MATH 417 10/14/2015

Note Title 10/14/2015

<u>Quiz</u>: Let $f : \underset{B}{\mathbb{R}^3} \to \underset{C}{\mathbb{R}^3}$ be the linear transformation given by

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ x-z \\ y+z \end{pmatrix}.$$

The basis B: $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ | C: $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

Find the matrix ${}_B f_C$.

Challenge (not graded): Let $V \subseteq \mathbb{R}^3$ be the plane $x+y+2z=0$

$W \subseteq \mathbb{R}^3$ be the plane $x+y+z=0$

Let $B: \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ be a basis of V

$C: \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ be a basis of W

Find ${}_B f_C$ where $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ y+z \\ -x-2y-z \end{pmatrix}$

Solution of Challenge:

$$f\begin{pmatrix}2\\0\\-1\end{pmatrix}=\begin{pmatrix}2\\-1\\-1\end{pmatrix} \qquad f\begin{pmatrix}0\\2\\-1\end{pmatrix}=\begin{pmatrix}2\\1\\-3\end{pmatrix}$$

$$\left(C \mid f(B)\right) = \left(\begin{array}{cc|cc}1 & -1 & 2 & 2\\ -1 & 0 & -1 & 1\\ 0 & 1 & -1 & -3\end{array}\right) \sim \left(\begin{array}{cc|cc}1 & -1 & 2 & 2\\ 0 & -1 & 1 & 3\\ 0 & 1 & -1 & -3\end{array}\right) \sim$$

$$\sim \left(\begin{array}{cc|cc}1 & -1 & 2 & 2\\ 0 & 1 & -1 & -3\\ 0 & 0 & 0 & 0\end{array}\right) \overset{\uparrow 1}{\sim} \left(\begin{array}{cc|cc}1 & 0 & 1 & -1\\ 0 & 1 & -1 & -3\\ 0 & 0 & 0 & 0\end{array}\right)$$

$${}_{B}f_{C} = \begin{pmatrix}1 & -1\\ -1 & -3\end{pmatrix}$$

The consistency of even/odd permutations.

An intrinsic characterization of an even or odd permutation:

$$\begin{aligned} 1 &\longrightarrow 5 \\ 2 &\longrightarrow 4 \\ 3 &\longrightarrow 1 \\ 4 &\longrightarrow 3 \\ 5 &\longrightarrow 2 \end{aligned}$$

Which pairs are reversed by the permutation?

$\cancel{\{1,2\}}$, $\cancel{\{1,3\}}$, $\{1,4\}$, $\{1,5\}$,
$\{2,3\}$, $\{2,4\}$, $\{2,5\}$, $\{3,4\}$,
$\{3,5\}$, $\{4,5\}$

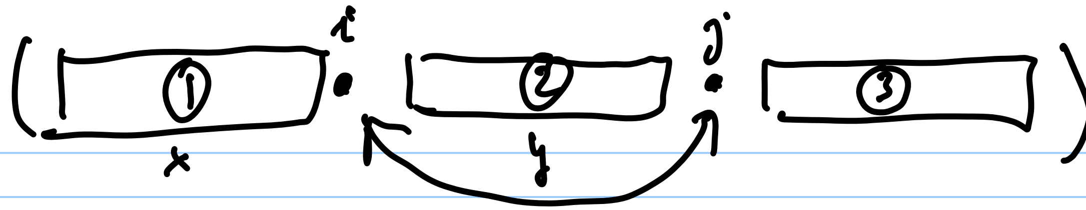
$$\binom{5}{2} = \frac{5\cdot 4}{2} = 10$$

$$\binom{n}{2} = \frac{n\cdot(n-1)}{2}$$

8 reversed: even

(Double-check: $\begin{matrix}5\\4\\1\\3\\2\end{matrix} \to \begin{matrix}1\\4\\5\\3\\2\end{matrix} \to \begin{matrix}1\\2\\5\\3\\4\end{matrix} \to \begin{matrix}1\\2\\3\\5\\4\end{matrix} \to \begin{matrix}1\\2\\3\\4\\5\end{matrix}$ ✓ even)

For consistency, we need to prove that if we switch two elements in a permutation, the number of pairs reverse will change by an odd number.



How does it affect the reversal of pairs?

- A pair in zones 1, 2, 3 (not containing i or j) is not affected.
- A pair $\{x, i\}$ or $\{x, j\}$ where x is in zone 1 or 3 is not affected.
- A pair $\{y, i\}$ or $\{y, j\}$ where y is in zone 2 is reversed!!! ← an even number

$\sim \{i', j'\}$ is reversed

Total number of reversals is odd.

