MATH 417 11/16/2015

Note Title 11/16/2015

Theorem: If $A$ is a Hermitian matrix ($A = \bar{A}^T$) then

① $A$ is diagonalizable and all its eigenvalues are real

② Eigenspaces with respect to different eigenvalues are orthogonal (using complex dot product)

③ There exists a unitary matrix $Q$ ($Q^{-1} = \bar{Q}^T$) such that $Q^{-1} A Q = D$ is diagonal.

Proof: (1) $Au = \lambda u \quad u \neq 0$

$\bar{u}^T \bar{A}^T = \bar{\lambda} \bar{u}^T$

$\|$

$\bar{u}^T A$

Eigenvalues are real.

$\cdot (\cdot\cdot)$

$\bar{u}^T A u = \lambda \underbrace{\bar{u}^T u}_{u \cdot u > 0 \in \mathbb{R}}$

$\| (\cdot\cdot).$

$\bar{\lambda} \bar{u}^T u$

$\therefore \lambda = \bar{\lambda}$

$\therefore \lambda \in \mathbb{R}.$

(2) $Au = \lambda u \qquad \lambda, \mu \in \mathbb{R}$

$Av = \mu v \quad \} \quad \underbrace{\bar{v}^T \bar{A}^T}_{\bar{v}^T A} = \mu \bar{v}^T$

$$\bar{v}^T A u \overset{\cdot(\cdot\,\cdot)}{=} \lambda \bar{v}^T u = \lambda\, v \cdot u$$

$(\cdot\,\cdot)\cdot \quad \|$

$$\mu \bar{v}^T u = \mu\, v \cdot u$$

if $\lambda \neq \mu$
then $\boxed{v \cdot u = 0}$

$$\mathbb{C} = \{ a + bi \mid a, b \in \mathbb{R} \}$$

$\mathbb{C}$ is a real vector space

$\underbrace{\mathbb{C} \cong \mathbb{R}^2}_{\text{as a real vector space}}$ basis: $1, i$

$$(a_1+b_1 i, \dots, a_n+b_n i)\cdot(c_1+d_1 i, \dots, c_n+d_n i) =$$

$$= (a_1+b_1 i)(c_1-d_1 i) + \dots + (a_n+b_n i)(c_n-d_n i) =$$

$$= a_1c_1 + b_1d_1 + \dots + a_nc_n + b_nd_n + i(\dots ? \dots)$$

$\therefore$ $\mathbb{C}^n \cong \mathbb{R}^{2n}$ as real vector spaces

if vectors in $\mathbb{C}^n$ are complex-orthogonal then the corresponding vectors in $\mathbb{R}^{2n}$ are orthogonal.

(not always vice versa).

(3) Note that a square matrix is unitary if and only if its columns are complex-orthogonal and of length 1.

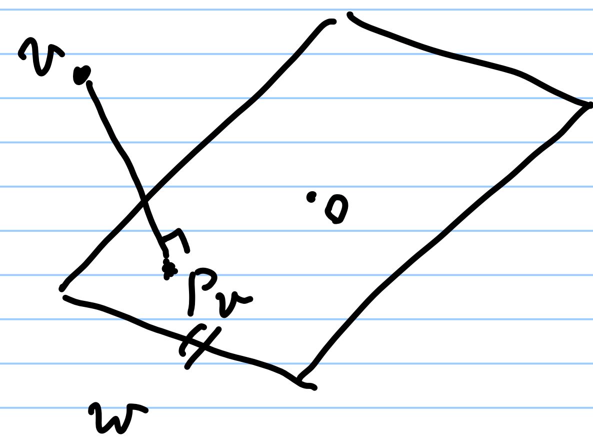
$$Q = \begin{pmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & & \vdots \\ q_{n1} & \cdots & q_{nn} \end{pmatrix} \qquad u_1 = \begin{pmatrix} q_{11} \\ \vdots \\ q_{n1} \end{pmatrix}, \dots, u_n = \begin{pmatrix} q_{1n} \\ \vdots \\ q_{nn} \end{pmatrix}$$

$\bar{Q}^T Q = I$ is equivalent to $\underset{\underset{u_j \cdot u_k}{\parallel}}{\bar{u}_j^T u_k} = 1$ if $j = k$

$= 0$ if $j \neq k$.

$\square$

# Orthogonal projection formula




$\swarrow$ plane through the origin in $\mathbb{R}^3$

One can find a formula for orthogonal projection onto any vector subspace $W \subseteq \mathbb{R}^n$.

we find a basis of $W$:

$$B: \; w_1 = \begin{pmatrix} a_{11} \\ \vdots \\ \vdots \\ a_{n1} \end{pmatrix}, \ldots, w_p = \begin{pmatrix} a_{1p} \\ \vdots \\ \vdots \\ a_{np} \end{pmatrix}$$

$p \leq n$

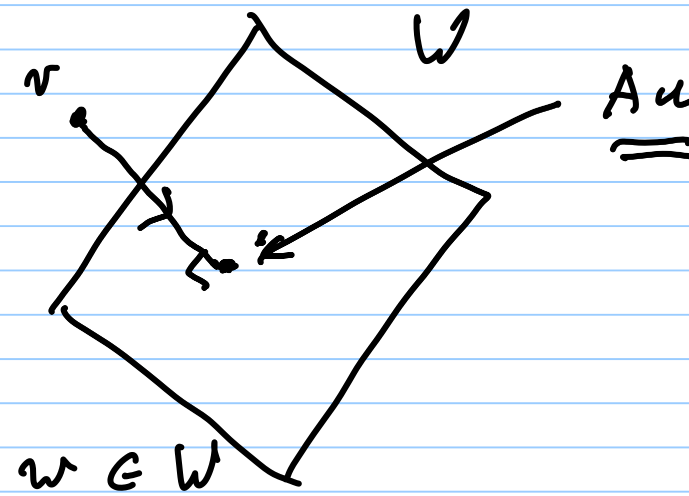
(interested in $p < n$)

