Note Title 12/9/2015

MATH 417 12/9/2015

Exam: (1) Eigenvalues, eigenvectors, eigenspaces
algebraic and geometric multiplicity

(2) Complex numbers, complex arithmetics,
$(+, \cdot, \div, -)$
complex eigenvalues and eigenspaces

(3) Jordan canonical form $\Leftarrow$ generalised
eigenspace dimensions

4. Symmetric matrices – principal axes

5. Gramm-Schmidt orthogonalisation, ~~orthogonal reduced echelon form~~, orthogonal matrices

6. orthogonal Projection formula, Gramm matrix

7. singular values ($\sqrt{\text{eigenvalues of the Gramm matrix}}$)

Review:

① Find the eigenvalues an eigenspaces of the matrix

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 7 \end{pmatrix}.$$

and their algebraic and geometric multiplicities.

Solution:

$$\chi_A(x) = \det\begin{pmatrix} x-2 & -3 \\ -2 & x-7 \end{pmatrix} = (x-2)(x-7) - 6 =$$

characteristic polynomial

$$= x^2 - 9x + 8 =$$

$$= (x-1)(x-8)$$

how many times the root occurs in factoring the polynomial

Eigenvalues : 1, 8

Both have algebraic and geometric multiplicity 1.

dimension of eigenspace

$1 \leq$ geometric multiplicity $\leq$ algebraic multiplicity

$A = \begin{pmatrix} 2 & 3 \\ 2 & 7 \end{pmatrix}$

$\boxed{\lambda = 1}$ $\quad \lambda I - A = \begin{pmatrix} -1 & -3 \\ -2 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}$

$\left\langle \begin{pmatrix} -3 \\ 1 \end{pmatrix} \right\rangle$

eigenspace spanned by $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$\boxed{\lambda = 8}$ $\quad \lambda I - A = \begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1/2 \\ 0 & 0 \end{pmatrix}$

eigenspace spanned by $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\rightarrow \left\langle \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\rangle$.

Find the eigenvalues, eigenspaces, algebraic & geometric multiplicity of

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

Solution:

$$\det(Ix - A) = \begin{pmatrix} x-2 & -1 & -1 \\ 0 & x-2 & -1 \\ 0 & 0 & x-3 \end{pmatrix} = (x-2)(x-2)(x-3)$$

Eigenvalues: $\boxed{\begin{matrix} 2 \\ 2 \end{matrix}}, \boxed{\begin{matrix} 3 \\ 1 \end{matrix}}$

Alg. multiplicities: 2, 1

$\lambda = 2$

$$(2I - A) = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Eigenspace : $\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$

Geometric multiplicity of 2 : 1.

$\lambda = 3$ Geometric multiplicity of 3 : 1

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \qquad (3I - A) = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Eigenspace: $\left\langle \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\rangle$

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2 Complex case:

$$(3 - 2i) - (-2 - 6i) =$$

$$= 5 + 4i$$

$$(3-2i)\cdot(-2-6i) = -6-18i+4i-12 =$$

$$= -18-14i$$

rationalize denominator; complex conjugate

$$\frac{3-2i}{-2-6i} = \frac{(3-2i)(-2+6i)}{(-2-6i)(-2+6i)} =$$

$$= \frac{-6+18i+4i+12}{4+36} =$$

$$(a-bi)(a+bi)$$
$$= a^2+b^2$$

$$= \frac{6 + 22i}{40} = \frac{3}{20} + \frac{11}{20} i$$

$$\begin{array}{r} 36 \\ +484 \\ \hline 520 \end{array} = 13 \cdot 40$$

<u>Check:</u> $\frac{9+4}{4+36} = \frac{13}{40}$ ✓

Find the eigenvalues of

$$A = \begin{pmatrix} 1 & -1 \\ +1 & +1 \end{pmatrix}$$

<u>Solution:</u>

$$\det(x\underline{I} - A) = \det \begin{pmatrix} x-1 & 1 \\ -1 & x-1 \end{pmatrix} = x^2 - 2x + 2$$

$$\frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

Eigenspaces : $(1+i)\underline{I} - \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \sim \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}$

