Euler characteristic

Terminology: A finite CW-complex means it
has finitely many cells (equivalently, it is compact).

If $V$ is a finite-dimensional vector space
(over a field $F$) $f : V \to V$ is a homomorphism
of vector spaces, then we can define

\[ [f] \quad \text{(ordered)} \]

If we choose a basis $B$ of $V$ and the matrix
of $f$ with respected to \( b \) in the source and target is

\[
A = \begin{pmatrix}
  a_{11} & \cdots & a_{1n} \\
  \vdots & \ddots & \vdots \\
  a_{n1} & \cdots & a_{nn}
\end{pmatrix},
\]

then \( t \cdot f = a_{11} + \cdots + a_{nn} \). This is reversed under the same change of base in the source and target.

\[
X(A) = \det \left( t \cdot \mathbf{I} - A \right) \in F[t]
\]

minus the coefficient of \( X(A) \) at \( t^{m-1} \).

But \( X(B^{-1}AB) = \det \left( t \cdot \mathbf{I} - B^{-1}AB \right) = \det B^{-1} \det \left( t \cdot \mathbf{I} - A \right) B = (\det B)^{-1} \det \left( t \cdot \mathbf{I} - A \right) \det B = \det \left( t \cdot \mathbf{I} - A \right) = X(A) \).
Another fact about the trace: If I have a morphism of short exact sequences of finite dimensional vector spaces

\[ 0 \to U \to V \to W \to 0 \]

\[ \begin{array}{lll} f & \mid & g \\
\end{array} \]

\[ 0 \to U \to V \to W \to 0 \]

(really, an endomorphism) then

\[ \text{tr } g = \text{tr } f + \text{tr } h. \]

The SES splits: Let be a basis of \( V \) which contains a basis of \( U \). Then the matrix of \( g \) is
The sum of diagonal elements is $h_f + h_k$.

**Proposition:** Suppose I have a self-chain map $f$ of a finite chain complex $C$ (non-trivial on finitely many dimensions).

\[
0 \to C_N \xrightarrow{d_N} \cdots \xrightarrow{d_{M-1}} C_M \xrightarrow{d_M} C_{M+1} \to \cdots
\]

\[
0 \to C_N \xrightarrow{f_N} \cdots \xrightarrow{f_{M-1}} C_M \xrightarrow{f_M} C_{M+1} \to \cdots
\]
of vector spaces over $k$. (The differentials are homomorphisms of vector spaces.) Then

\[ \sum (-1)^k f_k = \sum (-1)^k h_k f \]

\[ h_k f : H_k C \to H_k C \]

a vector space over $k$.

Proof: Induction on $N - M$.

\[ 0 \to \ker d_N \to C_N \to C_N / \ker d_N \to 0 \]

\[ H_N f \downarrow \quad \begin{array}{c} \tilde{f}_N \\ \downarrow \end{array} \]

\[ 0 \to \ker d_N = H_N C \to C_N \to C_N / \ker d_N \to 0 \]

induced map
By the homomorphism theorem,

\[ d_N : C_N / \ker d_N \rightarrow C_{N-1} \]

\[ \text{Ker } d_N \hookrightarrow \text{Im } d_N \leq \ker d_{N-1} : C_{N-1} \rightarrow C_{N-2} \]
$d_{N-1} \rightarrow C_{N-2}$

$C_{N-1} \rightarrow A_{N-1}$

$\text{Im } d = \text{Im } d_{N-1}$

$\text{Ker } d = H_{N-1}$

(you can check this.)

Apply the induction hypothesis to the chain complex

$\cdots \rightarrow 0 \rightarrow A_{N-1} \rightarrow C_{N-2} \rightarrow C_{N-3} \rightarrow \cdots \rightarrow C_{M-1} \rightarrow 0$

$\partial_{N-1} \downarrow \quad \varphi_{N-2} \downarrow \quad \varphi_{N-3} \downarrow \quad \eta$

$\cdots \rightarrow 0 \rightarrow A_{N-1} \rightarrow C_{N-2} \rightarrow C_{N-3} \rightarrow \cdots \rightarrow C_{M-1} \rightarrow 0$
\[ \sum_{k \leq N-1} (-1)^k \cdot h \cdot f_k = \sum_{k \leq N-2} (-1)^k \cdot h \cdot f_k + (-1)^{N-1} \cdot h \cdot g_{N-1} \]

\[ h \cdot g_{N-1} + h \cdot f_N = h \cdot f_{N-1} \]

\[ h \cdot H_N^f + h \cdot f_N = h \cdot f_N \]

\[ h \cdot f_N = h \cdot f_N - h \cdot H_N^f \]

\[ h \cdot g_{N-1} = h \cdot f_{N-1} - h \cdot f_N = h \cdot f_{N-1} - h \cdot f_N + h \cdot H_N^f \]

Substitute this to \( \theta \), we get what we are trying to prove. \( \square \)
For a finite CW-complex $X$ and a continuous map $f : X \to X$, define

$$
\tilde{f} = \sum_{k} (-1)^{k} f_{*} (H_{k}(X) \otimes \mathbb{Q})
$$

Recall: For an abelian group $A$, $A \otimes \mathbb{Q}$ is a vector space over $\mathbb{Q}$. For $M, N$ abelian groups, $M \otimes_{\mathbb{Z}} N = \text{the universal}$

\[ \begin{array}{c}
\text{pairable for distributive} \\
\text{multiplications}
\end{array} \]

\[ \begin{array}{c}
\text{such that} \\
\text{$A \otimes \mathbb{Q}$}
\end{array} \]

Functor $A_{\mathbb{Q}} \times A_{\mathbb{Q}} \to A_{\mathbb{Q}}$
A finitely generated abelian group $A = \mathbb{Z}^k \oplus \mathbb{Z}/k_1 \oplus \cdots \oplus \mathbb{Z}/k_n$

$f \circ \text{Id}_A$

The Euler characteristic $(X = \text{a finite CW-complex})$
We have proved: $\dim X \leq \text{num}_k \text{ of k-cells}$.

$$\chi(X) = \sum_{k=0} \left| I_k \right| (-1)^k$$

(by the proposition, applied to the complex $C^{\text{cell}}(X)$)

(If $\text{Id}: V \to V$ = $\dim V$).

$\chi(X)$ is a homotopy invariant.

Notice: If we have a covering space

$p: Y \to X$ $X$ connected

from the $CW$-complex
such that \(|p^*(x)| = m\) (\(m\)-fold covering \(m\)-sheeted covering)

then \(Y\) is a finite \(C\) -complex,
(a covering of a cell is a cell)

The \# of \(k\)-cells of \(Y\)

\[X(Y) = m \cdot X(X)\]

Examples:
- \(X(CP^n) = m + 1\)
- \(X(RP^n) = 1\) when \(n\) even
- \(X(RP^n) = 0\) when \(n\) odd
\[ X(s^n) = \begin{cases} 2 & \text{when } n \text{ even} \\ 0 & \text{when } n \text{ odd} \end{cases} \]