HW: Prove the 5-lemma stating that if in the diagram

\[
\begin{array}{ccc}
A & \rightarrow & B & \rightarrow & C & \rightarrow & D & \rightarrow & E \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
A' & \rightarrow & B' & \rightarrow & C' & \rightarrow & D' & \rightarrow & E'
\end{array}
\]

the horizontal sequences are exact and the marked vertical arrows are isomorphisms, so is the remaining vertical arrow.
0 \to C \to D \to E \to 0

SES of chain complexes is long exact sequence in homology.

0 \to X \to Y \to Z \to 0

\begin{align*}
0 & \to \cdot \\
\downarrow & \downarrow \\
0 & \to \cdot
\end{align*}

What is the homology of the chain complex $\mathbb{Z} \to \mathbb{Z} \to X \to Y \to 0$?
Continuing our proof of excision in singular homology:

Recall that we constructed a natural chain map

\[ \text{sd}: C_X \to C_X \]

and a natural chain homotopy \( h: \text{Id} \simeq \text{sd} \) (level-wise)

Proposition: Let \( \mathcal{U} \) be an open cover of a space \( X \) and define

\[ C_\mathcal{U}(X) \]
to be the subcomplex of $C_X$ such that
\[
(C_y(X))_\mu := \mathbb{Z}\{ \sigma : \Delta_\mu \to X \mid \sigma(\Delta_\mu) \subseteq U \text{ for some } U \in \mathcal{F} \}
\]

Then the inclusion
\[
j : C_y(X) \subseteq C(X)
\]
induces an isomorphism in homology.

Proof: Depends on the following
Lemma: Suppose \( c \in C_n X \), \( d \in C_{n-1} X \). Then there exists an element \( x \in C_{n+1} X \) such that \( c + dx \in (C \varphi(X))_n \).

Why does the lemma suffice to prove the proposition?

We may write down a SES of chain complexes

\[ 0 \to C \varphi(X) \to C(X) \to C X / C \varphi X \to 0 \quad (*) \]

The lemma precisely states that \( H_n (C X / C \varphi(X)) = 0 \).

Then the LES corresponding to \( (*) \) gives

\[ 0 \to H_n C \varphi X \to H_n C X \to 0 \quad \text{exact} \]

\[ \text{this is an isomorphism.} \]
Proof of the lemma: Let $c \in C_{n+1}X$, $dc \in (C_{n}X)_{m-1}$.

$$c = \sum_{i=1}^{2} k_{i} \sigma_{i} \quad \sigma_{i} : \Delta_{m} \rightarrow X.$$ 

($\exists m$) $sd^{m}c \in C_{n}X$ (#) (by the Lebesgue number theorem)

in $\Delta_{m}$ applied to 

$$\{ \sigma_{i}^{-1}(U) \mid U \in \mathcal{G} \}$$

The images of $sd(v) \quad v = Id : \Delta_{n} \rightarrow \Delta_{n}$

have diameter $\leq \frac{m-1}{m} \text{diam} (\Delta_{m})$

We have, however, a homotopy

$$H : Id \rightarrow sd^{m}c$$

This needs proof:

$$f \approx g, \quad g \leq h$$
Then \( f \circ h \) or \( f \circ h \) directly in \( S\mathbb{d}^n \).

\[
dhc + [Hdc] = [\mathbb{d}^n c + c]
\]

\[
\delta (C_y (X)) \text{ because } dc \circ (C_y (X))_{n-1}.
\]

\[
\therefore dhc = c + \frac{c}{2} \circ (C_y (X))_n.
\]

How does the Proposition that

\[
y : C_y X \rightarrow C X \text{ induces } \equiv \in H^1
\]
prove the excision axiom?

Next time.