k|^2 = m^2$, and ϕ_k are defined by (18.15).

Remark. There is also an expression for $\Delta(x-y)$ in terms of its Fourier transform. Let us find it. First notice that

$$\theta(t-t') = \int_{-\infty}^{\infty} \frac{e^{-ix(t-t')}}{2\pi(x+i\epsilon)}dx.$$

for some small positive ϵ. To see this we extend the contour from the line Im $z = \epsilon$ into the complex z plane by integrating along a semi-circle in the lower-half plane when $t > t'$. There will be one pole inside, namely $z = -i/\epsilon$ with the residue 1. By applying the Cauchy residue theorem, and noticing that the integral over the arc of the semi-circle goes to zero when the radius goes to infinity, we get

$$\int_{-\infty}^{\infty} \frac{e^{-ix(t-t')}}{2\pi(x+i\epsilon)}dx = -i.$$

The minus sign is explained by the clockwise orientation of the contour. If $t < t'$ we can take the contour to be the upper half-circle. Then the residue theorem tells us that the integral is equal to zero (since the pole is in the lower half-plane). Inserting this expression in (18.6), we get

$$\Delta(x-y) = -\frac{1}{(2\pi)^4}\int \frac{e^{-i(\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})+E_k(t-t')+s(t-t'))}d^3k ds}{2E_k(s+i\epsilon)} +$$

$$+\frac{1}{(2\pi)^4}\int \frac{e^{i(\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})+E_k(t-t')-s(t-t'))}d^3k ds}{2E_k(s-i\epsilon)}.$$

Change now s to $k_0 - E_k$ (resp. $k_0 + E_k$) in the first integral (resp. the second one). Also replace $\mathbf{k}$ with $-\mathbf{k}$ in the second integral. Then we can write

$$\Delta(x-y) = -\frac{1}{(2\pi)^4}\int \frac{e^{-i(\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})+k_0(t-t'))}d^3k dk_0}{2E_k(k_0-E_k+i\epsilon)} - \frac{1}{(2\pi)^4}\int \frac{e^{-i(\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})+k_0(t-t'))}d^3k dk_0}{2E_k(k_0+E_k-i\epsilon)}$$

$$= -\frac{1}{(2\pi)^4}\int \frac{e^{-i(\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})+k_0(t-t'))}d^3k dk_0}{k_0^2-E_k^2+i\epsilon} = -\frac{1}{(2\pi)^4}\int \frac{e^{-i(k\cdot(x-y)+k_0(t-t'))}d^3k dk_0}{|k|^2-m^2+i\epsilon},$$

where $k = (k_0, \mathbf{k}), |k| = k_0^2 - ||\mathbf{k}||^2$. Here the expression in the denominator really means $|k|^2 - m^2 + \epsilon^2 + i\epsilon(m^2 + ||\mathbf{k}||^2)^{1/2}$. Thus we can formally write the following formula for $\Delta(x-y)$ in terms of the Fourier transform:

$$-\Delta(x-y) = (2\pi)^2 F(1/(|k|^2 - m^2 + i\epsilon)). \tag{18.17}$$

Note that if we apply the Fourier transform to both sides of the equation

$$(\partial_\mu\partial^\mu + m^2)G(x-y) = \delta(x-y)$$

which defines the Green function, we obtain

$$(-k_0^2 + k_1^2 + k_2^2 + k_3^2 + m^2)F(G(x-y)) = F(\delta(x-y)) = (2\pi)^2.$$

Applying the inverse Fourier transform, we get

$$-\Delta(x-y) = (2\pi)^2 F(1/(|k|^2 - m^2)).$$

The problem here is that the function $1/(|k|^2 - m^2)$ is not defined on the hypersurface $|k|^2 - m^2 = 0$ in $\mathbb{R}^4$. It should be treated as a distribution on $\mathbb{R}^4$ and to compute its

Fourier transform we replace this function by adding $i\epsilon$. Of course to justify formally this trick, we have to argue as in the proof of the formula (18.14).

18.4 Let us arrive at the path integral starting from the Heisenberg picture of quantum mechanics. In this picture the states ϕ originally do not depend on time. However, in Lecture 5, we defined their time evolution by

$$\phi(x,t) = e^{-iHt}\phi(x).$$

Here we have scaled the Planck constant $\hbar$ to 1. The probability that the state $\psi(x)$ changes to the state $\phi(x)$ in time $t'-t$ is given by the inner product in the Hilbert space

$$\langle\phi(x,t)|\psi(x,t')\rangle := \langle\phi(x)e^{-iHt}, e^{-iHt'}\psi(x)\rangle = \langle\phi(x), e^{-iH(t-t')}\psi(x)\rangle. \tag{18.18}$$

Let $|x\rangle$ denote the eigenstate of the position (vector) operator $Q = (Q_1, Q_2, Q_3) : \psi \to (x_1, x_2, x_3)\psi$ with eigenvalue $x \in \mathbb{R}^3$. By Example 9, Lecture 6, $|x\rangle$ is the Dirac distribution $\delta(x-y) : f(y) \to f(x)$. Then we denote $\langle|x'\rangle, t'||x\rangle, t\rangle$ by $\langle x', t'|x, t\rangle$. It is interpreted as the probability that the particle in position x at time t will be found at position x' at a later time t'. So it is exactly $P((t,x),(t',x'))$ that we studied in section 1. We can write any ϕ as a Fourier integral of eigenvectors

$$\psi(x,t) = \int \langle\psi|x\rangle dx,$$

where following physicist's notation we denote the inner product $\langle\psi, |x\rangle\rangle$ as $\langle\psi|x\rangle$. Thus we can rewrite (18.18) in the form

$$\langle\phi|\psi\rangle = \int\int \langle\phi|x'\rangle\langle x'|e^{\frac{-iH(t'-t)}{\hbar}}|x\rangle\langle x|\psi\rangle dx dx'.$$

We assume for simplicity that H is the Hamiltonian of the harmonic oscillator

$$H = (P^2/2m) + V(Q).$$

Recall that the momentum operator $P = i\frac{d}{dx}$ has eigenstates

$$|p\rangle = e^{-ipx}$$

with eigenvalues p. Thus

$$e^{-iP^2}|p\rangle = e^{-ip^2}|p\rangle.$$

We take $\Delta t = t' - t$ small to be able to write

$$e^{-iH(t'-t)} = e^{-i\frac{P^2}{2m}\Delta t}e^{-iV(Q)\Delta t}[1 + O(\Delta t^2)]$$

(note that $\exp(A+B)\neq \exp(A)\exp(B)$ if $[A,B]\neq 0$ but it is true to first approximation). From this we obtain

$$\langle x',t'|x,t\rangle := \langle\delta(y-x'(t))|\delta(y-x(t))\rangle = e^{-iV(x)\Delta t}\langle x'|e^{-iP^2\Delta t/2m}|x\rangle =$$

$$= e^{iV(x)\Delta t}\int\int\langle x'|p'\rangle\langle p'|e^{-iP^2\Delta t/2m}|p\rangle\langle p|x\rangle dpdp' =$$

$$= \frac{1}{2\pi}e^{iV(x)\Delta t}\int\int e^{-i(p^2\Delta t/2m)}\delta(p'-p)e^{ip'x'}e^{-ipx}dpdp' =$$

$$= \frac{1}{2\pi}e^{-iV(x)\Delta t}\int e^{-ip^2\Delta t/2m}e^{ip(x-x')}dp = \Big(\frac{m}{2\pi i\Delta t}\Big)^{\frac{1}{2}}e^{im(x'-x)^2/2\Delta t}e^{i\Delta tV(x)}.$$

The last expression is obtained by completing the square and applying the formula (18.6) for the Gaussian integral. When $V(x)=0$, we obtain the same result as the one we started with in the beginning. We have

$$\langle x',t'|x,t\rangle = \Big(\frac{m}{2\pi i\Delta t}\Big)^{\frac{1}{2}}\exp(iS),$$

where

$$S(t,t') = \int_t^{t'}[\frac{1}{2}m\dot{x}^2 - V(x)]dt.$$

So far, we assumed that $\Delta t = t'-t$ is infinitesimal. In the general case, we subdivide $[t,t']$ into N subintervals $[t=t_1,t_2]\cup\ldots\cup[t_{N-1},t_N=t']$ and insert corresponding eigenstates $x(t_i)$ to obtain

$$\langle x',t'|x,t\rangle = \int\langle x_1,t_1|\ldots|x_N,t_N\rangle dx_2\ldots dx_{N-1} = \int[Dx]\exp(iS(t,t')), \tag{18.19}$$

where $[Dx]$ means that we have to understand the integral in the sense described in the beginning of section 18.2

We can insert additional functionals in the path integral (18.19). For example, let us choose some moments of time $t<t_1<\ldots<t_n<t'$ and consider the functional which assigns to a path $\gamma:[t,t']\to\mathbb{R}$ the number $\gamma(t_i)$. We denote such functional by $x(t_i)$. Then

$$\int[Dx]x(t_1)\ldots x(t_n)\exp(iS(t,t')) = \langle x',t'|Q(t_1)\ldots Q(t_n)|x,t\rangle. \tag{18.20}$$

We leave the proof of (18.20) to the reader (use induction on n and complete sets of eigenstates for the operators $Q(t_i)$). For any function $J(t)\in L_2([t,t'])$ let us denote by S^J the action defined by

$$S^J(\gamma) = \int_t^{t'}(\mathcal{L}(x,\dot{x}) + xJ(t))dt.$$

Let us consider the functional

$$J \to Z[J] = k\int [Dx]e^{iS^J}.$$

Then

$$i\frac{\delta^n}{\delta J(t_1)\dots\delta J(t_n)}|_{J=0}\int [Dx]e^{iS^J} = \langle x',t'|Q(t_1)\dots Q(t_n)|x,t\rangle. \tag{18.21}$$

Here $\frac{\delta}{\delta J(x_i)}$ denotes the (weak) functional derivative of the functional $J \to Z[J]$ computed at the function

$$\alpha_a(x) = \begin{cases} 1 & \text{if } x = a, \\ 0 & \text{if } x \neq a. \end{cases}$$

Recall that the weak derivative of a functional $F : C^\infty(\mathbb{R}^n) \to \mathbb{C}$ at the point $\phi \in C^\infty(\mathbb{R}^n)$ is the functional $DF(\phi) : C^\infty(\mathbb{R}^n) \to \mathbb{C}$ defined by the formula

$$DF(\phi)(h) = \frac{dF(\phi + th)}{dt} = \lim_{t\to 0}\frac{F(\phi+th) - F(\phi)}{t}.$$

If F has Frechét derivative in the sense that we used in Lecture 13, then the weak derivative exists too, and the derivatives coincide.

18.4 Now recall from the previous lecture the formula (17.22a) for the quantized free scalar field

$$\Psi(t,\mathbf{x}) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}\frac{1}{\sqrt{2E_k}}(a(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-E_\mathbf{k}t)} + a(\mathbf{k})^*e^{-i(\mathbf{k}\cdot\mathbf{x}-E_\mathbf{k}t)})d^3k.$$

We can rewrite it in the form

$$\Psi(t,\mathbf{x}) = \int_{\mathbb{R}^3}\frac{1}{\sqrt{2E_k}}(a(\mathbf{k})\phi_k(x) + a(\mathbf{k})^*\phi_k^*(x))d^3k,$$

where ϕ_k are defined in (18.15).

Define the *time-ordering* operator

$$T[\Psi(t,\mathbf{x})\Psi(t',\mathbf{x}')] = \begin{cases} \Psi(t,\mathbf{x})\circ\Psi(t',\mathbf{x}') & \text{if } t > t' \\ \Psi(t',\mathbf{x}')\circ\Psi(t,\mathbf{x}) & \text{if } t < t'. \end{cases}$$

Then, using (18.16), we immediately verify that

$$\langle 0|T[\Psi(t,\mathbf{x})\Psi(t',\mathbf{x}')]|0\rangle = -i\Delta(x-y). \tag{18.22}$$

This gives us another interpretation of the Green function for the Klein-Gordon equation.

For any operator A the expression

$$\langle 0|A|0\rangle$$

is called the *vacuum expectation value* of A. When $A = T[\Psi(t,\mathbf{x})\Psi(t',\mathbf{x}')]$ as above, we interpret it as the probability amplitude for the system being in the state $\Psi(t,\mathbf{x})$ at time t to change into state $\Psi(t',\mathbf{x}')$ at time t'.

