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Rationality of $\mathcal{R}_2$ and $\mathcal{R}_3$

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1. Introduction

Let $\mathcal{R}_g$ be the moduli space of genus g curves together with a non-trivial 2-torsion divisor class ϵ. In this paper we shall prove that the moduli spaces $\mathcal{R}_2$ and $\mathcal{R}_3$ are rational varieties. The rationality of $\mathcal{R}_4$ was proven by F. Catanese [3]. He also claimed the rationality of $\mathcal{R}_3$ but the proof was never published. The first published proof of rationality of $\mathcal{R}_3$ was given by P. Katsylo in [10]. Some years earlier A. Del Centina and S. Recillas [5] constructed a map of degree 3 from $\mathcal{R}_3$ to the moduli space $\mathcal{M}_4^{\mathrm{be}}$ of bi-elliptic curves of genus 4 and claimed that it could be used for proving the rationality of $\mathcal{R}_3$ based on the rationality of $\mathcal{M}_4^{\mathrm{be}}$ proven by F. Bardelli and Del Centina in [1]. In an unpublished preprint of 1990 I had shown that it is indeed possible. The present paper is based on this old preprint and also includes a proof of rationality of $\mathcal{R}_2$ which I could not find in the literature.

The relation between the moduli spaces $\mathcal{R}_3$ and $\mathcal{M}_4^{\mathrm{be}}$ is based on an old construction of P. Roth [13] and, independently, A. Coble [4]. Much later it had been rediscovered and generalized by S. Recillas [11], and nowadays is known as the trigonal construction. To each curve C of genus g together with a g_4^1 it associates a curve X of genus $g+1$ together with a g_3^1 and a non-trivial 2-torsion divisor class η. The Prym variety of the pair (X,η) is isomorphic to the Jacobian variety

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of C. When $g = 3$ and $g^1_4 = |K_C + \epsilon|$, the associated curve X turns out to be a canonical bi-elliptic curve of genus 4, the bi-elliptic involution τ switches the two g^1_3 on X, and the 2-torsion class η is coming from a 2-torsion divisor class on the elliptic quotient $X/(\tau)$. To make this paper self-contained we remind the construction following A. Coble.

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2. Rationality of $\mathcal{R}_2$

Let C be a genus 2 curve and $x_1, \dots, x_6$ be its six Weierstrass points. A non-trivial 2-torsion divisor class on C is equal to the divisor class $[x_i - x_j]$ for some $i \neq j$. The hyperelliptic series g^1_2 defines a degree 2 map $C \to \mathbb{P}^1$ and the images of the Weierstrass points are the zeroes of a binary form of degree 6. This defines a birational isomorphism between the moduli space $\mathcal{M}_2$ of genus 2 curves and the GIT-quotient $\mathbb{P}(V(6))//\mathrm{SL}(2)$, where $V(m)$ denotes the space of binary forms of degree m. A non-trivial 2-torsion divisor class is defined by choosing a degree 2 factor of the binary sextic. Thus the moduli space $\mathcal{R}_2$ is birationally isomorphic to the GIT-quotient $(\mathbb{P}(V(4)) \times \mathbb{P}(V(2)))//\mathrm{SL}(2)$ and the canonical projection $\mathcal{R}_2 \to \mathcal{M}_2$ corresponds to the multiplication map $V(4) \times V(2) \to V(6)$. At this point we may conclude by referring to Katsylo's result on rationality of fields of invariants of SL(2) in reducible representations [9]. However, we proceed by giving a more explicit proof.

Let $\mathcal{M}_2(2)$ be the moduli space of genus 2 curves together with a 2-level structure of its Jacobian (i.e. a choice of a symplectic basis in the space of 2-torsion points of the Jacobian). It is well-known that a 2-level structure is equivalent to an order of the set of the Weierstrass points and hence $\mathcal{M}_2(2)$ is birationally isomorphic to the GIT-quotient $P^6_1 = (\mathbb{P}^1)^6//\mathrm{SL}(2)$ (see, for example, [8]). The forgetful map $\mathcal{M}_2(2) \to \mathcal{M}(2)$ corresponds to the quotient map $P^6_1 \to P^6_1/S_6$, where the symmetric group S_6 acts naturally by permuting the factors. Under the natural isomorphism $\mathrm{Sp}(4, \mathbb{F}_2) \to S_6$ the stabilizer of a non-trivial 2-torsion point is conjugate to the subgroup $S_4 \times S_2$ of S_6. Thus we obtain a birational isomorphism

