MATH 417 11/13/2015

Example: Find the principal axes of

$$A = \begin{pmatrix} 3 & 1+i \\ 1-i & 4 \end{pmatrix}.$$

Solution:

$$\det(xI - A) = \det\begin{pmatrix} x-3 & -1-i \\ -1+i & x-4 \end{pmatrix} =$$

$$= (x-3)(x-4) - 2 = x^2 - 7x + 10 = (x-2)(x-5)$$

Eigenvalues: 2

$$\begin{pmatrix} -1 & -1-i \\ -1+i & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1+i \\ & \end{pmatrix}$$

eigenvector: $\begin{pmatrix} -1-i \\ 1 \end{pmatrix} = u$

normalize: $\begin{pmatrix} -(1+i)/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

5

$$\begin{pmatrix} 2 & -1-i \\ -1+i & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{-1-i}{2} \\ & \end{pmatrix}$$

$u \cdot v = (1-i)(-1)$ ✓
$+ 1 \cdot (-1-i) = 0$

$$\begin{pmatrix} -1 \\ -1+i \end{pmatrix} = v$$

normalise: $\begin{pmatrix} -1/\sqrt{3} \\ (-1+i)/\sqrt{3} \end{pmatrix}$

$$\bar{Q}^T A \underbrace{\begin{pmatrix} -(1+i)/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{3} & (-1+i)/\sqrt{3} \end{pmatrix}}_{Q} = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}.$$

unitary

(orientation of a unitary is always +, det is a complex number of length 1).

Example:

Find principal axes of: $A = \begin{pmatrix} 2 & 1+i & 2+i \\ 1-i & 3 & 3-i \\ 2-i & 3+i & 6 \end{pmatrix}$

$\mathrm{Re}\,(2+i)(1-i)(3+i) =$
$= 6 + 4 = 10$

$$\det(xI - A) = \begin{pmatrix} x-2 & -1-i & -2-i \\ -1+i & x-3 & -3+i \\ -2+i & -3-i & x-6 \end{pmatrix} =$$

$$= (x-2)(x-3)(x-6) - 2\cdot 10 - 5(x-3) - 2(x-6) - 10(x-2) =$$

$$= x^3 - 11x^2 + 36x - 36 - 20 - 5x + 15 - 2x + 12 - 10x + 20 =$$

$$= x^3 - 11x^2 + 19x - 9$$

$x = 1, 1, 9$

$$x^2 - 10x + 9$$

$$x - 1 \overline{\big)\, x^3 - 11x^2 + 19x - 9}$$

$$\underline{-x^3 + x^2}$$

$$-10x^2 + 19x - 9$$

$$\underline{10x^2 - 10x}$$

$$9x - 9$$

$$x - 1 \overline{\big)\, x^2 - 10x + 9}$$

...

$\boxed{\text{Eigenvalue } 1}$:

$$A = \begin{pmatrix} 2 & 1+i & 2+i \\ 1-i & 3 & 3-i \\ 2-i & 3+i & 6 \end{pmatrix}$$

$-1+i$

$\parallel$

$-(1-i)$

$-(2-i)$

$\parallel$

$-2+i$

$$\begin{pmatrix} 1 & 1+i & 2+i \\ 1-i & 2 & 3-i \\ 2-i & 3+i & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 1+i & 2+i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underset{u}{\begin{pmatrix} -1-i \\ 1 \\ 0 \end{pmatrix}}, \quad \underset{v}{\begin{pmatrix} -2-i \\ 0 \\ 1 \end{pmatrix}}$$

$(3+i)(2+i) =$

$= 5 + 5i$

$-6 - 6i + 5 + 5i$

$u \cdot v = (-1-i) \cdot (-2+i) = 3 + i$

$v \cdot v = 6$

keep v, replace u by $6\left(u - \frac{u \cdot v}{v \cdot v} v\right) =$

$$= 6u - (3+i) v = \begin{pmatrix} -1 - i \\ 6 \\ -3 - i \end{pmatrix}$$

$$(-1-i)(-2+i) = 3 + i$$

$$\begin{pmatrix} -1-i \\ 6 \\ -3-i \end{pmatrix} \cdot \begin{pmatrix} -2-i \\ 0 \\ 1 \end{pmatrix} = 0 \checkmark$$

<u>Normalise</u> :

$$\begin{pmatrix} (-1-i)/\sqrt{48} \\ 6/\sqrt{48} \\ (-3-i)/\sqrt{48} \end{pmatrix}, \begin{pmatrix} (-2-i)/\sqrt{6} \\ 0 \\ 1/\sqrt{6} \end{pmatrix}$$

$\boxed{\text{Eigenvalue: } 9}$

$$A = \begin{pmatrix} 2 & 1+i & 2+i \\ 1-i & 3 & 3-i \\ 2-i & 3+i & 6 \end{pmatrix}$$

$$\begin{pmatrix} 7 & -1-i & -2-i \\ -1+i & 6 & -3+i \\ -2+i & -3-i & 3 \end{pmatrix} \; -1 \sim \begin{pmatrix} 7 & -1-i & -2-i \\ 1 & 9+i & -6+i \\ -2+i & -3-i & 3 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 9+i & -6+i \\ 0 & -64-8i & 40-8i \\ 0 & 16-8i & -8-4i \end{pmatrix}$$

$(2-i)(9+i) =$

$= 19-7i$

$(2-i)(-6+i) =$

HW: 1 Find the principal axes of

$$A = \begin{pmatrix} 2 & 2+i \\ 2-i & 6 \end{pmatrix}$$

(2) Find the principal axes of

$$A = \begin{pmatrix} 3 & 1 & 1+i \\ 1 & 3 & 1+i \\ 1-i & 1-i & 5 \end{pmatrix}.$$