Solution of quiz: Free group on 2 generators, 

\[ \bigvee_m S^1 = S^1 \times \{0, \ldots, m-1\} / (*, *) \sim (*, *) \]

\[ x \wedge y = x \vee y / x \sim y \]

\[ \overline{\mu} (\bigvee_m S^1, *) = F \{ x_0, \ldots, x_{m-1} \} \]

The smash product

J. F. Adams

\[ X, Y \text{ one based} \]

\[ X \wedge Y = X \vee Y / (X \times Y) \]

\[ x_0, x_1, \ldots, x_{n-1} \]

\[ \varepsilon_i \text{ is the free generator of } \pi_1 (S^1 \vee \Sigma^n, *) \]
Proof by induction on $m$:

$x \rightarrow \bigvee s'$

$p \rightarrow \vdash_{n-1} (\text{pass to switch open neighborhoods to satisfy assumptions})$

SK theorem:

$\vdash_{n-1} F(x_0, \ldots, x_{n-2}) = \langle x_0, \ldots, x_{n-2} |\rangle$

$\vdash_{n-1} F(x_{n-1}) = \langle x_{0}, \ldots, x_{n+1} |\rangle$

$\langle x_{n-1} |\rangle$
\[ \Pi_i \left( \bigvee_{I} S^i, * \right) = F(x_i, \forall i \in I) . \]

\[ S^i \times I / (+, i') \sim (*, i'), \forall i' \in I \]

The (co)limit argument:

\[ \Pi_i \left( \bigvee_{I} S^i, * \right) \cong \text{colim}_{I} \Pi_i \left( \bigvee_{I} S^i, * \right) \]

\[ \forall \text{finite } F \in I \]

Ind dexing

\[ \text{not } [0,1]! \]

\[ \text{the source category of the diagram:} \]

\[ \text{Obj: finite limit of } I \quad \Pi_i \leq . \]
\[
\lim_{Z \to I} F(Z) \cong F(I) \quad \text{if } I \to \mathcal{Z}
\]

A graph is a pair of sets \((V, E)\) with two maps: \(S, T : E \to V\).

The topological realization
\(|\Gamma| := V \cup (E \times \mathcal{C}_0) \setminus \{(e,0) \sim s(e)
\}
\)

\((e,1) \sim t(e),\)

often we don't distinguish \(\Gamma, |\Gamma|\) (although we should).

We say \(\Gamma\) is connected when \(|\Gamma|\) is connected...

etc.

If \(\Gamma\) is a connected graph then \(|\Gamma| \cong \sqrt{S'}\).

A tree is a graph such that \(|\Gamma|\) does not contain a subgraph homeomorphic to \(S'\)...
(1) Derive a combinatorial characterization (without referring to topology) of a connected graph.

(2) Derive a combinatorial characterization of a tree.

A spanning tree of a connected graph is a maximal subgraph which is a tree.

\[ \left( \text{exp. 71} \right) \quad V' \subseteq V \quad \exists \quad E' \subseteq E \]
If \( T \) is a connected graph, then a spanning tree, \( T' \), of \( G \), along with the remaining edges not in the spanning tree, \( E \), is a homotopy equivalence.

Homework: Prove this rigorously.
\[
\pi_1\left(\Gamma, *, \epsilon\right) \cong F\left(\text{edges of } \Gamma \text{ not in } T\right) \backslash \mathcal{E}(\Gamma) \setminus \mathcal{E}(T).
\]

**Example:** Compact surfaces:

\[
\begin{align*}
\# \, S^1 \times S^1 &= 1 \quad \# \, \mathbb{RP}^2 &= m \quad m \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{represent} \quad S^1 \times S^1 : \quad \mathbb{RP}^2
\end{align*}
\]
Opposite sides identified

Cut out a disk:

\[
\begin{align*}
\# S^1 \times S^1
\end{align*}
\]

\[
\begin{align*}
\# \mathbb{R}^2
\end{align*}
\]
$$\pi_1(\#^m S^1 \times S^1) =$$

$$= \langle a_1, b_1, \ldots, a_m, b_m \mid a_1 b_1 a_1^{-1} b_1^{-1}, \ldots, a_m b_m a_m^{-1} b_m^{-1} \rangle$$

$$\pi_1(\#^m \mathbb{RP}^2) = \langle a_1, \ldots, a_m \mid a_1^2 a_2^2 \ldots a_m^2 \rangle$$

Sesquital Van Kampen:

$$\pi_1(\#^m S^1 \times S^1 \to \#^m \mathbb{RP}^2)$$

$$\mathfrak{s}^1 \to \mathfrak{v}^m$$

$$\mathfrak{s}^1 \to \mathfrak{v}^m \to \mathbb{D}^2$$