MATH 417 11/25/2015

Note Title 11/25/2015

$$A = \begin{pmatrix} 2 & 1+i & 2+i \\ 1-i & 3 & 3-i \\ 2-i & 3+i & 6 \end{pmatrix}$$

Q: Finish the 3×3 Hermitian matrix example from 11/13

$\lambda = 9$

$$\begin{pmatrix} 7 & -1-i & -2-i \\ -1+i & 6 & -3+i \\ -2+i & -3-i & 3 \end{pmatrix} \sim \begin{pmatrix} 7 & -1-i & -2-i \\ 1 & 9+i & -6+i \\ -2+i & -3-i & 3 \end{pmatrix}$$

-1

$$\begin{pmatrix} 1 & 9+i & -6+i \\ 7 & -1-i & -2-i \\ -2+i & -3-i & 3 \end{pmatrix} \begin{array}{l} \\ -7 \\ 2-i \end{array} \sim \begin{pmatrix} 1 & 9+i & -6+i \\ 0 & -64-8i & 40-8i \\ 0 & 16-8i & -8+8i \end{pmatrix} \sim$$

$$(9+i)(2-i) = 19-7i$$

$$(-6+i)(2-i) = -11+8i$$

$$\frac{(5-i)(-1-i)}{(-1+i)(-1-i)} = \frac{-6-4i}{2} = -3-2i$$

$$\sim \begin{pmatrix} 1 & 9+i & -6+i \\ 0 & -8-i & 5-i \\ 0 & 2-i & -1+i \end{pmatrix} \begin{array}{l} \\ \\ 3+2i \end{array}$$

$$(3+2i)(2-i) = 8+i$$

$$\sim \begin{pmatrix} 1 & 9+i & -6+i \\ 0 & 2-i & -1+i \\ 0 & 0 & 0 \end{pmatrix} 2 \frac{1}{2-i}$$

$$\frac{(-1+i)(2+i)}{(2-i)(2+i)} = \frac{-3+i}{5}$$

$$\sim \begin{pmatrix} 1 & 9+i & -6+i \\ 0 & 1 & -\frac{3+i}{5} \\ 0 & 0 & \underset{\underset{5}{\uparrow}}{0} \end{pmatrix}$$

$$\begin{pmatrix} 2+i \\ 3-i \\ 5 \end{pmatrix}$$

$$(-9-i)(3-i) + (6-i)5$$

$$= -28 + 6i + 30 - 5i = 2 + i$$

$$A = \begin{pmatrix} 2 & 1+i & 2+i \\ 1-i & 3 & 3-i \\ 2-i & 3+i & 6 \end{pmatrix} \qquad \begin{pmatrix} 2+i \\ 3-i \\ 5 \end{pmatrix} =$$

$18 + 9i$

$27 - 9i$

$$= \begin{pmatrix} 4+2i + 4+2i + 10+5i \\ 3-i + 9-3i + 15-5i \\ 5 + 10 + 30 \end{pmatrix}$$

45

✓

Normalise: $5 + 10 + 25 = 40$

$$\boxed{\text{principal axis:} \quad \begin{pmatrix} (2+i)/\sqrt{40} \\ (3-i)/\sqrt{40} \\ 5/\sqrt{40} \end{pmatrix}}$$

$$Q = \begin{pmatrix} (-1-i)/\sqrt{48} & (-2-i)/\sqrt{6} & (2+i)/\sqrt{40} \\ 6/\sqrt{48} & 0 & (3-i)/\sqrt{40} \\ (3-i)/\sqrt{48} & 1/\sqrt{6} & 5/\sqrt{40} \end{pmatrix}$$

An example of a <u>unitary matrix</u>.

$$Q^{-1} = \bar{Q}^{T}.$$

Q invertible

Q: similarity of matrix vs. <u>congruence</u> $Q^T A Q = B$

Similarity of matrices: A similar to B ← square matrices of same size

$$(\exists P) \qquad P^{-1}AP = B$$

Over $\mathbb{C}$, any matrix is similar to a matrix in Jordan form. The number, sizes and eigenvalues of blocks are uniquely determined.