If we write the operators $\Psi(t,\mathbf{x})$ in terms of the creation and annihilation operators, it is clear that the normal ordering $:\Psi(t,\mathbf{x})\Psi(t',\mathbf{x}'):$ differs from the time-ordering $T[\Psi(t,\mathbf{x})\Psi(t',\mathbf{x}')]$ by a scalar operator. To find this scalar, we compare the vacuum expectation values of the two operators. Since the operator in the normal ordering kills $|0\rangle$, the vacuum expectation value of $:\Psi(t,\mathbf{x})\Psi(t',\mathbf{x}'):$ is equal to zero. Thus we obtain

$$T[\Psi(t,\mathbf{x})\Psi(t',\mathbf{x}')] - :\Psi(t,\mathbf{x})\Psi(t',\mathbf{x}'): = \langle 0|T[\Psi(t,\mathbf{x})\Psi(t',\mathbf{x}')]|0\rangle.$$

The left-hand side is called the *Wick contraction* of the operators $\Psi(t,\mathbf{x})\Psi(t',\mathbf{x}')$ and is denoted by

$$\overbrace{\Psi(t,\mathbf{x})\Psi(t',\mathbf{x}')}\,. \tag{18.23}$$

It is a scalar operator.

If we now have three field operators $\Psi(x_1), \Psi(x_2), \Psi(x_3)$, we obtain

$$T[\Psi(x_1),\Psi(x_2),\Psi(x_3)] - :\Psi(x_1),\Psi(x_2),\Psi(x_3): =$$

$$= \frac{1}{2}\sum_{\sigma\in S_3}\langle 0|T[\Psi(x_{\sigma(1)})\Psi(x_{\sigma(2)})]|0\rangle\Psi(x_{\sigma(3)}).$$

For any n we have the following Wick's theorem:

Theorem 1.

$$T[\Psi(x_1),\ldots,\Psi(x_n)] - :\Psi(x_1)\cdots\Psi(x_n): =$$

$$= \sum_{\text{perm}}\langle 0|T[\Psi(x_{\sigma(1)})\Psi(x_{\sigma(2)})]|0\rangle :\Psi(x_{\sigma(3)}\cdots\Psi(x_{\sigma(n)}): +$$

$$+\sum_{\text{perm}}\langle 0|T[\Psi(x_{\sigma(1)})\Psi(x_{\sigma(2)})]|0\rangle\langle 0|T[\Psi(x_{\sigma(3)})\Psi(x_{\sigma(4)})]|0\rangle :\Psi(x_{\sigma(5)}\cdots\Psi(x_{\sigma(n)}: +\ldots$$

$$+\sum_{\text{perm}}\langle 0|T[\Psi(x_{\sigma(1)})\Psi(x_{\sigma(2)})]|0\rangle\cdots\langle 0|T[\Psi(x_{\sigma(n-1)})\Psi(x_{\sigma(n)})]|0\rangle,$$

where, if n is odd, the last term is replaced with

$$\sum_{\text{perm}}\langle 0|T[\Psi(x_{\sigma(1)})\Psi(x_{\sigma(2)})]|0\rangle\cdots\langle 0|T[\Psi(x_{\sigma(n-2)})\Psi(x_{\sigma(n-1)})]|0\rangle\Psi(x_{\sigma(n)}).$$

Here we sum over all permutations that lead to different expressions.

Proof. Induction on n, multiplying the expression for $n-1$ on the right by the operator $\Psi(x_n)$ with smallest time argument.

Corollary.

$$\langle 0|T[\Psi(x_1),\ldots,\Psi(x_n)]|0\rangle = \sum_{\text{perm}} \langle 0|T[\Psi(x_{\sigma(1)})\Psi(x_{\sigma(2)})]|0\rangle \cdots \langle 0|T[\Psi(x_{\sigma(n-1)})\Psi(x_{\sigma(n)})]|0\rangle$$

if n is even. Otherwise $\langle 0|T[\Psi(x_1),\cdots,\Psi(x_n)]|0\rangle = 0$.

The assertion of the previous theorem can be easily generalized to any operator valued distribution which we considered in Lecture 17, section 4.

The function

$$G(x_1,\ldots,x_n) = \langle 0|T[\Psi(x_1),\ldots,\Psi(x_n)]|0\rangle$$

is called the *n-point function* (for scalar field). Given any function $J(x)$ we can form the generating function

$$Z[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int J(x_1)\ldots J(x_n)\langle 0|T[\Psi(x_1),\ldots,\Psi(x_n)]|0\rangle d^4x_1\ldots d^4x_n. \qquad (18.24)$$

We can write down this expression formally as

$$Z[J] := \langle 0|T[\exp(i\int J(x)\Psi(x)d^4x)]|0\rangle.$$

We have

$$i^n G(x_1,\ldots,x_n) = \frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)} Z[J]|_{J=0}. \qquad (18.25)$$

18.5 In section 18.3 we used the path integrals to derive the propagation formula for the states $|\mathbf{x}\rangle$ describing the probability of a particle to be at a point $\mathbf{x}\in\mathbb{R}^3$. We can try to do the same for arbitrary pure states $\phi(\mathbf{x})$. We shall use functional integrals to interpret the propagation formula (18.18)

$$\langle\phi(t,\mathbf{x})|\psi(t',\mathbf{x})\rangle = \langle\phi(\mathbf{x}), e^{-iH(t-t')}\psi(\mathbf{x})\rangle. \qquad (18.26)$$

Here we now view $\phi(t,\mathbf{x}),\psi(t,\mathbf{x})$ as arbitrary fields, not necessary coming from evolved wave functions. We shall try to motivate the following expression for this formula:

$$\langle\phi_1(t,\mathbf{x})|\phi_2(t',\mathbf{x})\rangle = \int_{f_1}^{f_2} [D\phi]e^{iS(\phi)}, \qquad (18.27)$$

where we integrate over some space of functions (or distributions) ϕ on $\mathbb{R}^4$ such that $\phi(t,\mathbf{x}) = f_1(\mathbf{x}), \phi(t',\mathbf{x}) = f_2(\mathbf{x})$. The integration is *functional integration.*

For example, assume that

$$S(\phi) = \int_t^{t'} \Big(\int_{\mathbb{R}^3} \mathcal{L}(\phi) d^3x \Big) dt,$$

where

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2\phi^2) + J\phi$$

corresponds to the Klein-Gordon Lagrangian with added source term $J\phi$. We can rewrite $S(\phi)$ in the form

$$S(\phi) = \int \mathcal{L}(\phi) d^4x = \int (\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2\phi^2 + J\phi) d^4x =$$

$$= \int (-\frac{1}{2}\phi(\partial_\mu \partial^\mu + m^2)\phi + J\phi) d^4x = \int (-\frac{1}{2}\phi A \phi + J\phi) d^4x, \tag{18.28}$$

where

$$A = \partial_\mu \partial^\mu + m^2$$

is the D'Alembertian linear operator. Now the action functional $S(\phi(t, \mathbf{x}))$ looks like a quadratic function in ϕ. The right-hand side of (18.27) recalls the Gaussian integral (18.6). In fact the latter has the following generalization to quadratic functions on any finite-dimensional vector spaces:

Lemma 1. *Let $Q(x) = \frac{1}{2}x \cdot Ax + b \cdot x$ be a complex valued quadratic function on $\mathbb{R}^n$. Assume that the matrix A is symmetric and its imaginary part is positive definite. Then*

$$\int_{\mathbb{R}^n} e^{iQ(x)} d^n x = (2\pi i)^{n/2} \exp[-\frac{1}{2} i b \cdot A^{-1} b] (\det A)^{-1/2}.$$

Proof. First make the variable change $x \to x + A^{-1}b$, then make an orthogonal variable change to reduce the quadratic part to the sum of squares, and finally apply (18.6).

As in the case of (18.6) we use the right-hand side to define the integral for any invertible A.

So, as soon as our action $S(\phi)$ looks like a quadratic function in ϕ we can define the functional integral provided we know how to define the determinant of a linear operator on a function space. Recall that the determinant of a symmetric matrix can be computed as the product of its eigenvalues

$$\det(A) = \lambda_1 \ldots \lambda_n.$$

Equivalently,

$$\ln \det(A) = \sum_{i=1}^n \ln(\lambda_i) = -\frac{d(\sum_{i=1}^n \lambda_i^{-s})}{ds}\Big|_{s=0}$$

This suggests to define the determinant of an operator A in a Hilbert space V which admits a complete basis $(v_i), i = 0, 1 \ldots$ composed of eigenvectors of A with eigenvalues λ_i as

$$\det(A) = e^{-\zeta'_A(0)}, \tag{18.28}$$

where

$$\zeta_A(s) = \sum_{i=1}^{n} \lambda_i^{-s}. \tag{18.29}$$

The function $\zeta_A(s)$ of the complex variable s is called the *zeta function* of the operator A. In order that it makes sense we have to assume that no λ_i are equal to zero. Otherwise we omit 0 from expression (18.19). Of course this will be true if A is invertible. For example, if A is an elliptic operator of order r on a compact manifold of dimension d, the zeta function ζ_A converges for Re $s > d/r$. It can be analytically continued into a meromorphic function of s holomorphic at $s = 0$. Thus (18.29) can be used to define the determinant of such an operator.

Example. Unfortunately the determinant of the D'Alembertian operator is not defined. Let us try to compute instead the determinant of the operator $A = -\Delta + m^2$ obtained from the D'Alembertian operator by replacing t with it. Here Δ is the 4-dimensional Laplacian. To get a countable complete set of eigenvalues of A we consider the space of functions defined on a box B defined by $-l_i \leq x_i \leq l_i, i = 1, \ldots, 4$. We assume that the functions are integrable complex-valued fucntions and also periodic, i.e. $f(-l) = f(l)$, where $l = (l_1, l_2, l_3, l_4)$. Then the orthonormal eigenvectors of A are the functions

$$f_k(x) = \frac{1}{\Omega} e^{ik\cdot x}, \quad k = 2\pi(n_1/l_1, \ldots, n_4/l_4), \quad n_i \in \mathbb{Z},$$

where Ω is the volume of B. The eigenvalues are

$$\lambda_k = ||k||^2 + m^2 > 0$$

Introduce the *heat function*

$$K(x, y, t) = \sum e^{-\lambda_k t} f_k(x) \bar{f}_k(x).$$

It obeys the heat equation

$$A_x K(x, y, t) = -\frac{\partial}{\partial t} K(x, y, t).$$

Here A_x denotes the application of the operator A to the function $K(x, y, t)$ when y, t are considered constants. Since $K(x, y, 0) = \delta(x - y)$ because of orthonormality of f_k, we obtain that $\theta(t)K(x, y, t)$ is the Green function of the heat equation. In our case we have

$$K(x, y, t) = \theta(t) \frac{1}{16\pi^2 t^2} e^{-m^2 t - (x-y)^2/4t} \tag{18.30}$$

(compare with (18.16)).

Now we use that

$$\zeta_A(s) = \sum_k \lambda_k^{-s} = \sum_{i=1}^{n} \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-\lambda_k t} dt =$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} (\sum_k e^{-\lambda_k t}) dt = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} Tr[e^{-At}] dt.$$

Since

$$Tr(e^{-At}) = \int_B K(x,x,t) dx,$$

we have

$$Tr(e^{-At}) = \frac{\Omega}{16\pi^2 t^2} e^{-m^2 t}.$$

This gives for Re $s > 4$,

$$\zeta_A(s) = \frac{\Omega}{16\pi^2 \Gamma(s)} \int_0^\infty t^{s-3} e^{-m^2 t} dt = \frac{\Omega (m^2)^{2-s} \Gamma(s-2)}{16\pi^2 \Gamma(s)}.$$

After we analytically continue ζ to a meromorphic function on the whole complex plane, we will be able to differentiate at 0 to obtain

$$\ln \det(-\Delta + m^2) = -\zeta'(0) = \Omega \frac{1}{32\pi^2} m^4 (-\frac{3}{2} + \ln m^2).$$

We see that when the box B increases its size, the determinant goes to infinity.

So, even for the Laplacian operator we don't expect to give meaning to the functional integral (18. 27) in the infinite-dimensional case. However, we located the source of the infinity in the functional integral. If we set

$$N = \int e^{i\phi A \phi} D[\phi] = \det(A)^{-1/2}$$

then

$$N^{-1} \int e^{i\phi A\phi + J\phi} D[\phi] = \exp[-\frac{1}{2} i J(x) G(x,y) J(y)]$$

has a perfectly well-defined meaning. Here $G(x,y)$ is the inverse operator to A represented by its Green function. In the case of the operator $A = \Box + m^2$ it agrees with the propagator formula:

$$\exp(-\frac{i}{2} \int J(x)J(y) \langle 0|T[\Psi(\mathbf{x})\Psi(\mathbf{x})]|0\rangle d^3x d^3y = Z[J] = N \int e^{i\phi A\phi + J\phi} D[\phi] \qquad (18.32)$$

Exercises.