Put

$$\underset{n \times p}{A} = \begin{pmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{np} \end{pmatrix}$$

$$W = \{ Au \mid u \in \mathbb{R}^p \} \qquad u = \begin{pmatrix} b_1 \\ \vdots \\ b_p \end{pmatrix}$$

$$Au = b_1 w_1 + \cdots + b_p w_p .$$




$$(v - Au) \cdot w = 0 \quad \text{for all } w \in W$$

$$(v - Au) \cdot (Au') = 0 \quad \text{for all } u' \in \mathbb{R}^p$$

with matrix multiplication:

$$(Au')^T (v - Au) = 0$$

$$\underbrace{\underset{p\times n}{A^T} \; \underset{n\times p}{A}}_{p\times p}$$

$$(u')^T A^T (v - Au) = 0$$

for every $u' \in \mathbb{R}^p$

$$\therefore \quad A^T (v - Au) = 0$$

$$A^T v - A^T A u = 0$$

$(A^T A)^{-1}$ ?

Gramm matrix, it is invertible when the columns of A are linearly independent (will prove).

$$(A^TA)^{-1}A^Tv - u = 0$$

$$u = (A^TA)^{-1}A^Tv$$

$\therefore$ The image of the projection is

$$Au = A(A^TA)^{-1}A^Tv.$$

The matrix of orthogonal projection

onto the column space of $A$ is:

$$\boxed{A(A^TA)^{-1}A^T.}$$

Example: Find the matrix of the orthogonal projection onto

$$\left\langle \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} \right\rangle \subseteq \mathbb{R}^4.$$

$$A = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} \qquad A^TA = \begin{pmatrix} 1 & 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} = 1+4+1+9 = 15$$

$$AA^T = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} \cdot (1\ 2\ 1\ 3) = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 2 & 6 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 9 \end{pmatrix}$$

Answer to the problem: $\begin{pmatrix} 1/15 & 2/15 & 1/15 & 3/15 \\ 2/15 & 4/15 & 2/15 & 6/15 \\ 1/15 & 2/15 & 1/15 & 3/15 \\ 3/15 & 6/15 & 3/15 & 9/15 \end{pmatrix}.$

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<u>Example</u>: Find the matrix of orthogonal projection in $\mathbb{R}^4$ onto the column space of

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

<u>Solution</u>: Gramm matrix:

$$A^T A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 6 \end{pmatrix}$$

$$(A^T A)^{-1} = \frac{1}{9} \begin{pmatrix} 6 & -3 \\ -3 & 3 \end{pmatrix}$$

$$A (A^T A)^{-1} A = \frac{1}{9} \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 6 & -3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 0 & 3 \\ 6 & -3 \\ -3 & 3 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix} = \underbrace{\frac{1}{9}\begin{pmatrix} 6 & 0 & 3 & 3 \\ 0 & 6 & -3 & 3 \\ 3 & -3 & 3 & 0 \\ 3 & 3 & 0 & 3 \end{pmatrix}}_{\text{Answer}}$$

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HW: (1) Find the matrix of orthogonal projection in $\mathbb{R}^4$ onto

$$W = \left\langle \begin{pmatrix} 1 \\ 1 \\ -1 \\ 2 \end{pmatrix} \right\rangle.$$

(2) Find the matrix of orthogonal projection in $\mathbb{R}^4$ onto

$$W = \left\langle \begin{pmatrix} 1 \\ 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\rangle.$$

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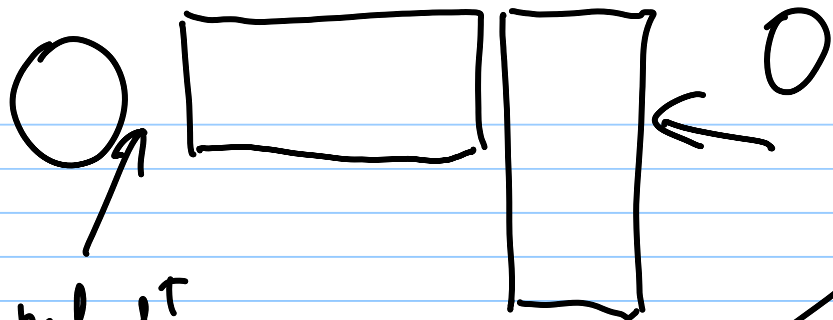
Why is the Gramm matrix invertible?

$A^T A$ where $A$ has independent columns.



put $A^T$
into
OREF
(using row operations

do not
affect
invertibility

use the same column
operation to put $A$
into $\underbrace{\text{OCEF}}_{(\text{OREF}^T)}$.

$\Downarrow$

$I$.

Correction to Friday's HW:

In the second problem, the matrix should be:

$$\begin{pmatrix} 3 & 1 & 1+i \\ 1 & 3 & 1+i \\ 1-i & 1-i & 4 \end{pmatrix}.$$