Example: Consider the surface E in $\mathbb{R}^4$ parametrized by

orientation: $dr \wedge ds$

$x = r+1$

$y = s$ $\qquad 0 \le r \le 1$

$z = rs$ $\qquad 0 \le s \le 1$

$t = r+s$

$\in \Lambda^2 V$ $\qquad$ V has basis dx, dy, dz, dt

function

$$\int_E \underbrace{x\, dy \wedge dt + y\, dx \wedge dy + dy \wedge dz}_{\text{differential 2-form}} =$$

$$= \int_{\substack{0 \le r \le 1 \\ 0 \le s \le 1}} (r+1)\, ds \wedge (dr + ds) + s\, dr \wedge ds + ds \wedge (s\, dr + r\, ds) =$$

$$\begin{aligned} dx &= dr \\ dy &= ds \\ dz &= s\, dr + r\, ds \\ dt &= dr + ds \end{aligned}$$

$$= \int_{\substack{0 \le r \le 1 \\ 0 \le s \le 1}} -(r+1)\, dr \wedge ds + s\, dr \wedge ds - s\, dr \wedge ds$$

$$= \int_{\substack{0 \le r \le 1 \\ 0 \le s \le 1}} -(r+1)\, dr \wedge ds = \int_{s=0}^{1} \left(\overbrace{\int_{r=0}^{1} -(r+1)\, dr}^{\left[-\frac{r^2}{2} - r\right]_0^1 = -\frac{3}{2}} \right) ds =$$

$$= \int_{s=0}^{1} -\frac{3}{2}\, ds = \underline{-\frac{3}{2}}$$

If $\mathbb{R}^n$, variables $x_1, \dots, x_n$. Let V be a vector space with basis $dx_1, \dots, dx_n$. A differential k-form is a sum of terms of the form
(function on $\mathbb{R}^n$) $\cdot$ element of $\Lambda^k V$.

A k-form can be integrated over a k-dimensional parametrized region in $\mathbb{R}^n$. The result only depends on parametrization up to sign (orientation)

↑ ordering the k parameters.

<u>Example</u>: Consider the 3-dimensional region E in $\mathbb{R}^4$

$$\left.\begin{array}{ll} x = r\cos s\cos u & 0 \le r \le 1 \\ y = r\cos s\sin u & 0 \le s \le \pi \\ z = r\sin s & 0 \le u \le 2\pi \\ t = r & \end{array}\right\} D$$

orientation: $dr \wedge ds \wedge du$

Integrate

$$x = r\cos s\cos u$$
$$y = r\cos s\sin u$$
$$z = r\sin s$$
$$t = r$$

$$dx = \cos s\cos u\,dr - r\sin s\cos u\,ds - r\cos s\sin u\,du$$

$$dy = \cos s\sin u\,dr - r\sin s\sin u\,ds + r\cos s\cos u\,du$$

$$dz = \sin s\,dr + r\cos s\,ds$$
$$dt = dr$$

$$\int_E t\, dx \wedge dy \wedge dz = \int_D r\,(\cos s \cos u\, dr - r \sin s \cos u\, ds - r \cos s \sin u\, du) \wedge$$

$$(\cos s \sin u\, dr - r \sin s \sin u\, ds + r \cos s \cos u\, du) \wedge$$

$$(\sin s\, dr + r \cos s\, ds) =$$

$$= \int_D r\,(-r \sin s \cos u\, ds)(r \cos s \cos u\, du) +$$

$$r\,(-r \sin s \sin u\, ds)(r \cos s \sin u\, du) \sin s\, dr$$

$$+\, r\,(\cos s \cos u\, dr)(r \cos s \cos u\, du)$$

$+ r(\cos s \sin u\, dr)(r \cos s \sin u\, du) \quad r \cos s\, ds$

$$= \int_D r^3 (-\sin s \cos s)\, dr\, ds\, du - r^2 \cos^2 s\, dr\, ds\, du =$$

$$= \underbrace{\int -\frac{1}{2} r^3 \sin 2s\, dr\, ds\, du}_{0} -$$

$0 \leq r \leq 1$
$0 \leq s \leq \pi$
$0 \leq u \leq 2\pi$

$$- \frac{1}{2} \int_{\substack{0 \leq r \leq 1 \\ 0 \leq s \leq \pi,\ 0 \leq u \leq 2\pi}} r^3\, dr\, ds\, du =$$

$\sin 2s = 2 \sin s \cos s$

$\cos 2s = 2\cos^2 s - 1$

$\cos^2 s = \frac{1}{2} + \frac{1}{2} \cos 2s$

$$\left.\begin{array}{l} 0 \leq r \leq 1 \\ 0 \leq s \leq \pi \\ 0 \leq u \leq 2\pi \end{array}\right\} D$$

$$= -2\pi^2 \cdot \frac{1}{2} \cdot \frac{1}{4} r^4 \Big|_0^1 = -\frac{\pi^2}{4}.$$

HW: (1) Let E be the 2-dimensional region in $\mathbb{R}^4$ given by

$$x = r^2 \qquad 0 \le r \le 1$$
$$y = r - s \qquad 0 \le s \le 1$$
$$z = s \qquad \text{orientation } dr \wedge ds$$
$$t = 2s$$

Calculate

$$\int_E z\,dx\wedge dy + x\,dz\wedge dt + dy\wedge dz$$

2. Let E be the 3-dimensional region in $\mathbb{R}^4$ given by

$$\begin{aligned} x &= r+s+u \\ y &= s+u \\ z &= 2r+s \\ t &= r+2u \end{aligned} \qquad \begin{aligned} &0\le r\le 1 \\ &0\le s\le 1 \\ &0\le u\le 1 \end{aligned}$$

orientation: $dr\wedge ds\wedge du$

Calculate
$$\int_E x\,dy\wedge dz\wedge dt + z\,dx\wedge dz\wedge dt$$