1. Consider the quadratic Lagrangian

$$L(x,\dot{x},t) = a(t(x^2 + b(t)\dot{x}^2 + c(t)x \cdot x + d(t)x + e(t) \cdot x + f(t).$$

Show that the Feynman propagator is given by the formula

$$K(a,b) = A(t_a,t_b)\exp[\frac{i}{\hbar}\int_{t_a}^{t_b} L(x_{cl},\dot{x}_{cl};t)dt],$$

where A is some function, and x_{cl} denotes the classical path minimizing the action.

2. Let $K(a,b)$ be propagator given by formula (18.14). Verify that it satisfies property (18.3).

3. Give a full proof of Wick's theorem.

4. Find the analog of the formula (18.24) for the generating function $Z[J]$ in ordinary quantum mechanics.

5. Show that the zeta function of the Hamiltonian operator for the harmonic oscillator is equal to the Riemann zeta function.

Lecture 19. FEYNMAN DIAGRAMS

19.1 So far we have succeeded in quantizing fields which were defined by Lagrangians for which the Euler-Lagrange equation was linear (like the Klein-Gordon equation, for example). In the non-linear case we no longer can construct the Fock space and hence cannot give a particle interpretation of quantized fields. One of the ideas to solve this difficulty is to try to introduce the quantized fields $\Psi(t,\mathbf{x})$ as obtained from free "input" fields $\Psi_{\text{in}}(t,\mathbf{x})$ by a unitary automorphism $U(t)$ of the Hilbert space

$$\Psi(t,\mathbf{x}) = U^{-1}(t)\Psi_{\text{in}}U(t). \tag{19.1}$$

We assume that Ψ_{in} satisfies the usual (quantum) Schrödinger equation

$$\frac{\partial}{\partial t}\Psi_{\text{in}} = i[H_0, \Psi_{\text{in}}], \tag{19.2}$$

where H_0 is the "linear part" of the Hamiltonian. We want Ψ to satisfy

$$\frac{\partial}{\partial t}\Psi = i[H, \Psi],$$

where H is the "full" Hamiltonian. We have

$$\frac{\partial}{\partial t}\Psi_{\text{in}} = \frac{\partial}{\partial t}[U(t)\Psi U(t)^{-1}] = \dot{U}(t)\Psi U^{-1} + U(t)(\frac{\partial}{\partial t}\Psi)U^{-1}(t) + U(t)\Psi\dot{U}^{-1}(t) =$$

$$= \dot{U}(U^{-1}\Psi_{\text{in}}U)U^{-1} + U[iH(\Psi,\Pi),\Psi]U^{-1} + U(U^{-1}\Psi_{\text{in}}U)\dot{U}^{-1} =$$

$$= \dot{U}U^{-1}\Psi_{\text{in}} + iU[H(\Psi,\Pi),\Psi]U^{-1} + \Psi_{\text{in}}U\dot{U}^{-1}. \tag{19.3}$$

At this point we assume that the Hamiltonian satisfies

$$U(t)H(\Psi,\Pi)U(t)^{-1} = H(U(t)\Psi U(t)^{-1}, U(t)\Pi U(t)^{-1}) = H(\Psi_{\text{in}}, \Pi_{\text{in}}).$$

Using the identity

$$0 = \frac{d}{dt}[U(t)U^{-1}(t)] = [\frac{d}{dt}U(t)]U(t)^{-1} + U(t)\frac{d}{dt}U^{-1}(t),$$

allows us to rewrite (19.3) in the form

$$[\dot{U}U^{-1} + iH(\Psi_{\text{in}}, \pi_{\text{in}}), \Psi_{\text{in}}] = i[H_0, \Psi_{\text{in}}].$$

This implies that

$$[\dot{U}U^{-1} + i(H(\Psi_{\text{in}}, \Pi_{\text{in}}) - H_0), \Psi_{\text{in}}] = 0.$$

Similarly, we get

$$[\dot{U}U^{-1} + i(H(\Psi_{\text{in}}, \Pi_{\text{in}}) - H_0), \Pi_{\text{in}}] = 0.$$

It can be shown that the Fock space is an irreducible representation of the algebra formed by the operators Ψ_{in} and Π_{in}. This implies that

$$\dot{U}U^{-1} + i(H(\Psi_{\text{in}}, \Pi_{\text{in}}) - H_0) = f(t)\mathbf{id}$$

for some function of time t. Note that the Hamiltonian does not depend on $\mathbf{x}$ but depends on t since Ψ_{in} and Π_{in} do. Let us denote the difference $H(\Psi_{\text{in}}, \Pi_{\text{in}}) - H_0 - f(t)\mathbf{id}$ by $H_{\text{int}}(t)$. Then multiplying the previous equation by $U(t)$ on the right, we get

$$i\frac{\partial}{\partial t}U(t) = H_{\text{int}}(t)U(t). \tag{19.4}$$

We shall see later that the correction term $f(t)\mathbf{id}$ can be ignored for all purposes.

Recall that we have two pictures in quantum mechanics. In the Heisenberg picture, the states do not vary in time, and their evolution is defined by $\phi(t) = e^{iHt}\phi(0)$. In the Schrödinger picture, the states depend on time, but operators do not. Their evolution is described by $A(t) = e^{iHt}Ae^{-iHt}$. Here H is the Hamiltonian operator. The equations of motion are

$$[A(t), H] = i\dot{A}(t), \quad \dot{\phi} = 0 \quad \text{(Heisenberg picture)},$$

$$H\phi = i\frac{\partial \phi}{\partial t}, \quad \dot{A} = 0 \quad \text{(Schrödinger picture)}.$$

We can mix them as follows. Let us write the Hamiltonian as a sum of two Hermitian operators

$$H = H_0 + H_{\text{int}}, \tag{19.5}$$

and introduce the equations of motion

$$[A(t), H_0] = \dot{A}(t), \quad H_{\text{int}}(t)\phi = i\frac{\partial \phi}{\partial t}.$$

One can show that this third picture (called the *interaction picture*) does not depend on the decomposition (19.5) and is equivalent to the Heisenberg and Schrödinger pictures. If we denote by Φ_S, Φ_H, Φ_I (resp. A_S, A_H, A_I) the states (resp. operators) in the corresponding pictures, then the transformations between the three different pictures are given as follows

$$\Phi_S = e^{-iHt}\Phi_H, \quad \Phi_I(t) = e^{iH_0t}\Phi_S(t),$$

$$A_S = e^{-iHt}A_H(t)e^{iHt}, \quad A_I = e^{iH_0t}A_S e^{-iH_0t}.$$

Let us consider the family of operators $U(t,t'), t, t' \in \mathbb{R}$, satisfying the following properties

$$U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3),$$

$$U^{-1}(t_1,t_2) = U(t_2,t_1),$$

$$U(t,t) = \mathbf{id}.$$

We assume that our operator $U(t)$ can be expressed as

$$U(t) = U(t,-\infty) := \lim_{t' \to -\infty} U(t,t').$$

Then, for any fixed t_0, we have

$$U(t) = U(t,-\infty) = U(t,t_0)U(t_0,-\infty) = U(t,t_0)U(t_0).$$

Thus equation (19.4) implies

$$(H_{\text{int}}(t) - i\frac{\partial}{\partial t})U(t,t_0) = 0 \tag{19.6}$$

with the initial condition $U(t_0,t_0) = \mathbf{id}$. Then for any state $\phi(t)$ (in the interaction picture) we have

$$U(t,t_0)\phi(t_0) = \phi(t)$$

(provided the uniqueness of solutions of the Schrödinger equation holds). Taking the Hermitian conjugate we get

$$U(t,t_0)^* H_{\text{int}}(t) + i\frac{\partial}{\partial t}U(t,t_0)^* = 0,$$

and hence, multiplying the both sides by $U(t,t_0)$ on the right,

$$U(t,t_0)^* H_{\text{int}}(t)U(t,t_0) + i\frac{\partial}{\partial t}U(t,t_0)^*U(t,t_0) = 0.$$

Multiplying both sides of (19.6) by $U(t,t_0)^*$ on the left, we get

$$U(t,t_0)^* H_{\text{int}}(t)U(t,t_0) + iU(t,t_0)^*\frac{\partial}{\partial t}U(t,t_0) = 0.$$

Comparing the two equations, we obtain

$$\frac{\partial}{\partial t}(U(t,t_0)^*U(t,t_0)) = 0.$$

Since $U(t,t_0)^*U(t,t_0) = \mathbf{id}$, this tells us that the operator $U(t,t_0)$ is unitary. We define the *scattering matrix* by

$$S = \lim_{t_0\to-\infty, t\to\infty} U(t,t_0). \tag{19.7}$$

Of course, this double limit may not exist. One tries to define the decomposition (19.5) in such a way that this limit exists.

Example 1. Let H be the Hamiltonian of the harmonic oscillator (with $\omega/2$ subtracted). Set

$$H_0 = \omega_0 a^*a, H_{\text{int}} = (\omega-\omega_0)a^*a.$$

Then

$$U(t,t_0) = e^{-i(\omega-\omega_0)(t-t_0)a^*a}.$$

It is clear that the S-matrix S is not defined.

In fact, the only partition which works is the trivial one where $H_{\text{int}}(t) = 0$.

We will be solving for $U(t,t_0)$ by iteration. Replace $H_{\text{int}}(t)$ with $\lambda H_{\text{int}}(t)$ and look for a solution in the form

$$U(t,t_0) = \sum_{n=0}^{\infty}\lambda^n U_n(t,t_0).$$

Replacing $H_{\text{int}}(t)$ with $\lambda H_{\text{int}}(t)$, we obtain

$$i\frac{\partial}{\partial t}U(t,t_0) = \sum_{n=0}^{\infty}\lambda^n\frac{\partial}{\partial t}U_n(t,t_0) = \lambda H_{\text{int}}(t)\sum_{n=0}^{\infty}\lambda^n\frac{\partial}{\partial t}U_n(t,t_0) = \sum_{n=0}^{\infty}\lambda^{n+1}H_{\text{int}}\frac{\partial}{\partial t}U_n(t,t_0).$$

Equating the coefficients at λ^n, we get

$$\frac{\partial}{\partial t}U_0(t,t_0) = 0,$$

$$i\frac{\partial}{\partial t}U_n(t,t_0) = H_{\text{int}}(t)U_{n-1}(t,t_0), \quad n \geq 1.$$

Together with the initial condition $U_0(t_0,t_0) = 1, U_n(t,t_0) = 0, n \geq 1$, we get

$$U_0(t,t_0) = 1, \quad U_1(t,t_0) = -i\int_{t_0}^{t} H_{\text{int}}(\tau)d\tau,$$

$$U_2(t,t_0) = (-i)^2\int_{t_0}^{t} H_{\text{int}}(\tau_2)\Big(\int_{\tau_1}^{\tau_2} H_{\text{int}}(\tau_1)d\tau_1\Big)d\tau_2,$$

$$U_n(t,t_0) = (-i)^n \int_{t_0}^{t} H_{\rm int}(\tau_n)\Big(\int_{t_0}^{\tau_n} \ldots \Big(\int_{t_0}^{\tau_3} H_{\rm int}(\tau_2)\Big(\int_{t_0}^{\tau_2} H_{\rm int}(\tau_1)d\tau_1\Big)d\tau_2\Big)\ldots\Big)d\tau_n =$$

$$= (-i)^n \int_{t_0\le\tau_1\le\ldots\le\tau_n\le t} T[H_{\rm int}(\tau_n)\ldots H_{\rm int}(\tau_1)]d\tau_1\ldots d\tau_n =$$

$$= \frac{(-i)^n}{n!}\int_{t_0}^{t}\ldots\int_{t_0}^{t} T[H_{\rm int}(\tau_n)\ldots H_{\rm int}(\tau_1)]d\tau_1\ldots d\tau_n.$$

After setting $\lambda = 1$, we get

$$U(t,t_0) = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{t_0}^{t}\ldots\int_{t_0}^{t} T[H_{\rm int}(t_n)\ldots H_{\rm int}(t_1)]dt_1\ldots dt_n. \qquad (19.8)$$

Taking the limits $t_0 \to -\infty, t \to +\infty$, we obtain the expression for the scattering matrix

$$S = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} T[H_{\rm int}(t_n)\ldots H_{\rm int}(t_1)]dt_1\ldots dt_n. \qquad (19.9)$$

We write the latter expression formally as

$$S = T\exp\Big(-i\int_{-\infty}^{\infty} : H_{\rm int}(t) : dt\Big).$$

Let ϕ be a state, we interpret $S\phi$ as the result of the scattering process. It is convenient to introduce two copies of the same Hilbert space $\mathcal{H}$ of states. We denote the first copy by $\mathcal{H}_{\rm IN}$ and the second copy by $\mathcal{H}_{\rm OUT}$. The operator S is a unitary map

$$S : \mathcal{H}_{\rm IN} \to \mathcal{H}_{\rm OUT}.$$

The elements of the each space will be denoted by $\phi_{\rm IN}, \phi_{\rm OUT}$, respectively. The inner product $P = \langle\alpha_{\rm OUT}, \beta_{\rm IN}\rangle$ is the transition amplitude for the state $\beta_{\rm IN}$ transforming to the state $\alpha_{\rm OUT}$. The number $|P|^2$ is the probability that such a process occurs (recall that the states are always normalized to have norm 1). If we take $(\alpha_i), i \in I$, to be an orthonormal basis of $\mathcal{H}_{\rm IN}$, then the "matrix element" of S

$$S_{ij} = \langle S\alpha_i|\alpha_j\rangle$$

can be viewed as the probability amplitude for α_i to be transformed to α_j (viewed as an element of $\mathcal{H}_{\rm OUT}$).