$$\mathcal{R}_2 \to P^6_1/(S_4 \times S_2).$$

It is well-known that the variety P_1^6 is isomorphic to the Segre cubic threefold V_3 defined in $\mathbb{P}^5$ by equations

$$F_1 = \sum_{i=0}^{5} t_i = 0, \quad F_3 = \sum_{i=0}^{5} t_i^3 = 0,$$

where the group S_6 acts by permuting the coordinates (see [8]). Let $\mathbb{C}[t_0, \ldots, t_5]$ be the projective coordinate ring. We have

$$P_1^6 \cong \mathrm{Proj}(\mathbb{C}[t_0, \ldots, t_5]/(F_1, F_3))^{S_4 \times S_2}.$$

Here $S_4 \times \{1\}$ acts by permuting the first 4 coordinates t_0, t_1, t_2, t_3 and $\{1\} \times S_2$ permutes the remaining coordinates. The ring $\mathbb{C}[t_0, \ldots, t_5]^{S_4 \times S_2}$ is freely generated by the symmetric functions

$$u_\alpha(t_0, t_1, t_2, t_3) = \sum_{i=0}^{3} t_i^\alpha, \alpha = 1, 2, 3, 4, \quad u_5 = t_4 + t_5, u_6 = t_4 t_5.$$

We have

$$F_1 = u_1 + u_5, \quad F_3 = u_3 + u_5^3 - 3u_5 u_6.$$

This allows us to eliminate u_3 and u_1 to obtain

$$(P_1^6)/(S_4 \times S_2) \cong \mathrm{Proj}(\mathbb{C}[u_2, u_4, u_5, u_6]) \cong \mathbb{P}(2, 4, 1, 2).$$

This proves the rationality of $\mathcal{R}_2$.

Remark 2.1. According to G. Salmon [14], p.203, the algebra of SL(2)-invariant polynomials on $V(2) \times V(4)$ is generated by 6 bi-homogeneous polynomials of bi-degrees $(0,3), (0,4), (3,0), (2,2), (1,2)$ and $(3,3)$. The square of the last invariant is a polynomial in the remaining invariants.

Let us give another proof of rationality of $\mathcal{R}_2$. Let $(C, \epsilon) \in \mathcal{R}_2$. Consider the map $C \to |2K_C + \epsilon|^* \cong \mathbb{P}^2$ given by the linear system $|2K_C + \epsilon|$. Its image Y is a plane singular quartic. It is easy to see that $|K_C + \epsilon|$ consists of a unique divisor, the sum of two distinct Weierstrass points $x_i + x_j$. The divisors $3x_i + x_j$ and $x_i + 3x_j$ belong to $2K_C + \epsilon$. The corresponding lines in $\mathbb{P}^2$ intersect at the singular point of Y whose pre-image in C consists of the points x_i, x_j. The tangent lines at the branches of the singular point intersect Y with multiplicity 4. This allows one to find an equation of Y in the form

$$t_0^2 t_1 t_2 + t_0 t_1 t_2 F_1(t_1, t_2) + F_4(t_1, t_2) = 0,$$

where F_1 and F_4 are homogeneous polynomials of degree 1 and 4, respectively. Replacing t_0 by an appropriate linear form $t_0 + at_1 + bt_2$, we may assume that $F_1 = 0$. Finally, by scaling the coordinates, we obtain that $\mathcal{R}_2$ is birationally isomorphic to the quotient of $V(4) \cong \mathbb{C}^5$ by a 2-dimensional torus. It is obviously a rational variety.

