Please fill out online teaching evaluations.

Test 3 due on Monday.

Theorem (Invariance of domain): Suppose $U, V \subseteq \mathbb{R}^m$, there exists a homeomorphism $\varphi : U \rightarrow V$. Then if $U$ is open in $\mathbb{R}^m$, so is $V$.

Example: $U \subset X$ open $\quad V \subset X$ not open $\quad U \subset V$. 

\begin{center}
\includegraphics[width=0.5\textwidth]{example_diagram}
\end{center}
Proof: Assume $U$ is open. Consider the 1-point compactification $R^m \subset S^m$.

Let $x \in U$. Let $B$ be a closed ball with center $x$ which is contained in $U$. Consider the composition $\psi$:

$$B \subset U \xrightarrow{\psi} V \cong R^m \subset S^m$$

$$\partial B = \partial_{R^m} B \subset S^{m-1}.$$ 

(For $S \subset X$, $\partial XS = \text{Closure of } S \setminus \text{Interior of } S.$)
Now consider

$$S^n \setminus \psi(\partial B).$$  \hspace{1cm} (1)

By Jordan’s theorem, (1) has exactly two connected components,

$$S^n \setminus \psi(\partial B) = \left( S^n \setminus \psi(B) \right) \sqcup \psi(B \setminus \partial B) \hspace{1cm} (2)$$

($$\psi : B \to S^n$$ is injective)

and

$$\psi : B \to \psi(B)$$ is a homeomorphism

(because $$B$$ is compact).

From last time,
\[ F_i (s^n \setminus \gamma (B)) = 0 \quad \forall i \]

In particular, \( s^n \setminus \gamma (B) \) is path-connected hence connected.

On the RHS of (2), the first \( \gamma \) summand is connected.
But the second disjoint summand is an image of a connected set hence also connected.

\[ \therefore \quad s^n \setminus \gamma (B) \quad \text{and} \quad \gamma (B \setminus \Theta B) \quad \text{are} \]
the connected component of \( (s^n \setminus \gamma (\Theta B)) \).

Therefore, they are open (in \( s^n \setminus \gamma (\Theta B) \), hence \( s^n \)).

Since \( \gamma (B \setminus \Theta B) \) is open, it is a neighborhood in \( s^n \) contained in \( V = \gamma (\Theta B) \).
of \( \Psi(x) \). But \( x \in U \) was arbitrary, hence
\( \forall \Psi(x) \). \( \Box \)

Some remarks on what I will talk about
in 695, and why.
(since omission of this class).

Recall the following statement from our proof
of the Jordan theorem:

**Theorem**: Let \( A \subset S^n \) such that \( A \cong S^k \)
$0 \leq k \leq m-1$. Then
\[
\tilde{H}_{n-k-1}(S^n \setminus A) \cong \mathbb{Z}, \quad \tilde{H}_{n}(S^n \setminus A) = 0, \quad n + m - b - 1.
\]

(A slight digression: For a knot $K$ ($k = 1$, $n = 3$)
\[
H_i(S^3 \setminus K) \cong \mathbb{Z}.
\]
\[
\text{Ab}(\pi_1(S^3 \setminus K)) \cong H_1(S^3 \setminus K) \cong \mathbb{Z}.
\]

The Theorem is a special case of something known as Alexander duality: we can “inflate” the homology of $A$ and $S^n \setminus A$ when $A \subset S^n$.\)
a homeomorphic image of a finite CW complex.

\[ H_k A \cong H_{n-k-1} (S^n \setminus A) \]

This certainly cannot be naturally isomorphic.

( in the category of closed subsets \( A \subseteq S^n \) which are images of compact CW complexes, and inclusions)

\[ A \subseteq B \implies (S^n \setminus B) \subseteq (S^n \setminus A) \]

the inclusion is reversed!

\[ \widetilde{H}_k (A) \to \widetilde{H}_k (B) \quad \widetilde{H}_{n-k-1} (S^n \setminus A) \subseteq \]
This important mathematical phenomenon is called **contravariance**.

In category theory, if $C$ is a category, we have an **opposite category** $C^{op}$ where we "reverse the arrows": reverse $S,T$ & order of composition.

A **contravariant functor** from a category $C$ to a category $D$ is a **functor** (covariant)
\[ C^{op} \to D \]
(equivalently, \( C \to D^{op} \)).

Let \( X \in C \) (\text{subset of } X_1) \). \( \exists \) a contravariant functor from the category to itself.

If we want to make Alexander duality functorial, then we need an analogue of homology which is a contravariant functor.

\( \wedge \) such a functor exists, and is called
Homology, we write a superscript.

\[ \tilde{H}_k(A) \equiv \tilde{H}^{n-k-1}(S^n, A) \]

Alexander duality.

The naive (slightly wrong) approach:

There is one extremely important source of

contravariant functors (representable functors):

If \( C \) is a category, \( a \in \text{Obj} \cdot C \)

\[ C(-, a) : C \text{op} \rightarrow \text{Sets} \]
$C(x, a)$ is the set of morphisms, i.e., $x \to a$.

There are some factors:

$\text{Hom}(?, A) : Ab \to Ab$

$x \mapsto \text{Hom}(X, A)$

an abelian group of homomorphisms
of ab. groups \( X \rightarrow A \) an abelian group with "additive addition of functions":

\[
(f + g)(x) = f(x) + g(x)
\]

\( f, g : X \rightarrow A. \)

Could we define:

\[ H^n(X) = \text{Hom} \left( H_n(X, \mathbb{Z}) \right) \]

\( \gamma \) have

\[ \mathbb{Z}/k \]

It is not quite right, for \( \gamma \) does not
behave well.

**Example:** $RP^n$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$H_i(RP^n, \mathbb{Z})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n$</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td>$n-1$</td>
<td>$0$</td>
</tr>
<tr>
<td>$n-2$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$0$</td>
<td>$\mathbb{Z}$</td>
</tr>
</tbody>
</table>

Homology: We lose information.

The correct construction: work on chain complexes.

Poincaré duality: If $M$ is a compact connected
oriented $n$-manifold \( \therefore \)

\[ H_k(P) \cong H_{n-k}(P) \]

\( \text{(Example: } n \text{ odd; } \mathbb{R}P^n \text{ inherits an orientation from } S^n. \) \]

\[ H_k(\mathbb{R}P^n) \]

\[
\begin{array}{cccc}
& \mathbb{Z} & & \\
\mathbb{Z}/2 & & & \\
0 & & & \\
0 & & & \\
\end{array}
\]

\[ \text{Homology of a space is a ring. } \]

\[ \bigwedge H^k \bigwedge H^{k+2} X \]
needed for moving deadline.

Remember: Test due Monday.
Evaluation, please. Thanks.