19.2 Let us consider the so-called Ψ^4-*interaction process*. It is defined by the Hamiltonian

$$H = H_0 + H_{\rm int},$$

where

$$H_0 = \frac{1}{2}\int_{\mathbb{R}^3} :\Pi^2 + (\partial_1\Psi)^2 + (\partial_2\Psi)^2 + (\partial_3\Psi)^2) + m^2\Psi^2 : d^3x,$$

$$H_{\text{int}} = \frac{f}{4!}\int :\Psi^4: d^3x.$$

Here Ψ denotes Ψ_{in}; it corresponds to the Hamiltonian H_0. Recall from Lecture 17 that

$$\Psi = \frac{1}{(2\pi)^{3/2}}\int (2E_k)^{-1/2}(e^{ik\cdot x}a(\mathbf{k}) + e^{-ik\cdot x}a(\mathbf{k})^*)d^3k,$$

where $k = (E_k, \mathbf{k}), |E_k| = m^2 + |\mathbf{k}|^2$.

We know that

$$H_0 = \int E_{\mathbf{k}}a(\mathbf{k})^*a(\mathbf{k})d^3k.$$

Take the initial state $\phi = |\mathbf{k}_1\mathbf{k}_2\rangle \in \mathcal{H}_{\text{IN}}$ and the final state $\phi' = |\mathbf{k}_1'\mathbf{k}_2'\rangle \in \mathcal{H}_{\text{OUT}}$. We have

$$S = 1 + (-if)/4!\int :\Psi(x)^4: d^4x + (-if)^2/2(4!)^2\int T[:\Psi(x)^4::\Psi(y)^4:]d^4xd^4y + \ldots.$$

The approximation of first order for the amplitude $\langle\mathbf{k}_1\mathbf{k}_2|S|\mathbf{k}_1'\mathbf{k}_2'\rangle$ is

$$\langle\mathbf{k}_1\mathbf{k}_2|S|\mathbf{k}_1'\mathbf{k}_2'\rangle_1 = -i\int\langle\mathbf{k}_1\mathbf{k}_2| :\Psi^4: |\mathbf{k}_1'\mathbf{k}_2'\rangle d^4k,$$

To compute $:\Psi^4:$ we have to select only terms proportional to $a(\mathbf{k}_1)^*a(\mathbf{k}_2)^*a(\mathbf{k}_1')a(\mathbf{k}_2')$. There are 4! such terms, each comes with the coefficient

$$e^{i(k_1+k_2-k_1'-k_2')\cdot x}/(E_{\mathbf{k}_1}E_{\mathbf{k}_2}E_{\mathbf{k}_1'}E_{\mathbf{k}_2'})^{\frac{1}{2}}(2\pi)^6,$$

where $k = (E_{\mathbf{k}}, \mathbf{k})$ is the 4-momentum vector. After integration we get

$$\langle\mathbf{k}_1\mathbf{k}_2|S|\mathbf{k}_1'\mathbf{k}_2'\rangle = \mathcal{M}\delta(k_1 + k_2 - k_1' - k_2'),$$

where

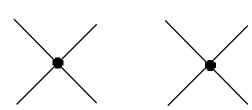
$$\mathcal{M} = -if/4(2\pi)^2(E_{\mathbf{k}_1}E_{\mathbf{k}_2}E_{\mathbf{k}_1'}E_{\mathbf{k}_2'})^{\frac{1}{2}}. \tag{19.10}$$

So, up to first order in f, the probability that the particles $\langle\mathbf{k}_1\mathbf{k}_2|$ with 4-momenta $\mathbf{k}_1, \mathbf{k}_2$ will produce the particles $\langle\mathbf{k}_1'\mathbf{k}_2'|$ with momenta $\mathbf{k}_1', \mathbf{k}_2'$ is equal to zero, unless the total momenta are conserved:

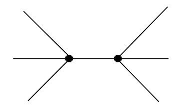
$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_1' + \mathbf{k}_2'.$$

since the energy $E_{\mathbf{k}}$ is determined by $\mathbf{k}$, we obtain that the total energy does not change under the collision.

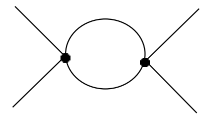
$$T[:\Psi^4(x)::\Psi^4(y):] =: \Psi^4(x)\Psi^4(y): \tag{19.11a}$$



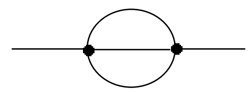
$$+16\,\overbrace{\Psi(x)\Psi(y)} : \Psi^3(x)\Psi^3(y) : \qquad (19.11b)$$



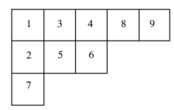
$$+72(\overbrace{\Psi(x)\Psi(y)})^2 : \Psi^2(x)\Psi^2(y) : \qquad (19.11c)$$



$$+96(\overbrace{\Psi(x)\Psi(y)})^3 : \Psi(x)\Psi(y) : \qquad (19.11d)$$



$$24(\overbrace{\Psi(x)\Psi(y)})^4 : \qquad (19.11e)$$



where the overbrace denotes the Wick contraction of two operators

$$\overbrace{AB} = T[AB] - : AB :$$

Here the diagrams are written according to the following rules:
(i) the number of vertices is equal to the order of approximation;
(ii) the number of internal edges is equal to the number of contractions in a term;
(iii) the number of external edges is equal to the number of uncontracted operators in a term.

We know from (18.22) that

$$\overbrace{\Psi(x)\Psi(y)} = \langle 0|T[\Psi(x)\Psi(y)]|0\rangle = -i\Delta(x-y), \qquad (19.12)$$

where $\Delta(x-y)$ is given by (18.16) or, in terms of its Fourier transform, by (18.17). This allows us to express the Wick contraction as

$$\overbrace{\Psi(x)\Psi(y)} = \frac{i}{(2\pi)^4}\int \frac{e^{-ik\cdot(x-y)}}{|k|^2 - m^2 + i\epsilon} d^4k. \qquad (19.13)$$

It is clear that only the diagrams (19.11c) compute the interaction of two particles. We have

$$\langle \mathbf{k}_1\mathbf{k}_2|S|\mathbf{k}_1'\mathbf{k}_2'\rangle_2 =$$

$$= \frac{f^2}{2!4!4!}\int \frac{d^4q_1d^4q_2}{(2\pi)^8}\int d^4xd^4y\frac{e^{-i(q_1+q_2)\cdot(x-y)}\langle \mathbf{k}_3\mathbf{k}_4| : \Psi(x)\Psi(x) :: \Psi(y)\Psi(y) : |\mathbf{k}_1\mathbf{k}_2\rangle}{(q_1^2 - m^2 + i\epsilon)(q_2^2 - m^2 + i\epsilon)}.$$

The factor $\langle \mathbf{k}_1\mathbf{k}_2| : \Psi(x)\Psi(x) :: \Psi(y)\Psi(y) : |\mathbf{k}_1'\mathbf{k}_2'\rangle$ contributes terms of the form

$$e^{i(k_1-k_1')x-(k_2-k_2')\cdot y}/4(E_{\mathbf{k}_1}E_{\mathbf{k}_2}E_{\mathbf{k}_1'}E_{\mathbf{k}_2'})^{\frac{1}{2}}(2\pi)^6.$$

Each such term contributes

$$\frac{f^2}{2!4!4!}\int \frac{d^4q_1d^4q_2}{(2\pi)^8 4(E_{\mathbf{k}_1}E_{\mathbf{k}_2}E_{\mathbf{k}_1'}E_{\mathbf{k}_2'})^{\frac{1}{2}}(2\pi)^6}\frac{1}{(q_1^2-m^2+i\epsilon)}\frac{1}{(q_2^2-m^2+i\epsilon)}\times$$

$$\int e^{i(k_1-k_1'-q_1-q_2)}d^4x\int e^{i(k_2-k_2'-q_1-q_2)}d^4y.$$

The last two integrals can be expressed via delta functions:

$$(2\pi)^8\delta(k_1-k_1'-q_1-q_2)\delta(k_2-k_2'+q_1+q_2).$$

This gives us again that

$$k_1+k_2=k_1'+k_2'$$

is satisfied if the collision takes place. Also we have $q_2=k_2-k_2'-q_1$, and, changing the variable q_1 by k, we get the term

$$\frac{f^2}{2!4!4!4(E_{\mathbf{k}_1}E_{\mathbf{k}_2}E_{\mathbf{k}_1'}E_{\mathbf{k}_2'})^{\frac{1}{2}}(2\pi)^6}\delta(k_1+k_2-k_1'-k_2')\int\frac{d^4k}{(k^2-m^2+i\epsilon)((k-q)^2-m^2+i\epsilon)},$$

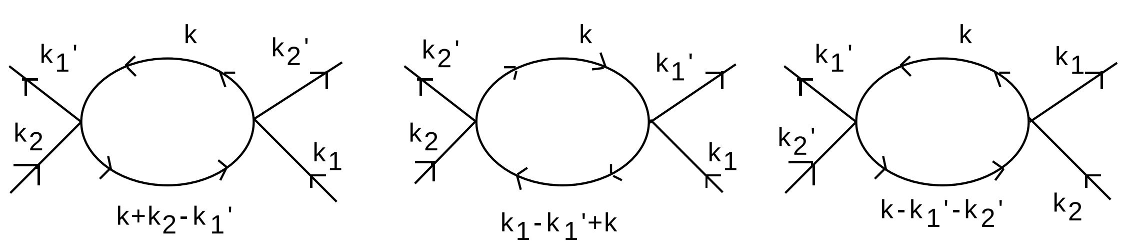
where $q=k_2-k_2'$. Other terms look similar but $q=k_2-k_1'$, $q=k_1'+k_2'$. The total contribution is given by

$$f^2[I(k_1'-k_1)+I(k_1'-k_2)+I(k_1'+k_2')]\delta(k_1+k_2-k_1'-k_2')/(E_{\mathbf{k}_1}E_{\mathbf{k}_2}E_{\mathbf{k}_1'}E_{\mathbf{k}_2'})^{\frac{1}{2}}(2\pi)^6,$$

where

$$I(q)=\frac{1}{2(2\pi)^4}\int\frac{d^4k}{(k^2-m^2+i\epsilon)((k-q)^2-m^2+i\epsilon)}. \qquad (19.14)$$

Each term can be expressed by one of the following diagram:



