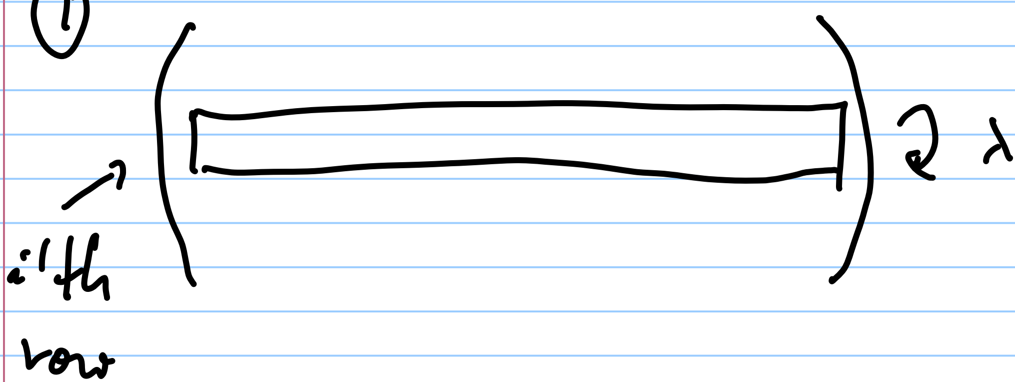
The reduced row echelon form (RREF):
of a matrix A

$$\mathrm{RREF}(A) = \begin{pmatrix} 1 & \boxed{\times\ \times} & 0 & \boxed{\times\ \times} & 0 & \boxed{*\ *} & \\ & & 1 & \boxed{\times} & 0 & \boxed{*\ *} & \cdots \\ & & & & 1 & \boxed{*\ *} & \\ & & & & & & \end{pmatrix}$$

The blank part is 0.

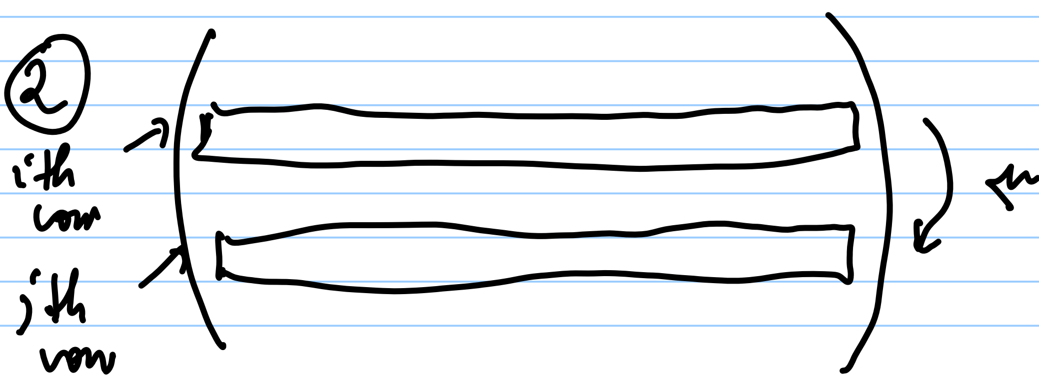
We can get from A to RREF(A) by performing <u>elementary row operations</u>:

(1)



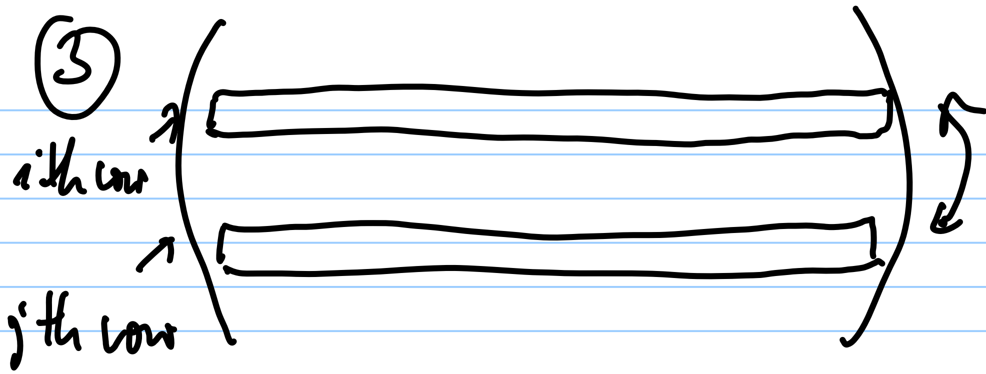
multiply the i'th row by $\lambda \neq 0$

