The degree of a map

Milnor: Topology from a differential viewpoint.

Let $M$ be a compact connected oriented $n$-manifold. The degree

$[M, S^n] \to \mathbb{Z}$

homotopy classes of maps $M \to S^n$

$\phi : M \to S^n$ continuous $\Rightarrow$ (in fact, $\epsilon$-close to)
a smooth map. For $f$ smooth, by Sard's theorem, there exists a point $x \in S^m$ such that $f(\cdot, x)$ is transverse

For every $y \in f^{-1}(x)$

$Df\big|_y : TM_y \to TS^m_x$ is onto

(in our case $\Rightarrow$

an isomorphism of vector spaces)

Orientation of $N$ (and $S^m$) enables us to define $\text{sign} \left( \det (Df\big|_y) \right) \in \{ +1, -1 \}$

$d \deg(f) := \sum_{y \in f^{-1}(x)} \text{sign} \left( \det (Df\big|_y) \right)$
Theorem (Hopf): Assume $m > 1$.

This map

$$\text{deg}: [\Pi, S^m] \to \mathbb{Z}$$

is well defined and it is a bijection.

Addendum on homology: For $\Pi = S^m$, a map $f: S^m \to S^m$ with $\text{deg}(f) = k \in \mathbb{Z}$ induces an endomorphism $k: H^*(S^m) \to H^*(S^m)$.

Proof (sketch) of the Theorem: The first issue is independence of choice (representative of homotopy...
clear, smooth approximation, transverse point).

1. Local in dependence on a point \( x \).

There exists a neighborhood \( U \) of \( x \) such that every point \( x' \in U \) is transverse, and the degree, as calculated using \( x' \) is the same as the degree calculated using \( x \).

\[ \{ y_1, \ldots, y_n \} = f^{-1}(x). \]

identified with open set of \( \mathbb{R}^n \) via coordinates.
Partial derivatives in neighborhoods of \( y_k \) exist, are continuous, \( \det Df \neq 0 \) on them. So we can apply the inverse function theorem in a neighborhood \( U \) of \( y_k \), \( f \) is 1-1 and \( \det Df \) does not change signs.

What remains to be proved is that there exists a neighborhood \( V \) of \( k \) such that

\[ f^{-1}(V) \subseteq U, U \cdots U U_k. \]

This is because \( M \) is compact.

2. Full homotopy invariance. "Relative" smooth approximation (moved the same way): If
$h: f \simeq g \quad f, g: M \rightarrow S^m$ are smooth

Then $h$ is $\epsilon$-close to a smooth map

$h': M \times [0, \epsilon] \rightarrow S^m$

such that $h'|_{M \times \{0\}} = f, \quad h'|_{M \times \{\epsilon\}} = g$.

Given $f, g$ smooth and $h: f \simeq g$ smooth, $x \in S^m$

In any neighborhood of $x$ there exist a transversal point of $h'$ (by Sard's theorem). Suppose $f \not \perp x$

$g \not \perp x$

we can find a neighborhood $V$ of $x$ such that

$\deg f, \deg g$ when calculated are $x \not \in V$
gives the same value as when calculated one by one. And now we can replace $x$ by $x' \in V$ and dual $h' \in V'$.

$h$ is $\pi \times [0,1] \to S$

$\therefore h^{-1}(x') \cong \dim n+1$

a submanifold of $\pi \times [0,1]$

of dimension 1 (with boundary).

compact.

Classification of 1-d compact smooth manifolds with boundary:

\[ \square \quad \square \quad \square \quad \square \quad \square \]

compact interval
(Proof: parametrization by arc length.)

\[ f \circ h^{-1}(x') = x \]

A point in a small neighborhood of \( x \).

\[ \deg f = \deg g \text{ when calculated using } x' \]

\[ \therefore \text{ also } x \]
3. What about a non-local change of $x$?

Let $x, x' \in S^n$ arbitrary such that $f(x, x')$. Is $\deg f$, as calculated by $x, x'$, the same?

There is a homotopy (through homeomorphisms)

$$f^n : S^n \to S^n$$

which moves $x$ to $x'$. (state using $\text{id} \sim y$)

$id \sim y$

$y : S^n \to S^n$ is a retraction,
\( \gamma(x) = x'. \)

Now \( f \circ \gamma \neq x'. \) Also, \( \deg y \circ f \), calculated using \( x' \), is equal to \( \deg f \), calculated using \( x \).

(this means match).

But also \( f \neq \gamma \circ f \), so \( \deg f = \deg y \circ f \), as calculated using \( x' \).

This furnishes the proof of \( \deg \) being well-defined.

\( \circ \) \( \deg \) is onto.
The key idea is that \( S^n \) becomes contractible after the removal of a single point.

**NEXT TIME**
No HW today, but there will be HW on Wed due on Fri.

On Wed: No Quiz.

On Fri: I will hand you the last take-home midterm, due last day of class.

(last week: interesting application such as Jordan's chain. In dimension n, invariance of domain.)