Unfortunately, the last integral is divergent. So, in order for this formula to make sense, we have to apply the renormalizationprocess. For example, one subtracts from this expression the integral $I(q_0)$ for some fixed value q_0 of q. The difference of the integrals will be convergent. The integrals (19.14) are examples of *Feynman integrals.* They are integrals of some rational differential forms over algebraic varieties which vary with a parameter. This is studied in algebraic geometry and singularity theories. Many topological and algebraic-geometrical methods were introduced into physics in order to study these integrals. We view integrals (19.14) as special cases of integrals depending on a parameter q:

$$I(q) = \int\limits_{\text{real} \quad k} \frac{d^4k}{\prod_i S_i(k,q)}, \tag{19.15}$$

where S_i are some irreducible complex polynomials in k, q. The set of zeroes of $S_i(k,q)$ is a subvariety of the complex space $\mathbb{C}^4$ which varies with the parameter q. The cycle of integration is $\mathbb{R}^4 \subset \mathbb{C}^4$. Both the ambient space and the cycle are noncompact. First of all we compactify everything. We do this as follows. Let us embed $\mathbb{R}^4$ in $\mathbb{C}^6 \setminus \{0\}$ by sending $\mathbf{x} \in \mathbb{R}^4$ to $(x_0, \dots, x_5) = (1, \mathbf{x}, |\mathbf{x}|^2) \in \mathbb{C}^6$. Then consider the complex projective space $\mathbb{P}^5(\mathbb{C}) = \mathbb{C}^6 \setminus \{0\}/\mathbb{C}^*$ and identify the image Γ' of $\mathbb{R}^4$ with a subset of the quadric

$$Q := z_5 z_0 - \sum_{i=1}^{4} z_i^2 = 0.$$

Clearly Γ' is the set of real points of the open Zariski subset of Q defined by $z_0 \neq 0$. Its closure Γ in Q is the set $Q(\mathbb{R}) = \Gamma' \cup \{(0,0,0,0,0,1)\}$. Now we can view the denominator in the integrand of (19.14) as the product of two linear polynomials $G = (x_5 - m^2 + i\epsilon)(x_5 - 2\sum_{i=1}^4 x_i q_i + q_5 - m^2 + i\epsilon)$. It is obtained from the homogeneous polynomial

$$F(z;q) = (z_5 - (m^2 - i\epsilon)z_0)(z_5 - 2\sum_{i=1}^{4} z_i q_i + q_5 z_0 - (m^2 - i\epsilon)z_0)$$

by dehomogenization with respect to the variable z_0. Thus integral (19.14) has the form

$$I(q) = \int\limits_{\Gamma} \omega_q, \tag{19.16}$$

where ω_q is a rational differential 4-form on the quadric Q with poles of the first order along the union of two hyperplanes $F(z;q) = 0$. In the affine coordinate system $x_i = z_i/z_0$, this form is equal to

$$\omega_q = \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{(x_5 - m^2 + i\epsilon)(x_5 - 2\sum_{i=1}^4 x_i q_i + q_5 - m^2 + i\epsilon)}.$$

The 4-cycle Γ does not intersect the set of poles of ω_q so the integration makes sense. However, we recall that the integral is not defined if we do not make a renormalization. The reason is simple. In the affine coordinate system

$$y_i = z_i/z_5 = x_i/(x_1^2 + x_2^2 + x_3^2 + x_4^2) = x_i(y_1^2 + y_2^2 + y_3^2 + y_4^2) = x_i|y|^2, \quad i = 1, \dots 4,$$

the form ω_q is equal to

$$\omega_q = \frac{d(y_1/|y|^2) \wedge d(y_2/|y|^2 \wedge d(y_3/|y|^2) \wedge d(y_4/|y|^2)}{(1-(m^2+i\epsilon)(\sum_{i=1}^4 y_i^2)(1-2\sum_{i=1}^4 y_i q_i + (q_5 - m^2 + i\epsilon)(\sum_{i=1}^4 y_i^2))}.$$

It is not defined at the boundary point $(0,\dots,0,1) \in \Gamma$. This is the source of the trouble we have to overcome by using a renormalization. We refer to [**Pham**] for the theory of integrals of the form (19.16) which is based on the Picard-Lefschetz theory.

19.3 Let us consider the n-point function for the interacting field Ψ

$$G(x_1,\dots,x_n) = \langle 0|T[\Psi(x_1)\cdots\Psi(x_n)]|0\rangle.$$

Replacing $\Psi(x)$ with $U(t)^{-1}\Psi_{\text{in}}U(t)$ we can rewrite it as

$$G(x_1,\dots,x_n) = \langle 0|T[U(t_1)^{-1}\Psi_{in}(x_1)U(t_1)\cdots U(t_n)^{-1}\Psi)in(x_n)U(t_n)]|0\rangle =$$

$$= \langle 0|T[U(t)^{-1}U(t,t_1)\Psi_{\text{in}}(x_1)U(t_1,t_2)\cdots U(t_{n-1},t_n)\Psi_{\text{in}}(x_n)U(t_n,-t)U(-t)]|0\rangle,$$

where we inserted $1 = U(t)^{-1}U(t)$ and $1 = U(-t)^{-1}U(-t)$. When $t > t_1$ and $-t < t_n$, we can pull $U(t)^{-1}$ and $U(-t)$ out of time ordering and write

$$G(x_1,\dots,x_n) =$$

$$\langle 0|U(t)^{-1}T[U(t,t_1)\Psi_{\text{in}}(x_1)U(t_1,t_2)\cdots U(t_{n-1},t_n)\Psi_{\text{in}}(x_n)U(t_n,-t)]U(-t)]|0\rangle,$$

when t is sufficiently large.

Take $t\to\infty$. Then $U(-\infty)$ is the identity operator, hence $\lim_{t\to\infty} U(-t)|0\rangle = |0\rangle$. Similarly $U(\infty) = S$ and

$$\lim_{t\to\infty} U(t)|0\rangle = \alpha|0\rangle$$

for some complex number with $|\alpha| = 1$. Taking the inner product with $|0\rangle$, we get that

$$\alpha = \langle 0|S|0\rangle = \langle 0|T\exp(i\int_{-\infty}^{\infty} H_{\text{int}}(\tau))d\tau|0\rangle.$$

Using that $U(t,t_1)\cdots U(t_n,-t) = U(t,-t)$, we get

$$G(x_1,\dots,x_n) = \alpha^{-1}\langle 0|T[\Psi_{\text{in}}(x_1)\cdots\Psi_{\text{in}}(x_n)\exp(-i\int_{-\infty}^{\infty} H_{\text{int}}(\tau)d\tau)]|0\rangle =$$

$$= \frac{\langle 0|T[\Psi_{\text{in}}(x_1)\cdots\Psi_{\text{in}}(x_n)\exp(-i\int_{-\infty}^{\infty} H_{\text{int}}(\tau))d\tau)]|0\rangle}{\langle 0|T\exp(i\int_{-\infty}^{\infty} H_{\text{int}}(\tau)d\tau)|0\rangle} =$$

$$= \frac{\sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \int\limits_{-\infty}^{\infty} \ldots \int\limits_{-\infty}^{\infty} \langle 0|T[\Psi_{\text{in}}(x_1)\cdots\Psi_{\text{in}}(x_n)H_{\text{int}}(t_1)\cdots H_{\text{int}}(t_m)]|0\rangle dt_1\ldots dt_m}{\sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \int\limits_{-\infty}^{\infty} \ldots \int\limits_{-\infty}^{\infty} \langle 0|T[H_{\text{int}}(t_1)\cdots H_{\text{int}}(t_m)]|0\rangle dt_1\ldots dt_m}. \tag{19.17}$$

Using this formula we see that adding to H_{int} the operator $f(t)\mathbf{id}$ will not change the n-point function $G(x_1,\ldots,x_n)$.

19.4 To compute the correlation function $G(x_1,\ldots,x_n)$ we use Feynman diagrams again. We do it in the example of the Ψ^4-interaction. First of all, the denominator in (19.17) is equal to 1 since $:\Psi^4:$ is normally ordered and hence kills $|0\rangle$. The m-th term in the numerator series is equal to $G(x_1,\ldots,x_n)$ is

$$G(x_1,\ldots,x_n)_m = \frac{(-i)^m}{m!}\langle 0|\int\ldots\int T[\prod_{i=1}^{n}\Psi_{\text{in}}(x_i)\prod_{i=1}^{m} :\Psi_{\text{in}}(y_i)^4:]d^4y_1\ldots d^4y_m]|0\rangle. \tag{19.18}$$

Using Wick's theorem, we write $\langle 0|T[\Psi_{\text{in}}(x_1)\ldots\Psi_{\text{in}}(x_n) : \Psi_{\text{in}}(y_1)^4 : \ldots \Psi_{\text{in}}(y_m)]|0\rangle$ as the sum of products of all possible Wick contractions

$$\overbrace{\Psi_{\text{in}}(x_i)\Psi_{\text{in}}(x_j)},\quad \overbrace{\Psi_{\text{in}}(x_i)\Psi_{\text{in}}(y_j)},\quad \overbrace{\Psi_{\text{in}}(y_i)\Psi_{\text{in}}(y_j)}.$$

Each x_i occurs only once and y_j occurs exactly 4 times. To each such product we associate a Feynman diagram. On this diagram $x_1,\ldots,x_n$ are the endpoints of the external edges (tails) and $y_1,\ldots,y_m$ are interior vertices. Each vertex is of valency 4, i.e. four edges originate from it. There are no loops (see Exercise 6). Each edge joining z_i with z_j ($z = x_i$ or y_j) represents the Green function $-i\Delta(z_i,z_j)$ equal to the Wick contraction of $\Psi_{\text{in}}(z_i)$ and $\Psi_{\text{in}}(z_j)$.

Note that there is a certain number $A(\Gamma)$ of possible contractions which lead to the same diagram Γ. For example, we may attach to an interior vertex 4 endpoints in 4! different way. It is clear that $A(\Gamma)$ is equal to the number of symmetries of the graph when the interior verices are fixed.

We shall summarize the *Feynman rules* for computing (19.19). First let us introduce more notation.

Let Γ be a graph as described in above (a *Feynman diagram*). Let E be the set of its endpoints, V the set of its interior vertices, and I the set of its internal edges. Let I_v denote the set of internal edges adjacent to $v \in V$, and let E_v be the set of tails coming into v. We set

$$i\Delta_F(p) = \frac{i}{|p|^2 - m^2 + i\epsilon}.$$

1. Draw all distinct diagrams Γ with $n = 2k$ endpoints $x_1,\ldots,x_n$ and m interior vertices $y_1,\ldots,y_m$ and sum all the contributions to (19.19) according to the next rules.

2. Put an orientation on all edges of Γ, all tails must be incoming at the interior vertex (the result will not depend on the orientation.) To each tail with endpoint $e \in E$ assign the 4-vector variable p_e. To each interior edge $\ell \in I$ assign the 4-variable variable k_ℓ.

3. To each vertex $v \in V$, assign the weight

$$A(v) = (-i)(2\pi)^4\delta^4(\sum_{\ell\in I_v, e\in E_v} \pm p_\ell \pm k_e).$$

The sign + is taken when the edge is incoming at vertex, and − otherwise.
4. To each tail with endpoint $e \in E$, assign the factor

$$B(e) = i\Delta_F(p_e).$$

5. To each interior edge $\ell \in I$, assign the factor

$$C(\ell) = \frac{i}{(2\pi)^4}\Delta_F(k_\ell)d^4k_\ell.$$

6. Compute the expression

$$\mathcal{I}(\Gamma) = \frac{1}{A(\Gamma)}\prod_{e\in E} B(e)\int \prod_{v\in V'} A(v)\prod_{\ell\in I} C(\ell). \tag{19.20}$$

Here V' means that we are taking the product over the maximal set of vertices which give different contributions $A(v)$.
7. If Γ contains a connected component which consists of two endpoints x_i, x_j joined by an edge, we have to add to $\mathcal{I}(\Gamma)$ the factor $i\Delta_F(p_i)(2\pi)^4\delta^4(p_i+p_j)$. Let $\mathcal{I}(\Gamma)'$ be obtained from $\mathcal{I}(\Gamma)$ by adding all such factors.

Then the Fourier transform of $G(x_1,\ldots,x_n)_m$ is equal to

$$\hat{G}(p_1,\ldots,p_n)_m = \sum_\Gamma (2\pi)^4\delta^4(p_1+\ldots+p_n)\mathcal{I}(\Gamma)'. \tag{19.20}$$

The factor in from of the sum expresses momentum conservation of incoming momenta. Here we are summing up with respect to the set of all topologicall distinct Feynman diagrams with the same number of endpoints and interior vertices.

As we have observed already the integral $\mathcal{I}(\Gamma)$ usually diverges. Let L be the number of cycles in γ. The $|I|$ internal momenta must satisfy $|V|-1$ relations. This is because of the conservation of momenta coming into a vertex; we subtract 1 because of the overall momentum conservation. Let us assume that instead of 4-variable momenta we have d-variable momenta (i.e., we are integrating over $\mathbb{R}^d$. The number

$$D = d(|I|-|V|+1) - 2|I|$$

is the degree of divergence of our integral. This is equal to the dimension of the space we are integrating over plus the degree of the rational function which we are integrating. Obviously the integral diverges if $D > 0$. We need one more relation among $|V|, |E|$ and $|I|$. Let N be the valency of an interior vertex ($N = 4$ in the case of Ψ^4 interaction). Since each interior edge is adjacent to two interior vertices, we have $N|V| = |E| + 2|I|$. This gives

$$D = d(\frac{N|V| - |E|}{2} - |V| + 1) - N|V| + |E| = d - \frac{1}{2}(d-2)|E| + (\frac{N-2}{2}d - N).$$

In four dimension, this reduces to

$$D = 4 - |E| + (N-4)|V|.$$

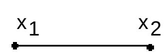
In particular, when $N = 4$, we obtain

$$D = 4 - |E|.$$

Thus we have only two cases with $D \geq 0$: $|E| = 4$ or 2. However, this analysis does not prove the integral converges when $D < 0$. It depends on the Feynman diagram. To give a meaning for $\mathcal{I}(G)$ in the general case one should apply the renormalization process. This is beyond of our modest goals for these lectures.

