Fundamental groupoid \[ \text{continuous maps of topological spaces} \]

A homotopy between two maps of spaces

\[ f : X \to Y \]
\[ g : X \to Y \]

We say \( f \) and \( g \) are homotopic, write \( f \simeq g \), such that \( h(x, 0) = f(x) \), \( h(x, 1) = g(x) \). A homotopy we write \( h(x, t) = h_t(x) \)
equivalence is a pair of maps \( f : X \to Y \), \( g : Y \to X \) with homotopies \( fg \sim Id_Y \), \( gf \sim Id_X \).

We also write \( X \simeq Y \). A space \( X \simeq \{x\} \) is called contractible.

Most of algebraic topology only distinguishes spaces up to homotopy equivalence.

**HW:** Prove that the Möbius band \( M \) is \( \simeq \) to an ordinary band \( B \).

Take the quotient equivalence relation satisfying

\[
M = \left[ 0,1 \right] \times \left[ 0,1 \right] / (x,0) \sim (1-x,1)
\]
\[ B = \left[ 0, 1 \right] \times \left[ 0, 1 \right] \left/ (x, 0) \sim (x, 1) \right. \]

How to prove that two spaces are not homeomorphic (or homotopy equivalent)?

The method: Device computable algebraic characteristics of spaces which are the same for homeomorphic (or homotopy equivalent) spaces. Such characteristics are called topological (resp. homotopical) invariants.
Often, they happen to be functors.

The fundamental groupoid $I = [0, 1]$

A path in a space $X$ is a map $w : I \to X$.

Beginning point of $w : w(0)$

End point of $w : w(1)$.

$I$ is contractible.

When speaking of a homotopy of paths $w \equiv \eta$

$w, \eta : I \to X$, we mean a homotopy $h : w \equiv \eta$

such that $h_t(0) = w(0)$, $h_t(1) = w(1)$

$\therefore = \eta(0)$, $\therefore = \eta(1)$.
In particular, if \( \omega, \eta \) are homotopic paths, then
\[
\omega(0) = \eta(0),
\]
\[
\omega(1) = \eta(1).
\]

Two paths are homotopic if there exist a homotopy between them. This relation of being homotopic is an equivalence relation.

**Definition:** The fundamental groupoid \( \pi_1(X) \) of a space \( X \) is a groupoid where
Obj $\pi(X) = X$ considered as a set.

The morphisms from $x \to y$ in $\pi(X)$

$\pi(X)(x,y) := \text{The set of equivalence classes } [\omega]$

(\text{hom}_{\pi(X)}(x,y))

\text{of paths from } x \to y$

\text{with respect to homotopy of paths.}

\text{means with beginning point } x$

\text{and point } y.

\text{Then}

$\text{Id}_X := \text{constant path } I \to X$

$\text{const}_X \quad t \mapsto x$

\text{Composition}
\[ T(\eta) = s(\eta) \]
\[ \eta \circ \omega = \eta \circ \phi \]

\[ g(t) = w(2t) \text{ when } 0 \leq t \leq \frac{1}{2} \]
\[ g(t) = \eta(2t-1) \text{ when } \frac{1}{2} \leq t \leq 1 \]

We must prove:
- unitality
- associativity
- inverse.

**HW:** Define a category Homotop
whose objects are spaces and morphisms are homotopy classes of maps. (composition should be given by composition of representatives of the classes.) Prove that your definition works.