PLÜCKER FORMULAS

Let $f : X \to \mathbb{P}^n$ be a map of a nonsingular projective curve X. It is given by an invertible sheaf $\mathcal{L}$ of some degree d and a basis $(s_0, \dots, s_n)$ of a linear subspace V of $H^0(X, \mathcal{L})$. In coordinate-free way, we have a linear subspace $V \subset H^0(X, \mathcal{L})$ of dimension $n + 1$ and the map $X \to \mathbb{P}(V^*)$ defined by $f(x)(s) = \mathbb{P}(\{s : s(x) = 0\})$ for any $s \in V$.

For any point $x \in X$ and $s \in V \setminus \{0\}$ we denote by $\nu_x(s)$ the largest power of the maximal ideal $\mathfrak{m}_{X,x}^i$ such that $s_x \in \mathfrak{m}_{X,x}^i \mathcal{L}_x$. Let $V_i = \nu_x^{-1}(\mathbb{Z}_{\geq i}), i \geq 0$. Since $\dim \mathfrak{m}_{X,x}^i / \mathfrak{m}_{X,x}^{i+1} = 1$, the quotient space V_i/V_{i+1} is of dimension ≤ 1. This implies that there is a unique sequence of nonnegative numbers $0 \leq \alpha_1 \leq \dots \leq \alpha_n$ such that

$$\{0\} \subset V_{n+\alpha_n} \subset \dots \subset V_{1+\alpha_1} \subset V, \tag{1}$$

where $\mathrm{codim} V_{i+\alpha_i} = i$. The linear projective subspace $PO_{i-1}(f, x) = \mathbb{P}(V_{i+\alpha_i}^{\perp})$ of $\mathbb{P}(V^*)$ of dimension $i - 1$ is called the *osculating (i-1)-plane* of f at the point x. The sequence of projective subspaces

$$PO_0(f, x) \subset \dots \subset PO_{n-1}(f, x) \subset \mathbb{P}(V^*) \tag{2}$$

is called the *osculating flag* of f at x with *ramification sequence* $(\alpha_1, \dots, \alpha_n)$. Here the subscript indicates the dimension of the subspace.

It is clear that $PO_0(f, x) = f(x)$. Let us choose projective coordinates in such a way that

$$PO_i(f, x) = \{T_{i+1} = \dots = T_n = 0\}, \quad i = 0, \dots, n-1. \tag{3}$$

Let $t_i = T_i/T_0$ be inhomogeneous coordinates, and let t be a local coordinate of X at x, the map f is given locally by

$$(x_1, \dots, x_n) = (t^{1+\alpha_1} g_1(t), t^{2+\alpha_2} g_2(t), \dots, t^{n+\alpha_n} g_n(t)), \tag{4}$$

where $g_i(0) \neq 0$. Under the homomorphism of ring

$$k[t_1, \dots, t_n] \to \mathcal{O}_{\mathbb{P}(V^*), f(x)} \to \mathcal{O}_{X,x}$$

the images of the functions t_i belong to $\mathfrak{m}_{X,x}^{i+\alpha_i}$. In particular, if $\alpha_1 > 0$, all functions belong to $\mathfrak{m}_{X,x}^2$ and hence the map f is ramified at x. If $\alpha_1 = 0$, and $f^{-1}(f(x)) = \{x\}$, then f is a closed embedding in an affine neighborhood of x. In this case, the osculating hyperplane $PO_{n-1}(f, x)$ intersects $f(X)$ at x with multiplicity $n + \alpha_n$. A point x is called a *hyperosculating point* if $\alpha_n > 0$. Clearly, all ramification points are hyperosculating points. An *honest osculating* point is a hyperosculating point with $\alpha_1 = \dots = \alpha_{n-1} = 0$.

Example 1. *Assume $n = 2$ and let $C \subset \mathbb{P}^2$ be an embedded plane curve with affine equation $F(x, y) = 0, F(0, 0) = 0$. Let X be its normalization and $f : X \to C \hookrightarrow \mathbb{P}^2$ be the composition such that $f(x) = (0, 0)$. Choose* *the system coordinates as in* (3)*. Then the coordinate line* $x = 0$ *intersects the curve with multiplicity* $1 + \alpha_1$ *at the origin and the coordinate line* $y = 0$ *intersects the curve with multiplicity* $2 + \alpha_2$*. An honest hyperosculating point with is called an inflection points (of order* α_n*). Obviously,* C *is singular at* $(0,0)$ *if* $\alpha_1 > 1$*. If* $(0,0)$ *is an ordinary cusp, then the osculating flag is formed by the point and the cuspidal tangent. The ramification sequence is* $(1,1)$*. If* $(0,0)$ *is a node and* x *corresponds to one of its branches, then the osculating flag is formed by the point and the corresponding tangent of the branch. It could be a hyperosculating point.*

