Take-Home Final Term:

Give it in class 3/17 (Monday)

Due in class 3/21 (Friday)

(may include Serre-Tate, Van Kampen)

descriptions of $f.g.$ abelian groups

otherwise, what we covered since last midterm.

5 problems.

Quiz this Wednesday. (graphs and groups)
Homology

\[ \Pi_1 \text{ sees no information about cells in a CW - } \infty. \]

above dimension 2

Suspension of a space:

\[ \Sigma X = X \times [0, 1] / (x, 0) \sim (y, 0) \]

\[ (x, 1) \sim (y, 1) \quad \text{for } x, y \in X \]

\[ \text{HW: Prove that } \Sigma^n \sim \Sigma^{n+1} \]
Concepts of algebraic topology which are preserved by suspension, are called stable.

Homology is stable. \((H_i)\) is unstable.

Knots are not stable.

(Upon suspension, a knot becomes trivial.)

There are different methods for defining homology (analogous to different methods of defining the integral.)
Singular homology - very similar to it.

We will now model

object into $X$, and

then do some algebra,

get groups.

very functional.

Why the weird name? Higher -

dimensional Hawaiian

earings have infinitely

many non-zero singular

homology groups, even

they are a subspace of $\mathbb{R}^n$

for a fixed $n$. 

Simplices, after

they are mapped to $X$, 

become "singular".
Definitions

The model - the standard $n$-simplex.

For all $x \in \mathbb{R}^n \cup \{0\}$

$$\Delta_n = \left\{ x = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i = 1 \right\}$$

$x_i \geq 0$

Face maps

$$\partial_i : \Delta^{n-1} \rightarrow \Delta^n$$

$i = 0, \ldots, n$
The set of angular $n$-simplices in a space $X$:

$$S_n X : = \{ \sigma : \Delta_n \to X \}$$

If $C$ is a category, $C^{op}$

continuous

$$\partial : S_m X \to S_{m-1} X.$$

$$(\sigma : \Delta_n \to X) \mapsto (\Delta_{n-1} \to \Delta_n \to X)$$

on $C \to C^{op}$

(equivalence class)
The group of singular chains

\[ C_m X = \mathbb{Z} S_m X \]

\[ \mathbb{Z} S = \text{The free abelian group on a set } S. \]

\[ = \text{Abelianization of the free group on } S \]

\[ F_S = \{ a : S \to \mathbb{Z} \mid F F \subseteq S \text{ for } a \} \]

We write \[ a = \sum_{s \in F} a(s) \cdot s. \]
We have finite linear combinations; no relations except those always true for linear combinations.

Example: in \( \mathbb{Z} \{a, b, c\} \), simplify

\[
4a - 2b + 3c - 2a + b - a - a
\]

\[
= 0. a - b + 3c = -b + 3c
\]

(\(-1\)b + 3c)

"formal" linear combinations.
Universal property - $\mathcal{Z}$ is the left adjoint to forget: $\text{Ab} \to \text{Set}$

$S \in \mathcal{Z}$

$\exists \text{ homomorphism of ab. groups as a linear combination}$

\[ \text{map of set} \]

\[ \text{abelian group} \]
Again, $C_n X = \mathcal{Z}(S_n X)$, I

Define a homomorphism of abelian groups

$\Delta_n = d_\sigma : C_n X \to C_{n-1} X$.

By universal property,

I precisely need to define

\[
\delta(\sigma) := \sum_{i=0}^{n} (-1)^i \partial_i \sigma,
\]

for $\sigma \in S_n X$. This means $\delta : D_n \to X$. 
The formula $\partial$ means the boundary.

Example: $\sigma = \text{Id} : \Delta_m \rightarrow \Delta_m$

\[ d(\sigma) = \sum_{i=0}^{n} (-1)^i \partial_i \]

$\partial_i : \Delta_{n-1} \rightarrow \Delta_n$ (indices and orientation!)
How do we know $\partial$ is a good formula? 