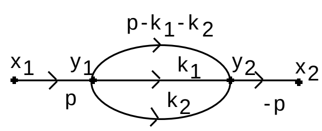
Examples 1. $n = 2$. The only possible diagrams with low-order contribution $m \leq 2$ and their corresponding contributions $\mathcal{I}(\Gamma)'$ for computing $\hat{G}(p, -p)$ are

$m = 0$:



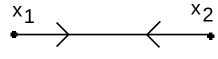
$$I(\Gamma)' = \frac{i}{|p_1|^2 - m^2 + i\epsilon}\delta(p_1 + p_2).$$

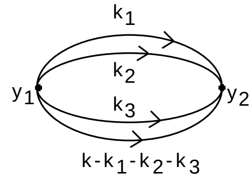
$m = 2$:



$$I(\Gamma)' = \frac{(-i\lambda)^2}{3!}\Big(\frac{i}{|p|^2 - m^2 + i\epsilon}\Big)^2 \int \frac{d^4k_1 d^4k_2}{(2\pi)^8}$$
$$\times \frac{i^3}{(|k_1|^2 - m^2 + i\epsilon)(|k_2|^2 - m^2 + i\epsilon)(|p - k_1 - k_2|^2 - m^2 + i\epsilon)}.$$

$m = 2$:



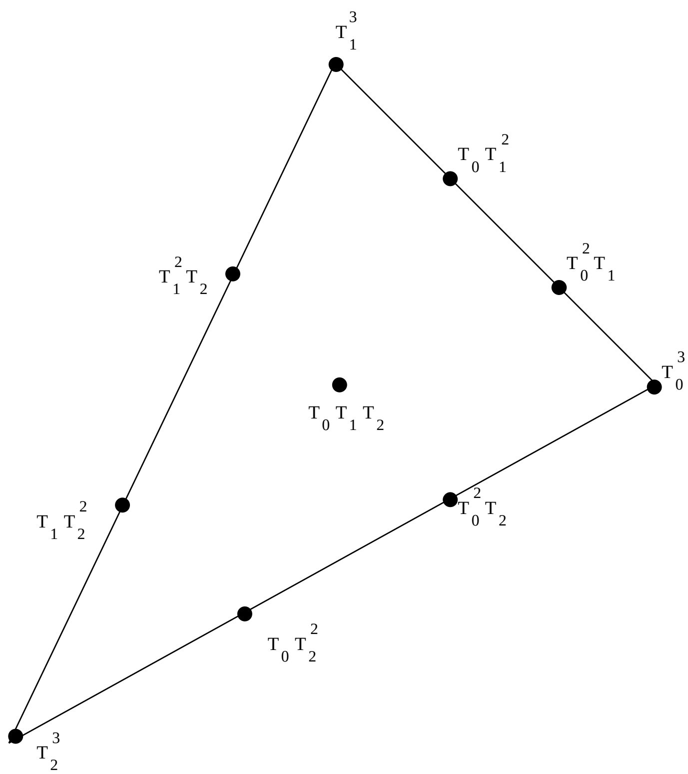


$$\mathcal{I}(\Gamma)' = \frac{(-i\lambda)^2}{4!}\frac{i}{|p|^2 - m^2 + i\epsilon}\int \frac{d^4k_1 d^4k_2 d^4k_3}{(2\pi)^{12}}$$

$$\times \frac{i^4}{(|k_1|^2 - m^2 + i\epsilon)(|k_2|^2 - m^2 + i\epsilon)(|k_3|^2 - m^2 + i\epsilon)(|k_1 + k_2 + k_3|^2 - m^2 + i\epsilon)}.$$

2. $n = 4$. The only possible diagrams with low-order contribution $m \leq 2$ and their corresponding contributions $\mathcal{I}(\Gamma)'$ for computing $\hat{G}(p_1, p_2, p_3, p_4)$ are

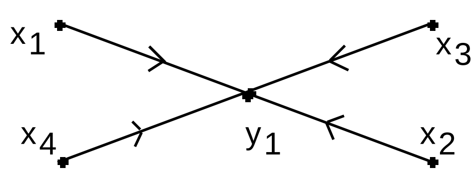
$m = 0:$



$$\mathcal{I}(\Gamma)' = i^2(2\pi)^2\frac{\delta(p_1 + p_2)\delta(p_3 + p_4)}{(|p_1|^2 - m^2 + i\epsilon)(|p_3|^2 - m^2 + i\epsilon)}.$$

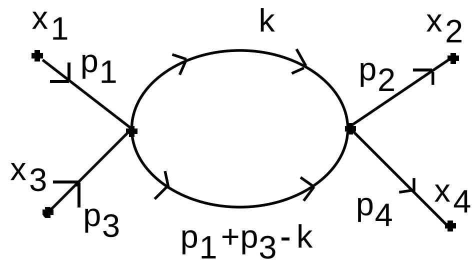
and similarly for other two diagrams.

$m = 1:$



$$\mathcal{I}(\Gamma)' = \frac{(2\pi)^4\delta^4(p_1 + p_2 + p_3 + p_4)i^4(-i)}{(|p_1|^2 - m^2 + i\epsilon)(|p_2|^2 - m^2 + i\epsilon)(|p_3|^2 - m^2 + i\epsilon)(|p_4|^2 - m^2 + i\epsilon)}.$$

$m = 2$: Here we write only one diagram of this kind and leave to the reader to draw the other diagrams and the corresponding contributions:



$$\mathcal{I}(\Gamma)' = \frac{1}{2}\prod_{i=1}^{4}\frac{i(-i)^2}{(|p_i|^2 - m^2 + i\epsilon)}\int\frac{d^4k(i)^4\delta(p_1+p_2-p_3-p_4)}{(2\pi)^4(|k|^2 - m^2 + i\epsilon)(|p_1+p_2-k|^2 - m^2 + i\epsilon)}$$

and similarly for other two diagrams.

Exercises.

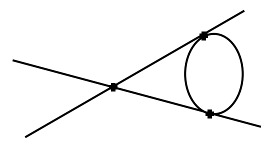
1. Write all possible Feynman diagrams describing the S-matrix up to order 3 for the free scalar field with the interaction given by $H_{\text{int}} = g/3!\Psi^3$. Compute the S-matrix up to order 3.

2. Describe the Feynman rules for the theory $\mathcal{L}(x) = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m_0^2\phi^2) - g\phi^3/3! - f\phi^4/4!$.

3. Describe Feynman diagrams for computing the 4-point correlation function for ψ^4 interaction of the free scalar of order 3.

4. Find the second order matrix elements of scattering matrix S for the Ψ^4-interaction.

5. Find the Feynman integrals which are contributed by the following Feynman diagram:



6. Explain why the normal ordering in $:\Psi_{\text{in}}^4(y_i):$ leads to the absence of loops in Feynman diagrams.

Lecture 20. QUANTIZATION OF YANG-MILLS FIELDS

20.1 We shall begin with quantization of the electro-magnetic field. Recall from Lecture 14 that it is defined by the 4-vector potential (A_μ) up to gauge transformation $A'_\mu = A_\mu + \partial_\mu \phi$. The Lagrangian is the Yang-Mills Lagrangian

$$\mathcal{L}(x) = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}(|\mathbf{E}|^2 - |\mathbf{H}|^2), \tag{20.1}$$

where $(F_{\mu\nu}) = (\partial_\mu A_\nu - \partial_\nu A_\mu)_{0\le\mu,\nu\le 3}$, and $\mathbf{E}, \mathbf{H}$ are the 3-vector functions of the electric and magnetic fields. The generalized conjugate momentum is defined as usual by

$$\pi_0(x) = \frac{\delta\mathcal{L}}{\delta(\delta_t A_0)} = 0, \qquad \pi_i(x) = \frac{\delta\mathcal{L}}{\delta(\delta_t A_i)} = -E_i, \quad i > 0. \tag{20.2}$$

We have

$$\frac{\partial A_\mu}{\partial t} = \frac{\partial A_0}{\partial x_\mu} + E_\mu.$$

Hence the Hamiltonian is

$$H = \int (\dot{A}_\mu \Pi_\mu - \mathcal{L})d^3x = \int \frac{1}{2}(|\mathbf{E}|^2 + |\mathbf{H}|^2) + \mathbf{E}\cdot\nabla A_0)d^3x. \tag{20.3}$$

Now we want to quantize it to convert A_μ and π_μ into operator-valued functions $\Psi_\mu(t,\mathbf{x})$, $\Pi_\mu(t,x)$ with commutation relations

$$[\Psi_\mu(t,\mathbf{x}), \Pi_\nu(t,\mathbf{x})] = i\delta_{\mu\nu}\delta(\mathbf{x}-\mathbf{y}),$$

$$[\Psi_\mu(t,\mathbf{x}), \Psi_\nu(t,\mathbf{x})] = [\Pi_\mu(t,\mathbf{x}), \Pi_\nu(t,\mathbf{x})] = 0. \tag{20.4}$$

As $\Pi_0 = 0$, we are in trouble. So, the solution is to eliminate the A_0 and $\Pi_0 = 0$ from consideration. Then A_0 should commute with all Ψ_μ, Π_μ, hence must be a scalar operator.

We assume that it does not depend on t. By Maxwell's equation $\text{div}\mathbf{E} = \nabla \cdot \mathbf{E} = \rho_e$. Integrating by parts we get

$$\int \mathbf{E} \cdot \nabla A_0 d^3x = \int A_0 \text{div}\mathbf{E} d^3x = \int A_0 \rho_e d^3x.$$

Thus we can rewrite the Hamiltonian in the form

$$H = \frac{1}{2}\int (|\mathbf{E}|^2 + |\mathbf{H}|^2)d^3x + \frac{1}{2}\int A_0\rho_e d^3x. \tag{20.5}$$

Here $\mathbf{H} = \text{curl } \mathbf{\Psi}$.

We are still in trouble. If we take the equation $\text{div}\mathbf{E} = -\text{div}\Pi = -\rho_e\mathbf{id}$ as the operator equation, then the commutator relation (20.4) gives, for any fixed $\mathbf{y}$,

$$[A_j(t,\mathbf{y}), \nabla\mathbf{\Pi}(t,\mathbf{x})] = -[A_j(t,\mathbf{x}), \rho_e\mathbf{id}] = i\sum_i \partial_i\delta(\mathbf{x}-\mathbf{y}) = 0.$$

But the last sum is definitely non-zero. A solution for this is to replace the delta-function with a function $f(\mathbf{x}-\mathbf{y})$ such that $\sum_i \partial_i f(\mathbf{x}-\mathbf{y}) = 0$. Since we want to imitate the canonical quantization procedure as close as possible, we shall choose $f(\mathbf{x}-\mathbf{y})$ to be the *transverse* δ*-function*,

$$\delta_{ij}^{\text{tr}}(\mathbf{x}-\mathbf{y}) = \frac{1}{(2\pi)^3}\int e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}(\delta_{ij} - \frac{k_ik_j}{|\mathbf{k}|^2})d^3k = i(\delta_{ij}\delta(\mathbf{x}-\mathbf{y}) - \partial_i\partial_j)G_\triangle(\mathbf{x}-\mathbf{y}). \tag{20.6}$$

Here $G_\triangle(\mathbf{x}-\mathbf{y})$ is the Green function of the Laplace operator $\triangle$ whose Fourier transform equals $\frac{1}{|\mathbf{k}|^2}$. We immediately verify that

$$\sum_i \partial_i\delta_{ij}^{\text{tr}}(\mathbf{x}-\mathbf{y}) = \partial_j\delta(\mathbf{x}-\mathbf{y}) - \triangle\partial_j G_\triangle(\mathbf{x}-\mathbf{y}) = \partial_j\delta(\mathbf{x}-\mathbf{y}) - \partial_j \triangle G_\triangle(\mathbf{x}-\mathbf{y}) =$$

$$= \partial_j\delta(\mathbf{x}-\mathbf{y}) - \partial_j\delta(\mathbf{x}-\mathbf{y}) = 0.$$

