MATH 417 11/30/2015

Note Title 11/30/2015

Differential forms

(what grad, curl, div
∇ $\nabla\times$ $\nabla\cdot$
become in higher dimension)

Remember the determinant formula:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

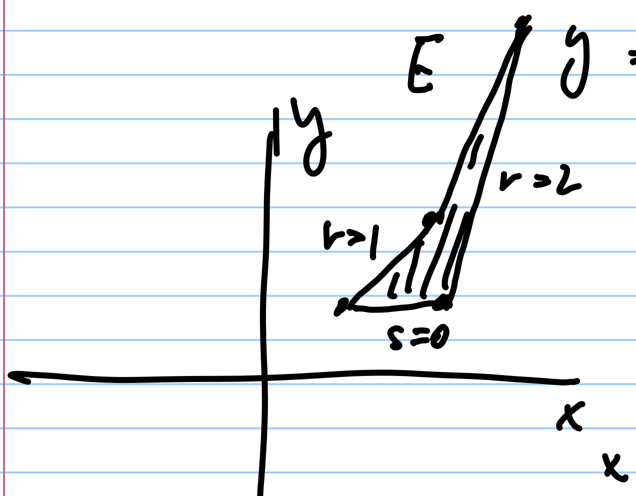
$+1$ σ even
-1 σ odd

$$\det A = \sum_{\substack{\sigma \text{ permutations} \\ \text{on } \{1,\dots,n\}}} \operatorname{sign}(\sigma)\, a_{1\sigma(1)} \cdot \ldots \cdot a_{n\sigma(n)}$$

Remember from calculus III how this relates to integration: Look at a double integral and substitution.

$$x = r + s \qquad 1 \le r \le 2$$

$$y = r \cdot s + 1 \qquad 0 \le s \le 1$$

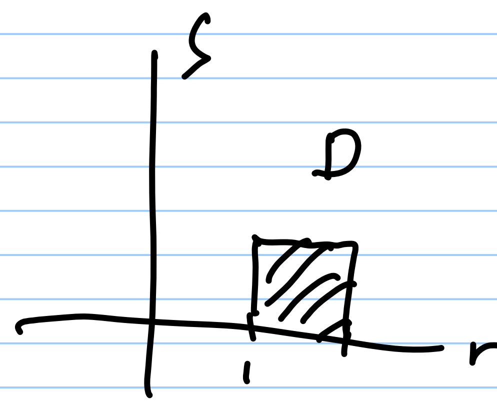


$r = 2$

$x = 2 + s$

$y = 2s + 1$

$r = 2$, $s = 1$: $x = 3$, $y = 5$



$r = 1$

$x = 1 + s$

$y = 1 + s$

$s = 0$ $\quad$ $s = 1$ $\quad$ $r = 2$ $\quad$ $x = 2$

$x = r$ $\quad$ $x = r+1$ $\quad$ $s = 0$ $\quad$ $y = 1$

$y = r+1$ $\quad$ $y = 2r+1$

Substitution in 2 variables lets us integrate a function of $f(x,y)$ over the region E by thinking of x, y as a function of r, s:

evaluate by r, s

$f(x,y) = x+y$

$$\int_E f(x,y)\,dx\,dy = \int_D f(x,y) \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{pmatrix} dr\,ds$$

$$\int_E (x+y)\,dx\,dy = \int_D (r+s+rs+1)\det\begin{pmatrix}1 & 1\\ s & r\end{pmatrix} dr\,ds =$$

$$= \int_D (r+s+rs+1)(r-s)\,dr\,ds$$

$$= \cdots$$

$$x = r+s$$

$$y = r\cdot s + 1$$

$$\frac{\partial x}{\partial r} = \frac{\partial (r+s)}{\partial r} = 1 \qquad \frac{\partial x}{\partial s} = 1$$

$$\frac{\partial y}{\partial r} = s \qquad \frac{\partial y}{\partial s} = r$$

Think of this as the chain rule:

$$dx = \frac{\partial x}{\partial r}\,dr + \frac{\partial x}{\partial s}\,ds$$

$$dy = \frac{\partial y}{\partial r}\,dr + \frac{\partial y}{\partial s}\,ds$$

$$\int_E f(x,y)\,dx\,dy = \int_D f(x,y)\left(\frac{\partial x}{\partial r}\,dr + \frac{\partial x}{\partial s}\,ds\right)\left(\frac{\partial y}{\partial r}\,dr + \frac{\partial y}{\partial s}\,ds\right)$$

functions of r,s

$$\frac{\partial x}{\partial r}\frac{\partial y}{\partial r}\,dr\,dr + \frac{\partial x}{\partial r}\frac{\partial y}{\partial s}\,dr\,ds + \frac{\partial x}{\partial s}\frac{\partial y}{\partial r}\,ds\,dr + \frac{\partial x}{\partial s}\frac{\partial y}{\partial s}\,ds\,ds$$

But I want is: $\left(\frac{\partial x}{\partial r}\frac{\partial y}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial y}{\partial r}\right) dr\,ds$

To make this equal: $dr\,dr = 0$ $\quad dr\,ds = -ds\,dr$

$ds\,ds = 0$

anticommutativity

if $r = s$, anticommutativity implies

$dr\,dr = -dr\,dr$

$\therefore 2\,dr\,dr = 0 \Rightarrow dr\,dr = 0.$

Sometimes we denote the product of differentials by the symbol $\wedge$:

$$\int_E f(x,y)\, dx \wedge dy = \int_D \underbrace{f(x,y)}_{\substack{\text{as a}\\ \text{function}\\ \text{of } r,s}} \left(\frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial s} ds\right) \wedge \left(\frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial s} ds\right).$$

With this formalism, we can explain all

vector calculus in all dimensions.

