

MATH 417

11/25/2015

Note Title

11/25/2015

$$A = \begin{pmatrix} 2 & 1+i & 2+i \\ 1-i & 3 & 3-i \\ 2-i & 3+i & 6 \end{pmatrix}$$

Q: Finish the 3×3 Hermitian matrix example from 11/13

$$\lambda = 9$$

$$\begin{pmatrix} 7 & -1-i & -2-i \\ -1+i & 6 & -3+i \\ -2+i & -3-i & 3 \end{pmatrix} \xrightarrow{J^{-1}} \begin{pmatrix} 7 & -1-i & -2-i \\ 1 & 9+i & -6+i \\ -2+i & -3-i & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 9+i & -6+i \\ 7 & -1-i & -2-i \\ -2+i & -3-i & 3 \end{pmatrix} \begin{matrix} \downarrow -7 \\ \\ \downarrow \end{matrix} \begin{matrix} \\ \\ 2-i \end{matrix} \sim \begin{pmatrix} 1 & 9+i & -6+i \\ 0 & -64-8i & 40-8i \\ 0 & 16-8i & -8+8i \end{pmatrix} \sim$$

$$(9+i)(2-i) = 19-7i$$

$$(-6+i)(2-i) = -11+8i$$

$$\frac{(5-i)(-1-i)}{(-1+i)(-1-i)} = \frac{-6-4i}{2} =$$

$$= -3-2i$$

$$(3+2i)(2-i) = 8+i$$

$$\sim \begin{pmatrix} 1 & 9+i & -6+i \\ 0 & -8-i & 5-i \\ 0 & 2-i & -1+i \end{pmatrix} \begin{matrix} \\ \\ \uparrow \end{matrix} \begin{matrix} \\ \\ 3+2i \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 9+i & -6+i \\ 0 & 2-i & -1+i \\ 0 & 0 & 0 \end{pmatrix} \cdot \frac{1}{2-i}$$

$$\frac{(-1+i)(2+i)}{(2-i)(2+i)} = \frac{-3+i}{5}$$

$$\sim \begin{pmatrix} 1 & 9+i & -6+i \\ 0 & 1 & \frac{-3+i}{5} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2+i \\ 3-i \\ 5 \end{pmatrix}$$

↑
5

$$(-9-i)(3-i) + (6-i)5$$

$$= -28 + 6i + 30 - 5i = 2 + i$$

$$A = \begin{pmatrix} 2 & 1+i & 2+i \\ 1-i & 3 & 3-i \\ 2-i & 3+i & 6 \end{pmatrix}$$

$$\begin{pmatrix} 2+i \\ 3-i \\ 5 \end{pmatrix} =$$

$$= \begin{pmatrix} 4+2i + 4+2i + 10+5i \\ 3-i + 9-3i + 15-5i \\ 5 + 10 + 30 \end{pmatrix}$$

$18+9i$
 $27-9i$
 45

Normalise: $5 + 10 + 25 = 40$

principal axes:

$$\begin{pmatrix} (2+i)/\sqrt{40} \\ (3-i)/\sqrt{40} \\ 5/\sqrt{40} \end{pmatrix}$$

$$Q = \begin{pmatrix} (-1-i)/\sqrt{49} & (-2-i)/\sqrt{6} & (2+i)/\sqrt{40} \\ 6/\sqrt{49} & 0 & (3-i)/\sqrt{40} \\ (3-i)/\sqrt{49} & 1/\sqrt{6} & 5/\sqrt{40} \end{pmatrix}$$

An example of a unitary matrix.

$$Q^{-1} = \bar{Q}^T$$

Q invertible
↓

Q: similarity of matrix vs. congruence $Q^T A Q = B$

Similarity of matrices: A similar to B \iff A, B square matrices of same size

$$(\exists P) \quad P^{-1}AP = B$$

Over \mathbb{C} , any matrix is similar to a matrix in Jordan form. The number, sizes and eigenvalues of blocks are uniquely determined.

For a symmetric ^{real} matrix A , we know that A is diagonalizable, eigenvalues real, eigenspaces orthogonal.

\therefore There exist an orthogonal matrix Q such that

$$Q^{-1} A Q \text{ is diagonal}$$

means:

$$Q^{-1} = Q^T$$

$$Q^T A Q$$



this is for centering quadratic
vector x

$$x^T A x = 1$$

If x is a column of Q then it is a principal axis. If we change to the basis of columns of A (orthonormal basis), then the quadric in coordinates z_1, \dots, z_n of this basis, becomes:

$$\lambda_1 z_1^2 + \dots + \lambda_n z_n^2 = 1$$

Q: Algebraic vs. geometric multiplicity of eigenvalues.

Let A be a square matrix.

$$\det(xI - A)$$

the characteristic
polynomial $\chi_A(x)$

} The roots of $\chi_A(x)$ are
eigenvalues. The multiplicity
of the root (# of times a
linear factor repeats) =
algebraic multiplicity
of the eigenvalue.