So we modify the commutation relation, replacing the delta-function with the transverse delta-function. However the equation

$$[\Psi_i(t,\mathbf{x}), \Pi_j(t,\mathbf{y})] = i\delta_{ij}^{\text{tr}}(\mathbf{x}-\mathbf{y}) \tag{20.7}$$

also implies

$$[\nabla\cdot\mathbf{\Psi}(\mathbf{t},\mathbf{x}), \Pi_j(t,\mathbf{y})] = 0.$$

This will be satisfied if initially $\nabla\cdot\Psi(\mathbf{x}) = 0$. This can be achieved by changing (A_μ) by a gauge transformation. The corresponding choice of A_μ is called the *Coulomb gauge*. In the Coulumb gauge the equation of motion is

$$\partial_\mu\partial^\mu\mathbf{A} = 0$$

(see (14.20)). Thus each coordinate $A_\mu, \mu = 1,2,3$, satisfies the Klein-Gordon equation. So we can decompose it into a Fourier integral as we did with the scalar field:

$$\mathbf{\Psi}(t,x) = \frac{1}{(2\pi)^{3/2}} \int \frac{1}{(2E_k)^{1/2}} \sum_{\lambda=1}^{2} \vec{\epsilon}_{\mathbf{k},\lambda}(a(\mathbf{k},\lambda)e^{-i\mathbf{k}\cdot\mathbf{x}} + a(\mathbf{k},\lambda)^* e^{i\mathbf{k}\cdot\mathbf{x}}).$$

Here the energy E_k satisfies $E_k = |\mathbf{k}|$ (this corresponds to massless particles (*photons*)), and $\vec{\epsilon}_{\mathbf{k},\lambda}$ are 3-vectors. They are called the *polarization vectors.* Since div$\mathbf{\Psi} = 0$, they must satisfy

$$\vec{\epsilon}_{\mathbf{k},\lambda} \cdot \mathbf{k} = 0, \quad \vec{\epsilon}_{\mathbf{k},\lambda} \cdot \vec{\epsilon}_{\mathbf{k},\lambda'} = \delta_{\lambda\lambda'}. \tag{20.8}$$

If we additionally impose the condition

$$\vec{\epsilon}_{\mathbf{k},1} \times \vec{\epsilon}_{\mathbf{k},2} = \frac{\mathbf{k}}{|\mathbf{k}|}, \qquad \vec{\epsilon}_{\mathbf{k},1} = -\vec{\epsilon}_{-\mathbf{k},1}, \quad \vec{\epsilon}_{\mathbf{k},2} = \vec{\epsilon}_{-\mathbf{k},2}, \tag{20.9}$$

then the commutator relations for Ψ_μ, Π_μ will follow from the relations

$$[a(\mathbf{k},\lambda), a(\mathbf{k}',\lambda')^*] = \delta(\mathbf{k}-\mathbf{k}')\delta_{\lambda\lambda'},$$

$$[a(\mathbf{k},\lambda), a(\mathbf{k}',\lambda')] = [a(\mathbf{k},\lambda)^*, a(\mathbf{k}',\lambda')^*] = 0. \tag{20.10}$$

So it is very similar to the Dirac field but our particles are massless. This also justifies the choice of $\delta_{ij}^{\mathrm{tr}}(\mathbf{x}-\mathbf{y})$ to replace $\delta(\mathbf{x}-\mathbf{y})$.

Again one can construct a Fock space with the vacuum state $|0\rangle$ and interpret the operators $a(\mathbf{k},\lambda)$, $a(\mathbf{k},\lambda)^*$ as the annihilation and creation operators. Thus

$$|0\rangle = a(\mathbf{k},\lambda)^*|0\rangle$$

is the state describing a photon with linear polarization $\vec{\epsilon}_{\mathbf{k},\lambda}$, energy $E_{\mathbf{k}} = |\mathbf{k}|^2$, and momentum vector $\mathbf{k}$.

The Hamiltonian can be put in the normal form:

$$H = \frac{1}{2}\int :|\mathbf{E}|^2 + |\mathbf{H}|^2 : d^3x = \sum_{\lambda=1}^{2} \int a(\mathbf{k},\lambda)^* a(\mathbf{k},\lambda) d^3k.$$

Similarly we get the expression for the momentum operator

$$\mathbf{P} = \frac{1}{2}\int : \mathbf{E} \times \mathbf{H} : d^3x = \frac{1}{2}\int \mathbf{k} a(\mathbf{k},\lambda)^* a(\mathbf{k},\lambda) d^3k.$$

We have the *photon propagator*

$$iD_{\mu\nu}^{\mathrm{tr}}(x',x) = \langle 0|T[\Psi_\mu(x')\Psi_\nu(x)]|0\rangle. \tag{20.11}$$

Following the computation of the similar expression in the scalar case, we obtain

$$D_{\mu\nu}^{\mathrm{tr}}(x',x) = \frac{1}{(2\pi)^4}\int d^4k \frac{e^{-ik\cdot(x'-x)}}{k^2+\epsilon i} \sum_{\lambda=1}^{2} \vec{\epsilon}_{\mathbf{k},\lambda}^{\nu} \cdot \vec{\epsilon}_{\mathbf{k},\lambda}^{\mu}. \tag{20.12}$$

To compute the last sum, we use the following simple fact from linear algebra.

Lemma. *Let $v_i = (v_{1i}, \dots, v_{ni})$ be a set of orthonormal vectors in n-dimensional pseudo-Euclidean space. Then, for any fixed a, b,*

$$\sum_{i=1}^{n} g_{ii} v_{ai} v_{bi} = g_{ab}.$$

Proof. Let X be the matrix with i-th column equal to v_i. Then X^{-1} is the (pseudo)-orthogonal matrix, i.e., $X^{-1}(g_{ij})(X^{-1})^t = (g_{ij})$. Thus

$$X(g_{ij})X^t = X(X^{-1}(g_{ij})(X^{-1})^t)X^t = (g_{ij}).$$

However, the ab-entry of the matrix in the left-hand side is equal to the sum $\sum_{i=1}^{n} g_{ii} v_{ai} v_{bi}$.

We apply this lemma to the following vectors $\eta = (1,0,0,0), \vec{\epsilon}_{\mathbf{k},1}, \vec{\epsilon}_{\mathbf{k},2}, \hat{k} = (0, \mathbf{k}/|\mathbf{k}|)$ in the Lorentzian space $\mathbb{R}^4$. We get, for any $\mu, \nu \geq 1$,

$$\sum_{\lambda=1}^{2} \vec{\epsilon}^{\,\nu}_{\mathbf{k},\lambda} \cdot \vec{\epsilon}^{\,\mu}_{\mathbf{k},\lambda} = -g_{\mu\nu} + \eta_\mu \eta_\nu - \hat{k}_\mu \hat{k}_\nu = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{|\mathbf{k}|^2}.$$

Plugging this expression in (20.12), we obtain

$$D^{\text{tr}}_{\mu\nu}(x', x) = \frac{1}{(2\pi)^4} \int \frac{e^{-ik\cdot(x'-x)}}{k^2 + \epsilon i} (\delta_{\mu\nu} - \frac{k_\mu k_\nu}{|\mathbf{k}|^2}) d^4 k. \tag{20.13}$$

This gives us the following formula for the Fourier transform of $D^{\text{tr}}_{\mu\nu}(x', x)$:

$$D^{\text{tr}}_{\mu\nu}(x', x) = F\Big(\frac{1}{k^2 + \epsilon i} (\delta_{\mu\nu} - \frac{k_\mu k_\nu}{|\mathbf{k}|^2})\Big)(x' - x). \tag{20.14}$$

We can extend the definition of $D^{\text{tr}}_{\mu\nu}$ by adding the components $D^{\text{tr}}_{\mu\nu}$, where μ or ν equals zero. This corresponds to including the component Ψ_0 of $\mathbf{\Psi}$ which we decided to drop in the begining. Then the formula $\text{div}\mathbf{E} = \triangle A_0 = \rho_e$ gives

$$\Psi_0(t, \mathbf{x}) = \int \frac{\rho(t, \mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d^3 y.$$

From this we easily get

$$D^{\text{tr}}_{00}(x, x') = \int e^{-ik\cdot(x'-x)} \frac{1}{|\mathbf{k}|^2} d^4 k = \frac{\delta(t - t')}{4\pi|\mathbf{x} - \mathbf{x}'|}. \tag{20.15}$$

Other components $D^{\text{tr}}_{0\mu}(x, x')$ are equal to zero.

20.2 So far we have only considered the non-interaction picture. As an example of an interaction Hamiltonian let us consider the following Lagrangian

$$\mathcal{L} = \frac{1}{2}(|\mathbf{E}|^2 - |\mathbf{H}|^2) + \psi^*(iD - m)\psi, \tag{20.16}$$

where $D = \gamma^\mu(i\partial_\mu + A_\mu)$ is the Dirac operator for the trivial line bundle equipped with the connection (A_μ). Recall also that ψ^* denote the fermionic conjugate defined in (16.11). We can write the Lagrangian as the sum

$$\mathcal{L} = \frac{1}{2}(|\mathbf{E}|^2 - |\mathbf{H}|^2) + (i\psi^*(\gamma^\mu\partial_\mu - m)\psi) + (i\psi^*\gamma^\mu A_\mu\psi).$$

Here we change the notation Ψ to A. The first two terms represent the Lagrangians of the zero charge electromagnetic field (photons) and of the Dirac field (electron-positron). The third part is the interaction part.

The corresponding Hamiltonian is

$$H = \frac{1}{2}\int : |\mathbf{E}|^2 + |\mathbf{H}|^2 : d^3x - \int \psi^*(\gamma^\mu(i\partial_\mu + A_\mu) - m)\psi d^3x, \tag{20.17}$$

Its interaction part is

$$H_{\text{int}} = \int (i\psi^*\gamma^\mu A_\mu\psi)d^3x. \tag{20.18}$$

The interaction correlation function has the form

$$G(x_1, \ldots, x_n; x_{n+1}, \ldots, x_{2n}; y_1, \ldots, y_m) =$$

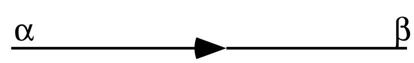
$$= \langle 0|T[\psi(x_1)\ldots\psi(x_n)\psi^*(x_{n+1})\ldots\psi^*(x_{2n})A_{\mu_1}(y_1)\ldots A_{\mu_m}(y_m)]|0\rangle. \tag{20.19}$$

Because of charge conservation, the correlation functions have an equal number of ψ and ψ^* fields. For the sake of brevity we omitted the spinor indices. It is computed by the formula

$$G(x_1, \ldots, x_n; x_{n+1}, \ldots, x_{2n}; y_1, \ldots, y_m) =$$

$$\frac{\langle 0|T[\psi_{\text{in}}(x_1)\ldots\psi^*_{\text{in}}(x_{n+1})\ldots\psi^*_{\text{in}}(x_{2n})A^{\text{in}}_{\mu_1}(y_1)\ldots A^{\text{in}}_{\mu_m}(y_m)\exp(iH_{\text{in}})|0\rangle}{\langle 0|T[\exp(iH_{\text{in}})]|0\rangle}. \tag{20.20}$$

Again we can apply Wick's formula to write a formula for each term in terms of certain integrals over Feynman diagrams. Here the 2-point Green functions are of different kind. The first kind will correspond to solid edges in Feynman diagrams:



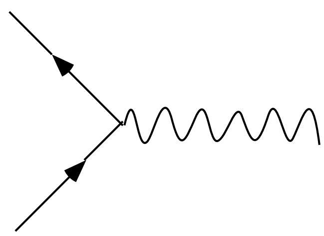
$$\langle 0|T[\psi_\alpha(x)\psi^*_\beta(x')|0\rangle = \overbrace{\psi_\alpha(x)\psi^*_\beta(x')} = \frac{i}{(2\pi)^4}\int e^{-ik\cdot(x'-x)}(k_\mu\gamma^\mu + i\epsilon)^{-1}_{\alpha\beta}d^4k.$$

We have not discussed the Green function of the Dirac equation but the reader can deduce it by analogy with the Klein-Gordon case. Here the indices mean that we take the $\alpha\beta$

entry of the inverse of the matrix $k_\mu\gamma^\mu - m + i\epsilon I_4$. The propagators of the second field correspond to wavy edges

$$\langle 0|T[A_\mu(x)A_\nu(x')|0\rangle = \overbrace{A_\mu(x)A_\nu(x')} = iD^{\mathrm{tr}}_{\mu\nu}(x',x).$$

Now the Feynman diagrams contain two kinds of tails. The solid ones correspond to incoming or outgoing fermions. The corresponding endpoint possesses one vector and two spinor indices. The wavy tails correspond to photons. The endpoint has one vector index $\mu = 1,\ldots,4$. It is connected to an interior vertex with vector index ν. The diagram is a 3-valent graph.