But let us first get some practice in pure algebra.

Let V be a vector space over $\mathbb{R}$ with basis denoted by $e_1, \dots e_n$. (you are welcome to think $V = \mathbb{R}^n$).

Now introduce the "$\wedge$" - product, enforcing
wedge
the rule of <u>anticommutation</u>.

$$e_i \wedge e_j = - e_j \wedge e_i$$

1 $\Lambda^0 \mathbb{R}^4$ $\boxed{1}$ $(e_i \wedge e_i = 0)$. $\boxed{\text{EXTERIOR ALGEBRA}}$

Consider $n = 4$: What basis element do I get?

4 $\boxed{e_1, e_2, e_3, e_4} \leftarrow \Lambda^1 \mathbb{R}^4$ $\binom{4}{1}$

$e_1 \wedge e_2 = - e_2 \wedge e_1$

6 $\boxed{e_1 \wedge e_2, \ e_1 \wedge e_3, \ e_1 \wedge e_4, \ e_2 \wedge e_3, \ e_2 \wedge e_4, \ e_3 \wedge e_4} \leftarrow \Lambda^2 \mathbb{R}^4$ $\binom{4}{2}$

$(1 < 2 \quad 1 < 3 \quad 1 < 4 \quad 2 < 3 \quad 2 < 4 \quad 3 < 4)$

4 $\boxed{e_1 \wedge e_2 \wedge e_3, \ e_1 \wedge e_2 \wedge e_4, \ e_1 \wedge e_3 \wedge e_4, \ e_2 \wedge e_3 \wedge e_4} \leftarrow \Lambda^3 \mathbb{R}^4$ $\binom{4}{3}$

1 $\boxed{e_1 \wedge e_2 \wedge e_3 \wedge e_4} \leftarrow \Lambda^4 \mathbb{R}^4$ $\binom{4}{4}$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Example: Simplify (express in the above basis):

$$(e_1 + 2e_2) \wedge (3e_2 + e_4) + e_3 \wedge (-e_1) =$$

$$= \underbrace{e_1 \wedge 3e_2}_{3e_1 \wedge e_2} + e_1 \wedge e_4 + \underbrace{2e_2 \wedge 3e_2}_{0} + 2e_2 \wedge e_4 \underbrace{- e_3 \wedge e_1}_{+e_1 \wedge e_3} =$$

$$= \underbrace{3e_1 \wedge e_2 + e_1 \wedge e_4 + 2e_2 \wedge e_4 + e_1 \wedge e_3}$$

Example:

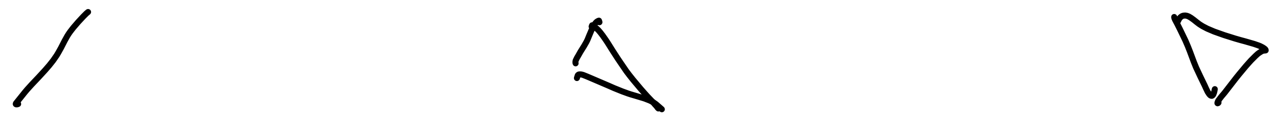
$$\det\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \overset{+1}{1\cdot 1\cdot 1} + \overset{9}{1\cdot 3\cdot 3} + \overset{12}{2\cdot 2\cdot 3} - \overset{6}{2\cdot 1\cdot 2} - \overset{2}{2\cdot 1\cdot 1} - \overset{9}{1\cdot 3\cdot 3}$$

$$= 5$$

$$(1e_1 + 1e_2 + 2e_3) \wedge (2e_1 + 1e_2 + 3e_3) \wedge (3e_1 + 3e_2 + e_3) =$$

$$= 1e_1 \wedge 1e_2 \wedge 1e_3 \; + \; 1e_2 \wedge 3e_3 \wedge 3e_1 \; + \; 2e_3 \wedge 2e_1 \wedge 3e_2$$

$$+ \; 2e_3 \wedge 1e_2 \wedge 3e_1 \; + \; 1e_2 \wedge 2e_1 \wedge 1e_3 \; + \; 1e_1 \wedge 3e_3 \wedge 3e_2$$



$$= (1\cdot1\cdot1 + 1\cdot3\cdot3 + 2\cdot2\cdot3 - 2\cdot1\cdot3 - 1\cdot2\cdot1 - 1\cdot3\cdot3)\, e_1\wedge e_2\wedge e_3 =$$

$$= 5\, e_1\wedge e_2\wedge e_3$$

$\therefore$ The wedge product calculates the determinant.

HW:

(1) Write a basis of $\Lambda^3 \mathbb{R}^5$ (use the basis e_1, e_2, e_3, e_4, e_5 of $\mathbb{R}^5$).

(2) Simplify in the exterior algebra over $\mathbb{R}^4$:

$$(e_1 - e_2 + e_4) \wedge (2e_1 + 3e_2 + e_3)$$

(use basis $e_i \wedge e_j$, $i < j$).

Due on Wed Dec 2.