Theorem: If $X$ is path-connected, locally path-connected SLGC (see last time) then the fiber functor $(\mathfrak{f} x \circ X)$

$\mathfrak{f} x \circ \operatorname{Cov}(X) \longrightarrow \pi_1(X, x) - \operatorname{set}$

Covering spaces, deck transformations

is an equivalence of categories.

Proof: Not yet done last time. Strategy: construct
an inverse \( \mathfrak{D} \), recalling that \( \mathfrak{G} \)-sets are disjoint unions of orbits. We really needed to construct

\[ \mathfrak{D} \left( \pi_1(X_1) / H \right). \]

How to construct \( \mathfrak{D} \) on morphisms: suppose we have an \( \pi_1(X_1) \)-equivariant map

\[ \varphi : \pi_1(X_1) / H \to \pi_1(X_1) / K. \]

What is \( \mathfrak{D}(\varphi) \)?

\[
\begin{array}{c}
\varphi \\
\downarrow \\
X \xrightarrow{f} X
\end{array}
\]

\[
\begin{array}{c}
\varphi \\
\downarrow \\
X \xrightarrow{f} X
\end{array}
\]
We also proved that

\[ \pi_1(Y, eH) = H \]

\[ \pi_1(\tilde{X}, eK) = K \]

We can also think of constructing \( Y \) as filling the diagram (of course, it is the same)

\[ \tilde{f} \]

\[ \text{(lifting \( f \))} \]
Recall the lifting criterion: The lift \( \tilde{f} \) exists if and only if

\[
\pi_1 \left( f, eH \right) \left( \pi_1 \left( \gamma, eH \right) \right) \subseteq \pi_1 \left( f_\ast, gK \right) \left( \pi_1 \left( \gamma, gK \right) \right) \]

\( f_\ast \)

Translated to isotropy groups:

\[
H \subseteq gKg^{-1}
\]

This is precisely equivalent to

\[
\pi_1 \left( X, x \right) / H \rightarrow \pi_1 \left( X, x \right) / K
\]

\( eH \rightarrow gK \)
being a $\pi_1 (X, x)$ - equivariant map.

By definition, $\text{Fib}_x \cdot \bar{\Phi} = \text{Id}_{\pi_1 (X, x)} - x$. 

\[ \bar{\Phi} \cdot \text{Fib}_x = \text{Id}_{\text{ Cov } (X)} \quad \text{(just watch the base point!)} \]

HW: Prove this.

Remark: A completely canonical version of this equivalence: (X locally path-connected, SLSC)

\[ \text{Fib} : \text{ Cov } (X) \rightarrow \text{Fun} (\pi_1 (X), \text{sets}) \]

\[ \in \text{ functors and natural transformations} \]
What is a "Γ-equivariant set" when Γ is a groupoid?

(Proof analogous.)

Example: If we have a universal cover of a nice space X (path connected, locally path-connected, 1LSC), you can calculate \( \pi_1(X, x) \) (because you've got it acting on a torsor):

\[ \rho = \exp(\text{Log}) : \mathbb{R} \to S^1 \]
(Universal because $\mathbb{R} \cong *$): $p^{-1}(1) = \mathbb{Z}$

a form of the fundamental group.

We know $[\omega] \in \pi_1(S^1, 1)$

$\omega(t) = e^{2\pi it}$.

Indeed, $\omega: \mathbb{Z} \to \mathbb{Z}$: $k \mapsto k+1 \in p^{-1}(1)$.

The only way this can happen is that...
\[ \pi_1(S^1) = \mathbb{Z} \{ [w] \} \quad \text{free abelian group} \]

by freeness

\[ \cong (\text{injective, surjective}) \quad \text{by torsion property} \]

This is an independent computation of \( \pi_1(S^1) \).

Example: Consider \( S^{n-1} = \{ z \in \mathbb{C}^n \mid \| z \| = 1 \} \).

\[ m > 1 \quad \text{Euclidean norm} \]
Consider $\mathbb{Z}/k$ acting on $\mathbb{S}^{2n-1}$: $z = (z_1, \ldots, z_m)$

$$j(z) = e^{2\pi i z/k} \cdot z = (e^{2\pi i z_1/k}, \ldots, e^{2\pi i z_m/k}).$$

We have a covering

$$\mathbb{S}^{2n-1} \rightarrow \mathbb{S}^{2n-1} / (\mathbb{Z}/k) = \mathbb{S}^m(k)$$

$m > 1$: universal cover

(slightly connected means nice & $\pi_1 = 0$).
\[ \pi_1 \left( \mathbb{L}_n(k) \right) \cong \mathbb{Z}/k. \]

**Example:**

For a compact surface \( X \), except when \( X \cong S^2 \) or \( X \cong \mathbb{R}P^2 \),

\[ \pi_1(\mathbb{L}_n(k)) \cong \mathbb{Z}/k. \]

\[ \text{SL}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{C}, \ ad - bc = 1 \right\} \]

\[ \text{Act}_+ \sim \mathbb{H} = \left\{ z \in \mathbb{C} \mid \text{Im} \ z > 0 \right\} \]

\[ \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \left( z \right) : = \frac{az + b}{cz + d}. \]
\[
\begin{pmatrix}
  -1 & 0 \\
  0 & -1 \\
\end{pmatrix}
\begin{pmatrix}
  2 \\
  1 \\
\end{pmatrix} = \frac{-2}{-1} = 2 \\
\]

\[PSL_2 \mathbb{R} := SL_2 \mathbb{R} / \langle \begin{pmatrix}
  0 & 1 \\
  1 & 0 \\
\end{pmatrix} \rangle.\]

\[\begin{pmatrix}
  0 & 1 \\
  -\bar{1} & \bar{0} \\
\end{pmatrix} \cdot \begin{pmatrix}
  1 \\
  -i \\
\end{pmatrix} = \begin{pmatrix}
  i \\
  i \\
\end{pmatrix} = \bar{z}.\]

We call \( \mathbb{H} \) the 2-dimensional hyperbolic space \( PSL_2 \mathbb{R} \) is group of isometries (define the standard Euclidean metric by \( y^2 \neq x + iy \)).
The point is: If $X$ is an orientable Riemann surface of genus $> 1$

$\# T \geq 1$

then there exists a subgroup $\Gamma \subset \text{PSL}_2(\mathbb{C})$ which acts freely (properly discontinuously) on $\mathbb{H}$ covering the action of $\text{PSL}_2(\mathbb{C})$, and

$\mathbb{H}/\Gamma \cong X$
\[ \Gamma \in \pi_1(X) \]

This is called geometrization. Geometrization (at least in some form) was recently proved for 3-manifolds (Perelman et al.).

Example: Graphs and groups.

"A covering space of a graph is a graph."

\[ \text{Moreover: If } \Gamma = (V,E) \text{ is a graph, then any covering of } |\Gamma| \text{ is, in an obvious}\]
A graph.

Theorem: A subgroup of a free group is free.

One can actually use this to write down the free generators of a subgroup of a free group.
Quit on Wed: Classification of f.g. abelian groups.

(watch the final commands and the Z's)