Example 2. *Let* $n = 1$*. Thus* $f : X \to \mathbb{P}^1$ *is a finite cover of some degree* d*. We have only one number* α_1 *and it is equal to* $e_x - 1$*, where* e_x *is the ramification index of* f *at* x*. A point is is a hyperosculating point and if and only if it is a ramification point of* f*.*

Let $G(i, V^*)$ be the Grassmann variety of linear i-dimensional subspaces of V^*, or, equivalently $(i-1)$-dimensional linear projective subspaces of $\mathbb{P}(V^*)$. We define the i-associated map

$$\tilde{f}^{(i)} : X \to G(i, V^*),\ x \mapsto PO_i(f, x), i = 0, \ldots, n.$$

In particular, $\tilde{f}^{(0)} = f$. The map $\tilde{f}^{(1)} : X \to G(2, V^*)$ is called the *Gauss map*. The map

$$\check{f} = \tilde{f}^{(n-1)} : X \to G(n, \mathbb{P}(V^*)) \cong G(1, \mathbb{P}(V)) \cong \check{\mathbb{P}}^n.$$

is called the *duality map*. We will see later the reason for this name.

Lemma 1. *The map* $\tilde{f}^{(i)} : X \to G(i, V^*)$ *is ramified at* x *if and only if* $\alpha_{i+1} - \alpha_i > 0$ *(*$\alpha_0 = 0$*).*

Proof. Recall from multilinear algebra that a linear k-dimensional subspace of a vector space is defined by the exterior product $\omega = v_1 \wedge \ldots \wedge v_k$ of its basis (it consists of all vectors v such that $\omega \wedge v = 0$). Thus we have an embedding

$$G(i, V^*) \hookrightarrow \mathbb{P}(\Lambda^i V^*).$$

If we choose a basis $e_0, \ldots, e_n$ in V^*, then the coordinates in $\Lambda^k V^*$ of the k-form ω are equal to $k \times k$-minors of the matrix whose ith row are the coordinates of v_i. Suppose our map is given in a neighborhood U of x by equations (4). Let $T_0 = 1$ be the subspace of the affine space V^*. We lift f to a map $U \to V^*$ using the same formulas

$$f(t) = (f_0(t), \ldots, f_n(t)) = (1, t^{1+\alpha_1} + \ldots, t^{2+\alpha_2} + \ldots, \ldots, t^{n+\alpha_n} + \ldots).$$

Let $\frac{d^{(s)}f}{dt^s}$ be the sth derivative of the vector function $f(t)$ (computed algebraically). The lift of $PO_i(f, x)$ to V^* is spanned by the first i linearly independent vectors in the sequence $f(t), \frac{df}{dt}, \ldots, \frac{d^{(s)}f}{dt^s}, \ldots$. (the first ones if the ramification sequence consists of zeroes). The coordinates of the vector $f(t) \wedge \ldots \frac{d^{(s)}f}{dt^s}$ are given by the matrix

$$\begin{pmatrix} 1+\ldots & t^{1+\alpha_1}+\ldots & \ldots & \ldots & t^{n+\alpha_n}+\ldots \\ 0 & (1+\alpha_1)t^{\alpha_1}+\ldots & \ldots & \ldots & (n+\alpha_n)t^{n+\alpha_n-1}+\ldots \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & \ldots & \ldots & \ldots & *t^{\alpha_n+i-1}+\ldots \end{pmatrix}$$

Note that when $t = 0$ we obtain that all minors are equal to zero, so it looks like the values of the ith asscoayed map is not defined at 0. However, observe that all minors are divisible by t^{α_1}, and when we cancel by this factor and take $t = 0$ we obtain the only nonzero minor corresponding to the first i columns. This is the Plücker coordinate of $e_0 \wedge e_1 \wedge \ldots \wedge e_i$. It is exactly our osculating $(i-1)$-plane corresponding to the point $t = 0$ with choice of coordinates as in (3).

