Theorem (Seifert-Van Kampen Theorem, v. 1):

Let \( X \) be a topological space, \( U, V \subseteq X \) open, \( X = U \cup V \).

Then the diagram of groupoids

\[
\begin{array}{ccc}
\pi(U \cap V) & \to & \pi(U) \\
\downarrow & & \downarrow \\
\pi(V) & \to & \pi(X)
\end{array}
\]

is a pushout of groupoids where the arrows are \( \pi(\text{inclusion}) \).
Comment: If we have a pushout of groupoids

\[
\begin{array}{ccc}
\Gamma_1 & \rightarrow & \Gamma_2 \\
\downarrow & & \downarrow \\
\Gamma_3 & \rightarrow & \Gamma_4
\end{array}
\]

and apply \( \text{Obj}(?) : \text{Groupoid} \rightarrow \text{Set} \).

"?" means an unlabeled variable.

Then you get a pushout of sets. (HW: prove this.)

[diagram:]

\[
\begin{array}{ccc}
\text{Obj} \Gamma_1 & \rightarrow & \text{Obj} \Gamma_2 \\
\downarrow & & \downarrow \\
\text{Obj} \Gamma_3 & \rightarrow & \text{Obj} \Gamma_4 \\
\end{array}
\]

\( S \)
Make $\mathcal{S}$ into a groupoid $\overline{\mathcal{S}}$ such that a functor $F : \overline{\mathcal{S}} \rightarrow \mathcal{S}$ for any groupoid $\mathcal{S}$ is determined uniquely by a map $\text{Obj} \mathcal{S} \rightarrow \mathcal{S}$. For $s,t \in \mathcal{S}$, $\overline{\mathcal{S}}(s,t) = \{ * \}$. Any morphism in $\overline{\mathcal{S}}$ then uniquely determines a morphism in $\mathcal{S}$.

$\mathbf{Set} \rightarrow \mathbf{Grpoid}$, (a "weak adjoint")

"getting ahead of ourselves."

Create a testing diagram of groupoids: 

\[
\begin{array}{c}
\overline{\mathcal{S}} \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
\mathcal{S} \end{array}
\]
Proof of the Theorem: Consider a testing diagram

\[
\begin{align*}
\pi(U \cup V) & \rightarrow \pi(U) \\
\downarrow & \downarrow \\
\pi(U) & \rightarrow \pi(X) \quad \text{(12k)} \\
\downarrow & \downarrow \\
V & \rightarrow X
\end{align*}
\]

Consistency check: We are OK on objects. As sets,

\[
\begin{align*}
U \cup V & \rightarrow U \\
\downarrow & \downarrow \\
V & \rightarrow X
\end{align*}
\]
On object, \( h(x) = f(x) \) if \( x \in U \),
\[ g(x) \text{ otherwise} \]

On morphisms, \( [\omega]: x \to y \in \text{ Mor } \pi(X) \)

A path \( \omega: I \to X \) \( \omega(0) = x, \omega(1) = y \).

By compactness, \( \exists \ 0 = t_0 < t_1 < \cdots < t_k = 1 \)

\( \omega \upharpoonright [t_i, t_{i+1}] \subseteq U \text{ a } \cdots \subseteq V \)

\( \forall \ i = 0, \ldots, k-1, \)

Let \( \omega_i \) be the path \( \omega \upharpoonright [t_i, t_{i+1}] \) linearly
reparametrized as \( I \to X \)
\[ I \to [t_i, t_{i+1}] \]
\[ \omega_i = \omega \circ \omega_i. \]

\[ \left[ \omega \right] = \left[ \omega_{k-1} \right] \circ \cdots \circ \left[ \omega_0 \right] \quad (\ast) \]

To prove "existence," it suffices to show that the
de\text{ \textquoteleft\textquoteleft} \text{existence} \text{ \textquoteleft\textquoteleft} does not depend on the choices. \text{\textquoteleft\textquoteleft} Homeotopy \text{\textquoteleft\textquoteleft} and the partition \{(s \to C \cdots \to t_k)\}. For partitions, use \text{\textquoteleft\textquoteleft}
To deal with homotopy of paths $h: w \sim \eta$

- $w(0) = \eta(0)$
- $w(1) = \eta(1)$
- $0 = s_0 \leq \ldots \leq s_k = 1$
- $0 = t_0 \leq \ldots \leq t_k = 1$

such that $[s_i, s_{i+1}] \times [t_j, t_{j+1}] \subseteq U \circ \ldots \subseteq V$
A good goal: \( \prod_i (S^i) \).

\[ S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \} \]

\[ U = S^- = \{ z \in S^1 \mid \text{Re } z < \frac{1}{2} \} \]
$V = S_+ \times \{ z \in S_1 \mid \text{Re} z > -\frac{1}{2} \}$

What about $U \cap V \cong \{ * \times * \}$

$U \cong +$, $V \cong +$

$\cong$ equivalent?

A 2-pushout of groupoids is a variation of the concept of a pushout which takes into account equivalence of categories (or better said, natural monomorphisms).

**Definition**: A diagram of groupoids

\[
\begin{array}{ccc}
\Gamma_1 & \xrightarrow{a} & \Gamma_2 \\
\downarrow b & & \downarrow c \\
\Gamma_3 & \xrightarrow{d} & \Gamma_4
\end{array}
\]
which commutes up to natural isomorphism
\[ \gamma : \text{co} \alpha \cong \text{do} \beta : \Gamma_1 \to \Gamma_2 \]

is called a 2-proarrow if for any

\[
\begin{array}{ccc}
\Gamma_1 & \xrightarrow{a} & \Gamma_2 \\
\downarrow & & \downarrow \\
\Gamma_3 & \xrightarrow{b} & \Gamma \\
\end{array}
\]

commutes up to natural \( \cong \) we can fill in
Uniqueness: Subject to this condition,

The 2-cell maps

This notion is "obviously" invariant under equivalence of categories.
Theorem: Suppose we have a pushout of groupoids

\[ \begin{array}{ccc}
\Gamma_1 & \rightarrow & \Gamma_2 \\
\downarrow i & & \downarrow j \\
\Gamma_3 & \rightarrow & \Gamma_4 \\
\end{array} \]  

(\ast)

where \( i \) and \( j \) are injective on objects. Then \((\ast)\)

is a 2-pushout (2-cell: \( \text{Id} \)).