Suppose now λ is an eigenvalue. Solution space of

$$\delta \mathbb{I} - A$$

is the eigenspace. (we should work over \mathbb{C}).

The dimension of the eigenspace is the geometric multiplicity.

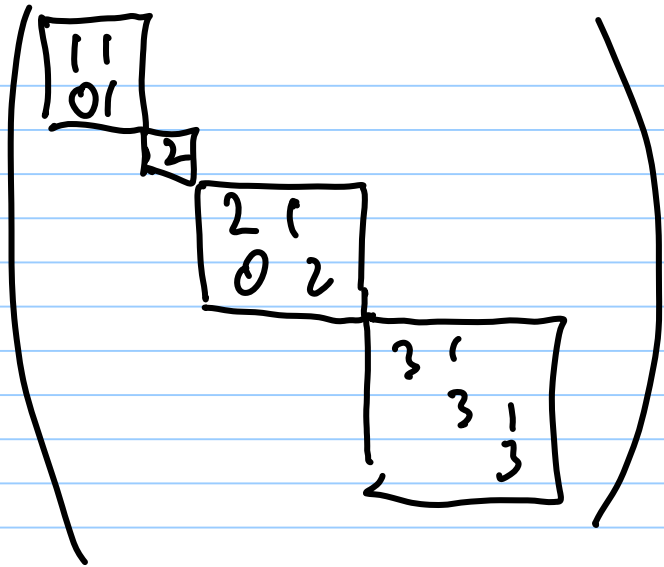
$$1 \leq \underbrace{\text{geometric multiplicity}} \leq \underbrace{\text{algebraic multiplicity}}$$



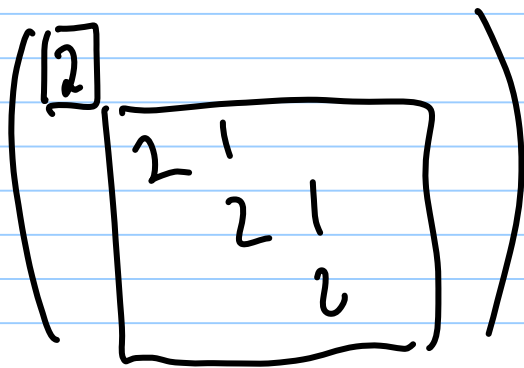
the number
of Jordan blocks
of that eigenvalue



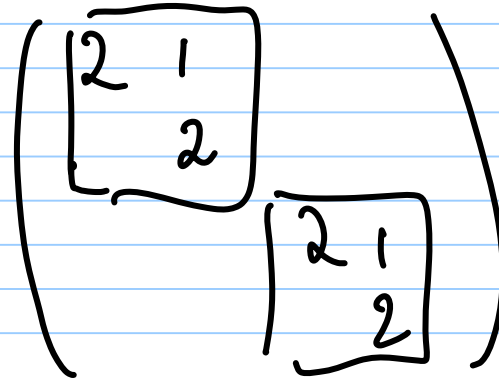
\sum sizes of the Jordan
blocks of that eigenvalue



| Eigenvalue | algebraic multiplicity | geometric multiplicity |
|------------|------------------------|------------------------|
| 1 | 2 | 1 |
| 2 | 3 | 2 |
| 3 | 3 | 1 |



not
similar



alg. multiplicity of 2 : 4

geom. multiplicity : 2

Q: Classify the quadric

$$3x^2 + 3y^2 + 10z^2 + 6xz + 6yz = 1$$

$$(x \ y \ z) \underbrace{\begin{pmatrix} 3 & 0 & 3 \\ 0 & 3 & 3 \\ 3 & 3 & 10 \end{pmatrix}}_A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1$$

Eigenvalues : 1, 3, 12 . If $Q^T A Q = \begin{pmatrix} 1 & & \\ & 3 & \\ & & 12 \end{pmatrix}$

and t, u, v are coordinates with respect to the orthonormal basis given by the columns of Q then the quadric becomes

$$t^2 + 3u^2 + 12v^2 = 1 \quad \text{ellipsoid}$$

Q: Example of sizes of Jordan blocks
Example: finding eigenvalue 3 of algebraic multiplicity 5
algebraic multiplicity Add up to size of matrix 5×5

| k | dim Ker $(3I - A)^k$ |
|-----|----------------------|
| 1 | 3 |
| 2 | 4 |
| 3 | 5 |
| 4 | 5 |

| | | | |
|---|---|---|----------------------|
| 0 | 3 | 2 | ← 2 blocks of size 1 |
| 3 | 1 | 0 | |
| 4 | 1 | 1 | |
| 5 | 0 | 1 | ← 1 block of size 3 |
| 5 | | | |