It is easy to see that the $i \times i$-minor Δ_1 which contains the least power of t is formed by the first i columns, and the the next least power occurs at the minor Δ_2 formed by the columns indexed by the set $\{1, 2, \ldots, i-1, i+1\}$. We have

$$\Delta_1 = at^{\alpha_0+\alpha_1+\ldots+\alpha_i} + \ldots,$$
$$\Delta_2 = bt^{\alpha_0+\ldots+\alpha_{i+1}+1} + \ldots,$$

where $a, b \neq 0$ and the other terms are divisible by higher power of t. This shows that after we factor Δ_1 from all Plücker coordinates of $\tilde{f}^{(i)}(t)$, the map is given by $\tilde{f}^{(i)} = (1, \frac{b}{a}t^{\alpha_{i+1}-\alpha_i+1} + \ldots, \ldots)$, where all coordinates starting from the third coordinates vanish with higher order. Thus passing to affine coordinates we see that all affine coordinates belong to $\mathfrak{m}_{X.x}^{\alpha_{i+1}-\alpha_i+1}$. In particular, the map $\tilde{f}^{(i)}$ is ramified if and only if $\alpha_{i+1} - \alpha_i > 0$. □

Let X^* be the image of X under the map $\check{f} : X \to \check{\mathbb{P}}^n$. We call the curve X^* the *dual curve* of X (with respect to the map f).

Lemma 2. *(Duality) Let us identify the dual of $\check{\mathbb{P}}^n$ with $\mathbb{P}^n$. Then the dual map $\check{\check{f}} : X \to \mathbb{P}^n$ of $\check{f}$ is equal to f.*

Proof. We will envoke some limit arguments so let us assume that $k = \mathbb{C}$. There is more technical proof which works if the characteristic is equal to zero. The idea is that the tangent line to a curve X at a point x is the limit of the chords $< x, y >$, where $y \in X$ and approaches x. In other words the limit of the subspace spanned by the osculating 0-plane at x and y is equal to the osculating 1-plane at x. Similarly, the limit of the 2-plane spanned by $PO_1(f, x)$ and y when y approaches x is equal to $PO_2(f, x)$, and so on. This follows from the interpretation of osculating planes in terms of derivative vectors which we used in the proof of Lemma 2. In particular, the limit of the span of $PO_{n-2}(f, x)$ and y is equal to the osculating hyperplane $PO_{n-1}(X, x)$. Also the limit of the intersection of $PO_{n-1}(f, y)$ with $PO_i(f, x)$ is equal to $PO_{i-1}(f, x)$. This is because

$PO_{n-1}(f,y)$ always contains $PO_{i-1}(f,y)$ when y approaches x, so the intersection $PO_{n-1}(f,y) \cap PO_i(f,x)$ will contain $PO_{i-1}(f,x)$ by continuity of the osculating flag (follows from the local continuity of the derivative vectors).

Now use the projective duality. Let

$$PO_0(f,x) \subset \dots PO_i(f,x) \subset PO_{n-1}(f,x)$$

be the osculating flag at x and

$$PO_0(\tilde{f},x) \subset \dots \subset PO_i(\tilde{f},x) \subset PO_{n-1}(\tilde{f},x)$$

be the osculating flag in $\check{\mathbb{P}}^n$ with respect to $\check{f}$. I claim that the dual space of $PO_{n-1-i}(f,x)$ is equal to $PO_i(\tilde{f},x)$. This of course proves the assertion since we get that the osculating hyperplane $PO_{n-1}(\check{f},x)$ corresponds by duality to the point $(\check{f}(x)$ and is equal to $PO_0(f,x) = f(x)$. We use induction on i. By definition, this is true for $i = 0$. Assume it is true for $i = m$. Then

$$\lim_{y \to x} < PO_m(\check{f},x), PO_0(\check{f},y) >= PO_{m+1}(\check{f},x).$$

The dual of span is the intersection, thus the dual of $PO_{m+1}(\check{f},x)$ is equal to

$$\lim_{y \to x} PO_{n-1-m}(f,x) \cap PO_{n-1}(f,x) = PO_{n-2-m}(f,x).$$

This proves the assertion. □

Corollary 1. *The degree of the maps $f : X \to f(X)$ and $\check{f} : X \to \check{f}(X)$ are equal. In particular, $\check{f} : X \to \check{f}(X)$ is a birational map if and only if $\check{f} : X \to \check{f}(X)$ is birational.*

Proof. Let $U \subset X$ be an open Zariski subset such that the restriction of f and $\check{f}$ to U is an unramified finite map of some degree d. Suppose $\check{f}(x) = \check{f}(y)$ for $x, y \in U$. Then $f(x) = (\check{f})(x) = (\check{f})(y) = f^{(}y)$. This shows that the fibres of f and $\tilde{f}$ over a point in $f(U) \cap \check{f}(U)$ have the same cardinality. □

Definition 1. *Let C be an irreducible curve in $\mathbb{P}^n$ and $\pi : X \to C$ be its normalization map. Let $f : X \to \mathbb{P}^n$ is the composition of π and the inclusion map $C \hookrightarrow \mathbb{P}^n$. Then the curve $\check{f}(X)$ is called the dual curve of C and is denoted by C^*.*

It follows from the previous corollary that the curves C and C^* are birationally isomorphic. The duality lemma also shows that $(C^*)^* = C$.