For a symmetric real matrix A, we know that A is diagonalizable, eigenvalues real, eigenspaces orthogonal.

$\therefore$ There exist an orthogonal matrix Q such that

$$Q^{-1} A Q \text{ is diagonal}$$

means: $Q^{-1} = Q^T$

$$\begin{matrix} Q^{-1} A Q \\ \shortparallel \\ Q^T A Q \end{matrix}$$

This is for centering quadrics.

vector x

$$x^T A x = 1$$

If x is a column of Q then it $\quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

is a principal axis. If we change to the basis of columns of Q (orthonormal basis), then the quadric in coordinates $z_1, \dots z_n$ of this basis, becomes:

$$\lambda_1 z_1^2 + \dots + \lambda_n z_n^2 = 1$$

Q: Algebraic vs. geometric multiplicity of eigenvalues.

Let A be a square matrix.

$\det(xI - A)$
"
the characteristic polynomial $\chi_A(x)$

The roots of $\chi_A(x)$ are eigenvalues. The multiplicity of the root (# of times a linear factor repeats) = algebraic multiplicity of the eigenvalue.

Suppose now λ is an eigenvalue. Solution space of

$$\lambda I - A$$

is the <u>eigenspace</u>. (we should work over $\mathbb{C}$).

The dimension of the eigenspace is the <u>geometric multiplicity</u>.

$1 \leq$ <u>geometric multiplicity</u> $\leq$ <u>algebraic multiplicity</u>

$\uparrow$ the number of Jordan blocks of that eigenvalue

$\uparrow$ Σ sizes of the Jordan blocks of that eigenvalue

$$\begin{pmatrix} 1 & 1 & & & & & & \\ 0 & 1 & & & & & & \\ & & 2 & & & & & \\ & & & 2 & 1 & & & \\ & & & 0 & 2 & & & \\ & & & & & 3 & 1 & \\ & & & & & & 3 & 1 \\ & & & & & & & 3 \end{pmatrix}$$

Eigenvalue	algebraic multiplicity	geometric multiplicity
1	2	1
2	3	2
3	3	1

$$\begin{pmatrix} 2 & & & \\ & 2 & 1 & \\ & & 2 & 1 \\ & & & 2 \end{pmatrix} \quad \underline{\text{not}} \quad \underline{\text{similar}} \quad \begin{pmatrix} 2 & 1 & & \\ & 2 & & \\ & & 2 & 1 \\ & & & 2 \end{pmatrix}$$

alg. multiplicity of 2 : 4

geometric multiplicity : 2

Q : Classify the quadric

$$3x^2 + 3y^2 + 10z^2 + 6xz + 6yz = 1$$

$$(x\ y\ z) \underbrace{\begin{pmatrix} 3 & 0 & 3 \\ 0 & 3 & 3 \\ 3 & 3 & 10 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1$$

Eigenvalues : 1, 3, 12 . If $Q^T A Q = \begin{pmatrix} 1 & & \\ & 3 & \\ & & 12 \end{pmatrix}$

and t, u, v are coordinates with respect to the orthonormal basis given by the columns of Q then the product becomes

$$t^2 + 3u^2 + 12v^2 = 1 \qquad \underline{\text{ellipsoid}}$$

Q: Example of sizes of Jordan blocks

<u>Example</u>: Single eigenvalue 3 of $\underbrace{\text{algebraic multiplicity}}$ 5

alg. multiplicities Add up to size of matrix 5×5

k	$\dim \operatorname{Ker}(3I-A)^k$
1	3
2	4
3	5
4	5

$$\begin{array}{cccl}
0 & & & \\
 & 3 & & \\
3 & & 2 & \leftarrow \text{2 blocks of size 1} \\
 & 1 & & \\
4 & & 0 & \\
 & 1 & & \\
5 & & 1 & \leftarrow \text{1 block of size 3} \\
 & 0 & & \\
5 & & &
\end{array}$$

$$\begin{pmatrix}
\boxed{3} & & & & \\
 & \boxed{3} & & & \\
 & & 3 & 1 & \\
 & & & 3 & 1 \\
 & & & & 3
\end{pmatrix}$$