\[ \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2 = \mathbb{R}P^2 \# (S^1 \times S^1) \]

Isomorphism of \( \pi_1 \):

\[ \langle a, b, c \mid a^2 b^{-1} c \rangle \cong \langle x, y, z \mid x^2 y^{-1} y^{-1} z^{-1} \rangle \]

My solution (?):

\[ (c^{-2} b^{-1} c). (c^{-1} b^{-1} a^{-1} b c^2). (c^{-1} b c^2). (c^{-2} b c) \]

\[ c^{-2} b^{-1} a^{-1} b c a b c \]
Going back: recovering $a, b, c$ from $x, y, z$

\[ xy = bc^2 \]

\[ x y z = b c^2 c^{-2} b^{-1} c = c \]

\[ b^{-1} = c^2 z c^{-1} = (xyz)^2 z(xyz)^{-1} \]

\[ a = xc^{-1} b^{-1} \]

We will talk the abelianization.

First adjoint functor

Example: The free group $S \rightarrow G$ is universal property.
\[ F : \text{Sets} \rightarrow \text{Gp} \quad \text{Free group functor} \]

\[ U : \text{Gp} \rightarrow \text{Set} \quad \text{forgetful functor} \quad (\text{the "underlying", set}) \]

\[ \text{Sets} \left( S, U \mathcal{G} \right) \cong \text{Gp} (FS, G) \]

for every set \( S \), every group \( G \), the universal property gives me a bijection. (natural)

\[ f \in \text{Hom}(\text{Sets}) \]
\[ f \colon S \rightarrow U \mathcal{G} \quad \Rightarrow \quad Ff \downarrow \quad \mathcal{G} \]

\[ \text{adj} \quad \text{adj} \quad FT \]
F is called **left adjoint** to G.

U is called **right adjoint** to F.

The definition is gotten by replacing set, set with arbitrary categories C, D. If the conditions are met, then
(HW)

(1) Prove that a left adjoint preserves colimits, and a right adjoint preserves limits. (OK to just
consider

(defined!) dual to wellhead.)

A Universal algebra — operations (n-ary for
possibly different n)

— axioms: equalities

e.g. [a, (b, c, 7)] + [b, (c, a, 7)]

+ [Ic, a], b] = 0

[a, b] = -[b, a] = 0
Distributivity of $[\cdot, \cdot, 0, +]$ operations: $+, [\cdot, \cdot, 0, 0, \ell, \ell, \ell, \ell]$. 

morphisms = homomorphisms

(maps preserving operations)

$x(x_1, \ldots, x_n) = (x_1, x_2, \ldots, x_n)$

a category of universal algebras

\[
\xymatrix{	ext{t} \ar[r]^U & \text{t}' \ar[d]^f \ar@/^1.5pc/[ld]^G} \quad \text{where I may have}
\]

left out some forgotten function, operations, etc.
Theorem: This \( U \) always has a left adjoint (i.e., as a right adjoint).

\[ U \dashv \text{Set} \]

Corollary: Forgetting functors of universal algebras always preserve limits! (e.g., product)

\[ \text{e.g., } \text{The categorical product of groups is the Cartesian product.} \]

\[ \text{U is not a left adjoint, so it does not preserve product.} \]
The abelianization map $\text{ab} : \text{Grp} \rightarrow \text{Ab}$

is the left adjoint to the forgetful functor $U : \text{Ab} \rightarrow \text{Grp}$. How to compute $\text{ab}$:

$$\text{ab}(F(S)) = \frac{\mathbb{Z}S}{r}$$
free group

\[ \text{free abelian group on } S \]

\[ \text{forget } \]

\[ A_b \to G_p \]

\[ \text{forget } \]

\[ \text{set } \]

\[ \text{law} \]

\[ \text{adjoint } \]

\[ \sum_{x \in S} y(x) \cdot x \leq \text{treat this as "formal sum" } \]

\[ Z \]

\[ \text{is the left adjoint of the forgetful } \]

\[ \text{functor } \]

\[ A_b \to G_p \]

\[ \text{she? means } \]

\[ \text{an unnamed variable. } \]

\[ \text{HW(2): If } C \to D \to E \text{ are functors and } \]

\[ F \text{ is left adjoint to } U, \] \[ G \text{ is left adjoint to } V \} \]
then $F_\alpha$ is left adjoint to $\mathcal{U}$.

\[ \theta_\alpha \langle S \mid R \rangle = \mathbb{Z}S/\langle \text{subgroup generated by the relations } R \rangle. \]

I recommend: switch to additive notation.

Examples: \[ A\beta (\pi_1 (\mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2)) = \prod_{(c_i)} \mathbb{Z} \]

\[ = \mathbb{Z} \{ a_1, \ldots, a_n \}/\langle 2a_1 + \ldots + 2a_n \rangle = \mathbb{Z}/2 \oplus \cdots \oplus \mathbb{Z} \]

change generators \[ x_i = a_1 + \ldots + a_n, \]

\[ x_i - a_i, \quad i = 2, \ldots, n \]

in $A\beta$ 

copies shift of two
\[ s \times s' = 1 \]

\[ \text{Ab}(\prod_{k} (T \# \cdots \# T)) = \bigoplus_{m} \mathbb{Z} \]

\[ \text{Not only } \pi_{1}, \text{ but also } \text{Ab } \pi_{1} \text{ classifies compact surfaces. } H_{1} \text{ in the first homology.} \]

The algorithm for determining whether two finitely generated abelian groups are isomorphic.
Take one group
\[ \langle x_1, \ldots, x_m \mid a_{i1}x_1 + \cdots + a_{in}x_n, \quad i = 1, \ldots, m \rangle \]

can be \( \sim \)

From the \( U \)-matrix
\[
\begin{pmatrix}
a_{11} & \cdots & a_{1m} \\
\vdots & \ddots & \vdots \\
a_{m1} & \cdots & a_{mm}
\end{pmatrix}
\]

elementary row
or column operations

\( \text{0 add a } \mathbb{Z}\text{-multiple of a row to another row} \)

\( \text{1} \) similarly for columns
2) multiply a row by -1.
3') -1 1 columns.

$k$'s

\[
\begin{pmatrix}
1, m_1,
\vdots,
m_k,
0, \ldots
\end{pmatrix}
\]

The group $\mathbb{G} = \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k} \oplus \mathbb{Z}$

\[
m_1, \ldots, m_k > 1
\]

Count by columns, don't count 1's, count 0's.
This algorithm is not on Test 1!

HW due on Monday 2/10!!