MATH 417 11/23/2015

Note Title 11/23/2015

Singular values

If A is a matrix, then its singular values are square root, in decreasing order, of
(non-increasing if degenerate)

eigenvalues of A^TA.

What do the singular values mean? Use principal axes: We find an orthogonal matrix Q such that

$$Q^T A^T A Q = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \qquad \lambda_1 \geq \cdots \geq \lambda_n \geq 0$$

$$\| \\ (AQ)^T AQ.$$

The linear transformation corresponding to Q does not change angles or distances. If the columns of Q are $u_1, \dots u_n$ $\Leftarrow$ they have length 1 and are orthogonal. Then

$$(Au_i)\cdot(Au_j) = (Au_i)^T Au_j = \begin{cases} 0 & \text{if } i \neq j \\ \lambda_i & \text{if } i = j. \end{cases}$$

This means $Au_1, \dots Au_n$ are orthogonal, have lengths $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \cdots, \sqrt{\lambda_n}$.

This is not the same thing as diagonalising A! (A may not even be square.) But we have found an orthonormal basis such that after applying A to this basis, we get again orthogonal vectors of lengths $\sqrt{\lambda_1} \geq \cdots \geq \sqrt{\lambda_n}$.

Starting with a vector u_1 on the unit sphere, applying A produces a vector of length at most $\sqrt{\lambda_1}$. Looking at a vector u_2 orthogonal to u_1, on the unit sphere, applying A produces a vector of length at most $\sqrt{\lambda_2}$ ... .

<u>Example</u>: Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

What is the maximal possible length of a vector

Au of $\|u\| = 1$?

Solution : This is the ~~top~~ singular value.

$$A^T A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 4 & 3 \end{pmatrix}$$

Calculating Eigenvalues of $\begin{pmatrix} 6 & 4 \\ 4 & 3 \end{pmatrix}$

$$\det \begin{pmatrix} x-6 & -4 \\ -4 & x-3 \end{pmatrix} = (x-6)(x-3) - 16 =$$

$$= x^2 - 9x + 2$$

$$\frac{9 \pm \sqrt{81-8}}{2} = \frac{9 \pm \sqrt{73}}{2}$$

Answer to problem: $\sqrt{\frac{9+\sqrt{73}}{2}}$.

Application of Hermitean matrices

Quantum mechanics

Our state (or the state of the system we are observing — usually small scale) is an element in a complex vector space: $\mathbb{C}^n$. Usually, in theory, n is infinite, but of course in any computation, it is finite. We can never observe our exact state. In fact, observable quantities are $n \times n$ <u>Hermitean matrices</u> M. When we measure something, we get one of the eigenvalues of M (recall, they are real numbers). There is randomness to the measurement you can get. Suppose, say, our actual state is a linear combination of two eigenvectors corresponding to eigenvalues $\lambda_1 \neq \lambda_2 \in \mathbb{R}$:

Our state: $v = a_1 v_1 + a_2 v_2$

v_1, v_2 are eigenvectors of eigenvalues λ_1, λ_2.

Replacing v by $\frac{v}{\|v\|}$, without loss of generality, $\|v\| = 1$. Then

$v_1 \perp v_2$

$$|a_1|^2 + |a_2|^2 = 1$$

These are the probabilities of measuring the value λ_1

and λ_2 in our observation. By doing the observation, we change the state to an eigenvector with the given eigenvalue.

↑ we make it true.

Collapse of state.

If you measure observables corresponding to Hermitean matrices which do not commute,

$$M_1 M_2 \neq M_2 M_1,$$

then the order of measurement affects the probabilities.

<u>Quantum encryption</u> - you can generate quantum randomness to generate practical unbreakable codes.

The quantum equation of motion is the Schrödinger equation. If u is the state of the system in $\mathbb{C}^n$, then

$$\frac{du}{dt} = -\frac{i}{\hbar} H u .$$

H is a particular observable (= Hermitean matrix) called the Hamiltonian. In mechanical systems, it can actually be written down from position and momentum.

it has the physical dimension of energy.

If H is a Hermitean matrix, what about iH?

<u>Example</u>: $\begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2i & i-1 \\ i+1 & 3i \end{pmatrix}$

$H = \bar{H}^T$ $\quad\quad$ $H = -\bar{H}^T$

skew Hermitian $\begin{pmatrix} -2i & 1-i \\ -1-i & -3i \end{pmatrix}$

The solution of the Schrödinger equation is a unitary transformation. The states are always transformed by a unitary matrix.

HW due 12/2

① Find two non-commuting Hermitian matrices. (Hint: the odds are overwhelmingly in your favor.)

(Next time: 11/25 Q & A.)