(2)



add $\mu \cdot$ the i'th row to the j'th row

③



swap the ith and jth row.

Using the elementary row operations, we can get $A \rightsquigarrow \mathrm{RREF}(A)$.

The basic strategy is the Gauss elimination method, but still, we may vary the strategy slightly.

The <u>uniqueness theorem</u> says that the RREF(A) only depends on A, not on the exact method we used.

<u>Example</u>: $A =$

Method 1

$$\begin{pmatrix} 2 & 1 & 0 & 3 \\ 1 & 1 & 1 & 4 \\ 3 & 2 & 1 & 8 \end{pmatrix} \quad \tfrac{1}{2}$$

$$\begin{pmatrix} 1 & 1/2 & 0 & 3/2 \\ 1 & 1 & 1 & 4 \\ 3 & 2 & 1 & 8 \end{pmatrix} \quad -1 \quad -3$$

Method 2

$$\begin{pmatrix} 2 & 1 & 0 & 3 \\ 1 & 1 & 1 & 4 \\ 3 & 2 & 1 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 4 \\ 2 & 1 & 0 & 3 \\ 3 & 2 & 1 & 8 \end{pmatrix} \quad -2 \quad -3$$

$$\begin{pmatrix} 1 & 1/2 & 0 & 3/2 \\ 0 & 1/2 & 1 & 5/2 \\ 0 & 1/2 & 1 & 7/2 \end{pmatrix} \ ②2$$

$$\begin{pmatrix} 1 & 1/2 & 0 & 3/2 \\ 0 & 1 & 2 & 5 \\ 0 & 1/2 & 1 & 7/2 \end{pmatrix} \begin{array}{l} -1/2 \\ -1/2 \end{array}$$

$$\begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{array}{l} -5 \quad 1 \end{array}$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$=$$

$$\begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & -1 & -2 & -5 \\ 0 & -1 & -2 & -4 \end{pmatrix} \ ②-1$$

$$\begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & -1 & -2 & -4 \end{pmatrix} \begin{array}{l} -1 \\ 1 \end{array}$$

$$\begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{array}{l} -5 \quad 1 \end{array}$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

! ! ? ? ? ?

$$\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = -1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 0 & 3 \\ 1 & 1 & 1 & 4 \\ 3 & 2 & 1 & 8 \end{pmatrix}$$

A *linear combination* of $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \left(-1 \cdot \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right)$$

$$\begin{pmatrix} -2 \\ -1 \\ -3 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

In what context can we talk about linear combinations?

<u>Example</u>: n- column vectors $\nearrow \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$

n- row vectors $\longrightarrow (2\ 2\ 4)$

$m \times n$ matrices

$$2 \times 3: \quad 3 \cdot \begin{pmatrix} 2 & 1 & 4 \\ 2 & 2 & 1 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1/2 \end{pmatrix} =$$

$$\begin{pmatrix} 6 & 3 & 12 \\ 6 & 6 & 3 \end{pmatrix} + \begin{pmatrix} -1 & -1 & -2 \\ -1 & 0 & -1/2 \end{pmatrix} = \begin{pmatrix} 5 & 2 & 10 \\ 5 & 6 & 5/2 \end{pmatrix}$$

A <u>vector space</u> (over $\mathbb{R}$) is a set V

with an operation $+$:

associativity

$$\text{Axioms} \begin{cases} x + (y+z) = (x+y) + z & x, y, z \in V \\ x + y = y + x & \text{commutativity} \\ 0 + x = x + 0 = x & 0 \in V \end{cases}$$