To prove the next result, we need to know what happens with the osculating flag when we compose f with a projection map $\mathbb{P}^n \to \mathbb{P}^{n-1}$. Recall, in coordinate-free way, a linear n-dimensional subspace $E \subset V$ defines the projection map $p : \mathbb{P}(V^*) \setminus \mathbb{P}(E^\perp) \to \mathbb{P}(E^*)$ defined by the restriction map $V^* \to E^*$. Suppose we have an osculating flag of linear subspaces (1). Intersecting E with this flag we get

$$0 \subset E \cap V_{n+\alpha_n} \subset \dots \subset E \cap V_{1+\alpha_1} \subset E.$$

Suppose E is in general position, i.e. $\dim E \cap V_{i+\alpha_i} = \dim E - i$. Then the flag

$$0 \subset E \cap V_{n-1+\alpha_{n-1}} \subset \ldots \subset E \cap V_{1+\alpha_1} \subset E$$

defines an osculating flag of $p \circ f$ at x. Its ramification sequence is equal to $(\alpha_1, \ldots, \alpha_{n-1})$. On the other hand, if E contains $V_{n+\alpha_n}$ (equivalently, the center of the projection $\mathbb{P}(E^\perp)$ is contained in the osculating hyperplane $PO_{n-1}(f,x)$), but intersects other subspaces transversally, then the ramification sequence for $p \circ f$ becomes $(\alpha_1, \ldots, \alpha_{n-2}, \alpha_n + 1)$. Finally note that the union of osculating k-planes $PO_k(f,x)$ is of dimension at most $1+k$. Thus we can always choose the center of projection not lying on any $PO_{n-2}(f,x)$.

Let $C \subset \mathbb{P}^n$ and $f : X \to \mathbb{P}^n$ be the composition of the inclusion map and the normalization map $\pi : X \to C$. Let O be a general point in $\mathbb{P}^n$ and $p : C \to \mathbb{P}^{n-1}$ be the projection with center at O. Any point $x \in X$ with $\alpha_{n-1} > 0$ is a hyperosculating point with respect to the composition $p \circ f : X \to \mathbb{P}^{n-1}$. Also, if O lies in an osculating hyperplane $PO_{n-1}(f,x)$, then x is a hyperosculating point $p \circ f$. The hyperplane in $\check{\mathbb{P}}^n$ corresponding to O intersects the dual curve C^* at the number of points equal to the degree of C^*. Each such point corresponds to a hyperplane in $\mathbb{P}^n$ which is an osculating hyperplane $PO_{n-1}(f,x)$ for some point x on X. Since the dual map $\check{f} : X \to C^* \hookrightarrow \check{\mathbb{P}}^n$ is birational, the point x is uniquely determined by the hyperplane $PO_{n-1}(f,x)$. By taking O general enough we may assume that O is contained only in osculating hyperplanes of non-hyperosculating points. Thus we obtain

Theorem 2. *The degree of the curve C^* is equal to the number of honest hyperosculating points $x \in X$ with respect to the composition of $\pi : X \to C \hookrightarrow \mathbb{P}^n$ and the projection $C \to \mathbb{P}^{n-1}$ from a general point minus the number of points $x \in X$ with ramification indices $0 = \alpha_1 = \ldots = \alpha_{n-2} < \alpha_{n-1}$.*

Example 3. *Let $n = 2$, $C \subset \mathbb{P}^2$ and $\pi : X \to C$ is the normalization map. A point $x \in X$ with $\alpha_1 > 0$ is a ramification point of $X \to C$. The set of hyperosculating points with respect to $f : X \to C \to \mathbb{P}^1$ is equal to the set of ramification points. By Hurwitz formula,*

$$2g - 2 + 2d = \sum_{x \in X} (e_x - 1),$$

where d is the degree of C, g is the genus of X, and e_x is the ramification index of f at x. Since $f = \check{f}$, we know from Lemma 1 that the ramification index at each point x is equal to $\alpha_2 - \alpha_1$. Suppose that C has at most nodes or ordinary cusps as singularities. Then for each x with $\pi(x)$ equal to a cusp, we have $\alpha_1 = 1$. For any other x we have $\alpha_1 = 0$. We can choose the center O of the projection general enough such that it does not lie on any osculating line with $\alpha_2 > 1$. Thus $e_x = 1$ at each ramification point, and we obtain

$$\deg C^* = 2g - 2 + 2d - \kappa = d(d-3) - 2\delta - 2\kappa + 2d - \kappa = d(d-1) - 2\delta - 3\kappa,$$

where δ is the number of nodes, and κ is the number of cusps.