3. The Coble-Roth map

Let $\mathcal{K}_3$ denote the moduli space of pairs $(C, (a, -a))$, where C is a curve of genus 3 and a is a divisor class of degree 0 on C. The projection to C fibres $\mathcal{K}_3$ over $\mathcal{M}_3$ with fibres isomorphic to the Kummer varieties of curves of genus 3. The Coble-Roth map is a rational map

$$cr : \mathcal{K}_3 \to \mathcal{R}_4$$

defined as follows. Assume that $a \neq 0$ and C is not hyperelliptic. Consider the natural map

$$\phi : |K_C + a| \times |K_C - a| \to |2K_C|, \ (D_1, D_2) \mapsto D_1 + D_2. \tag{3.1}$$

We can choose an isomorphism $|K_C \pm a| \cong \mathbb{P}^1$ and an isomorphism $|2K_C| \cong |\mathcal{O}_{\mathbb{P}^2}(2)|$, where $\mathbb{P}^2 = |K_C|^*$. Let V_3 be the determinant cubic parametrizing reducible conics in the space of conics $|\mathcal{O}_{\mathbb{P}^2}(2)|$. Using projective coordinates (u_0, u_1) and (v_0, v_1) on each copy of $\mathbb{P}^1$, we see that the map is given by a linear system of divisors of bi-degree $(1, 1)$. Thus the pre-image X of the cubic $\mathcal{D}_3$ is a divisor of bi-degree $(3, 3)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. For C general enough it is a smooth canonical curve of genus 4. It is also isomorphic to a section of V_3 by the 3-dimensional linear space, the linear span of the image of the map ϕ. As is well-known, the cubic V_3 admits a double cover ramified along its singular locus (parametrizing the irreducible components of singular conics). The restriction of the cover to the image of ϕ defines a non-ramified double cover of the curve X, hence a 2-torsion divisor class η on X.

A remarkable fact is that the Coble-Roth map is birational. This is proved as follows. Starting from a canonical curve $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ of genus 4 and a non-trivial 2-torsion divisor class η on X we identify the $|K_X + \eta|^*$ with $\mathbb{P}^2$. The image of X under this linear system $|K_X + \eta|$ is the Wirtinger sextic model of X (see [3]). For any point $a = ((\alpha_0, \alpha_1), (\beta_0, \beta_1)) \in \mathbb{P}^1 \times \mathbb{P}^1$ one defines the polar $P_a(X)$ of X

with respect to a by the formula:

$$P_a(X) = \sum_{i,j=0}^{1} \alpha_i \beta_j \frac{\partial^2 F}{\partial t_i \partial \tau_j} = 0,$$

where $F = F(t_0, t_1; \tau_0, \tau_1)$ is a bi-homogeneous equation of X. The set of polars of X generates a 3-dimensional linear system in $|\mathcal{O}_X(2)|$, where X is considered to be embedded in $\mathbb{P}^3 = |K_X|^*$. Now we can view any divisor $D \in |\mathcal{O}_X(2)|$ as a conic in the space $\mathbb{P}^2 = |\mathcal{O}_X(K_X + \varepsilon)|^*$. This defines a map:

$$P : \mathbb{P}^1 \times \mathbb{P}^1 \to |\mathcal{O}_{\mathbb{P}^2}(2)|, \quad a \to P_a(F).$$

It is given by a divisor W of type $(1,1,2)$ on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$. The projection of W to $\mathbb{P}^2$ is a conic bundle with the discriminant curve C of degree 4. The degree 2 cover of C parametrizing irreducible components of the fibres defines a non-trivial 2-torsion point ϵ on C. This defines the inverse map. We refer for the details to a paper of S. Recillas [12].

Now let us identify $\mathcal{R}_3$ with the closed subvariety of $\mathcal{K}_3$ contained in the locus of singular points of the fibres of $\mathcal{K}_3 \to \mathcal{M}_3$. For any $(C, \epsilon) \in \mathcal{R}_3$ the corresponding pair $(X, \eta) \in \mathcal{R}_4$ is invariant with respect to the involution σ induced by switching the factors in the map $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^5$ defined by the map (3.1), where $a = \epsilon$,

$$|K_C + \epsilon| \times |K_C + \epsilon| \to |2K_C|, \quad (D_1, D_2) \mapsto D_1 + D_2.$$