\underline{Lemma:} $d \circ d = 0$.

\[ C^d_\infty X \to C^d_{\infty-1} X \to C^d_{\infty-2} X \]

\[ \text{Proof:} \quad d \]

By the universal property, it suffices to show 

\[ d \circ (d \circ (\sigma)) = 0 \quad \sigma : C^d_{\infty} \to X \]

\[ \sigma \in C^d_{\infty} X. \]

(Note: it would suffice to consider \[ \sigma = \text{Id} : \delta^n \to \delta^n. \])
\[ d^d(\sigma) = \prod_{i=0}^{n} (-1)^i \delta^{\sigma_i \partial_i} (\sigma \circ \partial_i \circ \partial_j) = \]

\[ \sum_{i=0}^{n} (-1)^i \sum_{j=0}^{n-1} (-1)^j \delta^{\sigma_i \partial_i} (\sigma \circ \partial_i \circ \partial_j) = \Theta \]

\[ \partial_i \partial_j \left( (x_0, \ldots, x_{n-1}) \right) = \partial_i \left( (x_0, \ldots, x_{j-1}, \underbrace{0, \ldots, 0}_{\text{just barely can be empty}}, x_{j+1}, \ldots, x_{n-1}) \right) = \]

\[ \delta^{\sigma_i \partial_i} \]

\[ \begin{cases} \geq j \quad & \text{if } i \leq j \ \\ < j \quad & \text{others} \end{cases} \]
If \( i \leq j \implies i \cdot \tilde{e}_j = \tilde{e}_{j+1} \cdot \tilde{e}_i \), \( r' > j' \)

There is a bijection:

\[
\begin{align*}
\{ (i', j') \mid 0 \leq i', j' \leq n-1, \quad i' < j' \} & \equiv \\
\{ (i', j') \mid 0 \leq j' \leq n-1, \quad i' > j' \} & \equiv \{ (i', j') \mid 0 \leq i' \leq n-1, \quad j' = 1 \}
\end{align*}
\]

\( (i, j) \xrightarrow{y} (i', j') = (j+1, i') \)

\[
\sum_{i=0}^{n} \sum_{j=0}^{i-1} (-1)^{i+j} \sigma \circ \tilde{\sigma}_j \circ \tilde{e}_j = \sum_{(i, j) \in \mathcal{C}} (-1)^{i+j} \sigma \circ \tilde{\sigma}_j \circ \tilde{e}_j
\]
\[ + \sum_{(i', j') \in D} (-1)^{i' + j'} \circ \partial_{i' + j'} = 0 \]

Remember, if \( (i', j') = \varphi (i, j) \) then \( i_1 \circ \partial_{j'} = \)

\[ = (j' + 1, i_1) \]

\[ = 0 \circ \partial_{j'} \]

\[ i' + j' = i + j + 1 \]

terms are matched by the injection \( \varphi \) with opposite signs. \( \Box \)

\[ \text{Definition: A chain complex is a sequence} \]
of abelian group $C = (C_n)_{n \in \mathbb{Z}}$ with homomorphisms

$$d : C_n \to C_{n-1}$$

(called a differential) which satisfies $d \circ d = 0$.

> (evolve the $d$ in differential forms,

Of course that one goes up

> (co-chain complex).

Example: $C_X = (C_nX)_{n \in \mathbb{Z}} \, d$ is the singular chain complex of a space $X$. 
Definition: For a chain complex $C$, 

$$H_n C = \frac{\text{Ker } d : C_n \rightarrow C_{n-1}}{\text{Im } d : C_{n+1} \rightarrow C_n}$$

with homology (homology "in dimension $n" ) d_{n+1}$

$\text{Im } d_{n+1} \subseteq \text{Ker } d_n$ because $dd = 0$.

Singular homology of a space $X$ is defined by

$$H_n X := H_n (\mathcal{C}X).$$