Recall from Lemma 1 that a honest hyperosculating point $x \in X$ with respect to $f : X \to \mathbb{P}^n$ is a ramification point of $\check{f} : X \to C^* \hookrightarrow \check{\mathbb{P}}^n$. Take a general codimension 2 subspace Π in $\check{\mathbb{P}}^n$ and let $q : C^* \to \mathbb{P}^1 = L$ be the projection from Π (where L is the dual line of Π in $\mathbb{P}^n$). Consider the composition $q \circ \check{f} : X \to \mathbb{P}^1$. It is easy to see that it is given by $x \mapsto PO_{n-1}(f, x) \cap L$. Assume that $f : X \to C$ is birational. Thus the degree of the map $q \circ \check{f}$ is equal to the degree of the projection map q, and hence is equal to the degree of C^*. By Hurwitz formula we expect $2g - 2 + 2\deg(C^*)$ ramification points counting with multiplicities.

The ramification points of $q \circ \check{f}$ are ramification points of $\check{f} : X \to \mathbb{P}^n$, i.e. hyperosculating points of f with $\alpha_n > \alpha_{n-1}$, and the ramification points of $C^* \to L$ which are nonsingular points of C^*. By the duality, the tangent line $PO_1(\check{f}, x)$ of C^* at $\check{f}(x)$ is dual to $PO_{n-2}(f, x)$. The point $\check{f}(x)$ is a ramification point of the projection $C^* \to L$ if and only if the tangent line $PO_1(\check{f}, x)$ intersects the center of the projection, or, equivalently, the span of $PO_{n-2}(f, x)$ and L is a hyperplane. Let $p : \mathbb{P}^n \to \mathbb{P}^{n-1}$ be the projection from a general point on L. The image of $PO_{n-2}(f, x)$ is an osculating hyperplane of x with respect to the composition of $p \circ f$ such that $f(x) \in L$. If $n > 2$, the number of such osculating hyperplanes is equal to the degree of the dual curve of $p(C)$. Thus we obtain the following.

Theorem 3. *Assume $n > 1$. The number of hyperosculating points of $f : X \to C \hookrightarrow \mathbb{P}^n$ with $\alpha_n > \alpha_{n-1}$ (counting with appropriate multiplicities) is equal to the difference between $2g - 2 + 2\deg C^*$ and the number of points $x \in X$ such that there exists a hyperplane containing L and $PO_{n-2}(f, x)$. If $n > 2$, the latter number is equal to the degree of the dual curve of the projection of C from a general point in $\mathbb{P}^n$. If $n = 2$, it is equal to the degree of the curve C.*

Example 4. *Assume $n = 2$ and C has only nodes and cusps. A point $x \in X$ with $\alpha_1 > 0$ is a cusp and has ramification indices $(1, 1)$. So a cusp is not counting as a ramification point of $\check{f} : X \to C^*$. So all ramification points of this map correspond to honest hyperosculating points of C, i.e., inflection points. Its number is equal to*

$$2g - 2 + 2\deg C^* - d = 2g - 2 + 2(d(d-1) - 2\delta - 3\kappa) - d = 6(g-1) + 3d - 2\kappa.$$

Assume $n = 3$ and C is nonsingular. The projection $\bar{C}$ of C to $\mathbb{P}^2$ is a curve of genus g, degree d with $\delta = -g + (d-1)(d-2)/2$ nodes. The degree of the dual curve $\bar{C}^*$ is equal to $2g - 2 + 2d = d(d-1) - 2\delta$. By Theorem 2, the degree of C^* is equal to the number of honest hyperosculating points of $\bar{C}$ (since $\bar{C}$ does not have cusps) which is equal, by above, to $6(g-1) + 3d$. Thus the number of honest hyperosculating points is equal to

$$2g - 2 + 2deg(C^*) - (2g-2) - 2d = 12(g-1) + 4d = n(n+1)(g-1) + (n+1)d.$$

Proceeding by induction on n, we see that the rights-hand-side is the formula for the number of honest hyperosculating points for a nonsingular curve of degree d in $\mathbb{P}^n$.