How to compute homology of a CW-complex

(C = the first type of application: two spaces are not homotopy equivalent if they have non-isomorphic homology).

A CW-complex $X$ has a filtration

$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots \quad X = \bigcup X_n$

$I_m = \text{an indexing set, "set of } m\text{-cells.}"

$f_m : S^{n-1} \times I_m \to X_{n-1} \quad \text{"attaching map.}"

In short of space,

\[ f^n : \mathbb{S}^{m-1} \times I^n \rightarrow X_{m-1} \]

\[ \left\{ \begin{array}{c}
\leq x \times \text{Id} \\
\downarrow \leftarrow \\
0^n \times I^n \rightarrow X_m
\end{array} \right. \]

\( C W \)-pair \( (X, A) \)
of the same, \( \emptyset \rightarrow A \)
replaced by \( A \).

**Proposition 1:** \( H_k(X_m, X_{m-1}) = \begin{cases} \mathbb{Z} I^n & \text{if } k = m \\ 0 & \text{if } k \neq m. \end{cases} \)

**Proof:** Let \( x \in D^m \setminus S^{m-1} \) (whatever your model is).

Denote \( X^o \) to be the pushout.
Left column: 

\[ \text{Definition of a deformation which } \quad \xrightarrow{\text{inclusion}} \quad X^0 \]

\[ D^m \times I^m \rightarrow X^n \]

\[ H_k (X^n, X^0) \]

\[ \text{at } D_0^m = D^m \text{, discrete } \]

\[ X^n - X^{n-1} \sim D_0^m \times I^m \]

Right column:

\[ H_k (X^n, X^{n-1}) \]

\[ \text{of a deformation centred.} \]
\[ H_k(X \times I) = \bigoplus_{I} H_k(X) \]
\[ \text{define} \quad (\cong H_k(X) \otimes \mathbb{Z}^I) \]
\[ \left( C(X \times I) = \bigoplus_{I} C(X) \right) \]

\[ H_k(0, 1^{m-1}) = \mathbb{Z} \quad \forall \ k = m \]
\[ 0 \quad \text{else} \]

\[ \mathbb{Z}(I_m) \quad \forall \ k = n \]
\[ 0 \quad \text{else} \]

Do we have a differential?
$d = d_m : H_n(X_n, X_{n-1}) \rightarrow H_{n-1}(X_{n-1}, X_{n-2})$

this means $dd = 0$ (or $d_{n-1}d_n = 0$).

This $d$ can be just the connecting map of the LES in homology, assigned to

$0 \rightarrow C(X_{n-1}, X_{n-2}) \rightarrow C(X_n, X_{n-2}) \rightarrow C(X_n, X_{n-1}) \rightarrow 0$

In terms of the standard LES in homology (not using homotopy), by naturality, this is

$H_n(X_n, X_{n-1}) \rightarrow H_{n-1}(X_{n-1}) \rightarrow H_{n-1}(X_{n-1}, X_{n-2}) \rightarrow H_{n-2}(X_{n-2})$
Proposition: \( d_{m-1} \circ d_m = 0 \).

Proof:

\[
\begin{align*}
H_m(X_{n-1}, X_{n-1}) &\xrightarrow{\partial} H_{m-1}(X_{n-1}) &\xrightarrow{H_{m-1}(e)} &H_{m-1}(X_{n-1}, X_{n-2}) \\
&\quad \Downarrow 0 &\quad \Downarrow 2 &\\
\text{two consecutive maps in } &\alpha \in \text{LES.}
\end{align*}
\]

\[
\Downarrow \quad H_{m-2}(X_{n-2}) \\
H_{m-2}(X_{n-2}, X_{n-3}), \square
\]
Definition: \( \text{Cell}(X) : \quad \text{Cell}_n(X) = \mathbb{Z} \mathbb{I}^n \) \\
the CW chain complex of X \\

\[ d_n^{\text{cell}} = d_n : \mathbb{Z} \mathbb{I}^n \to \mathbb{Z} \mathbb{I}^{n-1} \]

\[ H_n(X_n, X_{n-1}) \quad H_{n-1}(X_{n-1}, X_{n-2}) \]

