Solutions to test:

1. \( \text{Top} \to \text{sets} \)

2. Left adjoint

\[ \text{set} (U \times X, S) \cong \text{Top} (X, c_S) \]

(a) \( X \to S \)

(b) \( X \to c_S \) continuous
\[ X = s - 1 \]

Continuous on \( S \) is a topology on \( S \) which makes every map \( f: X \rightarrow s - 1 \) continuous.
\[ S^1 \times [0,1] / \sim \]

\[ (t,0) \sim (t,0) \]

\[ x \mapsto a^2 \]

\[ a_{i,b} \mapsto a_{i,b}^{-1} \]
(a) \[ \boxed{2} \]

(b) \[ \langle a, b \mid a \ b \ a^{-1} \ b^{-1} \rangle \]
(c) Abelianization: \( \langle a, b \mid -2b \rangle \)

\( \mathbb{Z} \oplus \mathbb{Z}/2 \)

\( \mathbb{R}P^2 \# \mathbb{R}P^2 \)

(4) A relation will show up as a cycle in the graph and vice versa.

Covering space = mapping \( f: \tilde{X} \to X \) which

(loosely, \( \tilde{X} \) is the covering space, \( X \) satisfies the)
For every $x \in X$ there exist an open neighborhood $U_x = U \ni x$ (called a \textit{fundamental neighborhood}) such that

$$f^{-1}(U) = \bigsqcup_{i \in I} U_i, \quad U_i \subset Y$$

where

$$f|_{U_i} : U_i \to U$$

is a homeomorphism.

\underline{Note:} This concept has almost nothing to do with an open cover.
Exercise: Let $U \subset X$ be an open subset where $U$ is not closed. Prove that the inclusion $i: U \to X$ is not a covering.

Example: $f: \mathbb{R} \to S^1$

$$f(x) = \exp(2\pi i x)$$

is a covering. For any point $z \in S^1$, $S^1 \setminus \{z\}$ is a fundamental neighborhood.

$$f^{-1}(z) = x + \mathbb{Z} = \{x + n \mid n \in \mathbb{Z}\}$$

let $z = e^{2\pi i x}$
The intervals \((x+m, x+m+1)\), \(m \in \mathbb{Z}\) can serve as \(U_s'\) for \(U = S \backslash \{b\}\).

\[ f: (x+m, x+m+1) \rightarrow S \backslash \{b\} \]
Theorem: Let \( f: \mathcal{Y} \to \mathcal{X} \) be a covering.

Then the functor

\[
\pi(f): \pi(\mathcal{U}) \to \pi(\mathcal{X})
\]

has the following property: For any morphism

\( \alpha: x \to y \in \text{Mor} \pi(\mathcal{X}) \),

and any \( x \in \text{Obj} \pi(\mathcal{Y}) \) such that \( \pi(f)(x) = x \)

there exist a unique morphism \( \gamma: \tilde{x} \to \tilde{y} \in \text{Obj} \pi(\mathcal{U}) \)

such that \( \pi(f)(\gamma) = \alpha \). 

\( \gamma \) is determined by the data.
Proof: Lifting a path \( \tilde{w} : [0, 1] \rightarrow X \)

\[ \tilde{w}(0) = x \]

To a path \( \bar{w} : [0, 1] \rightarrow Y \) : By compactness of \([0, 1] \),

\[ t_0 = 0 < t_1 < \ldots < t_k = 1 \] such that

\[ \bar{w}([t_i, t_{i+1}]) \subset U \subset \text{a fundamental neighborhood} \]

Now proceed by induction on \( i \) to construct \( \tilde{w}/[0, t_i] \)

When \( \tilde{w}(t_i) \) is constructed, let \( \bar{w}([t_i, t_{i+1}]) \subset U \subset \text{fund. neighborhood} \)
Let \( \tilde{\omega}(t_i) \in V \), \( f|_V: V \cong U \). Let

\[
\tilde{\omega}|_{(t_i, t_{i+1})} = (f|_V)^{-1} \circ \omega|_{(t_i, t_{i+1})}.
\]

This completes the induction step. (Note: this definition is forced, so the path left unclear.)

Lifting a homotopy of paths (implies connectedness):

\[
h: [0, 1] \times [0, 1] \to X
\]
There exist \( 0 = u_0 < u_1 < \cdots < u_k = 1 \) \{compact\} with 0 = b_0 < b_1 < \cdots < t_c = 1 \}

And that \( \forall \frac{b_i}{u_i} \\ [u_i, u_{i+1}] \times [t_j, t_{j+1}] \subseteq U \)

Fundamental neighborhood.

\[ \uparrow \]

Proceed by induction on rectangle to

Lift homotopy to \( \tilde{h} : [0, 1] \times [0, 1] \to \tilde{Y} \).

Same argument as for lifting earlier.
Induction on $|i, j|$ in a decreasing manner in any linear way so that if $i' \leq i$, then $(i', j) \in (i, j')$.