Note Title 12/4/2014

# UM MATH CLUB 12/4/2014

## Regular Polytopes

All videos in this talk can be found at: skmathclub.blogspot.com

**Convex polytopes** : A convex sets in $\mathbb{R}^n$ has the property that $x, y \in S$ then the line segment between $x$ and $y$

is in S.

A convex polytope is the convex hull (smallest convex set containing) a finite set.

A "dual" view is equivalently a (bounded) solution of finitely many non-strict linear inequalities

$$a_1 x_1 + \cdots + a_n x_n \leq b$$

$$\vdots$$

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The dimension of a polytope P is the smallest dimension of a $\underbrace{\text{linear space}}_{\substack{\text{affine space} \\ \text{(shift of a vector} \\ \text{subspace)}}}$ containing it.

If $P$ is given by inequalities, we can turn some of them into equalities. That way, we obtain a face (in the broader sense of $P$).

Euler characteristic

$$\chi(P) = \sum_{i=0}^{n} (-1)^i \cdot \left( \begin{matrix} \# \text{ of } i\text{-dimensional} \\ \text{faces} \end{matrix} \right) = 1$$

$P \subset \mathbb{R}^n$

(count $P$ as its own face).

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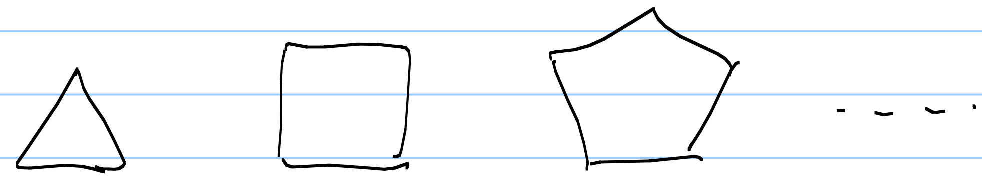
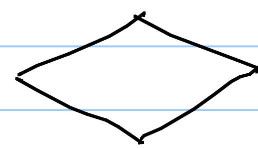
**Dimension 2:** A regular polytope is a regular polygon.

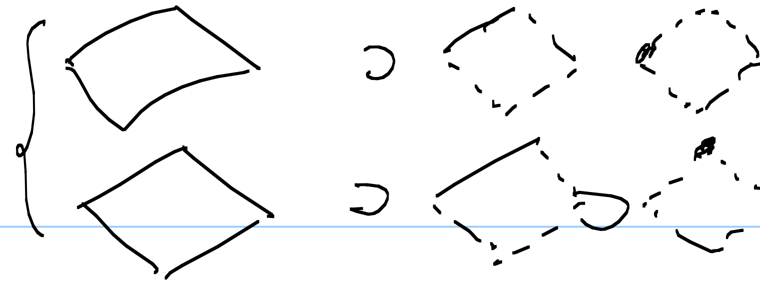


non-example:  .

A **symmetry** of a polytope $P$ is a mapping $P \to P$

which preserves distances.

$$P = P_k \supset P_{k-1} \supset \cdots \supset P_0$$



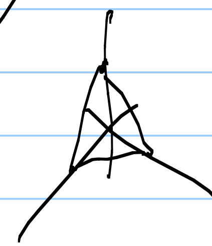
faces where $P_i$ has dimension $i$: Call this a flag. $P$ is regular if for any two flags there exists a symmetry which takes one to another.

In dim. 3: tetrahedron, cube, octahedron, dodecahedron, icosahedron.

## What about higher dimension?

tetrahedron $\circlearrowleft_{\text{dual}}$, cube $\xleftrightarrow[\text{dual}]{}$ octahedron

have generalisations in every dimension.

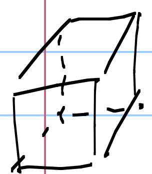


<u>simplex</u>: $\Delta_n \subset \mathbb{R}^{n+1}$

convex hull of $\left.\begin{array}{c} (1,0,\dots 0) \\ (0,1,\dots 0) \\ \vdots \\ (0,0,\dots,1) \end{array}\right\} n+1$

<u>hypercube</u>: convex hull of $\cdot$ in $\mathbb{R}^n$

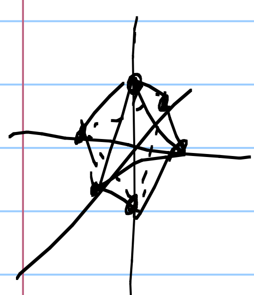
$$(\pm 1, \ldots, \pm 1) \quad \leftarrow 2^n \text{ points}$$



<u>"generalised octahedron"</u>: convex hull of

$$(0, \ldots 0, \pm 1, 0, \ldots 0) \quad \text{in } \mathbb{R}^n$$

$\nwarrow$ $2n$ points



The only other regular polytopes (other than 2 and 3) are in dimension 4. ← The 24-cell, 120-cell, 600-cell.

(Coxeter: Polytopes. ← Symmetries form a group.

special properties, he characterised "finite Coxeter groups").