So we can define (provisionally) \\
\[ H_n^{\text{cell}}(X) : = H_n \text{Cell}(X) \]

Theorem: We have \\
\[ H_n^{\text{cell}}(X) = H_n(X) \]

\[ H_m^{\text{cell}}(X_1, A) = H_m(X_1, A) \] for \( A \subset \text{CW-space} \)
This is natural in the category 
\((\text{CW-complexes}, \text{cell maps})\).

\[\text{Definition:}\]
A map \(f: X \rightarrow Y\) of \((\text{CW-complexes})\) is a cell map if \(f(X_m) \subseteq Y_m\).

\[\text{Theorem: Every (continuous) map of (CW-complexes)}\]
\[\text{is homotopic to cell map.} \]

Proof of the Theorem: Special cases which are important:
Lemma 1: \[ H_k (X_m) = H_k (X_m, X_m) = 0 \]
when \( m \leq n < k \).

Lemma 2: \[ H^l (X_m, X_m) = 0 \]
when \( H^l (X_m) \neq 0 \).

Resolution of the Theorem to the Lemmas.

Consider

\[ \text{(8)} \quad \text{Ker} (H_m (X_m, X_{m+1}) \to H_{m-1} (X_m, X_{m-1})) \]

\[ H_{m-1} (X_{m-2}) = 0 \quad \text{(Lemma 1)} \]
\[ H_n(K_n, X_{n-1}) \xrightarrow{\partial} H_{n-1}(X_{n-1}) \]

\[ \text{Image:} \downarrow H_{n-1}(C) \]

\[ H_{n-1}(X_{n-1}, X_{n-2}) \]

\[ \text{so (k) is exact} \]

\[ \ker (\partial : H_n(X_{n-1}, X_{n-2}) \to H_{n-1}(X_{n-1})) \]

\[ H_n(K_{n-1}, X_{n-2}) \rightarrow H_n(X_{n-2}) \rightarrow H_n(X_{n-1}, X_{n-2}) \xrightarrow{\partial} H_{n-1}(X_{n-1}) \]
So (**) is just $H_n(X_n)$!

What about

$\text{Im} \left( H_{n+1}(X_{n+1}, X_n) \right) \rightarrow H_n(X_n, X_{n-1})$?

\[ H_{n+1}(X_{n+1}, X_n) \xrightarrow{\text{?}} H_n(X_n) \xrightarrow{H_n(c)} H_n(X_n, X_{n-1}). \]

So we have shown

$H_n(X) \cong \text{Cohen} (\mathcal{C}_n ; H_{n+1}(X_{n+1}, X_n) - H_n(X_n)).$
\[ H_{n+1}(X_{n+1}, X_n) \xrightarrow{\partial} H_n(X_n) \]

by lemma \(2\)

Now we know that

\[ \text{Hunt}(X) = \text{Cohom}(H_{n+1}(X, X_n) \xrightarrow{\partial} H_n(X_n)) \]
Let
\[ H_{m+1}(X, X) \to H_n(X, X) \to H_n(X) \to H_n(X, X) \]

\[ \vdash \text{ Colim } \circ H_{m+1}(X, X) \cong H_n(X) \quad \text{by Lemma 2!} \]

This proves the theorem. \( \square \)

"A spectral sequence collapses."

Abhyankar - Hirzebruch

The Lemmas 1, 2 follow by induction on new from Proposition (for Lemma 2,}
the case of $X$ needs the limit axiom.

Induction step key point:

$$(X_m, X_m, X_{m-1}) :$$

$H_k(X_m, X_{m-1}) \rightarrow H_k(X_m, X_{m-1}) \rightarrow H_k(X_m, X_m)$

or use induction hypothesis.

$H(\phi) : 1) \text{ prove Lemma 1 }$ \hspace{1cm} 2) \text{ prove Lemma 2.}$
Quiz Wednesday.