Recall

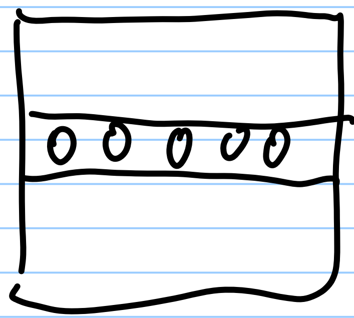
$$\det \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \sum_{\substack{\text{permutations} \\ \sigma : \{1, \dots n\} \to \{1, \dots, n\}}} \overset{\substack{+\text{ even}, -\text{ odd} \\ \downarrow}}{\operatorname{sign}(\sigma)} \underbrace{a_{1\sigma(1)} \cdots a_{n\sigma(n)}}_{\substack{\text{the rook arrangement} \\ \text{of the permutation}}}$$

The most important thing about the determinant

$\Rightarrow \det(AB) = \det(A)\det(B).$

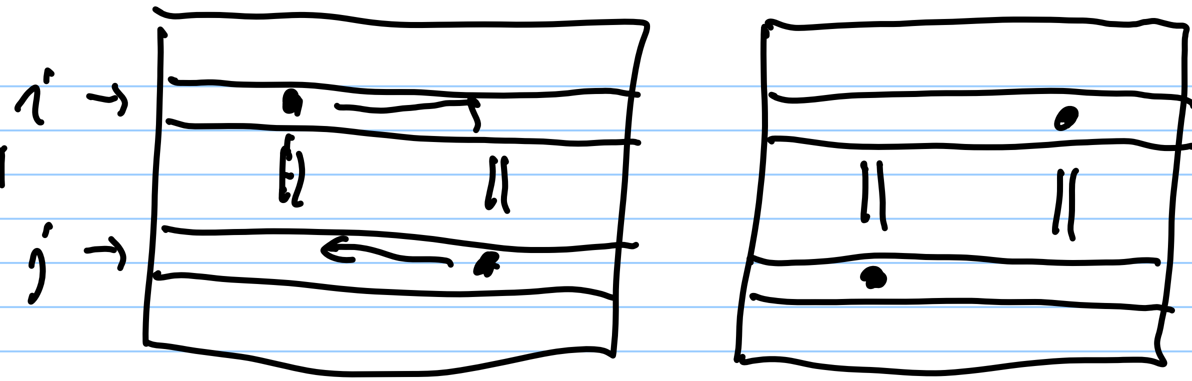
Relating determinants to row operations.

① If a matrix A has a zero row, then $\det(A) = 0$.



there will be a rook in the zero row.

② When two rows of the matrix A are the same then $\det A = 0$.



same rows of A

switch the coefs in these two rows:

in the determinant, you have the same monomials with opposite signs

they cancel out!

(3) If a matrix B is obtained from a matrix A by multiplying a row by λ then

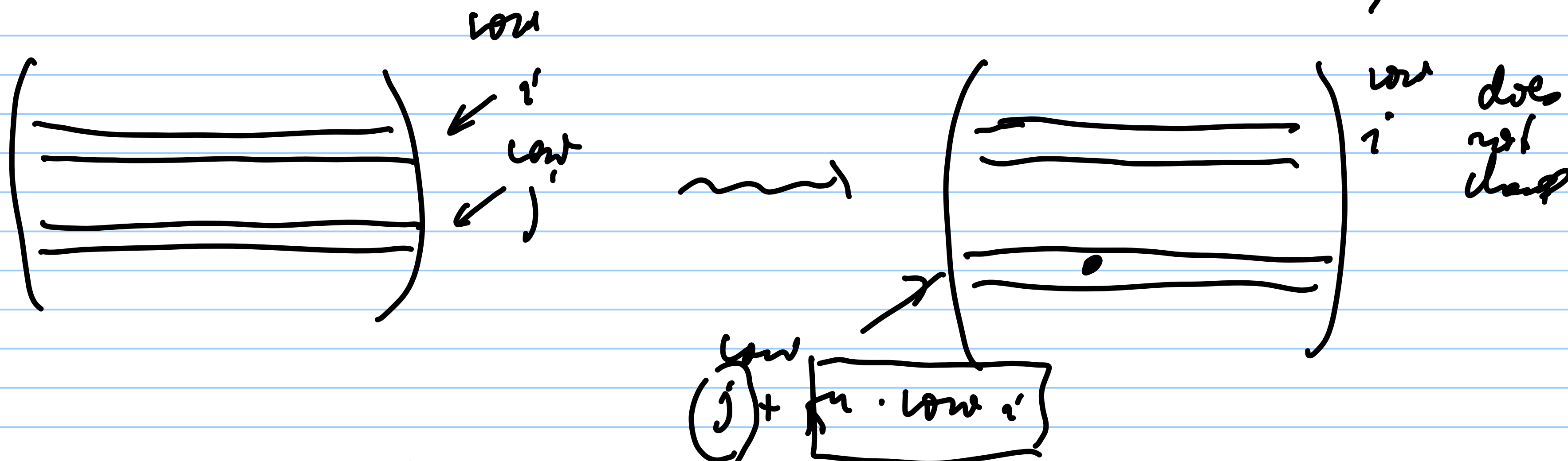
$$\det B = \lambda \det A$$

(every monomial goes up λ times)

What do row operations do to a determinant:

- multiply a row by λ $\rightsquigarrow$ determinant is multiplied by λ
- add a multiple of one row to another $\rightsquigarrow$ determinant is unchanged

• switch two rows ⇝ determinant changes sign /



each element in row j : the original element

this goes away in determinant { + μ · the element from row i and the same column by property 2 and 3.

HW:

$1 \mapsto 6$
$2 \mapsto 1$
$3 \mapsto 4$
$4 \mapsto 3$
$5 \mapsto 2$
$6 \mapsto 5$

How many pairs are reversed? (Is it even or odd?)