The quotient $X/(\sigma)$ is an elliptic curve E and the 2-torsion class η, being σ-invariant, is the pre-image of a 2-torsion divisor class on E. Let $\mathcal{R}_4^{\mathrm{be}}$ denote the moduli space of pairs (X, η), where X is a genus 4 curve together with a bi-elliptic involution σ and η is a σ-invariant non-trivial 2-torsion divisor class on X. The Coble-Roth map defines a rational map

$$\mathcal{R}_3 \to \mathcal{R}_4^{\mathrm{be}}.$$

Let us show that it is a birational map (see also [5]). Let X be a canonical curve of genus 4 on $\mathbb{P}^1 \times \mathbb{P}^1$. Suppose X has a bi-elliptic involution σ and $E = X/(\sigma)$ is an elliptic curve. The involution σ is induced by an automorphism $\tilde{\sigma}$ of $\mathbb{P}^1 \times \mathbb{P}^1$ (since X is canonically embedded in $\mathbb{P}^3$ and all non-singular quadrics in $\mathbb{P}^3$ are projectively isomorphic). Since the two g_3^1's of X induced by the rulings of $\mathbb{P}^1 \times \mathbb{P}^1$ are switched under σ, we obtain that $\tilde{\sigma}$ switches the two families of rulings of the quadric. This easily implies that $\tilde{\sigma}$ is conjugate to the switch involution of $\mathbb{P}^1 \times \mathbb{P}^1$. Thus we may assume that $\tilde{\sigma}$ is this involution. Then the factor $X/(\sigma)$

can be identified with a cubic curve E in $\mathbb{P}^1 \times \mathbb{P}^1/(\tilde{\sigma}) \cong \mathbb{P}^2$. Suppose η is a non-trivial 2-torsion divisor class on X that comes from the elliptic curve E. Then the pair (X, η) is invariant with respect to the involution $\tilde{\sigma}$, and the associated conic bundle $W \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ is invariant with respect to the involution $\tilde{\sigma} \times 1$. It follows from the construction that the two g^1_4's on the quartic discriminant curve C coincide. Since they complement each other in the bi-canonical linear system of C, each of them is equal to $|K_C + \varepsilon|$, where $2\varepsilon = 0$. This shows that $cr^{-1}(\mathcal{R}_4^{\mathrm{be}}) \subset \mathcal{R}_3$. Thus the Coble-Roth map defines a birational isomorphism

$$\mathcal{R}_3 \cong \mathcal{R}_4^{be}.$$

4. Rationality of $\mathcal{R}_3$

It remains to prove the rationality of $\mathcal{R}_4^{\mathrm{be}}$. It is a triple cover of the moduli space $\mathcal{M}_4^{be}$ of bi-elliptic curves of genus 4. The latter has a simple description. Each generic $X \in \mathcal{M}_4^{be}$ is uniquely determined by the isomorphism class of the following data: $(E, \mathcal{L}, s)$, where E is an elliptic curve, $\mathcal{L}$ is a degree 3 invertible sheaf on E, and $s \in H^0(E, \mathcal{L}^{\otimes 2})$. The isomorphism between triples $(E, \mathcal{L}, s)$ and $(E', \mathcal{L}', s')$ is induced by the isomorphisms between E and E'. If we use $\mathcal{L}$ to embed E in $\mathbb{P}^2$, this data is equivalent to the data (E, Q), where E is a cubic and Q is a conic on $\mathbb{P}^2$ that cuts out in E the divisor of zeroes of s. Here the isomorphism is induced by a projective transformation of $\mathbb{P}^2$. In this way we obtain a birational isomorphism

$$\mathcal{M}_4^{be} \cong V = |\mathcal{O}_{\mathbb{P}^2}(3)| \times |\mathcal{O}_{\mathbb{P}^2}(2)|/\mathrm{PGL}(3),$$

where the group acts diagonally. Similarly, we have a birational isomorphism

$$\mathcal{R}_4^{be} \cong \overline{|\mathcal{O}_{\mathbb{P}^2}(3)|} \times |\mathcal{O}_{\mathbb{P}^2}(2)|/\mathrm{PGL}(3), \tag{4.1}$$

where $\overline{|\mathcal{O}_{\mathbb{P}^2}(3)|}$ is the variety of pairs $(E, \eta), E \in |\mathcal{O}_{\mathbb{P}^2}(3)|, \eta \in {}_2\mathrm{Pic}(E) \setminus \{0\}$. There is a well-known birational PGL(3)-equivariant isomorphism

$$\overline{|\mathcal{O}_{\mathbb{P}^2}(3)|} \cong |\mathcal{O}_{\mathbb{P}^2}(3)|.$$