$(-x) + x = 0$

For $x \in V$, $\lambda \in \mathbb{R}$, we have $\lambda x \in V$:

$1 \cdot x = x \qquad x \in V$

$\mu \cdot (\lambda \cdot x) = (\mu \lambda) \cdot x$

$(\lambda + \mu) \cdot x = (\lambda \cdot x) + (\mu \cdot x)$

$\lambda \cdot (x + y) = (\lambda \cdot x) + (\lambda \cdot y) \qquad x, y \in V$

$-\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix}$

<u>Theorem</u>: ① $\underset{\mathbb{R}}{0} \cdot \underset{V}{x} = \underset{V}{0}$.

$\in$ "is an element of"

② $\underset{\underset{\mathbb{R}}{\cap}}{(-1)} \cdot \underset{\underset{V}{\cap}}{x} = \underset{\underset{V}{\in}}{-x}$

Proof: ① $0 \cdot x = (0+0) \cdot x = 0 \cdot x + 0 \cdot x \quad \Big| - 0 \cdot x$

$0 = 0 \cdot x$

② We must prove that

$(-1) \cdot x + x = 0$

$(-1) \cdot x + x = (-1) \cdot x + 1 \cdot x = ((-1) + 1) \cdot x = 0 \cdot x \overset{\text{By (1)}}{=} 0.$ $\square$

A *linear combination* of $x_1, \dots, x_n \in V$ where V is a vector space is an expression of the form

$$\lambda_1 x_1 + \dots + \lambda_n x_n \,, \quad \underbrace{\overset{\text{scalars}}{\lambda_1, \dots, \lambda_n}}_{\text{these are called coefficients.}} \in \mathbb{R}$$

A linear combination is the most general operation in a vector space.

More examples of vector spaces:

The set of solutions of a system of homogeneous linear equations:

$\uparrow$

the right hand side is 0.

Example:

$$x + 2y - 3z + t = 0$$
$$x + y + + 2t = 0$$

$$\begin{pmatrix} 1 & 2 & -3 & 1 \\ 1 & 1 & 0 & 2 \end{pmatrix} \quad -1$$

$$\begin{pmatrix} 1 & 2 & -3 & 1 \\ 0 & -1 & 3 & 1 \end{pmatrix} \quad ② \ -1$$

$$\begin{pmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & -3 & -1 \end{pmatrix} \quad -2$$

$$\begin{array}{c} \begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & -3 & -1 \end{pmatrix} \\ \begin{array}{cccc} x & y & z & t \\ & & \uparrow & \uparrow \\ & & A & B \end{array} \end{array}$$

$$\begin{array}{c} x \\ y \\ z \\ t \end{array} \begin{pmatrix} -3A-3B \\ 3A+B \\ A \\ B \end{pmatrix}$$

Add:

Example:

$A = 1$
$B = 2$
$$\begin{pmatrix} -9 \\ 5 \\ 1 \\ 2 \end{pmatrix}$$

$A = 3$
$B = 0$
$$\begin{pmatrix} -9 \\ 9 \\ 3 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -18 \\ 14 \\ 4 \\ 2 \end{pmatrix} \begin{matrix} \\ \\ \leftarrow A = 4 \\ \leftarrow B = 2 \end{matrix}$$

$$3 \cdot \begin{pmatrix} -9 \\ 5 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -27 \\ 15 \\ 3 \\ 6 \end{pmatrix}$$

$A = 3$

$B = 6$

Quiz on solving linear equations on Wed

HW: ← all HW due this Wed. in class (no late HW accepted grace period: until the grader picks it up)

Put to RREF:

$$\begin{pmatrix} 1 & -1 & 2 & 3 & 1 \\ 2 & -2 & 3 & 5 & 1 \\ 3 & -3 & 5 & 8 & 3 \end{pmatrix}.$$