MATH 417 10/30/2015

Note Title 10/30/2015

Eigenvalues

To calculate eigenvalues of a matrix A, we find the root of the polynomial $\det(xI - A)$.

factor

$x^2 - 2x + 1 = (x-1)^2 = (x-1)(x-1)$

Root: $x = 1$ of multiplicity 2.

if an eigenvalue is a root of multiplicity 2 or more, it is called <u>degenerate</u>.

(the multiplicity of the root is called the <u>algebraic</u> multiplicity of the eigenvalue).
always ≥ 1.

<u>Example</u>: Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 5 \end{pmatrix}.$$

$$\det \begin{pmatrix} x-2 & -1 & -2 \\ -1 & x-2 & -2 \\ -2 & -2 & x-5 \end{pmatrix} = (x-2)(x-2)(x-5) - 4 - 4$$

$$-4(x-2) - (x-5) - 4(x-2) =$$

$$= \cancel{x^3} - \cancel{9x^2} + \cancel{24x} - \cancel{20} - \cancel{8} - \cancel{4x} + \cancel{8} - \cancel{x} + \cancel{5} - \cancel{4x} + \cancel{8} =$$

$$= x^3 - 9x^2 + 15x - 7$$

$$
\begin{array}{r}
x^2 - 8x + 7 \\
x - 1 \enclose{longdiv}{x^3 - 9x^2 + 15x - 7} \\
-x^3 + x^2 \qquad\qquad \\
\hline
-8x^2 + 15x - 7 \\
8x^2 - 8x \qquad \\
\hline
7x - 7 \\
0
\end{array}
$$

$x = 1$
$x = 1$ } multiplicity 2 (a degenerate eigenvalue)
$x = 7$

$$\begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 5 \end{pmatrix} - \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix}$$

Solution space:

2 eigenvectors

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}}$$

$$\begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 5 \end{pmatrix} - \begin{pmatrix} 7 & & \\ & 7 & \\ & & 7 \end{pmatrix} = \begin{pmatrix} -5 & 1 & 2 \\ 1 & -5 & 2 \\ 2 & 2 & -2 \end{pmatrix} \qquad \boxed{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}$$

Solution space:

$$\begin{pmatrix} 1 & -5 & 2 \\ 0 & -24 & 12 \\ 0 & 12 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & -5 & 2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$p = 2$

We have a basis of eigenvectors

$$A \begin{pmatrix} -1 & -2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 7 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} -1 & -2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}^{-1} A \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 7 \end{pmatrix}}$$

A diagonalizable.

<u>Example:</u> Find the eigenvalues and eigenvectors of:

$$A = \begin{pmatrix} 4 & -1 \\ 4 & 0 \end{pmatrix}$$

$$\det \begin{pmatrix} x-4 & 1 \\ -4 & x \end{pmatrix} = (x-4)\,x + 4 = x^2 - 4x + 4$$

$$= (x-2)^2$$

degenerate eigenvalue 2 of algebraic multiplicity 2.

$$\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} : \boxed{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} \leftarrow \text{spans the eigenspace of } 2$$

$\boxed{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} = v$

1-dimensional

eigenspace: all the eigenvectors of a given eigenvalue, and 0.

$\underbrace{\text{all the eigenvectors of a given eigenvalue, and } 0.}_{\text{solution space of}}$

$\boxed{\text{We say: the } \underline{\text{geometric}} \text{ multiplicity of the eigenvalue } \lambda = 2 \text{ is } 1.}$

This matrix is <u>not</u> diagonalizable.

Generalized eigenvectors:

If A has an eigenvalue λ, then eigenvectors are solutions of $(A-\lambda I)v = 0$.

Generalized eigenvectors are solutions of $(A-\lambda I)^k v = 0$.

Turns out, we have always a basis of generalized eigenvectors.

$$\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\shortparallel$

$(A - \lambda I)$

$\lambda = 2$

In our case:

$(A - \lambda I)^2 = 0$

Another generalised eigenvector w:

$$(A - \lambda I) w = v$$

$$\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} w = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 2 & -1 & 1 \\ 4 & -2 & 2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1/2 & 1/2 \\ 0 & 0 & 0 \end{array}\right)$$

$$\begin{array}{ccc} & \uparrow & \\ & 0 & \end{array}$$

$$\boxed{w = \begin{pmatrix} +1/2 \\ 0 \end{pmatrix}}$$

$$\underbrace{(A-\lambda I)}_{\begin{pmatrix}2 & -1\\ 4 & -2\end{pmatrix}}\begin{array}{c}\begin{pmatrix}1 & +1/2\\ 2 & 0\end{pmatrix}\\ \begin{array}{cc}\uparrow & \uparrow\\ v & w\end{array}\end{array} = \begin{array}{c}\begin{pmatrix}1 & 1/2\\ 2 & 0\end{pmatrix}\\ \begin{array}{cc}\uparrow & \uparrow\\ v & w\end{array}\end{array}\cdot\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = \begin{array}{c}\begin{pmatrix}0 & 1\\ 0 & 2\end{pmatrix}\\ \begin{array}{cc}\uparrow & \uparrow\\ 0 & v\end{array}\end{array}$$

$$A\begin{pmatrix}1 & 1/2\\ 2 & 0\end{pmatrix} = \begin{pmatrix}1 & 1/2\\ 2 & 0\end{pmatrix}\cdot\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix} =$$

$$\begin{pmatrix}0 & 1\\ 0 & 2\end{pmatrix} + 2\cdot\begin{pmatrix}1 & 1/2\\ 2 & 0\end{pmatrix}$$

$$\begin{pmatrix} 1 & 1/2 \\ 2 & 0 \end{pmatrix}^{-1} A \begin{pmatrix} 1 & 1/2 \\ 2 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}}_{\text{Jordan block.}}$$

A most general Jordan block:

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ 0 & & & 1 \\ & & & \lambda \end{pmatrix}$$

has an eigenvector: $\underset{e_1}{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}$ $\underset{e_2}{\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}}$

$$J_k(\lambda) e_1 = \lambda e_1$$
$$J_k(\lambda) e_2 = \lambda e_2 + e_1$$
$$J_k(\lambda) e_3 = \lambda e_3 + e_2$$
$$\vdots$$
$$J_k(\lambda) e_k = \lambda e_k + e_{k-1}$$

There is an eigenvalue λ of algebraic multiplicity k, geometric multiplicity 1.

$$J_k(\lambda)\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}=\begin{pmatrix}1\\\lambda\\0\\\vdots\\0\end{pmatrix}$$

Generalised eigenvectors of $J_k(\lambda)$:

$$e_1, \ldots, e_k.$$

HW ① Diagonalise: $\begin{pmatrix}3&1&3\\1&3&3\\3&3&11\end{pmatrix}.$

(2) Find Q such that $Q^{-1}AQ$ is a Jordan block:

$$\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}.$$