It is defined by assigning to a plane cubic the Hessian invariant of the net of polar cubics (see [7] for details). This shows that

$$\mathcal{R}_3 \cong \mathcal{R}_4^{be} \cong \mathcal{M}_4^{be}.$$

It remains to use that the right-hand space is rational [1]. Recall that $\mathcal{M}_4^{\mathrm{be}}$ is isomorphic to the space of projective equivalence classes of pairs (F_3, Q_2), where

F_3 is a plane cubic and Q_2 is a plane conic. By fixing a conic, we see that $\mathcal{M}_4^{\rm be}$ is birationally isomorphic to the quotient $\mathbb{P}(S^3(V(2)))//\mathrm{SL}(2)$. The linear representation $S^3(V(2))$ of $\mathrm{SL}(2)$ decomposes as $V(6)+V(2)$ and we may apply Katsylo's result [9] to conclude the rationality. In fact, Bardelli and Del Centina prove the rationality directly by finding an appropriate subspace W of $V(6)+V(2)$ with stabilizer isomorphic to a subgroup $H=\mathbb{C}^*\rtimes\mathbb{Z}/2$ and computing explicitly the field of invariants of H on W.

Another possible approach to rationality of $\mathcal{M}_4^{be}$ (as indicated by the referee) consists of putting the cubic F_3 in the Hesse form $x_0^3+x_1^3+x_2^3+tx_0x_1x_2=0$. In this way $\mathcal{M}_4^{\rm be}$ becomes birationally isomorphic to the space $\mathbb{P}^1\times\mathbb{P}^5/G$, where $G\cong(\mathbb{Z}/3)^2\rtimes\mathrm{SL}(2,\mathbb{F}_3)$ is the Hesse group of order 216, the subgroup of $\mathrm{PGL}(3)$ leaving the Hesse pencil invariant. The proof of rationality of $\mathcal{M}_4^{\rm be}$ from [1] could be based on the explicit computation of invariants of the Hesse group.

Remark 4.1. It is well-known that a non-trivial 2-torsion class ϵ on a canonical curve C of genus 3 defines a family of everywhere tangent conics to C. This family is a conic in the space of conics and the quartic equation of C is the discriminant of this conic (see [4],§14 or [6], 6.2). In this way one obtains a birational isomorphism from $\mathcal{R}_3$ to the space of conics in $|2K_C|\cong\mathbb{P}^5$ modulo the group $\mathrm{PGL}(3)$ acting naturally on $|2K_C|$. Since each conic lies in a unique plane, we have a projection $\mathcal{R}_3\to G(3,6)$ to the Grassmannian $G(3,6)$ of planes in $\mathbb{P}^5$ modulo the action of the group $\mathrm{PGL}(3)$ with fibres isomorphic to the 5-dimensional space of conics in a given plane. By intersecting a plane with the discriminant cubic V_3 we obtain a birational isomorphism $G(3,6)/\mathrm{PGL}(3)$ and the space $\mathcal{R}_1$. This gives a fibration $\mathcal{R}_3\to\mathcal{R}_1$ with $\mathbb{P}^5$ as fibres. The rationality of $\mathcal{R}_3$ would follow if one can prove that this fibration is a projective bundle. I do not know how to prove it. Note that this fibration is birationally isomorphic to the fibration

$$\mathcal{R}_4^{be}\cong\overline{|\mathcal{O}_{\mathbb{P}^2}(3)|}\times|\mathcal{O}_{\mathbb{P}^2}(2)|/\mathrm{PGL}(3)\to\overline{|\mathcal{O}_{\mathbb{P}^2}(3)|}/\mathrm{PGL}(3)\cong\mathcal{R}_1$$

(see (4.1)). However, the group $\mathrm{PGL}(3)$ acts on the first factor with non-trivial stabilizer of a general point, so one cannot conclude that this fibration is a $\mathbb{P}^5$ bundle.

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