Lemma: If \( w \) is a word in \( x, x' \), \( x \in S \) and we keep canceling \(-x x'\) \(-x x'\) in two different ways, we get the same reduced word in the end.

Proof:

\[
\begin{align*}
w & \xrightarrow{\text{canceling sequence } 0} \xrightarrow{\text{canceling sequence } 0} \\
& \xrightarrow{\text{canceling sequence } 0} \xrightarrow{\text{canceling sequence } 0}
\end{align*}
\]

Induction on the \# of steps of whichever \( 0, 0 \) has \( \leq \) steps. (say, \( \Box \)).
Case 1: Both 1, 2 start with the same step.

Induction hypothesis.

Case 2: The first step of 1 is different from first step of 2. I will modify 1 to a sequence of cancellations 1, which does start with the same step as 2.

"The first step of 1 is the first step of 2.

After this first step, I decide to follow the steps of 0 until I get stuck. Note this must happen, otherwise the two elements cancelled in the first step.
of 0 would remain uncancelled.

I can get stuck (at using the steps of 0 in 0')
when one of the element I am attempting to
cancel is no longer there (because of the extra
step I took in the beginning).

Possibility: 0 I am attempting to do exactly
the step I already did in the beginning.

Just drop if - am in the same position
as after this step in 0. Go on doing same
steps as 0 to complete 0' - terminate in
same word as 0. By induction hypothesis,
(1) Terminates in the same word as (2).
In this case, I am done.

(2) (1) is not calling for exactly same step as the first step of (1), rather, is attempting to use only one of the elements cancelled in the first step of (1), and cancel it with the adjacent element on the other side:

\[
\begin{array}{c}
\text{---} \\
\times \times' \\
\text{---}
\end{array}
\]

\[
\begin{array}{c}
\text{---} \\
\times \times \times \\
\text{---}
\end{array}
\]

4 possibilities
"I am cancelling these two factors again, skip this step!

In \( x^1 x^1 = x x^1 \), it does not matter which term you cancel, just assuming you do cancel exactly for the first step of (1), produces the same result.

So I can reduce to possibility (0)."

Recall factorization of a group \( G \) by a normal subgroup \( H \):

\[ H \triangleleft G \quad : \quad g^{-1} H g = H \]
\[ \text{left triangle} \quad \| \quad \text{for all } g \in G \),

\[ \{ g^{-1} h g \mid h \in H \} \]

\[ G/H = \{ gH \mid g \in G \} \]

When \( H \triangleleft G \),

\[ \text{left coset } = \text{right coset} \]

\[ h \in H \leftrightarrow g h g^{-1} \in H \]

\[ gh = (ghg^{-1})g \quad \text{etc.} \]

A group: \( G, H, g H : = g, g H, h \).

\[ g_1 g_2 = g_2^{-1} h g_2 \text{ for } h \in H \]
Observation: Every group $G$ is a factor of a free group.

\[ f: \mathcal{G} \rightarrow G \]

\[ \mathcal{G} \rightarrow \mathcal{G} \]

Let $H := \ker f = \{ x \in \mathcal{F} \mid f(x) = e \}$. 

The homomorphism theorem: $\mathcal{F}G/H \cong G$

\[ yH \mapsto g \]

\[ \langle S \mid R \rangle \leq \text{means } FS / \text{the smallest normal subgroup containing } R, \text{ a set of words in } x, x^{-1}, x \in S \text{ if } R \subseteq FS \]
(generator)  (defining relations)

@ smalllet refers to the following: an intersection of (any number of) normal subgroups is a normal subgroup.

---

Examples: \( \langle a, b \mid ab = ba \rangle \)

\[
ab (ba)^{-1} = ab a^{-1} b^{-1}
\]

\( \langle a, b \mid ab^{-1} a^{-1} b^{-1} \rangle = \mathbb{Z} \times \mathbb{Z} \)

\( \langle a, b \mid a^6, b^2, bab^{-1} a \rangle \subset \text{dihedral group} \)
a presentation of $G$

$bab^{-1} = a^{-1}$

If $G \leq \langle R | S \rangle$ where $R$ is finite, then we say $G$ is finitely generated, and if both $R, S$ are finite, $G$ is finitely presented.

We will soon see that the Seifert–Van Kampen theorem gives us presentations of $\pi_1(X)$ when $X = \bigcup_i U_i$, all $U_i \cap U_j$ are contractible.

Example:

A knot is a 1-dimensional submanifold of $\mathbb{S}^3$ diffeomorphic to $S^1$. 
We say two knots are isomorphic if there exist an isotopy \( h : S^3 \to S^3 \) a homotopy through diffeomorphisms which connects one to the other.

\[ \mathcal{C} \neq \mathcal{O} \]

(Proof: thanks, Benjamin!)

Theorem: let \( K_1, K_2 \subset S^3 \) be knots. Then \( K_1 \equiv K_2 \) are isomorphic if and only if

\[ \pi_1(S^3 \setminus K_1) \cong \pi_1(S^3 \setminus K_2). \]

This is decidable by the Haken algorithm.
The difficulty:

Theorem: Let \( S_1, S_2 \) be finite sets, \( R_1, C \subseteq S_1 \) finite subsets. There does not exist an algorithm which would decide whether
\[
\langle S_1, R_1 \rangle \cong \langle S_2, R_2 \rangle.
\]

Proposition: Let
\[
\begin{array}{c}
S_1 \\
\downarrow
\end{array} 
\begin{array}{c}
S_2
\end{array}
\]

\( \rightarrow \)

1
Be a diagram of groups. Then the pushout exist. Suppose we have presentations
\[ G_i = \langle S_i | R_i \rangle \quad i = 1, 2, 3. \]

Then the presentation of the pushout is:
\[ \langle S_2 \sqcup S_3 | R_2 \cup R_3 \cup \{ u v^{-1} \} \rangle. \]

\[ u (\text{e.g. } u) \text{ is a word in } S_2 \quad \text{(e.g. } S_2) \text{ unionized } \]
\[ \psi(x) (\text{e.g. } \psi(x)), x \in S_1 \]

Proof: Next time. □