MATH 417 11/2/2015

Note Title 11/2/2015

No quiz Wed.

A Jordan block is of the form

$$J_n(\lambda) = \underbrace{\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}}_{n} n$$

$n=3 \quad \lambda = 2$

$$J_3(2) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$J_n(\lambda)$ has only one eigenvalue λ which has

$\underline{\text{algebraic multiplicity } n}$ (it has multiplicity n as a root of the polynomial $\underbrace{\det(Ix - A)}_{\text{this is called the } \underline{\text{characteristic polynomial}}}$)

$A = J_n(\lambda)$

In fact, the eigenspace has dimension 1:

$$\begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \lambda \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$J_n(\lambda) \cdot e_1 = \lambda e_1$$

but the eigenspace is spanned by e_1, hence has dimension 1. We say the <u>geometric multiplicity</u> of the eigenvalue λ is 1.

The concept of direct sum of $\underbrace{\text{matrices}}_{\text{(square)}}$ A and B.

$$A \oplus B = \begin{pmatrix} \boxed{A} & 0 \\ 0 & \boxed{B} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 & 3 \\ 3 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$$

QAQ^{-1}

<u>Theorem</u>: Every complex matrix A is similar to a matrix which is a <u>direct sum of one, two or more Jordan blocks</u>. The sizes and numbers of Jordan blocks are uniquely determined by the matrix A.

This is called the Jordan canonical form

Examples of matrices in Jordan form:

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i \end{pmatrix}$$

Example:

What are all the possible Jordan forms of 4×4 matrices which have a single eigenvalue 2 of algebraic multiplicity 4?

$$A_1=\begin{pmatrix} \boxed{\begin{matrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{matrix}} \end{pmatrix}, \quad A_2=\begin{pmatrix} \boxed{\begin{matrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{matrix}} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & \boxed{2} \end{pmatrix}, \quad A_3=\begin{pmatrix} \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} \end{pmatrix},$$

switching the blocks is <u>not</u> considered different (it is a similar matrix)

$$A_4 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

solution space

	A_1 $j=1$	A_2 $j=2$	A_3 $j=3$	A_4 $j=4$	A_5 $j=5$
$\dim \operatorname{Ker}(2I - A_j)$	1	2	2	3	4
$\dim \operatorname{Ker}((2I - A_j)^2)$	2	3	4	4	4
$\dim \operatorname{Ker}((2I - A_j)^3)$	3	4	4	4	4
$\dim \operatorname{Ker}((2I - A_j)^4)$	4	4	4	4	4

subspace $\to \langle e_1 \rangle = \operatorname{Ker}(2I - A_1)$

spanned $\to \langle e_1, e_2 \rangle = \operatorname{Ker}(2I - A_1)^2$

by $\to \langle e_1, e_2, e_3 \rangle = \operatorname{Ker}(2I - A_1)^3$

the given $\qquad \vdots$

vectors

number of

What is the pattern?

$\dim \operatorname{Ker}(\lambda I - A) =$ # Jordan blocks with eigenvalue λ

$\dim \operatorname{Ker}(\lambda I - A)^2 - \dim \operatorname{Ker}(\lambda I - A) =$ # Jordan blocks with eigenvalue λ of size ≥ 2

$\dim \ker (\lambda I - A)^3 - \dim \ker (\lambda I - A)^2 = \#$ Jordan blocks with eigenvalue λ of size ≥ 3.

$\vdots$

<u>Example</u>: Let

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ 2 & 2 & -1 \end{pmatrix}.$$

Find the Jordan form of A.

<u>Solution</u>:

$$\chi(A) = \det \begin{pmatrix} x-2 & -1 & 1 \\ -1 & x-2 & 1 \\ -2 & -2 & x+1 \end{pmatrix} = (x-2)(x-2)(x+1) + 2 + 2$$

characteristic polynomial

$$+ 2(x-2) + 2(x-2) - (x+1) =$$

$$= x^3 - 3x^2 + 4 + 4 + 2x - 4 + 2x - 4 - x - 1$$

$$= x^3 - 3x^2 + 3x - 1 = (x-1)^3.$$

$$1 \cdot I - A = \begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ -2 & -2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 \\ & 0 & \\ & & \end{pmatrix}$$

$\dim \ker(I - A) = 2$

$\dim \ker(I - A)^2 = 3$

$$(I-A)^2 = \begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ -2 & -2 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \checkmark$$

2 blocks

$3-2=1$ block of size ≥ 2

Jordan form:

$$\begin{pmatrix} \boxed{\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \boxed{1} \end{pmatrix}.$$

<u>Example</u>: Suppose I have a 10×10 matrix A with an eigenvalue 4 of algebraic multiplicity 6 and an eigenvalue i of algebraic multiplicity 4.

Suppose

	$\dim \operatorname{Ker}(4I - A)^k$	$\dim \operatorname{Ker}(iI - A)^k$
$k = 1$	3	2
2	5	4
3	6	4
4	6	4
5	6	4
6	6	4

What is the Jordan form:

$\lambda = 4$: $3-0 = \boxed{3}$ blocks

$5-3 = \boxed{2}$ blocks ≥ 2

$6-5 = \boxed{1}$ block ≥ 3

0

$3, 2, 1$

one block of each size

$0 \;>\; 3$

$3 \;>\; 2 \;>\; 1$ size 1

$5 \;>\; 1 \;>\; 1$ size 2

$6 \;>\; 0 \;>\; 1$ size 3

6

$0 \;>\; 2$

$2 \;>\; 2 \;>\; 0$

$4 \;>\; 0 \;>\; 2$

4

2 blocks of size 2

$$\begin{pmatrix} \boxed{\begin{matrix}4&1&0\\0&4&1\\0&0&4\end{matrix}} & & & & \\ & \boxed{\begin{matrix}4&1\\0&4\end{matrix}} & & \text{\Large 0} & \\ & & \boxed{4} & & \\ & \text{\Large 0} & & \boxed{\begin{matrix}i&1\\0&i\end{matrix}} & \\ & & & & \boxed{\begin{matrix}i&1\\0&i\end{matrix}} \end{pmatrix}.$$

HW: Find the Jordan form of

(1)

$$A = \begin{pmatrix} 2 & -1 & 1 & -1 \\ -2 & 1 & 2 & -2 \\ -4 & -4 & 7 & -4 \\ -1 & -1 & 1 & 2 \end{pmatrix}$$

② Suppose a 9×9 matrix A has two eigenvalues $1+i$ and $1-i$ and the dimensions of generalized eigenspaces are :

$\operatorname{Ker}\left(((1+i)I - A)^k\right)$ $\qquad\qquad$ $\operatorname{Ker}\left(((1-i)I - A)^k\right)$

$k = 1$ $\qquad$ 3 $\qquad\qquad\qquad\qquad$ 2

2	5	2
3	7	2
4	7	2
5	7	2
6	7	2
7	7	2