$\lambda = 1+i$

$$\left\langle \begin{pmatrix} i \\ 1 \end{pmatrix} \right\rangle$$

$\lambda = 1 - i$ $\quad \left\langle \begin{pmatrix} -i \\ 1 \end{pmatrix} \right\rangle$

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3 Jordan canonical form from <u>generalized</u> (!)
<u>eigenspace</u> dimensions:

solution spaces of $(\lambda I - A)^k$

Problem: Let A be a 10×10 with eigenvalues 2 and 4. The dimensions of generalised eigenspaces are: $\dim(\lambda I - A)^k$

$k \backslash \lambda$	2	4
1	3	1
2	6	2
3	7	2
4	8	2
5	8	2

$\boxed{\lambda = 2}$

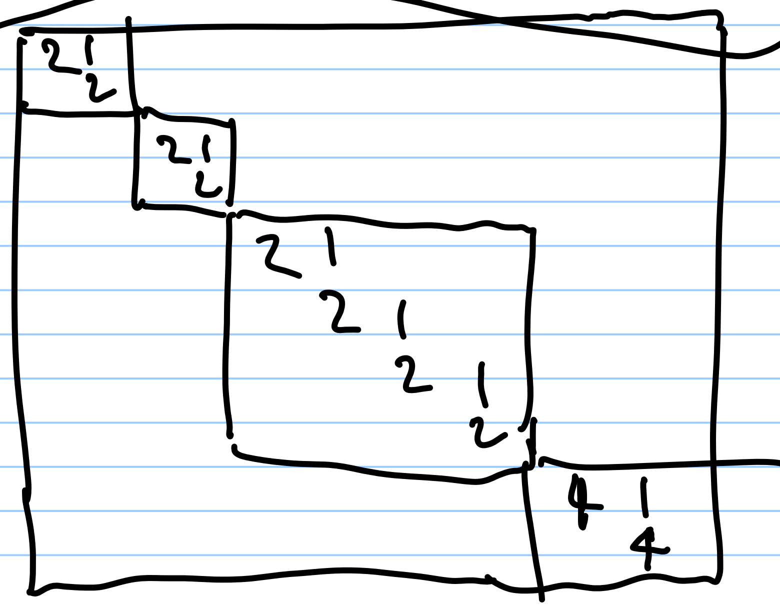
0			
	3		
3		0	← size 1
	3		
6		2	← 2 blocks size 2
	1		
7		0	
	1		
8		1	← 1 block size 4
	0		
8			

start with 0

$\boxed{\lambda = 4}$

0		
	1	
1		0
	1	
2		1
	0	
2		

1 block size 2



Always 2 differences

Another example $A = 12 \times 12$ matrix

eigenvalues	i	$2-i$
$\dim(\lambda I - A)^k$		
$k = 1$	3	3
2	5	4
3	6	5
4	6	6
5	6	6
6	6	6
7	6	6

Find JNF

i			
0			
	3		
3		1	size 1
	2		
5		1	size 2
	1		
6		1	size 3
	0		
6			

$2-i$			
0			
	3		
3		2	size 1
	1		
4		0	
	1		
5		0	
	1		
6		1	size 4
	0		
6			

$$
\begin{pmatrix}
i & & & & & & & & & & & \\
 & i & 1 & & & & & & & & & \\
 & & i & & & & & & & & & \\
 & & & i & 1 & & & & & & & \\
 & & & & i & 1 & & & & & & \\
 & & & & & i & & & & & & \\
 & & & & & & 2-i & & & & & \\
 & & & & & & & 2-i & & & & \\
 & & & & & & & & 2-i & 1 & & \\
 & & & & & & & & & 2-i & 1 & \\
 & & & & & & & & & & 2-i & 1 \\
 & & & & & & & & & & & 2-i
\end{pmatrix}
$$