We leave to the reader to state the rest of Feynman rules similar to these we have done in the previous lecture.

20.3 The disadvantage of the quantization which we introduced in the previous section is that it is not Lorentz invariant. We would like to define a relativistic n-point function. We use the path integral approach. The formula for the generating function should look like

$$Z[J] = \int [d[(A_\mu)] \exp(i(\mathcal{L} + J^\mu A_\mu), \tag{20.21}$$

where $\mathcal{L}$ is a Lagrangian defined on gauge fieleds. Unfortunately, there is no measure on the affine space $\mathcal{A}(\mathfrak{g})$ of all gauge potentials (A_μ) on a principal G-bundle which gives a finite value to integral (20.21). The reason is that two connections equivalent under gauge transformation give equal contributions to the integral. Since the gauge group is non-compact, this leads to infinity. The idea is to cancel the contribution coming from integrating over the gauge group.

Let us consider a simple example which makes clear what we should do to give a meaning to the integral (20.21). Let us consider the integral

$$Z = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} e^{-(x-y)^2}dydx.$$

Obviously the integral is divergent since the function depends only on the difference $x-y$. The latter means that the group of translations $x \to x+a, y \to y+a$ leave the integrand

and the measure invariant. The orbits of this group in the plane $\mathbb{R}^2$ are lines $x - y = c$. The orbit space can be represented by a curve in the plane which intersects each orbit at one point (a gauge slice). For example we can choose the line $x + y = 0$ as such a curve. We can write each point (x, y) in the form

$$(x,y) = (\frac{x-y}{2}, \frac{y-x}{2}) + (\frac{x+y}{2}, \frac{x+y}{2}).$$

Here the first point lies on the slice, and the second corresponds to the translation by (a, a), where $a = \frac{x+y}{2}$. Let us make the change of variables $(x, y) \to (u, v)$, where $u = x - y, v = x + y$. Then $x = (u + v)/2$, $y = (u - v)/2$, and the integral becomes

$$Z = \frac{1}{2}\int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} e^{-u^2} du\Big) dv = \sqrt{\pi}\int_{-\infty}^{\infty} da. \tag{20.22}$$

The latter integral represents the integral over the group of transformations. So, if we "divide" by this integral (equal to ∞), we obtain the right number.

Let us rewrite (20.22) once more using the delta-function:

$$Z = \int_{-\infty}^{\infty} da \int_{-\infty}^{\infty} 2\delta(x+y)e^{-(x-y)^2} dxdy = \int_{-\infty}^{\infty} da \int_{-\infty}^{\infty} 2e^{-4x^2} dx = \int_{-\infty}^{\infty} da \int_{-\infty}^{\infty} e^{-z^2} dz.$$

We can interpret this by saying that the delta-function fixes the gauge; it selects one representative of each orbit. Let us explain the origin of the coefficient 2 before the delta-function. It is related to the choice of the specific slice, in our case the line $x + y = 0$. Suppose we take some other slice $S : f(x, y) = 0$. Let $(x, y) = (x(s), y(s))$ be a parametrization of the slice. We choose new coordinates (s, a), where a is is uniquely defined by $(x, y) - (a, a) = (x', y') \in S$. Then we can write

$$Z = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-h(s)}|J| dads, \tag{20.23}$$

where $x - y = x'(s) - y'(s) = h(s)$ and $J = \frac{\partial(x,y)}{\partial(s,a)}$ is the Jacobian. Since $x = x'(s) + a, y = y'(s) + a$, we get

$$J = \begin{vmatrix} \frac{\partial x}{\partial s} & 1 \\ \frac{\partial y}{\partial s} & 1 \end{vmatrix} = \frac{\partial x}{\partial s} - \frac{\partial y}{\partial s} = \frac{\partial x'}{\partial s} - \frac{\partial y'}{\partial s}.$$

Now we use that

$$0 = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial s} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial s} = (\frac{\partial f}{\partial x'}, \frac{\partial f}{\partial y'}) \times (-\frac{\partial y'}{\partial s}, \frac{\partial x'}{\partial s}).$$

Let us choose the parameter s in such a way that

$$(\frac{\partial f}{\partial x'}, \frac{\partial f}{\partial y'}) = (-\frac{\partial y'}{\partial s}, \frac{\partial x'}{\partial s}).$$

One can show that this implies that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(f(x,y))dxdy=\int_{-\infty}^{\infty}ds.$$

This allows us to rewrite (20.23)

$$Z=\int_{-\infty}^{\infty}da\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(f(x,y))\Big|\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\Big|e^{-(x-y)^2}dxdy.$$

Now our integral does not depend on the choice of parametrization. We succeeded in separating the integral over the group of transformations. Now we can redefine Z to set

$$Z=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(f(x,y))\Big|\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\Big|e^{-(x-y)^2}dxdy.$$

Now let us interpret the Jacobian in the following way. Fix a point (x_0,y_0) on the slice, and consider the function

$$f(a)=f(x_0+a,y_0+a) \tag{20.24}$$

as a function on the group. Then

$$\frac{df}{da}\Big|_{f=0}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\Big|_{f=0}.$$

20.4 After the preliminary discussion in the previous section, the following definition of the path integral for Yang-Mills theory must be clear:

$$Z[J]=\int d[A_\mu]\Delta_{FP}\delta(F(A_\mu))\exp\Big[i\int(\mathcal{L}(A)+J^\mu A_\mu)d^4x\Big]. \tag{20.25}$$

Here $F:\mathcal{A}(\mathfrak{g})\to \mathrm{Maps}(\mathbb{R}^4\to\mathfrak{g})$ is a functional on the space of gauge potentials analogous to the function $f(x,y)$. It defines a slice in the space of connections by fixing a gauge (e.g. the Coulomb gauge for the electromagnetic field). The term Δ_{FP} is the *Faddeev-Popov* determinant. It is equal to

$$\Delta_{FP}=\Big|\frac{\delta F(g(x)A_\mu(x))}{\delta g(x)}\Big|_{g(x)=1}\Big|, \tag{20.26}$$

where $g(x):\mathbb{R}^4\to G$ is a gauge transformation.

To compute the determinant, we choose some local parameters θ^a on the Lie group G. For example, we may choose a basis η_a of the Lie algebra, and write $g(x)=\exp(\theta^a(x)\eta_a)$. Fix a gauge $F(A)$. Now we can write

$$F(g(x)A(x)))_a=F(A)_a+\int\sum_a M(x,y)_{ab}\theta^b(y)d^4y+\dots, \tag{20.27}$$

where $M(x,y)$ is some matrix function with

$$M(x,y)_{ab} = \frac{\delta F_a(x)}{\delta \theta^b(y)}.$$

To compute the determinant we want to use the functional analog of the Gaussian integral

$$(\det M)^{-1/2} = \int\int d\phi e^{-\phi(x)M(x)\phi(x)} d\phi.$$

For our purpose we have to use its anti-commutative analog. We shall have

$$\det(M) = \int DcDc^* \exp\Big(i \int c^*(x)M(x,y)c(y)d^4x\Big), \tag{20.28}$$

where $c(x)$ and $c^*(y)$ are two fermionic fields (*Faddeev-Popov ghost fields*), and integration is over Grassmann variables. Let us explain the meaning of this integral. Consider the *Grassmann variables* α satisfying

$$\alpha^2 = \{\alpha, \alpha\} = 0.$$

Note that $\frac{dx}{dx} = 1$ is equivalent to $[x, \frac{d}{dx}] = 1$, where x is considered as the operator of multiplication by x. Then the Grassmann analog is the identity

$$\{\frac{d}{d\alpha}, \alpha\} = 1.$$

For any function $f(\alpha)$ its Taylor expansion looks like $F(\alpha) = a + b\alpha$ (because $\alpha^2 = 0$). So $\frac{d^2 f}{d\alpha^2} = 0$, hence

$$\{\frac{d}{d\alpha}, \frac{d}{d\alpha}\} = 0.$$

We have $\frac{df}{d\alpha} = -b$ if a, b are anti-commuting numbers. This is because we want to satisfy the previous equality.

There is also a generalization of integral. We define

$$\int d\alpha = 0, \qquad \int \alpha d\alpha = 1.$$

This is because we want to preserve the property

$$\int_{-\infty}^{\infty} \phi(x)dx = \int_{-\infty}^{\infty} \phi(x+c)dx$$

of an ordinary integral. For $\phi(\alpha) = a + b\alpha$, we have

$$\int \phi(\alpha)d\alpha = \int (a+b\alpha)d\alpha = a\int d\alpha + b\int \alpha d\alpha =$$

$$= \int (a + (\alpha + c)b) d\alpha = (a + bc) \int d\alpha + b \int \alpha d\alpha.$$

Now suppose we have N variables α_i. We want to perform the integration

$$I(A) = \int \exp(\sum_{i,j=1}^{N} \alpha_i A_{ij} \beta_j) \prod_{i=1}^{N} d\alpha_i d\beta_i,$$

where α_i, β_i are two sets of N anti-commuting variables. Since the integral of a constant is zero, we must have

$$I(A) = \int \frac{1}{N!} \Big(\sum_{i,j=1}^{N} \alpha_i A_{ij} \beta_j \Big)^N) \prod_{i=1}^{N} d\alpha_i d\beta_i.$$

The only terms which survive are given by

$$I(A) = \int \Big(\sum_{\sigma \in S_N} \epsilon(\sigma) A_{1\sigma(1)} \dots A_{N\sigma(N)} \Big) \prod_{i=1}^{N} d\alpha_i d\beta_i = \det(A).$$

Now we take $N \to \infty$. In this case the variables α_i must be replaced by functions $\psi(x)$ satisfying

$$\{\phi(x), \phi(y)\} = 0.$$

Examples of such functions are fermionic fields like half-spinor fields. By analogy, we obtain integral (20.28).

20.5 Let us consider the simplest case when $G = U(1)$, i.e., the Maxwell theory. Let us choose the Lorentz-Landau gauge

$$F(A) = \partial^\mu A_\mu = 0.$$

We have $(gA)_\mu = A_\mu + \frac{\partial \theta}{\partial x_\mu}$ so that

$$F(gA) = \partial^\mu A_\mu + \partial^\mu \partial_\mu \theta.$$

Then the matrix M becomes the kernel of the D'Alembertian operator $\partial^\mu \partial_\mu$. Applying formula (20.28), we see that we have to add to the action the following term

$$\int \int c^*(x) \partial^\mu \partial_\mu c(y) d^4x d^4y,$$

where $c^*(x), c(x)$ are scalar Grassmann ghost fields. Since this does not depend on A_μ, it gives only a multiplicative factor in the S-matrix, which can be removed for convenience.

On the other hand, let us consider a non-abelian gauge potential $(A_\mu) : \mathbb{R}^4 \to \mathfrak{g}$. Then the gauge group acts by the formula $gA = gAg^{-1} + g^{-1}dg$. Then

$$M(x,y)_{ab} = \frac{\delta F(gA(x))}{\delta\theta^b(y)}\Big|_{g(x)=1} = \frac{\delta}{\delta\theta^b(y)}\Big[\frac{\delta F(A)_a}{\delta A^c_\mu(x)} D^\mu_{cd}\theta^d(x)\Big]_{\theta=0} =$$

$$= \frac{\delta F(A)_a}{\delta A^c_\mu(x)} D^\mu_{cb}\Delta(x-y),$$

where

$$D^\mu_{cd} = \partial^\mu\delta_{ab} + C_{abc}A^c_\mu,$$

and C_{abc} are the structure constants of the Lie algebra $\mathfrak{g}$. Choose the gauge such that

$$\partial^\mu A^a_\mu = 0.$$

Then

$$\frac{\delta F(A)}{\delta A^c_\mu} = \delta^\mu.$$

This gives, for any fixed y,

$$M(x,y)_{ab} = [\partial_\mu\partial^\mu\delta_{ab} + C_{abc}A^c_\mu\partial^\mu]\delta(x-y), \tag{20.29}$$

which we can insert as an additional term in the action:

$$\int d^4x(\alpha^*)^a(\delta_{ab}\partial_\mu\partial^\mu + C_{abc}A^c_\mu)\alpha^b.$$

Here α^*, α are ghost fermionic fields with values in the Lie algebra.