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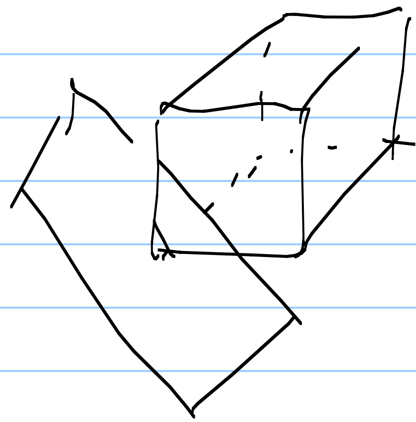
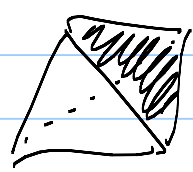
cells = 3-dim. faces.

cut the 4-dimensional solid } 3-dim. intersection by a 3-dimensional linear space.)

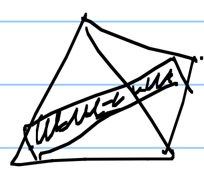


Each cell will reveal its cut by a plane.

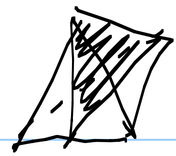
600-cell: the cells are tetrahedra.



type 1 cell



type 2 cell



type 3 cell.

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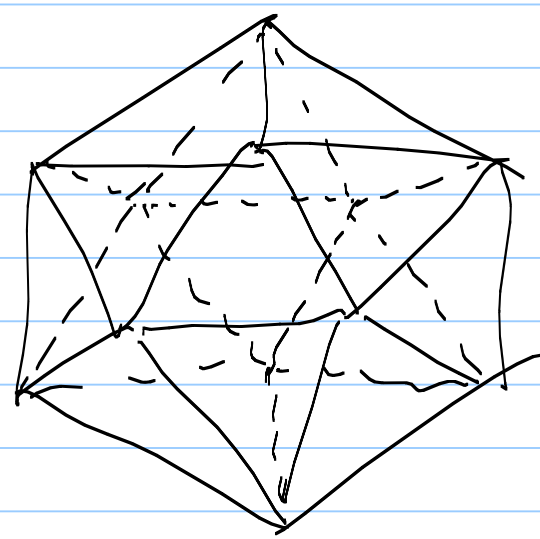
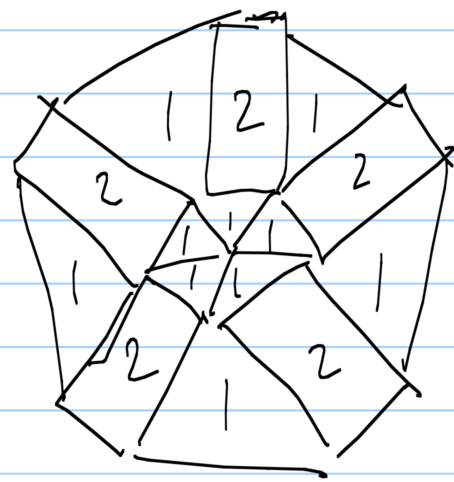
viewing 600-cell from a corner:

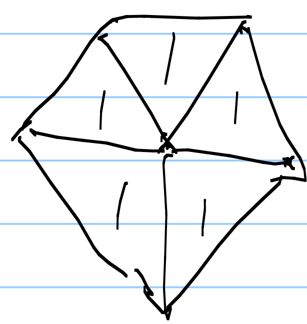
1. An icosahedron



20 type 1 cells.

2. Near a corner





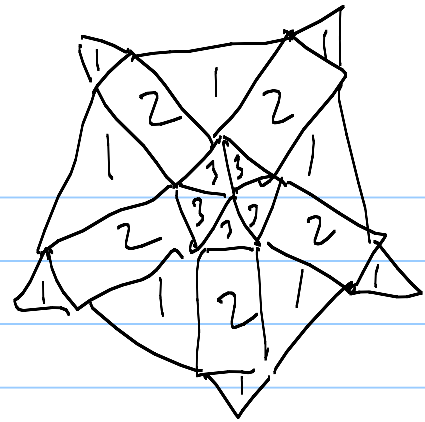
All new:
New cells:

30 type 2

20 type 1 (dim.)

60 expanding type 1

count so far: $\boxed{130}$



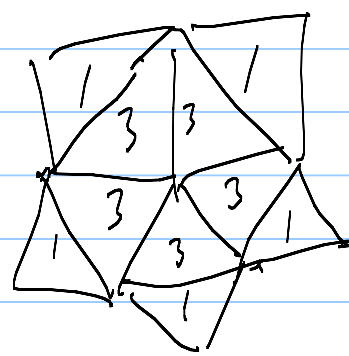
60 type 2

60 (dim) type 1

$\boxed{\text{60 type 3}}$

20 (exp.) type 1

running total $\boxed{330}$



↓ going back

$+ 330 - 60$

$= \boxed{600}.$