MATH 417 11/6/2015

Note Title 11/6/2015

The matrix

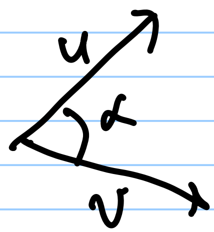
$$Q = \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$$

any two different ones have dot product 0.

has columns which are orthogonal and have length 1. } orthonormal basis.

Two column vectors $u = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, $v = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

have dot product $u \cdot v = u^T v = v^T u = a_1 b_1 + \cdots + a_n b_n$



$$u \cdot u = \|u\|^2$$

$$\cos\alpha = \frac{u \cdot v}{\|u\| \, \|v\|}$$

$$QQ^T = Q^T Q = I$$

$$Q = \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$$

$$Q^TQ = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In other words, the matrix Q satisfies

$$Q^{-1} = Q^T$$

Such a matrix is called an <u>orthogonal matrix</u>.

Proposition: If Q is an orthogonal matrix $n\times n$ then for any two column vectors u, v $n\times 1$

$$(Qu)\cdot(Qv) = u\cdot v$$

Comment:

This means, the linear transformation $\mathbb{R}^n \to \mathbb{R}^n$ with matrix Q with respect to the standard basis in the domain and codomain preserves lengths and angles. Such a linear transformation is called an <u>isometry</u>. Conversely, an isometry which keeps the origin fixed is a linear transformation given by an orthogonal matrix.

To be realisable by a move "in the real world" $n=3$ we have to have $\det Q = 1$.

We always have $(\det Q)^2 = \det Q \det Q^T =$

$$= \det QQ^T = \det I = 1.$$

$\det Q = \pm 1$.

Proof:

$$(Qu)\cdot(Qv) = (Qu)^T Qv = u^T \underbrace{Q^T Q}_{I} v =$$

$$= u^T I v = u^T v = u \cdot v. \quad \square$$

To summarise: If A is a symmetrical real matrix ($a_{ij} = a_{ji}$) then A is diagonalisable by a real orthogonal matrix:

$$Q^{-1} A Q = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Q is a real matrix $\quad QQ^T = I$

$$Q^{-1} = Q^T.$$

The elements of the orthonormal basis consisting of eigenvectors are called <u>principal axes</u>.

this has applications in geometry, probability and statistics.

What to do if the matrix has degenerate eigenvalues

symmetric real

Example: Find principal axes.

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 3 & 2 \\ 2 & 2 & 6 \end{pmatrix}.$$

"Cheat": The product of eigenvalues is the determinant.

$$\det A = 3\cdot3\cdot6 + 1\cdot2\cdot2 + 1\cdot2\cdot2 - 2\cdot3\cdot2 - 1\cdot1\cdot6 - 2\cdot2\cdot3$$

$$= 54 + 4 + 4 - 12 - 6 - 12 =$$

$$= 62 - 30 = 32$$

$$\det(xI - A) = \det\begin{pmatrix} x-3 & -1 & -2 \\ -1 & x-3 & -2 \\ -2 & -2 & x-6 \end{pmatrix}$$

$$= (x-3)(x-3)(x-6) - 4 - 4 - 4(x-3) - (x-6) - 4(x-3)$$

$= x^3 - 12x^2 + 45x - 54 - 8 - 9x + 30 =$

$= x^3 - 12x^2 + 36x - 32$

$8 - 48 + 72 - 32 = 0$

$\boxed{x = 2, 2, 8}$

$$
\begin{array}{r}
x^2 - 10x + 16 \\
x - 2 \,\big)\, \overline{x^3 - 12x^2 + 36x - 32} \\
-x^3 + 2x^2 \\
\hline
-10x^2 + 36x - 32 \\
10x^2 - 20x \\
\hline
16x - 32
\end{array}
$$

$$\begin{array}{r} x - 8 \\ x-2 \;\big)\; \overline{x^2 - 10x + 16} \\ -x^2 + 2x \quad\; \\ \hline -8x + 16 \end{array}$$

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 3 & 2 \\ 2 & 2 & 6 \end{pmatrix}.$$

$$2I - A = \begin{pmatrix} -1 & -1 & -2 \\ -1 & -1 & -2 \\ -2 & -2 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 \\ & 0 & \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

these vectors are <u>not</u> orthogonal !!!

Orthogonalising a basis: Gram-Schmidt orthogonalisation process

Orthogonal row echelon form: transpose the vectors so we can work with rows:

$$\begin{pmatrix} -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \end{pmatrix} \text{ RREF}$$

$$2$$

My approach: keep away from fractions

$$\begin{array}{c} u_1 \\ u_2 \end{array} \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \end{pmatrix}$$

don't worry about length 1 except in the end

$u_1 \cdot u_1 = 1$

$$u_2 \cdot u_2 = 5$$

replace u_1 by $(5u_1 - u_2)$

$$(5u_1 - u_2) \cdot u_2 = 5 u_1 \cdot u_2 - u_2 \cdot u_2 = 0$$

$$\begin{pmatrix} 10 & -2 & -4 \\ 0 & 2 & -1 \end{pmatrix}$$

lengths $\sqrt{120}$, $\sqrt{5}$

Normalize and transposing back:

$$\begin{pmatrix} 10/\sqrt{120} \\ -2/\sqrt{120} \\ -4/\sqrt{120} \end{pmatrix}, \begin{pmatrix} 0 \\ 2/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 3 & 2 \\ 2 & 2 & 6 \end{pmatrix}.$$

$$\lambda = 8$$

$$\begin{pmatrix} 5 & -1 & -2 \\ -1 & +5 & -2 \\ -2 & -2 & 2 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & 6 & -3 \\ 0 & -6 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & 6 & -3 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{array}{c} \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ & & \end{pmatrix} \\ \uparrow \\ 2 \end{array}$$

$$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \xrightarrow{\text{normalize}} \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}$$

Principal axes:

$$Q = \begin{pmatrix} 10/\sqrt{120} & 0 & 1/\sqrt{6} \\ -2/\sqrt{120} & 2/\sqrt{5} & 1/\sqrt{6} \\ -4/\sqrt{120} & -1/\sqrt{5} & 2/\sqrt{6} \end{pmatrix} \quad \begin{pmatrix} 10/\sqrt{120} \\ -2/\sqrt{120} \\ -4/\sqrt{120} \end{pmatrix}, \begin{pmatrix} 0 \\ 2/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}$$

$$Q^{-1} A Q = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}. \qquad Q^{-1} = Q^{T}$$

HW: Find the principal axes of the matrix:

$$A = \begin{pmatrix} 5 & 1 & 3 \\ 1 & 5 & 3 \\ 3 & 3 & 13 \end{pmatrix}.$$